HEAVY MOVING AVERAGES AND THEIR APPLICATION IN ECONOMETRIC FORECASTING

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Abstract. This paper presents the heavy ordered weighted moving average (HOWMA) operator. It is an aggregation operator that uses the main characteristics of two well-known techniques: the heavy OWA and the moving averages. Therefore, this operator provides a parameterized family of aggregation operators from the minimum to the total operator and includes the OWA operator as a special case. It uses a heavy weighting vector in the moving average formulation and it represents the information available and the knowledge of the decision maker about the future scenarios of the phenomenon, according to his attitudinal character. Some of the main properties of this operator are studied, including a wide range of families of HOWMA operators such as the heavy moving average and heavy weighted moving average operators. The HOWMA operator is also extended using generalized and quasi-arithmetic means. An example concerning the foreign exchange rate between US dollars (USD) and Mexican pesos (MXN) is also presented.

Keywords: Heavy moving averages, OWA operator, HOWA operator, econometric forecasting, exchange rate, economy.

JEL Classification: C43, C44, C53, C58.

Introduction

The ordered weighted averaging (OWA) operator introduced by Yager (1988) is an aggregation method whose use has become common (Beliakov et al. 2007; Calvo et al. 2002). Since the introduction of this operator, it has been used for many applications (Kacprzyk & Zadrożny, 2009; Merigó & Gil-Lafuente, 2010; Zhou & Chen, 2013) and in a wide range of frameworks (Belles et al. 2013; Merigó, 2010; Xu & Da, 2003).

This paper focuses on the heavy OWA (HOWA) and the moving average (MA). The HOWA operator (Yager, 2002) is an operator that provides a wide class of aggregation operators, for which the sum of the weights

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can range from 1 to ∞. This operator has a parameterized family of aggregation operators, which includes the minimum, the OWA and the total operators. The HOWA operator has been studied by using fuzzy measures (Yager, 2003), fuzzy numbers (Merigó et al. 2014a) and distance measures (Merigó et al. 2014b). Additional extensions have been developed by Merigó and Casanovas (2010; 2011a).

The moving average is a usual average that moves towards some part of the whole sample. Its main advantage is that it permits the consideration of the results of some part of the sample and allows one to make comparisons when modifying a selected part of the sample. It is used to solve time-series smoothing problems, and has been applied in economics and statistics, because it permits the forecasting of future results using historical data (Evans, 2002).

The aim of this paper is to analyze the use of moving averages with heavy aggregation operators in econometric forecasting. The main advantage of this operator is that it unifies the historical data with the information available and the knowledge of the decision maker, allowing for improved forecasting of the future in economics and statistics.

The main concepts of this new extension have been developed, along with a range of particular cases, including the heavy moving average (HMA), heavy weighted moving average (HWMA), arithmetic mean, median and weighted media HOWMA.

A further generalization of the HOWMA operator has also been developed using quasi-arithmetic means (Merigó & Gil-Lafuente, 2009; Merigó & Casanovas, 2011b), forming the quasi-arithmetic HOWMA (Quasi-HOWMA) operator. Its main advantage is that it includes a wide range of aggregation operators, including the HOWMA operator, quadratic aggregations and geometrics aggregations. Thus, it can represent the information in a more complete way, taking into account a wide range of complexities.

An application of the new approach in economic forecasting problems regarding the USD/MXN exchange rate is also developed. We use information from 1994-2014 to generate the econometric models and use the HOWMA operator to forecast the future values of the variables included in the models. It is seen that with the application of this operator, it is possible to decrease the forecasting error and generate different scenarios in a simpler way.

The remainder of the paper is organized as follows. In Section 1, we review the moving averages and some aggregation operators. Section 2 introduces the HOWMA operator, and Section 3 develops the generalized HOWMA operator. Section 4 explains the steps for using heavy aggregation operators in econometric forecasting, and Section 5 presents an application of the HOWMA operator in a USD/MXN exchange rate 2015 forecast. Section 6 summarizes the main conclusions of the paper.

1. Preliminaries

In this section, we briefly review some basic concepts to be used throughout the paper. We analyze the OWA operator, the heavy aggregation operators, the moving averages and the generalized aggregation operators.

1.1 OWA operator

The OWA operator was introduced by Yager (1988) and since then has been used in many applications (Emrouznejad and Marra (2014), Yager & Kacprzyk (1997), Yager et al. (2011)). In the following, we provide a definition of the OWA operator as introduced by Yager (1988).

Definition 1. An OWA operator of dimension n is an application $F: R^n \rightarrow R$ with an associated weight vector $w = [w_1, w_2, \dots, w_n]^T$, with $w_j \in [0, 1]$, $1 \leq i \leq n$, and

$$\sum_{i=1}^{n} w_{j=} w_1 + w_2 + \cdots + w_n = 1, \tag{1}$$

where

$$F(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j \, b_j, \tag{2}$$

with b_j denoting the jth largest element of the collection $a_1 a_2, \dots, a_n$.

Note that for the reordering step we distinguish between two types of operator: the descending OWA (DOWA) operator and the ascending OWA (AOWA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DOWA operator and w^*_{n-j+1} the jth weight of the AOWA operator.

1.2 Heavy aggregation operators

The heavy OWA (HOWA) operator (Yager, 2002) is an extension of the OWA operator. This operator is useful when the available information is not bounded by the maximum or the minimum operator of the usual OWA operator. The main difference between the OWA and HOWA operators is that the sum of the weights of the OWA operator must be 1. This restriction does not exist for the HOWA operator: the sum of the weights can range from 1 to n. In the following, we provide the definition of the HOWMA operator suggested by Yager (2002).

Definition 2. A heavy aggregation operator is an extension of the OWA operator that allows the sum of weights to be up to n. Thus, an HOWA operator is an application $R^n \rightarrow R$ which is associated with a weight vector w with $w_j \in [0,1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, so that

$$HOWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j\, b_j, \tag{3}$$

where b_j is the jth largest element of the collection $a_1 a_2, \dots, a_n$.

Note that the HOWA operator is bounded by the sum of all arguments, that is, the minimum and total operators. It is also a monotonic and commutative function. Similar to the case of the OWA operator, we can distinguish between the descending HOWA (DHOWA) operator and the ascending HOWA (AHOWA) operator.

One of the main characteristics of the HOWA operator is that the weight vector can range from $-\infty$ to ∞. Thus, the sum of the weights in the weighting vector w is unbounded: $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$. In this case, we can drastically under- or over-estimate the results according to the available information.

In addition, Yager (2002) introduced a characterizing parameter, called the beta value of the vector W, for the HOWA operator according to the magnitude of the weighting vector W. This new parameter is defined as $\beta(W) = (|W| - 1)/(n - 1)$. Because $|W| \in [1, n]$, it follows that $\beta \in [0,1]$. That is why if $\beta = 1$, we get the total operator, and if $\beta = 0$, we get the usual OWA operator.

Also note that if $w_j = 1/n$ for all j, then we obtain the heavy weighted average (HWA), which is defined by Merigó & Casanovas (2011) as follows.

Definition 3. A heavy weighted average (HWA) operator of dimension n is a mapping $HWA: R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $w_i \in [0,1]$ and $1 \leq \sum_{i=1}^{n} w_i \leq n$, such that

$$HWA\ (a_1, a_2, \dots, a_n) = \left(\sum_{i=1}^{n} w_i a_i\right), \tag{4}$$

where $a_i$is the ith argument of aggregation.

Note that like the HOWA operator, the weighting vector can range from $-\infty$ to ∞, so the sum of the weights is unbounded: $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$.

1.3 Moving averages

A moving average is a usual average that moves toward some part of the whole sample. More generally, the moving average can be seen as a moving aggregation operator. This method is used for solving time-series smoothing problems (Yager, 2008; 2013) and has been applied extensively in economics and statistics (Evans, 2002). The main advantage of using moving average in this context is the possibility of forecasting future results based on historical data. The moving average, according to Kenney and Keeping (1962), can be defined as follows.

Definition 4. Given $\{a_i\}_{i=1}^{N}$, the moving average of dimension n is defined as the sequence $\{s_i\}_{i=1}^{N-n+1}$ obtained by taking the arithmetic mean of the sequence of n terms, such that

$$s_i = \frac{1}{n} \sum_{j=i}^{i+n-1} a_j. \tag{5}$$

The usual moving average can be extended to the weighted moving average (WMA) by using weighted averages. The WMA is defined by Merigó & Yager (2013) as follows.

Definition 5. A weighted moving average (WMA) of dimension m, according to Merigó and Yager (2013), is a mapping $WMA: R^m \to R$ that has an associated weighting vector W of dimension m with $W = \sum_{i=1+t}^{m+t} w_i = 1$ and $w_i \in [0,1]$, such that

$$WMA\ (a_{1+t}, a_{2+t}, \ldots, a_{m+t}) = \sum_{i=1+t}^{m+t} w_i a_i, \tag{6}$$

where a_i is the ith argument, m is the total number of arguments considered from the whole sample and t indicates the movement of the average with respect to the initial analysis. Note that if $w_i = 1/m$ for all i, the WMA becomes the MA aggregation.

Another extension of the moving average is obtained when we combine it with the OWA operator, generating the ordered weighted moving average (OWMA). Merigó and Yager (2013) defined the OWMA as follows.

Definition 6. An ordered weighted moving average (OWMA) of dimension m is a mapping $OWMA: R^m \to R$ that has an associated weighting vector W of dimension m with $W = \sum_{j=1+t}^{m+t} w_j = 1$ and $w_j \in [0,1]$, such that

$$OWMA\ (a_{1+t}, a_{2+t}, \dots, a_{m+t}) = \sum_{j=1+t}^{m+t} w_j b_j, \tag{7}$$

where b_j is the jth largest argument of the a_i, m is the total number of arguments considered from the whole sample and t indicates the movement of the average with respect to the initial analysis.

1.4 Generalized aggregation operators

The generalized aggregation operators are operators that provide a general formulation for a wide range of cases by using generalized and quasi-arithmetic means (Zhao et al. 2010; Zhou & Chen, 2010). In this paper, we define the quasi-arithmetic mean because it includes the generalized mean as a particular case. The weighted quasi-arithmetic mean (Quasi-WA) is defined by Merigó & Yager (2013) as follows.

Definition 7. A Quasi-WA operator of dimension n is a mapping $Quasi - WA: R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0,1]$, such that

$$Quasi - WA(a_1, a_2, \dots, a_n) = g^{-1}\left(\sum_{i=1}^{n} w_i g(a_i)\right), \tag{8}$$

where *g(b)* is a strictly continuous monotone function. Note that if $w_i = 1/n$ for all i, the QWA becomes the simple quasi-arithmetic mean. Moreover, if $g(a) = a^{\lambda}$, the QWA becomes the weighted generalized mean (WGM).

The QWA operator can be extended by using the OWA operator, to obtain the ordered weighted quasi-arithmetic mean (Quasi-OWA). Fodor et al. (1995) defined it as follows.

Definition 8. A Quasi-OWA operator of dimension n is a mapping $Quasi - OWA: R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} g(b_j) = 1$ and $w_j \in [0,1]$, and such that

$$Quasi - OWA\ (a_1, a_2, \dots, a_3) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j)\right), \tag{9}$$

where b_j is the jth largest of the arguments a_i, and *g(b)* is a strictly continuous monotone function.

The moving average can be generalized by using generalized aggregation operators such as the generalized mean and the quasi-arithmetic mean, to obtain the generalized moving average (GMA) and the quasi-arithmetic moving average (Quasi-MA). Like in the case of the Quasi-WA, we define the Quasi-MA because it includes the GMA as a special case. The weighted quasi-arithmetic moving average (Quasi-WMA) can be defined as follows (Merigó & Yager, 2013).

Definition 9. A Quasi-WMA of dimension *m* is a mapping $Quasi - WMA: R^m \rightarrow R$ that has an associated weighting vector *W* of dimension *m* with $W = \sum_{i+1+t}^{m+t} w_i = 1$ and $w_i \in [0,1]$, such that

$$Quasi - WMA\ (a_{1+t}, a_{2+t}, \dots, a_{m+t}) = g^{-1}\left(\sum_{i=1+t}^{m+t} w_i g(a_i)\right), \tag{10}$$

where a_i is the *i*th argument, *m* is the total number of arguments considered from the whole sample, *t* indicates the movement of the average with respect to the initial analysis and *g(b)* is a strictly continuous monotone function.

It is important to note that the Quasi-WMA becomes the Quasi-MA when $w_i = 1/m$ for all *i*. Additionally, the Quasi-WMA becomes the Quasi-GWMA when $g(a) = a^{\lambda}$.

Another extension of the Quasi-WMA operator is obtained using the OWA operator, generating the ordered weighted quasi-arithmetic moving average (Quasi-OWMA). This extension can be defined as follows (Merigó & Yager, 2013).

Definition 10. An ordered weighted quasi-arithmetic moving average (Quasi-OWMA) of dimension *m* is a mapping $Quasi - OWMA : R^n \rightarrow R$ that has an associated weighting vector *W* of dimension *m* with $W = \sum_{j=1+t}^{m+t} w_j = 1$ and $w_j \in [0,1]$, such that

$$Quasi - OWMA(a_{1+t}, a_{2+t}, \dots, a_{m+t}) = g^{-1}\left(\sum_{j=1+t}^{m+t} w_j g\left(b_j\right)\right), \tag{11}$$

where b_j is the *j*th largest argument of the a_i, *m* is the total number of arguments considered from the whole sample, *t* indicates the movement of the average with respect to the initial analysis and *g(b)* is a strictly continuous monotone function.

2. Heavy moving averages

2.1. Theoretical foundations

The heavy ordered weighted moving average (HOWMA) operator is an extension of the OWA operator. The HOWMA operator combines the characteristics of the HOWA operator with the characteristics of the moving average. In general, this operator uses a moving average with an associated weighting vector that ranges from $-\infty$ to ∞. This is useful when we know that the result of the moving average will become lower or higher based on the expectations of the future.

Definition 11. Given a sequence $\{a_i\}_{i=1}^{N}$, the HOWMA operator is defined as the operator that acts on the sequence $\{s_i\}_{i=1}^{N-n+1}$, which is multiplied by a heavy weighting vector, according to

$$HOWMA(s_i) = \sum_{j=1+t}^{m+t} w_j b_j, \tag{12}$$

where b_j is the jth largest element of the collection $\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{n}$, and W is an associated weighting vector of dimension m that satisfies $1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ and $w_i \in [0,1]$. Observe that here we can also expand the weighting vector to the range $-\infty$ to ∞. Thus, the weighting vector w becomes unbounded: $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$.

We can see that, by allowing the sum of the weights to be greater than one, we are accepting the possibility that some part of the information is independent. From the reordering step, we have to distinguish between the descending HOWMA (DHOWMA) operator and the ascending HOWMA (AHOWMA) by using $w_j = w_{n-j+1}^*$, where w_j is the jth weight of the DHOWMA and w_{n-j+1}^* the jth weight of the AHOWMA

The HOWMA operator is monotonic and commutative for both the DHOWMA and the AHOWMA operators. It is monotonic because if $a_i \geq d_i$, for all i, then $HOWMA(a_1, \dots, a_n) \geq HOWMA(d_1, \dots, d_n)$. It is commutative because any permutation of the arguments has the same evaluation. Note that the HOWMA operator is bounded by the minimum and the total operators when it ranges between 1 and n. If the weighting vector ranges between $-\infty$ and ∞, then the HOWMA is not bounded.

When analyzing the magnitude of the weighting vector $|W|$ for the HOWMA operator, we can determine the beta value of the vector W used by Yager (2002) in the HOWA operator. The beta value is defined as $\beta(W) = (|W| - 1)/(n - 1)$. Since $|W| \in [1, n]$, it follows that $\beta \in [0,1]$. That is why if $\beta = 1$, we get the total operator, and if $\beta = 0$, we get the usual moving average.

Note that if $w_i = 1/m$ for all i, the HOWMA becomes the heavy moving average (HMA), which can be defined as follows.

Definition 12. Given $\{a_i\}_{i=1}^{N}$, the heavy moving average (HMA) is defined as the operator acting on the sequence of moving averages $\{s_i\}_{i=1}^{N-n+1}$ of dimension n, by taking the arithmetic mean of the sequence of n terms with associated weight H, such that

$$HMA(s_i) = \left(\frac{1}{n}\sum_{j=1}^{i+n-1} a_j\right)H. \tag{13}$$

Note that it is possible to expand the weight H from $-\infty$ to ∞. Additionally, the HMA can be extended by using weighted averages, to obtain the heavy weighted moving average (HWMA). It can be defined as follows.

Definition 13. A heavy weighted moving average (HWMA) of dimension m is a mapping $HWMA: R^m \to R$ that has an associated weighting vector W of dimension m that satisfies $1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ and $w_i \in [0,1]$:

$$HWMA(a_{1+t}, a_{2+t}, \dots, a_{m+t}) \sum_{i=1+t}^{m+t} w_i a_i. \tag{14}$$

2.2 Families of the HOWMA operator

In this section we consider different types of HOWMA operators by choosing different manifestations of the weighting vector. First of all, we consider the cases in which the HOWMA operator becomes a OWMA, minimum OWMA, maximum OWMA, total operator, arithmetic mean HOWMA, median HOWMA and Olympic HOWMA (Merigó & Gil-Lafuente, 2009).

a) The OWMA operator is obtained when $\beta = 0$.

b) The minimum OWMA operator is obtained when $w_n = 1, w_j = 0$ for all $j \neq b$ and $\beta = 0$.

c) The maximum OWMA operator is obtained when $w_1 = 1$, $w_j = 0$, for all $j \neq 1$ and $\beta = 0$.

d) If $w_i = 1/m$ for all i, the OWMA operator becomes the MA aggregation operator.

e) The total operator is obtained when $\beta = 1$.

f) If $w_j = 1/n$ for all j and $p_i = 1/n$ for all i, the arithmetic mean HOWMA is obtained.

g) To obtain the median HOWMA operator: If n is odd we assign $w_{(n+1)/2} = 1$, and $w_{j*} = 0$ for all others. If n is even we assign, for example, $w_{n/2} = w_{(n/2)+1} = 0.5$, and $w_{j*} = 0$ for all others.

h) To obtain the weighted median HOWMA operator: Select the kth largest argument b_k such that the sum of the weights from 1 to k is equal to or greater than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

i) The general Olympic-HOWMA operator is obtained if $w_j = 0$ for $j = 1, 2, \ldots, k, n, n-1, \ldots, n-k+1$; and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$.

j) The centered-HOWMA operator is obtained if the HOWMA operator is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive.

To understand this approach, we present a very simple numerical example that shows how to convert the usual moving averages in HOWMA.

Example 1. Assume that inflation of the last 6 months is $a = [0.4,0.2,0.3,0.5,0.1,0.4]$. With this information, we want to forecast the inflation for the next month. According to the information available, we have a pessimistic economic scenario, so we assume a Heavy weighted vector: $W = [0.05,0.05,0.15,0.20,0.20,0.40]$. The sum of the weights is 1.05, reflecting the prediction that inflation will increase during the next months. The aggregation can be solved as

$$HOWMA = (0.1 * 0.05) + (0.2 * 0.05) + (0.5 * 0.15) + (0.4 * 0.20) + (0.4 * 0.20) + (0.5 * 0.4) = 0.42$$

From the generalized perspective of the reordering step, it is possible to distinguish between the descending HOWMA (DHOWMA) and the ascending HOWMA (AHOWMA).

3. Generalized heavy moving averages

The Quasi-HWA operator generalized the HWA operator by using quasi-arithmetic means. It can be defined as follows.

Definition 14. A Quasi-HWA is defined as the operator acting on the moving average $\{s_i\}_{i=1}^{N-n+1}$ of dimension n of a given sequence $\{a_i\}_{i=1}^{N}$, obtained by taking the arithmetic mean of the sequence of n terms with associated weight H, such that

$$Quasi - HMA(s_i) = g^{-1}\left(\frac{1}{n}\sum_{j=i}^{i+n-1} g(a_j)\right)H, \qquad (15)$$

where $g(a_j)$ is a strictly continuous monotonic function. Note that it is possible to expand the weight H from $-\infty$ to ∞.

The Quasi-HWMA operator generalized the HMWA operator by using quasi-arithmetic means. It can be defined as follows.

Definition 15. A Quasi-HWMA of dimension m is a mapping $Quasi - HWMA: R^m \rightarrow R$ that has an associated weighting vector W of dimension m that satisfies $1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ and $w_i \in [0,1]$:

$$Quasi - HWMA(a_{1+t}, a_{2+t}, \dots, a_{m+t}) = g^{-1} \sum_{i=1+t}^{m+t} w_i g(a_i), \qquad (16)$$

where $g(a_i)$ is a strictly continuous monotonic function. Note that it is possible to expand the weights by using a weighting vector that ranges from $-\infty$ to ∞. Thus, the weighting vector w becomes unbounded: $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$.

The Quasi-HOWMA operator is an aggregation operator that uses quasi-arithmetic means in the HOWMA operator. It can be defined as follows.

Definition 16. A Quasi-HOWMA operator of dimension n is a mapping $HOWMA: R^n \rightarrow R$ that has an associated weighting vector W of dimension n that is defined by a given sequence $\{a_i\}_{i=1}^{N}$, from which a new sequence $\{s_i\}_{i=1}^{N-n+1}$ is obtained, which is multiplied by a heavy weighting vector, such that

$$Quasi - HOWMA(s_i) = g^{-1} \sum_{j=1+t}^{m+t} w_j g(b_j), \qquad (17)$$

where b_j is the jth largest element of the collection $\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{n}$, W is an associated weighting vector of dimension m that satisfies $1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ and $w_i \in [0,1]$, and $g(b)$ is a strictly continuous monotone function.

In Tables 1-3 we briefly present some of the main particular cases of the Quasi-HMA, Quasi-HWMA and Quasi-HOWMA operators (e.g., Merigó & Gil-Lafuente, 2009).

Table 1. Families of generalized HOWMA operators

Particular case	Quasi-HOWMA
$w_i = \dfrac{1}{n}, for\ all\ i$	Quasi-arithmetic heavy moving average (Quasi-HMA)
$g(b) = b^{\lambda}$	Generalized HOWMA
$g(b) = b$	HOWMA
$g(b) = b^{2}$	Heavy ordered weighted moving quadratic average (HOWMQA)
$g(b) \rightarrow b^{\lambda}, for\ \lambda \rightarrow\ 0$	Heavy ordered weighted moving geometric average (HOWMGA)
$g(b) = b^{-1}$	Heavy ordered weighted moving harmonic average (HOWMHA)
$g(b) = b^{3}$	Heavy ordered weighted moving cubic average (HOWMCA)
$g(b) \rightarrow b^{\lambda}, for\ \lambda \rightarrow \infty$	Heavy maximum
$g(b) \rightarrow b^{\lambda}, for\ \lambda \rightarrow \infty$	Heavy minimum
Etc.	

Table 2. Families of generalized HWMA operators

Particular case	Quasi-HWMA
$w_i = \dfrac{1}{n}, for\ all\ i$	Quasi-arithmetic heavy moving average (Quasi-HMA)
$g(a) = a^{\lambda}$	Generalized HWMA
$g(a) = a$	HWMA
$g(a) = a^{2}$	Heavy weighted moving quadratic average (HWMQA)
$g(a) \rightarrow a^{\lambda}, for\ \lambda \rightarrow\ 0$	Heavy weighted moving geometric average (HWMGA)
$g(a) = a^{-1}$	Heavy weighted moving harmonic average (HWMHA)
$g(a) = a^{3}$	Heavy weighted moving cubic average (HWMCA)
$g(a) \rightarrow a^{\lambda}, for\ \lambda \rightarrow \infty$	Heavy maximum
$g(a) \rightarrow a^{\lambda}, for\ \lambda \rightarrow \infty$	Heavy minimum
Etc.	

Table 3. Families of generalized HMA operators

Particular case	Quasi-HMA
$g(a) = a^{\lambda}$	Generalized HMA
$g(a) = a$	HMA
$g(a) = a^{2}$	Heavy moving quadratic average (HMQA)
$g(a) \rightarrow a^{\lambda}, for\ \lambda \rightarrow\ 0$	Heavy moving geometric average (HMGA)
$g(a) = a^{-1}$	Heavy moving harmonic average (HMHA)
$g(a) = a^{3}$	Heavy moving cubic average (HMCA)
$g(a) \rightarrow a^{\lambda}, for\ \lambda \rightarrow \infty$	Heavy maximum
$g(a) \rightarrow a^{\lambda}, for\ \lambda \rightarrow \infty$	Heavy minimum
Etc.	

4. Econometric forecasting with heavy moving averages

One of the main problems with the fundamental models for forecasting exchange rates is that they have not been successful in the short-term, having similar behavior to a random walk (e.g., Bacchetta & Van Wincoop, 2013; Boyer & Young, 2005; Cheung et al. 2005; Phillips, 2003). This indicates that, within the limitations of the econometric models, the variables used exhibit complex behavior. Thus, to forecast, it is necessary to make assumptions on the future behavior of the variables.

Information aggregation operators can integrate the knowledge of the decision maker into the model. Therefore, the use of HOWMA operators, instead of the typical time series methods, helps to forecast the future behavior of the variables in the econometric model.

To use HOWMA operators in econometric forecasting, it is necessary to perform a number of steps to obtain the best results:

Step 1. Identify the variables that will be used in the econometric model.

Step 2. Select the person who will be the decision maker.

Step 3. Ask the decision maker about the future behavior of the variables, i.e., if they will increase or decreased in the following months.

Step 4. Ask the decision maker how many months should be taken into account in the moving average.

Step 5. Request that the decision maker indicate the importance of each month in the forecast, within a range of [0, 1], taking into account that the sum will be influenced by the information provided in step 3.

Step 6. Use the HOWMA operator to forecast the future values of the variables.

Step 7. Integrate the results into the econometric model.

5. Forecasting the USD/MXN exchange rate for 2015

To evaluate the forecasting capacity of the operator, in this section, we will compare the results of the time series and the HOWMA operator using three traditional models to forecast the exchange rate. That will serve as the variables that will be used in the econometric models (Step 1).

Definition 17. The purchasing power parity (PPP) model, in its relative version, postulates that the variations in the exchange rate in a given period must be equal to the inflation differential; that is, the weaker currency should depreciate (Dornbusch, 1985; Rogoff, 1996; Taylor and Taylor, 2004).

Definition 18. The theory of interest rate parity (IRP), according to Aliber (1973), Fama (1984), Flood & Lessard (1986) and McCallum (1993), says that in free money markets, the spread of interest rates should equal the discount or premium future, so there is parity if the difference between interest rates offsets the forward premium of the stronger currency.

Definition 19. The theory of the balance of payments (BoP), proposed by Dornbusch (1979), says that the exchange rate is adjusted to the balance of inflows and outflows from international transactions in goods, services and assets, so that the current account is affected by the exchange rate due to relative price changes and hence by the competitiveness. On the other hand, the capital account is affected by the expectations of investors and the interest rate.

The process is defined as follows.. For the generation of the econometric models, information from 1994 to 2004 was used. Additionally, in the PPP model and IRP model, a transformation of data was performed, replacing the original data with the logarithm of the same. In the BoP model, such a process was not performed because some of the current account balance and foreign investment data are negative, so it is not possible to use the logarithm. Within each model, two additional variables were added. The first is the exchange rate with a lag. This is in order to smooth the model. The second variable added is the volatility, in order to identify the effect this has on the forecast rate.

The econometric models are as follows.

PPP model

$$tc_F = 0.107 + 0.914tc_{-1} + 0.0235v - 0.0195pci_{EUA} + 0.0408pci_{MEX}$$

S = 0.0133561   R-square = 99.0%   R-square (adjusted) = 99.0%

IRP model

$$tc_F = 0.0908 + 0.960tc_{-1} + 0.0255v - 0.00019i_{EUA} - 0.00228i_{MEX}$$

S = 0.0135118   R-square = 99.0%   R-square (adjusted) = 98.9%

BoP model

$$TC_F = 0.175 + 0.978TC_{-1} + 4.59V - 0.000041BC - 0.000023IEC - 0.000016IED + 0.00226R$$

S = 0.276848   R-square = 98.8%   R-square (adjusted) = 98.7%

In these models, tc_F is the forward exchange rate; tc_{-1} is the spot exchange rate with a lag; v is the volatility; pci_{EUA} is the USA price consumer index; pci_{MEX} is the MEX price consumer index; i_{EUA} is the USA interest rate; and i_{MEX} MEX interest rate. All of these variables are expressed in logarithm. In addition, TC_F is the forward exchange rate; TC_{-1} is the spot exchange rate with a lag; BC is the trade account balance;

IEC is the foreign portfolio investment; IED is the direct foreign investment; and R is the international reserves.

Step 2. The decision maker is a person who works in a bank, in the area of foreign investment. The goal of this step is to select a decision maker.

Step 3. The future behavior of the variables is shown in Table 4.

Table 4. Future behavior of the variables used in the econometric models

Variable	Behavior
TC_{-1}	Increase
V	Increase
PCI_{EUA}	Increase
PCI_{MEX}	Increase
i_{EUA}	Increase
i_{MEX}	Increase
BC	Decrease
IEC	Increase
IED	Increase
R	Decrease

Step 4. According to the decision maker, the number of months that impacts the forecast is 6.

Step 5. For the variables with increasing behavior, the weight vector $w = [0.05,0.05,0.15,0.15,0.4]$ was used, and for those with decreasing behavior, $w = [0.05,0.05,0.15,0.15,0.3]$ was used. Note that the sum of the weights for increasing behavior is 1.05 and for decreasing behavior is 0.95.

Step 6. The future values of the variables, obtained using the HOWMA operator, are presented in Table 5.

Table 5. Forecast of the variables using the HOWMA operator.

Month	TC_{-1}	V	PCI_{EUA}	PCI_{MEX}	i_{EUA}	i_{MEX}	BC	IEC	IED	R
01-15	14.4992	0.0100	249.6487	120.8966	0.0290	3.0007	12.2305	5,460.2904	6,614.8223	182.6809
02-15	14.7972	0.0115	254.2081	123.2515	0.0289	3.0561	161.0501	6,048.5773	6,940.0326	182.2990
03-15	15.0891	0.0125	258.9632	125.6829	0.0286	3.1171	240.1395	6,802.3056	7,150.4474	181.7453
04-15	15.4344	0.0133	263.7477	128.1325	0.0299	3.1790	137.2865	7,333.7766	7,339.1926	180.5131
05-15	15.7441	0.0142	269.3665	130.8735	0.0306	3.2424	87.5350	6,746.6993	7,492.6916	179.0111
06-15	16.0878	0.0150	274.8823	133.5705	0.0314	3.3082	194.6262	7,086.8599	7,620.5484	176.2247
07-15	16.4145	0.0145	280.8940	136.4263	0.0319	3.3812	170.4372	7,313.0197	7,772.5810	172.4796
08-15	16.7589	0.0149	286.7039	139.2608	0.0326	3.4514	181.6626	7,484.9042	7,943.4761	171.4899
09-15	17.1081	0.0153	292.6779	142.1683	0.0332	3.5232	184.7545	7,664.8380	8,112.0037	170.1357
10-15	17.4672	0.0156	298.7693	145.1323	0.0340	3.5966	168.6325	7,825.8587	8,281.2341	168.2936
11-15	17.8315	0.0159	305.0293	148.1703	0.0347	3.6719	170.1998	7,957.9195	8,453.5652	166.2764
12-15	18.2046	0.0163	311.4001	151.2654	0.0354	3.7486	174.3331	8,132.0878	8,629.7130	163.9596

Step 7. Using the information on the future behavior of the variables, we can forecast the USD/MXN exchange rate for 2015 for each of the models (see Table 6).

Table 6. Forecast for the USD/MXN exchange rate in 2015using HOWMA

Date	Spot exchange rate	PPP model	Error	IRP Model	Error	BoP Model	Error
01-15	14.6808	14.4197	-0.2611	14.4255	-0.2553	14.5561	-0.1360
02-15	14.9230	14.4971	-0.4259	14.4952	-0.4278	14.5631	-0.3629
03-15	15.2136	14.8056	-0.4080	14.8092	-0.4044	14.8342	-0.3734
04-15	15.2208	15.1007	-0.1201	15.1098	-0.1110	15.1095	-0.0940
05-15	15.2475	15.4455	0.1980	15.4622	0.2147	15.4608	0.2407
06-15	15.4692	15.7565	0.2873	15.7792	0.3100	15.7469	0.3221
07-15	15.9225	16.0647	0.1422	16.0965	0.1740	16.0656	0.2047
08-15	16.5032	16.3817	-0.1215	16.4208	-0.0824	16.3778	-0.0523
09-15	16.8519	16.7113	-0.1406	16.7585	-0.0934	16.7060	-0.0603
10-15	16.5813	17.0465	0.4652	17.1021	0.5208	17.0394	0.5572
11-15	16.6325	17.3888	0.7563	17.4533	0.8208	17.3816	0.8627
12-15	17.0365	17.7368	0.7003	17.8103	0.7738	17.7273	0.8198
Average	15.8569	15.9462	0.0893	15.9769	0.1200	15.9640	0.1071

In Table 7, we show the results of the forecast obtained using multiplicative decomposition of the time series.

Table 7. Forecast of the USD/MXN exchange rate for 2015 using time series

Date	Spot exchange rate	PPP model	Error	IRP model	Error	BoP model	Error
01-15	14.6808	15.6757	0.9949	15.6435	0.9627	15.6344	0.9536
02-15	14.9230	15.6272	0.7042	15.5716	0.6486	15.8959	0.9729
03-15	15.2136	15.2111	-0.0025	15.1264	-0.0872	15.4197	0.2061
04-15	15.2208	15.2666	0.0458	15.1832	-0.0376	15.4226	0.2018
05-15	15.2475	15.4326	0.1851	15.3492	0.1017	15.6026	0.3551
06-15	15.4692	15.7162	0.2470	15.6489	0.1797	15.8848	0.4156
07-15	15.9225	15.8889	-0.0336	15.8342	-0.0883	15.9721	0.0496
08-15	16.5032	15.6731	-0.8301	15.6062	-0.8970	15.6735	-0.8297
09-15	16.8519	15.6922	-1.1597	15.6222	-1.2297	15.8393	-1.0126
10-15	16.5813	15.8567	-0.7246	15.7903	-0.7910	15.7810	-0.8003
11-15	16.6325	15.7631	-0.8694	15.6985	-0.9340	15.7541	-0.8784
12-15	17.0365	16.0779	-0.9586	16.0393	-0.9972	16.2798	-0.7567
Average	15.8569	15.6568	-0.2001	15.5928	-0.2641	15.7633	-0.0936

Note that if we compare the results presented in Tables 6 and 7, it is clear that with the HOWMA operator the forecasting error is reduced drastically in the PPP model and the IRP model. In the case of the BoP model, the results obtained using time series are slightly better than those obtained using the HOWMA operator. It is important to observe that the HOWMA operator is based on the expectations of the decision makers. This is why it is easier to generate new scenarios using the weighting vector: it can include the historical data, along with the expectations for the future and the knowledge of the experts in the field. The same cannot be done using time series because time series include only information about the past. Thus, the time series technique has limited applicability in topics such as exchange rates that have high volatility and uncertainty.

6. Conclusions

This paper has introduced a new extension of the OWA operator, called the heavy ordered weighted moving average (HOWMA) operator. Basically, this operator uses the main characteristics of two techniques: the heavy OWA (HOWA) operator and the moving average (MA). Therefore, this operator uses historical information and combines it with a weighting vector according to the knowledge of the decision maker and the information available. They key issue with the heavy aggregation is to identify whether there is overlap between the data. The objective is to identify if the aggregation is an average, where the sum of the weights is equal to one, or if the data are partially independent, which implies that the aggregation should be carried out partially, using weights with a sum greater or lower than one.

We have defined and analyzed this new operator and studied some of its main properties. We have also developed a wide range of families of HOWMA operators, such as the arithmetic mean, median and weighted median, and the quasi-arithmetic and generalized HOWMAs. In addition, some other interesting particular cases have been obtained, including the HWA and HWMA.

An application of the new approach to an econometric forecasting problem has also been developed. We have seen that using the HOWMA operator instead of time series to forecast the future behavior of the variables leads to different results, which are adapted to the attitudinal character of the decision maker and decrease the forecasting error. It is important to note that by using the HOWMA operator, it is easier to construct different scenarios that will help the decision maker to have a better understanding of the situation.

In future research, we expect to develop new extensions of the OWA operator by considering other characteristics in the aggregation, such as the possibility of using moving averages with induced heavy aggregation operators (Merigó and Casanovas, 2011a), linguistic variables (Aggarwal, 2016; Xian et al. 2016), intuitionistic fuzzy sets (Yu, 2015), probabilistic aggregation operators (Merigó, 2010), Bonferroni means (Dutta, 2015; Verma, 2015) or distance measures.

Conflict of interest statement

All the Authors declare: “We have no conflict of interest to declare”

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