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THE ISOMETRY GROUP OF SEMI-RIEMANNIAN MANIFOLDS

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Abstract

This work presents two important subjects of modern mathematics, Lie Groups and semi-Riemannian Geometry, and shows a beautiful theorem that arises as a combination of both matters: the isometry group of a semi-Riemannian manifold is a Lie group. The structure of the proof presented is as follows. First, we introduce a theorem by Palais [1], which gives a sufficient condition for a group G of diffeomorphisms acting on a smooth manifold M to be a Lie group: that the set of all vector fields on M which generate global 1-parameters subgroups of G generates a finite-dimensional Lie algebra. Then we show that this result can be applied to the isometry group of semi-Riemannian manifolds, by proving that the set of all complete Killing vector fields generates a finite-dimensional Lie algebra.

Resum

Aquest treball presenta dos temes importants de la matemàtica moderna, els Grups de Lie i la Geometria semi-Riemanniana, i mostra un bonic teorema que en sorgeix com a combinació: el grup d'isometries d'una varietat semi-Riemanniana és un grup de Lie. L'estructura de la demostració seguida és la següent. Primer, introduïm un resultat per Palais [1], que dóna una condició suficient per tal que un grup de difeomorfismes G que actua sobre una varietat diferenciable M sigui un grup de Lie: que el conjunt de tots els camps vectorials de M que generen subgrups globals uniparamètrics de G genera un àlgebra de Lie de dimensió finita. A continuació apliquem aquest resultat al grup d'isometria de varietats semi-Riemannianes, demostrant que l'àlgebra de Lie generada pels camps Killing complets té dimensió finita.

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Introduction

The main goal of this work is to prove that the isometry group of a semi-Riemannian manifold is a Lie group, and to do so in a self-contained way, assuming no prior knowledge of Lie Group Theory or Semi-Riemannian Geometry on the part of the reader. Familiarity with Topology, Abstract Algebra, Euclidean Geometry, and Multivariable Calculus is expected, as they are subjects usually covered in any undergraduate Mathematics program, but nevertheless we include an appendix with some important results for completeness, as this text may also be of interest to theoretical physicists with weaker training on abstract mathematics. Not included in the appendix but also necessary for the understanding of some vital results are the Existence and Uniqueness Theorems for initial value problems of ordinary differential equations. As for Differential Geometry, it is covered on the first chapter of this work. However, some acquaintance with the subject is expected too, so a few proofs shall be skipped or sketched for the sake of brevity.

The motivations behind this work are *manifold*. First, to give a concise yet wide enough presentation on the subjects of Lie Groups and semi-Riemannian Geometry, while showing a beautiful combination of their results. Both subjects are of utmost value in modern mathematics and theoretical physics, yet unluckily not covered enough during undergraduate training besides some low-dimensional Riemannian Geometry. On the one hand, Lie groups are manifolds endowed with a group structure compatible with the smooth manifold structure. From this combination of structures arise powerful geometrical results. Moreover, Lie Groups have an *infinitesimal* counterpart, their Lie algebras, which allow one to study Lie groups from a local and linearized point of view. On the other hand, semi-Riemannian Geometry deals with manifolds furnished with a metric tensor of arbitrary index. A metric tensor is a smooth choice of a scalar product (i.e. a symmetric non-degenerate bilinear form) on each tangent space of the manifold. Hence, semi-Riemannian Geometry includes Riemannian Geometry as the particular case in which the metric tensor is required to be positive-definite, in addition to the properties stated above.

Our second motivation is to introduce the notion of isometry in a broad sense, whilst also highlighting its importance. Linear isometries of $\mathbb{R}^n$ (seen as an inner product vector space) are isomorphisms which preserve distances and angles between vectors. Generalizing this concept to semi-Riemannian Geometry, it translates to isometries being diffeomorphisms of semi-Riemannian manifolds which preserve the metric tensor. Hence, isometric manifolds are equivalent from the point of view of semi-Riemannian Geometry, since also all concepts that derive from the metric, such as geodesics and curvature, are preserved. The set of all isometries of a semi-Riemannian manifold is easily seen to form a group under the composition operation. It is also straightforward to give it a natural topology which turns it into a topological group: the compact-open topology.

Our last goal, but not the least important, is to present a complete and self-contained proof of the fact that the isometry group of a semi-Riemannian manifold has the structure of a Lie group. Again, this result is a generalization of the known result on $\mathbb{R}^n$. Seen as an inner product vector space, the linear isometries of $\mathbb{R}^n$ form the orthogonal group $O(n, \mathbb{R})$. This group consists of rotations, and rotations composed with a reflection; and it is a Lie group. Therefore, it is only natural to ask if the set of all isometries of any semi-Riemannian manifold has a natural Lie group structure compatible with the compact-open topology, and in this work we shall prove that indeed that is the case. It is not a trivial matter, as the proof requires sound knowledge on both Lie groups and semi-Riemannian manifolds. Therefore, graduate level books on semi-Riemannian geometry usually only give the statement of the theorem, without a proof and, similarly, books devoted mainly to Lie groups do not present enough semi-Riemannian geometry to prove the theorem either. There are of course notable exceptions, such as Michor's [2] *Topics in differential geometry*, but they are deeper books for the advanced reader.

The first general result on isometry groups of semi-Riemannian manifolds was published by Myers and Steenrod in 1939 [3]. They proved a particular case of the theorem we present: the isometry group of a Riemannian manifold, that is, a manifold with a positive-definite metric tensor, is a Lie group. Their proof is based on one of the many beautiful properties which hold for Riemannian manifolds but cannot be generalized to metrics with arbitrary index: Riemannian manifolds can be described as metric spaces by defining a notion of distance, and the topology the manifold gets as a metric space coincides with the one it already has as a differentiable manifold. The distance between two points is defined as the infimum of the length of curves joining both points. In order to prove their result, Myers and Steenrod showed that (Riemannian) isometries defined as diffeomorphisms preserving the metric tensor are equivalent to (metric space) isometries defined as diffeomorphisms preserving distance.

Generalizing the result of Myers and Steenrod, in 1953 Nomizu [4] proved that the group of transformations of a differentiable manifold which preserve an affine connection is a Lie group, by applying a version of Myers and Steenrod's theorem to the bundle of linear frames of such manifold. This result by Nomizu already proves our goal, since every semi-Riemannian manifold has a unique affine connection associated, called the Levi-Civita connection. However, we will take a different path.

Finally, in 1957, Palais [1] obtained a beautiful general theorem from which these results and many others related can be derived: *Let G be a group of diffeomorphisms of a differentiable manifold M. Let S be the set of vector fields which generate global 1-parameter subgroups of transformations in G. If S generates a finite dimensional Lie algebra, then G is a Lie group.* The proof was presented by Palais on a booklet of over a hundred pages, which is one of the reasons many modern books prefer to skip it. However, there is an excellent 1963 paper by Chu and Kobayashi [5] wherein the authors review many results concerning groups of transformations of a manifold. It includes a short, self-contained proof of the theorem of Palais, from which they derive all other results. Kobayashi then wrote a book [6] on transformation groups in 1972, again concerning this subject, which we will use as a reference on this matter.

Our path shall be the following. After presenting some basic notions of Differential Geometry in an introductory chapter, we then move on to Lie groups. Special attention is given to the definition of the exponential map, as it is one of the most important features of Lie Group Theory, and it is a fundamental tool we will need. The final part of the chapter on Lie groups presents their application as transformation groups, that is, Lie groups that act smoothly on a differentiable manifold. The last result of this section is the aforementioned theorem of Palais, presented with an extended version of Kobayashi's proof. The third chapter deals with semi-Riemannian geometry. It is not meant to be a complete account of the matter, and so only those results necessary for our goal are presented in detail. The final chapter is entirely devoted to isometries and their infinitesimal counterparts: Killing vector fields. We will prove that the flow of a Killing vector field is a 1-parameter group of isometries, and that every 1-parameter group of isometries is the flow of a given Killing vector field. Afterwards, we shall see that the set of all complete Killing vector fields generates a finite dimensional Lie algebra, which will allow us to use Palais' result, concluding that the isometry group of a semi-Riemannian manifold is a Lie group. We end this work by showing some interesting examples and further results on isometries.

Notation

We will denote the set of real numbers by $\mathbb{R}$. If $n > 0$ is a natural number, let $\mathbb{R}^n = \mathbb{R} \times \overset{n}{\cdots} \times \mathbb{R} = \{(a_1, ..., a_n) \mid a_i \in \mathbb{R}, i = 1, ..., n\}$.

The canonical coordinate functions for $\mathbb{R}^n$ will be denoted by the function $r_i : \mathbb{R}^n \to \mathbb{R}$, defined by $r_i(a_1, ..., a_n) = a_i$.

Let $U \subseteq R^n, V \subseteq R^m$ be open sets. We say that a map $f : U \to V$ is smooth (or $\mathcal{C}^\infty$) if all of the partial derivatives $\partial^k f / \partial r_{i_1} \cdots \partial r_{i_k}$ exist and are continuous.

We will use Einstein's summation convention: whenever a mathematical expression carries a repeated index, one as a super-index and the other as a sub-index, a summation must be understood. For example, $X_i Y^i = \sum_i X_i Y^i$, or $g^{ij} g_{ij} = \sum_{i,j} g^{ij} g_{ij}$.

Chapter 1

Differentiable Manifolds

Differential Geometry is the combination of calculus, multilinear algebra and geometry that allows us to generalize concepts such as differentiation, curve, or vector; from $\mathbb{R}^n$ to a geometrical object called a manifold. A manifold is essentially a space which is locally similar to an Euclidean space, so that it can be mapped, and which has enough structure so that the concepts mentioned above can be carried over, in a way that they are coordinate independent. Manifold theory is devoted to the study of such objects with no other structure, and mainly to the properties which are invariant under diffeomorphisms.

Later on we will want to add additional structure to these objects. A manifold equipped with a group structure smoothly compatible to the manifold structure will be a Lie group, as we will see in Chapter 2; and in Chapter 3 a manifold furnished with a metric tensor will become a semi-Riemannian manifold.

In this chapter we will present some basic notions of differentiable geometry. Some familiarity with the subject is expected, even if its only on low-dimensional submanifolds of $\mathbb{R}^3$, as we will skip some proofs for the sake of brevity. We refer the reader with no such previous knowledge to [7]. The main references for this chapter are [2], [8], [9], and [10], although the style followed is more similar to that of the introductory chapters of [11] and [12], which is more straightforward.

1.1 Basic Definitions

Let M be a topological space.

Definition 1.1. *A* chart *on M is a pair (U, φ) where $U \subseteq M$ is open and $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism of U onto an open subset $\varphi(U)$ of $\mathbb{R}^n$.*
We say that charts (U, φ) and (V, ψ) are $\mathcal{C}^\infty$-related *if whenever $U \cap V \neq 0$, the maps $(\varphi \circ \psi^{-1}) : \psi(U \cap V) \to \varphi(U \cap V)$ and $(\psi \circ \varphi^{-1}) : \varphi(U \cap V) \to \psi(U \cap V)$ are $\mathcal{C}^\infty$.*

Definition 1.2. *An* atlas *Ψ on M is a collection of charts $\Psi = \{(U_\alpha, \varphi_\alpha)\}$ such that $\{U_\alpha\}$ is an open cover of M and every pair of charts is $\mathcal{C}^\infty$-related. A* differentiable structure *on M is an atlas Φ which is maximal: if a chart (V, ψ) is $\mathcal{C}^\infty$-related to every chart $(U_\alpha, \varphi_\alpha) \in \Phi$, then also $(V, \psi) \in \Phi$. The functions $x^i = r^i \circ \varphi$, where $r^i = \mathbb{R}^n \to \mathbb{R}$ denote the canonical coordinate functions $r^i(a^1, ..., a^n) = a^i$, are called the* coordinate functions on U for φ.

Proposition 1.3. *Let Ψ be an atlas on M. Then, there exists a unique differentiable structure Φ on M such that $\Psi \subseteq \Phi$.*

Proof. $\Phi = \{(U, \varphi) \text{ chart on } M \mid (U, \varphi) \text{ is } \mathcal{C}^\infty\text{-related to every chart in } \Psi\}$. □

Definition 1.4. *A* smooth manifold of dimension n *is a pair (M, Φ) of a second countable Hausdorff topological space M and a differentiable structure Φ on M such that each chart in Φ takes its values in $\mathbb{R}^n$.*

Indistinctly we will talk about smooth manifolds, $\mathcal{C}^\infty$-manifolds, differentiable manifolds, or simply, manifolds. Smoothness will be always implied. A differentiable structure will be always implied too, even if it is not specified. We could have relaxed the definition by only asking for $\mathcal{C}^r$-differentiability, or by dropping the Hausdorff or second countability requirements, but such manifolds are uncommon and harder to deal with. For example, the fact that every smooth manifold admits a Riemannian metric is a direct consequence of second countability and the Hausdorff condition, through the existence of partitions of unity. In summary, whenever we talk about a manifold M, we will mean a second countable Hausdorff smooth manifold (M, Φ).

Remark 1.5. Notice that in order to specify a differentiable structure on M, one need only specify an atlas on M. Proposition 1.3 directly gives us the differentiable structure.

Remark 1.6. Open subsets V of a manifold (M, Φ) are also smooth manifolds, with differentiable structure $\Phi_V = \{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V}) \mid (U_\alpha, \varphi_\alpha \in \Phi)\}$. Also, if (M_1, Φ_1) and (M_2, Φ_2) are smooth manifolds, then (M, Φ), with Φ defined as the maximal set containing all product charts $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)$, with $(U_\alpha, \varphi_\alpha) \in \Phi_1$ and $(V_\beta, \psi_\beta) \in \Phi_2$, is also a manifold.

Example 1.7.

1. Obviously $\mathbb{R}^n$ is a manifold, with atlas $\{Id\}$.
2. Any real vector space V is also a manifold, as all real vector spaces are isomorphic to $\mathbb{R}^n$ for some n through a coordinate choice.
3. The n-dimensional sphere, $S^n = \{a \in \mathbb{R}^{n+1} \mid a_1^2 + \cdots + a_n^2 = 1\}$ is a manifold. The stereographic projections $S^n \to \mathbb{R}^n$ from the north and south pole are an atlas on S^n.
4. The set of all $n \times n$ invertible matrices $Gl(n, \mathbb{R}) := \{A \in M_{n\times n}(\mathbb{R}) \mid \det A \neq 0\}$ is a manifold, with an induced differentiable structure as an open subset of $\mathbb{R}^{n^2}$.

Definition 1.8. *We say that a function $f : M \to \mathbb{R}$ is* smooth *if its coordinate expression $(f \circ \varphi^{-1}) : \varphi(U) \to \mathbb{R}$ is $\mathcal{C}^\infty$ for all charts (U, φ) on M, in the usual calculus sense. We will denote the set of all smooth functions on M by $\mathcal{F}(M)$. Similarly, any map $\phi : M \to N$ between two manifolds is also called* smooth *if $(\varphi_N \circ \phi \circ \varphi_M^{-1})$, defined on a suitable open subset of $\mathbb{R}^{dimM}$, is $\mathcal{C}^\infty$ for all charts (U_M, φ_M) and (U_N, φ_N) on M and N, respectively. We say that a map $\phi : M \to N$ between two manifolds is a* diffeomorphism *if it is smooth, bijective, and its inverse map ϕ^{-1} is also smooth.*

Proposition 1.9. *The set of all diffeomorphisms* Diff(M) *of a manifold M onto itself is a group.*

Proof. The composition of maps is associative, and it is trivial to prove that the identity map, the inverse map, and compositions of diffeomorphisms are all diffeomorphisms. □

As we can see from the definitions above, in order to see whether a mathematical object is smooth on a manifold, one need only check if its coordinate expression is smooth for all charts in the differentiable structure of the manifold. A *partition of unity* is a collection of functions which allows one to do the opposite: to build global concepts from local ones by "stitching" them smoothly so that they become independent of the local chart.

Definition 1.10. *A* partition of unity *on a manifold M is a collection $\{f_\alpha\}_{\alpha\in A}$ of smooth functions $f_\alpha \in \mathcal{F}(M)$ such that*

- $0 \le f_\alpha(p) \le 1$ *for all $\alpha \in A$ and $p \in M$.*
- $\{\text{supp } f_\alpha\}_{\alpha\in A}$ *is locally finite, where* supp $f_\alpha = \overline{f_\alpha^{-1}(\mathbb{R} \setminus \{0\})}$.
- $\sum_\alpha f_\alpha(p) = 1$ *for all $p \in M$.*

We say that $\{f_\alpha\}_{\alpha\in A}$ is subordinate *to the cover $\{U_\beta\}_{\beta\in B}$ if for all $\alpha \in A$ there exists a $\beta \in B$ such that* supp $f_\alpha \subseteq U_\beta$.

Proposition 1.11. *Manifolds are locally compact.*

Proof. Let p be a point of an n-dimensional manifold M. By the definition of manifold, there exists a chart (U, φ) such that $U \subseteq M$ is an open neighborhood of p and $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism. Then $V := \varphi(U)$ is also an open neighborhood of $\varphi(p) \in \mathbb{R}^n$, and therefore there exists an $\varepsilon > 0$ such that $\varphi(p) \in B_\varepsilon(\varphi(p)) \subseteq V$, where $B_\varepsilon(\varphi(p))$ is the open ball of center $\varphi(p)$ and radius ε. Therefore also $\varphi(p) \in \overline{B_{\varepsilon/2}(\varphi(p))} \subseteq B_\varepsilon(\varphi(p)) \subseteq V$, for the closed ball of radius $\varepsilon/2$. By Heine-Borel, as $\overline{B_{\varepsilon/2}(\varphi(p))}$ is closed and bounded, it is also compact. Therefore, taking the inverse map, $p \in \varphi^{-1}\left(\overline{B_{\varepsilon/2}(\varphi(p))}\right)$ which is compact, proving that every $p \in M$ has a compact neighborhood. □

Proposition 1.12. *Every (second countable, Hausdorff, smooth) manifold is paracompact.*

Proof. Direct application of the result above and Lemma A.12. □

Lemma 1.13. *Let U be an open neighborhood of a point p in a differentiable manifold M. Then, there is a function $\theta \in \mathcal{F}(M)$, called a* bump function *at p, such that*

1. $0 \le \theta \le$ *on M.*
2. $\theta = 1$ *on some neighborhood V of p.*
3. *supp $\theta \subseteq U$.*

Proof. Let $f : \mathbb{R} \to [0, 1]$ be defined such that $f(t) = e^{-1/t}$ if $t > 0$ and $f(t) = 0$ if $t \le 0$. Next, choose $\varepsilon > 0$ and define $g(t) = f(t)/(f(t) + f(\varepsilon - t))$, and finally define $b(t) = g(t + 2\varepsilon)g(t - 2\varepsilon)$. This function b is a bump function on $\mathbb{R}$ at 0, it takes the value of 1 for $|x| \le \varepsilon$ and 0 for $|x| \ge 2\varepsilon$. Then the desired θ is $\theta = (b \circ x^1) \cdots (b \circ x^n)$ for some small enough ε, where $x^1, ..., x^n$ are the coordinate functions of a chart with $\varphi(p) = 0$. □

Theorem 1.14. *Let M be a smooth manifold with atlas $\Phi = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$. Then there exists a countable partition of unity $\{f_i \mid i \in \mathbb{N}\}$ subordinate to the open cover $\{U_\alpha\}$ with compact support for each for each i.*

Proof. See Warner [8] Theorem 1.11, p. 10. □

1.2 Tangent Vectors and Differentials

There are many ways in which one can define tangent vectors. The usual orthodox way is to define them as a derivation of the algebra of *germs* of functions (equivalence classes of functions which coincide in neighborhoods of a point), as it is done in Warner [8]. Physicist, on the other hand, like to describe tangent vectors as n-tuples which transform *contravariantly* under coordinate changes, see Curtis and Miller [12]. One may also define tangent vectors as equivalence classes of smooth curves on M, for the equivalence relation of their local coordinate expression having the same derivative, as done in Gallot, Hulin and Lafontaine [13]. For a thorough review of these different definitions, their relations and their subtleties, see Lee [10] chapter 2. Our approach will be that of O'Neill [11]. It is a similar yet perhaps less formal approach than that of Warner, but it may be easier to the non-expert reader, as it needs less prior knowledge on algebras and derivations. It first defines tangent vectors by axiomatizing their properties and then it proves an equivalence relation to show that indeed the tangent vector is a local object on M.

Definition 1.15. *Let M be a smooth manifold, and $p \in M$ a point. A* tangent vector *to M at p is a function $v : \mathcal{F}(M) \to \mathbb{R}$ such that*

- $v(af + bg) = av(f) + bv(g)$ *for all $a, b \in \mathbb{R}$ and for all $f, g \in \mathcal{F}(M)$.*
- $v(fg) = v(f)g(p) + f(p)v(g)$ *for all $f, g \in \mathcal{F}(M)$.*

Notice that this last property resembles the well known Leibnizian property for differentiating a product of real functions $f, g \in \mathcal{F}(\mathbb{R})$, which is $(fg)' = f'g + fg'$. This property is key in relating the concept of vector fields to that of directional derivatives of functions seen in multivariable calculus.

Definition 1.16. *The set of all tangent vectors at a point p on M is denoted by T_pM and called the* tangent space *to M at p.*

It is easy to check that it is vector space over $\mathbb{R}$ if we define, for all $v, w \in T_pM$, $f \in \mathcal{F}(M)$ and $a, b \in \mathbb{R}$,

$$(av + bw)(f) = av(f) + bw(f).$$

Let us now see that tangent vectors are local objects:

Proposition 1.17. *Let M be a smooth manifold and let p be a point in M. Let $v \in T_pM$, and let $f, g, h \in \mathcal{F}(M)$. Let $c \in \mathbb{R}$ be a constant.*

1. *If there exists a neighborhood U of p such that $f(q) = g(q)$ for all $q \in U$, then $v(f) = v(g)$.*
2. *If there exists a neighborhood V of p such that $h(q) = c$ for all $q \in V$, then $v(h) = 0$.*

Proof. (1.): We have $f(q) = g(q)$ for all $q \in U$. This is equivalent to $\tilde{h}(q) := f(q) - g(q) = 0$. As tangent vectors are linear, $v(\tilde{h}) = v(f) - v(g)$, so if we prove that $v(\tilde{h}) = 0$ we will have finished. Be Lemma 1.13, let θ be a bump function at p with support in U. As $\tilde{h}$ is 0 in U and θ is 0 outside U, $\tilde{h}\theta = 0$ on all of M. Obviously $v(\tilde{h}\theta) = v(0) = 0$, as $v(0) = v(0 + 0) = v(0) + v(0)$. Therefore $0 = v(\tilde{h}\theta) = v(\tilde{h})\theta(p) + \tilde{h}(p)v(\theta) = v(\tilde{h})$.

(2.): As h coincides in all U_p with the constant function c defined on all of M, by (1.) their image by v will coincide. Now, if 1 is the constant function of value 1, then $v(1) = v(1 \cdot 1) = v(1)1 + 1v(1) = 2v(1)$. Hence $v(1) = 0$, so $v(h) = v(c \cdot 1) = cv(1) = 0$. $\square$

Definition 1.18. *Let (U, φ) be a chart at p in M, with coordinate functions $x^1, ...x^n$. For each $i \in \{1, ..., n\}$ we define the* tangent vector at $p \in M$ in the x^i coordinate direction, *denoted by $\frac{\partial}{\partial x^i}\big|_p$, as*

$$(\frac{\partial}{\partial x^i}\big|_p)(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial r^i}\big|_{\varphi(p)}$$

for all smooth functions f on a neighborhood of p.

It is straightforward to prove that $\frac{\partial}{\partial x^i}\big|_p : \mathcal{F}(M) \to \mathbb{R}$ is in fact a tangent vector in the sense of Def. 1.15. We will also use the following notation: $\partial_i = \frac{\partial}{\partial x^i}$.

Theorem 1.19. *The set $\{\frac{\partial}{\partial x^i}\big|_p \mid i = 1, ..., n\}$ forms a basis of T_pM. Therefore* dim $T_pM = n$. *Under such basis, each tangent vector $v \in T_pM$ can be written as*

$$v = v^i \frac{\partial}{\partial x^i}\big|_p$$

where $v^i = v(x^i)$ are called the coordinates *of v on the basis $\{\frac{\partial}{\partial x^i}|_p\}$.*
If (U, ψ) is another chart at p in M with coordinate functions $y^1, ..., y^n$, so that $v = v(y^i)\frac{\partial}{\partial y^i}\big|_p$, the way in which the coordinates of v transform is

$$v(x^i) = \Big(\frac{\partial x^i}{\partial y^j}\Big)\big|_p v(y^j).$$

We say that tangent vectors behave contravariantly *under change of coordinates.*

Proof. See O'Neill [11] Theorem 1.12, p.8. $\square$

Definition 1.20. *Let $\phi : M \to N$ be a smooth map between manifolds. Let p be a point in M. The* differential map *(also sometimes called the* tangent map*) of ϕ at p is the linear map $d\phi_p : T_pM \to T_{\phi(p)}M$ which fulfills $(d\phi_p(v))(f) = v(f \circ \phi)$ for all $v \in T_pM$ and $f \in \mathcal{F}(M)$.*

We shall omit the subscript p whenever it is understood from the context.

Remark 1.21. Let $\phi : M \to N$ be a smooth map, and let $p \in M$. If (U, φ) and (V, ψ) are charts on M and N with coordinate functions $x^1, ..., x^n$ and $Y^1, ..., y^m$ respectively, then

$$d\phi(\frac{\partial}{\partial x^i}\big|_p) = \frac{\partial(y^j \circ \phi)}{\partial x^i}\big|_p \frac{\partial}{\partial y^j}\big|_{\phi(p)}.$$

Proposition 1.22. *Let $\psi : M_1 \to M_2$ and $\phi : M_2 \to M_3$ be smooth mappings between smooth manifolds. Then, for all $p \in M_1$, $d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$.*

Proof. Indeed, for all $v \in T_pM_1$ and for all $f \in \mathcal{F}(M_3)$, we have

$$\big(d(\psi \circ \phi)_p(v)\big)(f) = v(f \circ \psi \circ \phi) = d\phi_p(v)(f \circ \psi) = \Big(d\psi_{\phi(p)}\big(d\phi_p(v)\big)\Big)(f). \qquad \square$$

Definition 1.23. *Let M be a smooth manifold. A* submanifold *of M is a pair (N, ϕ) with $\phi : N \to M$ smooth such that ϕ is injective and $d\phi_p$ is non-singular for each $p \in M$.*

Let's see now the generalized version of the inverse function theorem:

Theorem 1.24. *Let $\phi : M \to N$ be a smooth mapping, and let $p \in M$. Then, $d\phi_p : T_pM \to T_{\phi(p)}N$ is a linear isomorphism if and only if there exists an open neighborhood U of p such that $\phi|_U : U \to \phi(U)$ is a diffeomorphism.*

Proof. ($\Leftarrow$): Trivial. ($\Rightarrow$): *Sketch.* Choose coordinate systems (U, φ) about $p \in M$ and (V, ψ) about $\phi(p) \in N$ such that $\phi(U) \subseteq V$. Then we can apply the Inverse Function Theorem of Multivariable Calculus (Theorem A.21) to $\psi \circ \phi \circ \varphi^{-1}$. $\square$

1.3 Vector Fields

Definition 1.25. *Let M be a manifold. We define the* tangent bundle *TM as the disjoint union of all tangent spaces. That is,*

$$TM := \bigsqcup_{p \in M} T_pM.$$

It is a family of tangent spaces parametrized by M, with the natural projection map $\pi : TM \to M$ given by $\pi(v) = p$ for all $v \in T_pM$. Therefore it may be often useful to write the elements of TM as (p, v), with $p \in M$ and $v \in T_pM$.
One can easily build a differentiable structure for TM starting from the one in M. Let Φ be the differentiable structure of M. Now, for every chart $(U_\alpha, \varphi_\alpha) \in \Phi$ with coordinate functions $x^1, ..., x^n$, consider the chart $\left(\pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha\right)$ on TM, where $\tilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ is given by

$$\tilde{\varphi}_\alpha(v) = \left(x^1(\pi(v)), ..., x^n(\pi(v)), dx^1(v), ..., dx^n(v)\right),$$

for every tangent vector $v \in \pi^{-1}(U_\alpha)$. Then, the differentiable structure $\tilde{\Phi}$ on TM will be the maximal collection containing the set

$$\{\left(\pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha\right) \mid (U_\alpha, \varphi_\alpha) \in \Phi\}.$$

Definition 1.26. *A* vector field *X on M is a lifting $X : M \to TM$ which assigns to each point $p \in M$ a vector $X_p \in T_pM$. For all $f \in \mathcal{F}(M)$, we denote by $X(f)$ the function which sends each $p \in M$ to the value $X_p(f)$. We say that a vector field X is* smooth *in the usual manifold sense. Equivalently, X is smooth if and only if $X(f)$ is a smooth function on M for all $f \in \mathcal{F}(M)$. We denote the set of all smooth vector fields on M by $\mathfrak{X}(M)$.*

Proposition 1.27. *$\mathfrak{X}(M)$ is a vector space over $\mathbb{R}$ a module over $\mathcal{F}(M)$.*

Proof. Straightforward, if for all vector fields $X, Y \in \mathfrak{X}(M)$ and $p \in M$ we define $(aX + bY)_p = aX_p + bY_p$ for all $a, b \in \mathbb{R}$; and $(fX)_p = f(p)X_p$ for all $f \in \mathcal{F}(M)$. $\square$

Remark 1.28. Generalizing the coordinate expression of tangent vectors at a point, for any vector field X on a chart (U, φ) with coordinate functions $x^1, ..., x^n$ we can write

$$X = X^i \frac{\partial}{\partial x^i} = X^i \partial_i \ ,$$

where $X^i = X \circ x^i : M \to R$ are the coordinate functions for X. The vector field X is smooth if and only if all X^i functions are smooth. Also then the coordinate expression for the function $X(f)$ is

$$X(f) = X^i \frac{\partial f}{\partial x^i} \ .$$

Definition 1.29. *A* derivation *on $\mathcal{F}(M)$ is a map $D : \mathcal{F}(M) \to \mathcal{F}(M)$ such that for all $a, b \in \mathbb{R}$ and $f, g \in \mathcal{F}(M)$,*

1. $D(af + bg) = aD(f) + bD(g)$ *($\mathbb{R} -$ linearity).*
2. $D(fg) = D(f)g + fD(g)$ *(Leibnizian).*

Remark 1.30. The combination of the definitions of smooth vector field and tangent vector show that any smooth vector field $X \in \mathfrak{X}(M)$ defines a derivation $f \mapsto X(f)$. Conversely any derivation D on $\mathcal{F}(M)$ defines a vector field X by setting $X_p(f) = D(f)(p)$.

Definition 1.31. *Let $\phi : M \to N$ be a smooth mapping. We say that vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are ϕ-*related *if $d\phi(X_p) = Y_{\phi(p)}$ for all $p \in M$.*

One could think that for any vector field X on M there is a unique vector field Y on N such that X and Y are ϕ-related, but in general that is not true. We need ϕ to be a diffeomorphism. In that case we can define the following.

Definition 1.32. *Let $\phi : M \to N$ be a diffeomorphism. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. We define the* push-forward *$\phi_*(X) \in \mathfrak{X}(N)$ and the* pull-back *$\phi^*(Y) \in \mathfrak{X}(N)$ by*

$$\phi_*(X) = d\phi \circ X \circ \phi^{-1}, \qquad \phi^*(Y) = d\phi^{-1} \circ Y \circ \phi.$$

That is, for $p \in M$ and $q \in N$,

$$\big(\phi_*(X)\big)(q) = d\phi(X_{\phi^{-1}(q)}), \qquad \big(\phi^*(Y)\big)(p) = d\phi^{-1}(Y_{\phi(p)}).$$

Integral curves

Definition 1.33. *A smooth map $\alpha : I \subseteq \mathbb{R} \to M$ is called a* smooth curve *on M. Its* velocity vector *is defined by $\alpha'(t) := d\alpha_t(\frac{d}{dt})$.*

Be careful not to be confused with the notation, as we will use t to denote both the point t in the manifold $I \subseteq \mathbb{R}$, and its coordinate function, which is the identity.

Remark 1.34. The tangent vector applied to a function $f \in \mathcal{F}(M)$ is

$$\alpha'(t)(f) = \frac{d(f \circ \alpha)}{dt}(t).$$

The coordinate expression of α'on a chart (U, φ) with coordinate functions $x^1, ..., x^n$ is

$$\alpha'(t) = \frac{d(x^i \circ \alpha)(t)}{dt} \partial_i \Big|_{\alpha(t)}$$

Definition 1.35. *Let $\alpha : I \subseteq \mathbb{R} \to M$ be a smooth curve and X a smooth vector field on M. We say that α is an* integral curve *of X if its velocity vector $\alpha'(t)$ is equal to $X_{\alpha(t)}$ for all $t \in I$.*

We can regard vector fields as first order differential equations on a manifold: if we express the definition of integral curve in terms of local coordinates, given a chart (U, φ) with coordinate functions $x^1, ...x^n$, we obtain

$$\frac{d(x^i \circ \alpha)}{dt} = (X^i \circ \varphi^{-1})(x^1 \circ \alpha, ..., x^n \circ \alpha),$$

where $X^i = X(x^i)$. The Existence and Uniqueness Theorem for the solution of a system of first order ODEs, given an initial condition, yields the following result:

Theorem 1.36. *Let $X \in \mathfrak{X}(M)$ be a smooth vector field on a differentiable manifold M. Then, for all $p \in M$ there exist an interval $I_p \subseteq \mathbb{R}$ and a smooth curve $\alpha_p : I_p \to M$, such that*

1. *$0 \in I_p$ and $\alpha_p(0) = p$.*
2. *α_p is an integral curve of X.*
3. *α_p is maximal: if $\beta : J_p \to M$ is a smooth curve on M fulfilling (1.) and (2.), then $J_p \subseteq I_p$ and $\beta = \alpha_p|_{J_p}$.*

Now, we define a map $Fl^X : \mathcal{D} \to M$, with $\mathcal{D} = \bigcup_{p \in M} I_p \times \{p\}$ by $Fl^X(t,p) = Fl^X_t(p) := \alpha_p(t)$. We call it the flow *of X.*

4. *$\mathcal{D}$ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$.*
5. *For each $t \in \mathbb{R}$, the map $Fl^X_t : \mathcal{D}_t \to \mathcal{D}_{-t}$, where $\mathcal{D}_t = \{p \in M \mid t \in I_p\}$ is open, is a diffeomorphism with inverse Fl^X_{-t}.*
6. *For every $t, s \in \mathbb{R}$ the domain of $Fl^X_t \circ Fl^X_s$ is contained in $\mathcal{D}_{t+s}$, and is equal if s and t have the same sign. Moreover, whenever $Fl^X_t \circ Fl^X_s$ is well defined, $Fl^X_t \circ Fl^X_s = Fl^X_{t+s}$. That is, for all $p \in M$,*

$$Fl^X_{t+s}(p) = Fl(t+s, p) = Fl^X(t, Fl^X(s,p)) = (Fl^X_t \circ Fl^X_s)(p).$$

Proof. See Warner [8] (Theorem 1.48 p.37) for a detailed exposition on how to properly apply the Existence and Uniqueness Theorem of solutions of systems of ODE's. □

Definition 1.37. *We say that a vector field $X \in \mathfrak{X}(M)$ is* complete *if its flow is defined globally. That is, if Fl^X is defined for all $t \in \mathbb{R}$ regardless of the point $p \in M$, i.e., if $I_p = \mathbb{R}$ for all $p \in M$.*

Remark 1.38. Notice that the flow Fl^X of a complete vector field X defines a 1-parameter abelian group of diffeomorphisms of M onto itself, also called a *1-parameter group of transformations* of M, since

1. $Fl^X_0(p) = p$ for all $p \in M$ so $Fl^X_0 = Id_M$.
2. $Fl^X_t \circ Fl^X_s = Fl^X_{t+s}$ for all $t, s \in \mathbb{R}$, so they commute.
3. $(Fl^X_t)^{-1} = Fl^X_{-t}$.

If X is not complete, we will call the local flow Fl^X the *local* 1-parameter group of X.

Proposition 1.39. *Every vector field X on a compact manifold M is complete.*

Proof. For each $p \in M$, let U_p be a neighborhood of p, and let $\varepsilon_p > 0$ such that the vector field X generates a local 1-parameter group of local transformations Fl^X on $(-\varepsilon_p, \varepsilon_p) \times U_p$. Since M is compact, the open covering $\{U_p \mid p \in M\}$ admits a finite subcovering $\{U_{p_i} \mid i = 1, ..., k\}$. Now we can choose $\varepsilon = \min\{\varepsilon_{p_i} \mid i = 1, ..., k\}$. That way, the flow Fl^X is well defined on all $(-\varepsilon, \varepsilon) \times U_{p_i}$, and as $\bigcup_i U_{p_i} = M$, it is therefore defined on $(-\varepsilon, \varepsilon) \times M$, and hence on all $\mathbb{R} \times M$. □

Lie Bracket

Definition 1.40. *Let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields on M. We define the* Lie bracket *of X and Y as the vector field $[X, Y]$ on M such that, for all $p \in M$ and for all $f \in \mathcal{F}(M)$*

$$[X, Y]_p(f) = X_p\big(Y(f)\big) - Y_p\big(X(f)\big).$$

Proposition 1.41. *The bracket of two smooth fields is also a smooth vector field; and the bracket operation anti-commutes and fulfills the Jacobi identity. That is, for all $X, Y, Z \in \mathfrak{X}(M)$,*

1. $[X, Y] \in \mathfrak{X}(M)$.
2. $[X, Y] = -[Y, X]$.
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Proof.

1. Since X, Y are smooth vector fields, $X(f), Y(f)$ are smooth functions on M, and therefore $X\big(Y(f)\big), Y\big(X(f)\big)$ are also smooth functions on M.
2. $[X, Y] = XY - YX = -(YX - XY) = -[Y, X]$.
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = [X, Y]Z - Z[X, Y] + [Y, Z]X - X[Y, Z] + [Z, X]Y - Y[Z, X] = XYZ - YXZ - ZXY + ZYX + YZX - ZYX - XYZ + XZY + ZXY - XZY - YZX + YXZ = 0$.

□

Proposition 1.42. *If $X, Y \in \mathfrak{X}(M)$ and $f, g \in \mathcal{F}(M)$. Then*

$$[fX, gY] = fg[X, Y] + f\big(X(g)\big)Y - g\big(Y(f)\big)X.$$

Proof. By applying both sides of the equation to any smooth function $h \in \mathcal{F}(M)$ one can easily check that they match. □

Proposition 1.43. *Let (U, φ) be a chart on M with coordinate functions $x^1, ...x^n$. Then, the local coordinate expression on U for $[X, Y]$, given $X, Y \in \mathfrak{X}(M)$, is*

$$[X, Y] = \left[X^i \partial_i, Y^j \partial_j\right] = \left(X^k \frac{\partial Y^l}{\partial x^k} - Y^k \frac{\partial X^l}{\partial x^k}\right) \partial_l.$$

Proof. Again check that both sides of the equation coincide for any $h \in \mathcal{F}(M)$. □

Lemma 1.44. *Let $p \in M$, $X \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$. Then,*

$$X(f)(p) = \frac{d}{dt}\Big|_0 (f \circ Fl_t^X)(p) = \lim_{t \to 0} \frac{1}{t} \left[f(Fl_t^X(p)) - f(p) \right]$$

Proof. Obvious from the definition of flow. Taking $\alpha(t) := Fl_t^X(p)$, which is an integral curve of X, so $X = \alpha'$, the result is immediate from 1.34. □

Proposition 1.45. *Let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields on a manifold M. Let $p \in M$. Then,*

$$[X, Y]_p = \lim_{t \to 0} \frac{1}{t} \left[dFl_{-t}^X (Y_{Fl_t^X(p)}) - Y_p \right].$$

Proof. Let $f \in \mathcal{F}(M)$ be a smooth function on M. Now, for a fixed $p \in M$, consider the mapping $\theta : \mathbb{R}^2 \to \mathbb{R}$ defined by $\theta(t, s) := Y_{Fl_t^X(p)}(f \circ Fl_s^X)$, which is locally defined near zero for both variables t and s (Theorem 1.36). Notice that $\theta(t, 0) = Y_{Fl_t^X(p)}(f)$ and $\theta(0, s) = Y_p(f \circ Fl_s^X)$, so

$$\begin{aligned}
\frac{\partial}{\partial t}\theta(0,0) &= \frac{\partial}{\partial t}\Big|_0 (Y_{Fl_t^X(p)}(f)) = \frac{\partial}{\partial t}\Big|_0 \Big(Y(f)(Fl_t^X(p)) \Big) = X_p(Y(f)), \\
\frac{\partial}{\partial s}\theta(0,0) &= \frac{\partial}{\partial s}\Big|_0 (Y_p(f \circ Fl_s^X)) = Y_p\Big(\frac{\partial}{\partial s}\Big|_0 (f \circ Fl_s^X) \Big) = Y_p(X(f)),
\end{aligned}$$

where the last equalities hold by the previous Lemma. But also,

$$\frac{\partial}{\partial u}\Big|_0 \theta(u, -u) = \frac{\partial}{\partial u}\Big|_0 \Big(Y_{Fl_u^X(p)}(f \circ Fl_{-u}^X) \Big) = \frac{\partial}{\partial u}\Big|_0 \Big(dFl_{-u}^X (Y_{Fl_u^X(p)}) \Big)(f). \qquad \square$$

1.4 Tensor Fields

In the previous section we have presented and studied vector fields. In the same way as vectors of a real vector space have their dual counterpart, the linear forms; also vector fields have their respective dual, the one-forms. Within this last section of Chapter 1, we will first introduce these objects, and then we will generalize both vector fields and one-forms to the notion of tensor fields. Since many results concerning tensor fields are generalizations of known results for vector fields or multilinear algebra, we shall omit many proofs, and refer the reader to O'Neill [11], Chapter 2.

Definition 1.46. *Let M be a smooth manifold, and let $p \in M$. The dual space T_pM^* of the tangent space T_pM is called the* cotangent space *of M at p. Its elements are linear maps $T_pM \to \mathbb{R}$. Similarly to TM, the* cotangent bundle *is defined as $TM^* = \bigcup_{p \in M} T_pM^*$, and also has a natural description as a manifold.*

Definition 1.47. *A* one-form ω *on a manifold* M *is a lift* $\omega : M \to TM^*$*, which assigns to each point* $p \in M$ *a linear form* ω_p *from the cotangent space* T_pM^**.*

Let X be a vector field on M. Since each linear form ω_p sends each tangent vector X_p to a value in $\mathbb{R}$, we can define a function $\omega(X) : M \to \mathbb{R}$ by $\omega(X)(p) = \omega_p(X_p)$. We say that a one form ω is *smooth* if for all smooth vector field $X \in \mathfrak{X}(M)$ the function $\omega(X)$ is smooth. We will denote by $\mathfrak{X}^*(M)$ the set of all smooth one-forms on M. Similarly to vector fields, by defining

$$(\omega + \vartheta)_p = \omega_p + \vartheta_p \qquad (f\omega)_p = f(p)\omega_p$$

$\mathfrak{X}^*$ becomes a module over $\mathcal{F}(M)$.

Remark 1.48. The differential map of a function $f \in \mathcal{F}(M)$ can be regarded as a one-form by identifying the tangent space of $\mathbb{R}$ with $\mathbb{R}$ itself. We shall also denote it by df, and it fulfills $df(v) = v(f)$ for every tangent vector $v \in T_pM$.

Remark 1.49. For the special case of f being a coordinate function of a chart (U, φ) on M, $f = x^i$, their differentials $dx^1, ..., dx^n$ are called the *coordinate one-forms* on U. They are dual to the coordinate basis $\partial_1, ..., \partial_n$, as $dx^i(\partial_j) = \partial x^i/\partial x^j = \delta^i_j$. Therefore, the coordinate expressions for one-forms ω and differentials df are

$$\omega = \omega(\partial_i)dx^i = \omega_i dx^i, \qquad df = \frac{\partial f}{\partial x^i}dx^i.$$

Also, as opposed to vectors, one-forms behave *covariantly* under a change of coordinates $x^i \to y^i$, as $\omega(\frac{\partial}{\partial y^i}) = \frac{\partial x^j}{\partial y^i}\omega(\frac{\partial}{\partial x^j})$.

Definition 1.50. *A* smooth tensor field T *of type* (r, s) *on a manifold* M *is an* $\mathcal{F}(M)$*-multilinear map* $A : (\mathfrak{X}^*(M))^r \times (\mathfrak{X}(M))^s \to \mathcal{F}(M)$*. That is,* A *produces a smooth function when evaluated on* r *one-forms and* s *vector fields,* $A(\omega^1, ..., \omega^r; X_1, ..., X_s) \in \mathcal{F}(M)$*. We denote the set of all smooth* (r, s) *tensor fields on* M *as* $\mathfrak{T}^r_s(M)$*.*

Remark 1.51. Notice that smooth functions are tensor fields of type $(0, 0)$, vector fields are $(1, 0)$ tensor fields, and one-forms are tensor fields of type $(0, 1)$.

Proposition 1.52. *Let* (U, φ) *be a chart in* M *with coordinate functions* $x^1, ...x^n$*. If* $A \in \mathfrak{T}^r_s(M)$*, its local coordinate expression on* U *is*

$$A = A^{i_1,...,i_r}_{j_1,...,j_s}\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}, \qquad A^{i_1,...,i_r}_{j_1,...,j_s} = A(dx^{i_1}, ..., dx^{i_r}; \partial_{j_1}, ..., \partial_{j_s}).$$

Definition 1.53. *A* contraction C *is an operation that shrinks an* (r, s) *tensor field* A *to an* $(r-1, s-1)$ *tensor field* CA*. If* $1 \le k \le r$ *and* $1 \le l \le s$*, the contraction* C^k_l *acting on the tensor field* $A = A^{i_1,...,i_r}_{j_1,...,j_s}\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$ *is defined by*

$$\begin{aligned} C^k_l A = A^{i_1,...,i_{k-1},m,i_{k+1},...,i_r}_{j_1,...,j_{l-1},m,j_{l+1},...,j_s}\, &\partial_{i_1} \otimes \cdots \otimes \partial_{i_{k-1}} \otimes \partial_{i_{k+1}} \otimes \cdots \otimes \partial_{i_r} \\ &\otimes dx^{j_1} \otimes \cdots \otimes dx^{j_{k-1}} \otimes dx^{j_{k+1}} \otimes \cdots \otimes dx^{j_s}. \end{aligned}$$

Definition 1.54. *A* tensor derivation $\mathcal{D}$ *on a smooth manifold* M *is a set of* $\mathbb{R}$*-linear functions* $\{\mathcal{D}^r_s : \mathfrak{T}^r_s \to \mathfrak{T}^r_s\}_{\{r\geq 0, s\geq 0\}}$ *such that for any tensors* A, B *and any contraction* C,

1. $\mathcal{D}(A \otimes B) = \mathcal{D}A \otimes B + A \otimes \mathcal{D}B$.
2. $\mathcal{D}(CA) = C\mathcal{D}(A)$.

Comparing to Def. 1.29 we can see how the case for $(0, 0)$ tensors matches the definition of derivations on $\mathcal{F}(M)$.

Proposition 1.55. *Let* D *be a tensor derivation on* M *and let* $A \in \mathfrak{T}^r_s$*. Then,*

$$\begin{aligned}\mathcal{D}\Big(A(\omega^1, ..., \omega^r; X_1, ..., X_s)\Big) &= (\mathcal{D}A)(\omega^1, ..., \omega^r; X_1, ..., X_s) \\ &\quad + A(\mathcal{D}\omega^1, ..., \omega^r; X_1, ..., X_s) + \cdots + A(\omega^1, ..., \mathcal{D}\omega^r; X_1, ..., X_s) \\ &\quad + A(\omega^1, ..., \omega^r; \mathcal{D}X_1, ..., X_s) + \cdots + A(\omega^1, ..., \omega^r; X_1, ..., \mathcal{D}X_s).\end{aligned}$$

Theorem 1.56. *Given a smooth vector field* $X \in \mathfrak{X}(M)$*, there is a unique tensor derivation* $£_X$ *such that*

1. $£_X f = X(f)$.
2. $£_X Y = [X, Y]$.

We shall call it the Lie derivative *relative to* X.

In Lemma 1.44 and Prop. 1.45 we have seen that the Lie derivative relative to X of functions and vector fields can be interpreted as their rate of change along the flow of X. In fact, this property can be extended to all tensor fields on M, and therefore many books choose to define the Lie derivative in terms of flows and then prove that it is indeed a tensor derivation (e.g. [2] or [13]). We shall give a proof of this equivalence for $(0, 2)$ tensor fields in Chapter 4.

Let us end this chapter showing how the Lie derivative relative to the bracket $[X, Y]$ of two vector fields behaves. We will write $[£_X, £_Y] := £_X \circ £_Y - £_Y \circ £_X$.

Proposition 1.57. *Let* $X, Y \in \mathfrak{X}(M)$ *and* $f \in \mathcal{F}(M)$*. Then,* $£_{[X,Y]}f = [£_X, £_Y]f$.

Proof. $£_X(£_Y f) - £_Y(£_X f) = X(Y(f)) - Y(X(f)) = (XY)(f) - (YX)(f) = [X, Y](f)$. ☐

Proposition 1.58. *Let* $X, Y, Z \in \mathfrak{X}(M)$*. Then,* $£_{[X,Y]}Z = [£_X, £_Y]Z$.

Proof.

$$\begin{aligned}£_X(£_Y Z) - £_Y(£_X Z) &= \Big[X, [Y, Z]\Big] - \Big[Y, [X, Z]\Big] = [X, YZ - ZY] - [Y, XZ - ZX] \\ &= XYZ - \cancel{XZY} - \cancel{YZX} + ZYX - YXZ + \cancel{YZX} + \cancel{XZY} - ZXY \\ &= [X, Y]Z - Z[X, Y] = \Big[[X, Y], Z\Big] = £_{[X,Y]}Z.\end{aligned}$$ ☐

Following these two propositions, and using the product rule (Prop. 1.55), one can prove similar results for the Lie derivative of any tensor field, finally proving the result $£_{[X,Y]} = [£_X, £_Y]$ by induction over the tensor ranks.

Chapter 2

Lie Groups

Informally, a Lie group is a mathematical structure which is both a group and a smooth manifold in which the group operation is smooth.

Lie groups are named after Sophus Lie, a Norwegian mathematician who laid the foundations for studying continuous transformation groups during the late 1880s. This application of Lie groups is described in section 4 of this chapter, and it is of utmost importance in both mathematics and theoretical physics. Lie's original goal was to build an equivalent to Galois theory for differential equations, which he achieved: the first Lie groups arose as symmetry groups of transformations of the variables involved in differential equations, that is, transformations that would send solutions to solutions. And the first Lie algebras described as such where the *infinitesimal* versions of such transformations. Nowadays this has been generalized to Lie transformation groups describing how a manifold can be transformed smoothly through a Lie group, thus providing a natural model for the concept of continuous symmetry. Its other relations to physics lie on representation theory (which we shall not consider), widely used in particle physics, and on Emmy Noether's theorem, which states that for every differentiable symmetry on a physical system, that is, for every differentiable transformation which leaves the Lagrangian of the system invariant, there is a corresponding conservation law.

Simultaneously and independently, the German mathematician Wilhelm Killing had also began to study Lie groups and Lie algebras. His works considered Lie groups regardless of their possible action on manifolds, as it is done today. However, it was not until the beginning of the XXth century that the great French mathematician Élie Cartan set the grounds for modern Lie theory, extending the theory to global terms.

We begin this chapter defining both Lie groups and Lie algebras. We show that for every Lie group G there is a Lie algebra $\mathfrak{g}$ related to the set of left-invariant vector fields on G. We then define the exponential map, which interprets the Lie algebra $\mathfrak{g}$ as an infinitesimal correspondent of the Lie group G. The sequence followed is standard to almost all books on Lie Group Theory. Our main references are [2], [8] and [14]. The interested reader will find more information in [15], which is the classical reference on Lie Group Theory. We end this Chapter introducing Lie transformation groups and proving the theorem by Palais mentioned in in the introduction on which we will build our proof that the isometry group of a semi-Riemannian manifold is a Lie group.

2.1 Basic Definitions: Lie Groups and Lie Algebras

Definition 2.1. *A* Lie group *G is a differentiable manifold which is also endowed with a group structure such that the map*

$$\begin{aligned} G \times G &\longrightarrow G \\ (x, y) &\longmapsto xy^{-1} \end{aligned}$$

is smooth. The symbol e *will usually denote the identity element of the Lie group.*

Proposition 2.2. *Let G be a Lie group. Then,*

1. *The map $x \mapsto x^{-1}$ of $G \to G$ is smooth.*
2. *The map $(x, y) \mapsto xy$ of $G \times G \to G$ is smooth.*

Proof.

1. The inverse map is the composition of smooth maps $x \mapsto (e, x) \mapsto ex^{-1} = x^{-1}$, so it is also smooth.
2. The multiplication map is the composition of smooth maps $(x, y) \mapsto (x, y^{-1}) \mapsto x(y^{-1})^{-1} = xy$, so it is also smooth.

□

Example 2.3.

1. The real space $\mathbb{R}^n$ is a Lie group under vector addition.
2. The group $Gl(n, \mathbb{R})$ of all $n \times n$ invertible real matrices is a n^2-dimensional Lie group under matrix multiplication. Its manifold structure comes from being a submanifold of $\mathbb{R}^{n^2}$. Many subgroups of $Gl(n, \mathbb{R})$ are Lie groups, such as the example below.
3. The orthogonal group $O(n, \mathbb{R}) = \{A \in Gl(n, \mathbb{R}) \mid A^{-1} = A^t\}$ is also a Lie group under matrix multiplication. It is compact, and has two connected components. Its identity component is $O^+(n) = SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) \mid \det A = +1\}$, the special orthogonal group (which is also a Lie group by itself), and $O^-(n) = \{A \in O(n, \mathbb{R}) \mid \det A = -1\}$. $O(n, \mathbb{R})$ has dimension $n(n-1)/2$.

Definition 2.4. *Let G be a Lie group, and let $a \in G$. We define the* left translation by a, *$\mu_a : G \to G$ by $x \mapsto \mu_a(x) = ax$; and the* right translation by a, *$\mu^a : G \to G$ by $x \mapsto \mu^a(x) = xa$.*

Clearly such maps are diffeomorphisms of G onto G. Therefore their respective differential maps, $d\mu_a : T_xG \to T_{ax}G$ and $d\mu^a : T_xG \to T_{xa}G$, are linear isomorphisms. Be careful with the notation: here the subscript a under $d\mu$ does not denote the point at which $d\mu$ acts, but that $d\mu_a$ is the differential map of μ_a, the left translation by a. Throughout all this section, we shall omit the subscripts of differential maps for the sake of clarity when using the notation d. We shall introduce a different notation when subscripts are needed.

Definition 2.5. *A smooth vector field X on G is called* left-invariant *(respectively* right-invariant*) if for each $a \in G$, X is μ_a-related (respectively μ^a-related) to itself. Recalling definition 1.31, that means that X is left-invariant if*

$$d\mu_a \circ X = X \circ \mu_a.$$

Informally, this translates to X being left-invariant if "its evaluation at $\mu_a(x)$ is the same as its evaluation at x transported by $d\mu_a$ to $\mu_a(x)$". Therefore left-invariant vector fields can be represented by their evaluation at any point, as left-translation by $d\mu_a$ will define them on the rest of G. In particular, we shall consider their evaluation at e.

Notice also that we have defined left-invariant vector fields as being smooth vector fields. In fact, we need not have done so, as all μ_a-related vector fields on a Lie group must necessarily be smooth. (See Warner [8] Proposition 3.7).

Proposition 2.6. *Let G be a Lie group and $\mathfrak{g}$ the set of left-invariant vector fields on G. Then $\mathfrak{g}$ is a real vector space (a subspace of $\mathfrak{X}(G)$), and it is isomorphic to the tangent space at the identity, T_eG, through its evaluation at the identity.*

Proof. That $\mathfrak{g}$ is a real vector space is immediate from the linearity of the tangent map $d\mu_a$ and the definition of $\mathfrak{X}(G)$ as a vector space. In order to see that $\mathfrak{g} \cong T_eG$ we have to see that the evaluation map $X \mapsto X_e$ from $\mathfrak{g}$ to T_eG is both injective and surjective. That it is a vector space morphism is again immediate from the definition of $\mathfrak{X}(G)$ as a vector space.

Now, if $X_e = Y_e$, then for all $z \in G$ we have $X_z = d\mu_z(X_e) = d\mu_z(Y_e) = Y_z$, therefore $X = Y$ as they coincide on every point z of G. So the map is injective. Finally, let $v \in T_eG$, and define a vector field X such that $X_z = d\mu_z(v)$ for each $z \in G$. Then $X_e = d\mu_e(v) = id(v) = v$, and since $X_{xy} = d\mu_{xy}(v) = d\mu_x d\mu_y(v) = d\mu_x(X_y)$ for all $x, y \in G$, X is left-invariant. Hence, the evaluation map is also surjective. $\square$

Definition 2.7. *A* Lie algebra *over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ equipped with a bilinear operator, the* bracket *operator $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that for all $X, Y, Z \in \mathfrak{g}$,*

1. $[X,Y] = -[Y,X]$.
2. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

We will usually denote Lie algebras by their vector space $\mathfrak{g}$ without specifying their bracket operation. However, one must be careful, different bracket operations applied to the same vector space can give rise to different Lie algebras.

Example 2.8.

1. The real line $\mathbb{R}$ is a Lie algebra taking all brackets to be identically 0.
2. In fact, any vector space becomes a Lie algebra if all brackets are set equal to 0. These kind of Lie algebras are called *abelian* Lie algebras.
3. The set $\mathfrak{gl}(n, \mathbb{R})$ of all $n \times n$ real matrices is a Lie algebra if we set $[A, B] = AB - BA$ for all $A, B \in \mathfrak{gl}(n, \mathbb{R})$.
4. The set of all $n \times n$ skew-symmetric matrices $\mathfrak{o}(n, \mathbb{R}) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^t = -A\}$ is a Lie algebra with the same bracket as $\mathfrak{gl}(n, \mathbb{R})$. It has dimension $n(n-1)/2$.
5. The vector space $\mathfrak{X}(M)$ of all smooth vector fields on a manifold M forms a Lie algebra under the Lie bracket operation defined in 1.40.

It is straightforward to check both properties of Def. 2.7 in all cases above. The last example is immediate from Proposition 1.41.

Proposition 2.9. *The vector space $\mathfrak{g}$ of all left-invariant vector fields of a Lie group G forms a Lie algebra under the bracket operation on vector fields.*

Proof. We first have to prove that the bracket of two left-invariant vector fields is also left-invariant, that is, given $X, Y \in \mathfrak{g}$ and for all $x \in G$, to see that $d\mu_x \circ [X,Y] = [X,Y] \circ \mu_x$. Then, to see that it forms a Lie algebra is immediate from the fact that the bracket operation of vector fields anti-commutes and fulfills the Jacobi identity, as seen in Prop. 1.41. So, we must show that $d\mu_x([X,Y]_y)(f) = [X,Y]_{xy}(f)$, for all $y \in G$ and $f \in \mathcal{F}(M)$:

$$\begin{aligned} d\mu_x([X,Y]_y)(f) &= [X,Y]_y(f \circ \mu_x) \\ &= X_y(Y(f \circ \mu_x)) - Y_y(X(f \circ \mu_x)) \\ &= X_y((d\mu_x \circ Y)(f)) - Y_y((d\mu_x \circ X)(f)) \\ &= X_y(Y(f) \circ \mu_x)) - Y_y(X(f) \circ \mu_x)) \\ &= d\mu_x(X_y)(Y(f)) - d\mu_x(Y_y)(X(f)) \\ &= X_{xy}(Y(f)) - Y_{xy}(X(f)) \\ &= [X,Y]_{xy}(f). \end{aligned}$$

□

Definition 2.10. *The Lie algebra $\mathfrak{g}$ of left-invariant vector fields of a Lie group G is called the* Lie algebra of the Lie group *G.*

As we have seen in Prop. 2.6 we can identify $\mathfrak{g}$ with T_eG, so T_eG with the corresponding bracket operation is also sometimes called the Lie algebra of the Lie group G. We will use either meaning depending on the context.

Example 2.11.

1. The Lie algebra of the Lie group $Gl(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{R})$.
2. The Lie algebra of the Lie groups $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ is $\mathfrak{o}(n, \mathbb{R})$.

2.2 Homomorphisms and Subgroups

Within this section on homomorphisms, we will use the notation $T_x\phi$ to denote the differential map $d\phi_x : T_xG \to T_{\phi(x)}H$ of a smooth map $\phi : G \to H$. This is done in order to avoid confusion between subindexes, as the tangent map of the left-multiplication map $\mu_a : G \to G$ will appear quite frequently in what follows, and the following proofs require keeping track of the point at which the map is acting.

Definition 2.12. *Let G, H be Lie groups. A map $\phi : G \to H$ is a* Lie group homomorphism *if ϕ is a group homomorphism of the abstract groups (i.e., $\phi(xy) = \phi(x)\phi(y)\ \forall x, y \in G$), and is also a smooth map between manifolds. A Lie group homomorphism is called an* isomorphism *if it is injective and surjective. A Lie group homomorphism of G onto itself is called an* automorphism.

Proposition 2.13. *Let $\phi : G \to H$ be a Lie group homomorphism. Let $x \in G$ Then,*

1. $\phi(e_G) = e_H$.
2. $\phi(x^{-1}) = (\phi(x))^{-1}$.

Proof.

1. $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G) \;\Rightarrow\; \phi(e_G) = e_H$.
2. $e_H = \phi(e_G) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}) \;\Rightarrow\; \phi(x^{-1}) = (\phi(x))^{-1}$.

□

Example 2.14. For $a \in G$, we define the *adjoint* map $\mathrm{ad}_a : G \to G$ by $x \mapsto \mathrm{ad}_a(x) = axa^{-1}$. It is an inner automorphism of G, since for all $x, y \in G$

$$\mathrm{ad}_a(xy) = axya^{-1} = axeya^{-1} = axa^{-1}aya^{-1} = (\mathrm{ad}_a(x))(\mathrm{ad}_a(y)).$$

Definition 2.15. *A map $\psi : \mathfrak{g} \to \mathfrak{h}$ is a* Lie algebra homomorphism *if it is a linear map between vector spaces, and it also preserves brackets: $\psi([X,Y]) = [\psi(X), \psi(Y)]\ \forall X, Y \in \mathfrak{g}$.*

Remark 2.16. There is a natural relation between homomorphisms of Lie groups and homomorphisms of their Lie algebras. As we have seen in 2.13, a Lie group homomorphism $\phi : G \to H$ maps the identity e_G of G to the identity e_H of H. Therefore the tangent map $T_{e_G}\phi$ of ϕ linearly sends $T_{e_G}G$ into $T_{e_H}H$. As we have seen in 2.6, $\mathfrak{g} \cong T_{e_G}G$ and $\mathfrak{h} \cong T_{e_H}H$, so the tangent map induces a linear transformation of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{h}$ of H. Therefore we can define $\phi' : \mathfrak{g} \to \mathfrak{h}$ such that for all $X \in \mathfrak{g}$, $\phi'(X) \in \mathfrak{h}$ is the unique left-invariant field on H that fulfills $(\phi'(X))_{e_H} = T_{e_G}\phi(X_{e_G})$.

Theorem 2.17. *Let $\phi : G \to H$ be a homomorphism between Lie groups, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their respective Lie algebras. Then,*

1. *For all left-invariant vector field $X \in \mathfrak{g}$, X and $\phi'(X)$ are ϕ-related.*
2. *$\phi' : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. To prove (1.) we have to see that for all $x \in G$, $T_x\phi(X_x) = (\phi'(X))_{\phi(x)}$:

$$\begin{aligned} T_x\phi(X_x) = T_x\phi(T_{e_G}\mu_x(X_e)) &= T_{e_G}(\phi \circ \mu_x)(X_{e_G}) = T_{e_G}(\mu_{\phi(x)} \circ \phi)(X_{e_G}) \\ &= T_{e_H}\mu_{\phi(x)}(T_{e_G}\phi(X_{e_G})) = T_{e_H}\mu_{\phi(x)}((\phi'(X))_{e_H}) = (\phi'(X))_{\phi(x)}. \end{aligned}$$

Where the third equality is fulfilled due to ϕ being a homomorphism.

Now, to prove (2.) we only need to see that the Lie bracket operation is preserved, as linearity has already been discussed in 2.16. That is, we need to see that $\phi'([X,Y]) = [\phi'(X), \phi'(Y)]$ for all $X, Y \in \mathfrak{g}$. In order to do so, let's see first that $[X,Y]$ is ϕ-related to $[\phi'(X), \phi'(Y)]$. We must show that for all $x \in G$ and for all $f \in \mathcal{F}(G)$, $T_x\phi([X,Y]_x)(f) = [\phi'(X), \phi'(Y)]_{\phi(x)}(f)$. Following a similar path as in 2.9, and using the result just derived in (1.), we can write

$$\begin{aligned} \phi'([X,Y])_{\phi(x)} &= T_x\phi([X,Y]_x)(f) \\ &= [X,Y]_x(f \circ \phi) \\ &= X_x(Y(f \circ \phi)) - Y_x(X(f \circ \phi)) \\ &= X_x(\phi'(Y)(f) \circ \phi)) - Y_x(\phi'(X)(f) \circ \phi)) \\ &= T_x\phi(X_x)(\phi'(Y)(f)) - T_x\phi(Y_x)(\phi'(X)(f)) \\ &= \phi'(X)_{\phi(x)}(\phi'(Y)(f)) - \phi'(Y)_{\phi(x)}(\phi'(X)(f)) \\ &= [\phi'(X), \phi'(Y)]_{\phi(x)}(f). \end{aligned}$$

So we see that $[X,Y]$ is ϕ-related to $[\phi'(X),\phi'(Y)]$. In particular, if we take the previous equality at $x = e_G$, we have that $[\phi'(X),\phi'(Y)]_{e_H} = T_{e_G}\phi([X,Y]_{e_G})$. But by the definition of ϕ' (Remark 2.16), $\phi'([X,Y])$ is the unique left-invariant vector field on H whose value at e_H is $T_{e_G}\phi([X,Y]_{e_G})$, so $\phi'([X,Y]) = [\phi'(X),\phi'(Y)]$. □

Example 2.18. Let $a \in G$. The automorphism $\mathrm{ad}_a : G \to G$ defined on Example 2.14 induces a Lie algebra automorphism $\mathrm{ad}'_a : \mathfrak{g} \to \mathfrak{g}$. We have $\mathrm{ad}_a(x) = axa^{-1} = \mu_a\mu^{a^{-1}}(x) = \mu^{a^{-1}}\mu_a(x)$. As X is left-invariant, $T_x(\mu^{a^{-1}}\mu_a)(X_x) = (T_{ax}\mu^{a^{-1}})(T_x\mu_a(X_x)) = T_{ax}\mu^{a^{-1}}(X_{ax})$, so $\mathrm{ad}'_a(X)$ is the unique left-invariant field on G that fulfills $(\mathrm{ad}'_a(X))_e = T_a\mu^{a^{-1}}(X_a)$.

Definition 2.19. *Let G be a Lie group. (H,ϕ) is a* Lie subgroup *of the Lie group G if*

1. *H is a Lie group.*
2. *(H,ϕ) is a submanifold of G.*
3. *$\phi : H \to G$ is a Lie group homomorphism.*

Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra *if the bracket induced from $\mathfrak{g}$ fulfills $[X,Y] \in \mathfrak{h}$ for all $X,Y \in \mathfrak{h}$, guaranteeing that $\mathfrak{h}$ is also a Lie algebra.*

Proposition 2.20. *Let (H,ϕ) be a Lie subgroup of G. Let $\mathfrak{h}$ and $\mathfrak{g}$ be their respective Lie algebras. Then, $\phi' : \mathfrak{h} \to \phi'(\mathfrak{h}) \subseteq \mathfrak{g}$ is a Lie algebra isomorphism.*

Proof. Immediate from 2.17. □

Thus, for every Lie subgroup (H,ϕ) of a Lie group G there is a Lie subalgebra of its Lie algebra $\mathfrak{g}$ isomorphic to the Lie algebra $\mathfrak{h}$. The converse is also true for connected Lie subgroups, and therefore there is a one-to-one correspondence between connected Lie subgroups of a Lie group and subalgebras of its Lie algebra. We refer the reader to Warner [8] 3.19 for a proof of this theorem, presented below.

Theorem 2.21. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\tilde{\mathfrak{h}} \subseteq \mathfrak{g}$ be a subalgebra. Then, there is a unique (up to isomorphism) connected Lie subgroup (H,ϕ) of G with Lie algebra $\mathfrak{h}$ such that $\phi'(\mathfrak{h}) = \tilde{\mathfrak{h}}$.*

Definition 2.22. *Let S be a set of vectors of a Lie algebra $\mathfrak{g}$. We say that the Lie algebra $\mathfrak{h}$ is* generated *by S if $\mathfrak{h} \subseteq \mathfrak{g}$ is the smallest Lie algebra that contains S.*

By the smallest Lie algebra that contains S, we mean that for all Lie algebra $\mathfrak{k}$ such that $S \subseteq \mathfrak{k}$, also $\mathfrak{h} \subseteq \mathfrak{k}$ as a subalgebra. Notice that $\mathfrak{h}$ must contain all linear combinations of vectors of S, and all brackets of linear combinations of vectors of S (which are linear combinations of brackets of vectors of S, by bilinearity of the bracket operation), where the bracket defined on $\mathfrak{h}$ is the one induced by $\mathfrak{g}$.

2.3 The Exponential Map

The exponential map is a fundamental tool in Lie Group Theory, which further characterizes the strong existing relation between Lie groups and Lie algebras. From now on, we shall go back to the d notation for the differential map, omitting subscripts whenever their presence is understood from context.

Definition 2.23. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. A* 1-parameter subgroup *of G is a Lie group homomorphism $\phi : \mathbb{R} \to G$.*

Visually, ϕ is a smooth curve in G that preserves the group structure, that is, $\phi(s+t) = \phi(s)\phi(t)$ and therefore $\phi(0) = e$.

Lemma 2.24. *Let G be a Lie group. Let $\alpha : \mathbb{R} \to G$ be a smooth curve with $\alpha(0) = e$. Let $X \in \mathfrak{g}$ be a left-invariant vector field on G. Then, the following are equivalent:*

1. *α is a 1-parameter subgroup with $X_e = \frac{d\alpha(t)}{dt}\big|_0$.*
2. *$\alpha(t) = Fl^X(t, e)$, for all $t \in \mathbb{R}$.*
3. *$x\alpha(t) = Fl^X(t, x)$, (i.e. $\mu^{\alpha(t)} = Fl^X_t$), for all $t \in \mathbb{R}$ and $x \in G$.*

Informally, equivalence (1.) $\iff$ (2.) of this lemma states that the integral curves starting at e generated by flows of left-invariant vector fields on G are always 1-parameter subgroups of G, and *vice versa*.

Proof. (1.) $\Rightarrow$ (3.) : Given $x \in G$, differentiation of the product $x\alpha(t)$ yields

$$\begin{aligned}\frac{d}{dt}x\alpha(t) = \frac{d}{ds}\Big|_0 x\alpha(t+s) = \frac{d}{ds}\Big|_0 x\alpha(t)\alpha(s) &= \frac{d}{ds}\Big|_0 \mu_{x\alpha(t)}\alpha(s) \\ &= d\mu_{x\alpha(t)}\frac{d\alpha(s)}{ds}\Big|_0 = d\mu_{x\alpha(t)}(X_e) = X_{x\alpha(t)},\end{aligned}$$

where the second equality holds because α is a 1-parameter subgroup. The formula above defines a differential equation, from which together with the initial condition $X_e = \frac{d\alpha(t)}{dt}\big|_0$ we obtain $x\alpha(t) = Fl^X(t, x)$, by existence and uniqueness of solutions (Theorem 1.36).

(3.) $\Rightarrow$ (2.) : Obvious, substituting $x = e$ in (3.).

(2.) $\Rightarrow$ (1.) : α is an integral curve of X, so $\frac{d\alpha(s)}{ds} = X_{\alpha(s)}$. Hence, $X_e = \frac{d\alpha(t)}{dt}\big|_0$, since $\alpha(0) = Fl^X(0, e) = e$. Moreover,

$$\frac{d}{ds}\alpha(t)\alpha(s) = \frac{d}{ds}\mu_{\alpha(t)}\alpha(s) = d\mu_{\alpha(t)}\frac{d}{ds}\alpha(s) = d\mu_{\alpha(t)}d(X_{\alpha(s)}) = X_{\alpha(t)\alpha(s)},$$

which again defines a differential equation. We set $\alpha(t)\alpha(0) = \alpha(t)$ as its initial condition. The solution is $\alpha(t)\alpha(s) = Fl^X(s, \alpha(t)) = Fl^X_s Fl^X_t(e) = Fl^X(t+s, e) = \alpha(t+s)$, so α is a homomorphism and therefore a 1-parameter subgroup. $\square$

Corollary 2.25. *Left-invariant vector fields on a Lie group are always complete.*

Proof. Any locally defined 1-parameter subgroup (hence a homomorphism $\mathbb{R} \to G$) can be extended to a globally defined one simply by multiplication: $\alpha(nt) = (\alpha(t))^n$. This gives us that the integral curve of a left-invariant vector field X going through e is defined for all $t \in \mathbb{R}$. For the integral curves $Fl^X(t, x)$ starting at $x \neq e$, one need only use property (3.) of the previous Lemma 2.24, which yields $Fl^X_t(x) = xFl^X_t(e)$ therefore proving that this integral curve is also defined for all real t. $\square$

Remark 2.26. The previous lemma and corollary can be adapted and applied also to right-invariant vector fields on G.

Definition 2.27. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The* exponential mapping *of the Lie group is a map* $\exp : \mathfrak{g} \to G$ *defined by*

$$\exp(X) = Fl^X(1, e) = \alpha_X(1)$$

for all $X \in \mathfrak{g}$, where α_X is the 1-parameter subgroup of G with $\frac{d\alpha(t)}{dt}|_0 = X_e$.

Theorem 2.28.

1. $\exp$ *: $\mathfrak{g} \to G$ is smooth.*
2. $\exp\ (tX) = Fl^X(t, e)$ *for all $t \in \mathbb{R}$.*
3. $Fl^X(t, x) = x \cdot \exp(tX)$.
4. $\exp$ *is a diffeomorphism from a neighborhood of* 0 *in* $\mathfrak{g}$ *onto a neighborhood of e in G.*

Proof.

1. Let $X \in \mathfrak{g}$ and let $x \in G$. Build the manifold $\mathfrak{g} \times G$, and define a smooth vector field $\mathfrak{X}$ in $\mathfrak{g} \times G$ by $\mathfrak{X}_{(X,x)} = (0_X, X_x)$. Then, this vector field defines a smooth flow $Fl^{\mathfrak{X}} : \mathbb{R} \times \mathfrak{g} \times G \to \mathfrak{g} \times G$, which projected onto G gives $\pi_G(Fl^{\mathfrak{X}}(t, (X, e)) = \alpha_X(t)$, which is smooth as it is a composition of smooth maps.
2. $\exp\ (tX) = Fl^{tX}(1, e) = Fl^X(t, e)$.
3. Directly from (3.) of the previous Lemma 2.24.
4. If we regard $\mathfrak{g}$ as a vector space manifold, its tangent space at 0 can be identified as $\mathfrak{g}$ itself, $T_0\mathfrak{g} \cong \mathfrak{g}$. So the differential map $d\exp$ can be seen as acting on $\mathfrak{g}$ rather than $T_0\mathfrak{g}$. And, since $d\exp(X) = \frac{d}{dt}|_0 \exp(0 + tX) = \frac{d}{dt}|_0 Fl^X(t, e) = X$, therefore $d\exp$ $: \mathfrak{g} \cong T_0\mathfrak{g} \to T_eG \cong \mathfrak{g}$ is the identity and hence an isomorphism, so we can apply the Inverse Function Theorem.

□

Example 2.29. For $Gl(n, \mathbb{R})$ and all its classical subgroups, we have

$$\exp(A) = e^A := \mathbb{1} + A + \frac{A^2}{2} + \cdots + \frac{A^k}{k!} + \cdots .$$

Proposition 2.30. *Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$. Let $\phi : G \to H$ be a Lie group homomorphism. Then, for all $X \in \mathfrak{g}$,*

$$\phi(\exp(X)) = \exp(\phi'(X)).$$

Proof. For each $X \in \mathfrak{g}$, let $\alpha_X : \mathbb{R} \to G$ be the 1-parameter subgroup of G defined by $\alpha_X(t) = \exp(tX)$. It is a Lie group homomorphism, and the corresponding Lie algebra homomorphism is $\alpha'_X : T_0\mathbb{R} \cong \mathbb{R} \to \mathfrak{g}$, defined by $\alpha'_X(\frac{d}{dt}) = X$. Therefore, the map $(\phi \circ \alpha_X) : \mathbb{R} \to H$ is a Lie group homomorphism with corresponding Lie algebra homomorphism $(\phi \circ \alpha_X)' : \mathbb{R} \to \mathfrak{h}$ defined by $(\phi \circ \alpha_X)'(\frac{d}{dt}) = \phi'(X)$. Now, if $\beta_{\phi'(X)} : \mathbb{R} \to H$ is the 1-parameter subgroup of H defined by $\beta_{\phi'(X)}(t) = \exp(t\phi'(X))$, then also $\beta'_{\phi'(X)}(\frac{d}{dt}) = \phi'(X)$. Therefore $\beta'_{\phi'(X)} = (\phi \circ \alpha_X)'$, from which follows that $\beta_{\phi'(X)} = \phi \circ \alpha_X$ by uniqueness (Remark 2.16). Hence, $\exp(\phi'(X)) = \beta_{\phi'(X)}(1) = (\phi \circ \alpha_X)(1) = \phi(\exp(X))$. □

Example 2.31. Again, given $a \in G$, consider the Lie group automorphism $\mathrm{ad}_a : G \to G$ with Lie algebra automorphism $\mathrm{ad}'_a : \mathfrak{g} \to \mathfrak{g}$ described in previous examples. Then, the previous proposition yields, for all $X \in \mathfrak{g}$,

$$\exp(\mathrm{ad}'_a(X)) = \mathrm{ad}_a(\exp(X)) = a\exp(X)a^{-1}.$$

Proposition 2.32. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, for all $X, Y \in \mathfrak{g}$,*

$$[X,Y] = \lim_{t\to 0} \frac{1}{t}[ad'_{exp(tX)}(Y) - Y].$$

Proof. In this proof we shall omit the subscript indicating the point at which the differential map is acting for the sake of clarity. Recalling Prop. 4.7,

$$\begin{aligned}[X,Y]_e &= \lim_{t\to 0}\frac{1}{t}[dFl^X_{-t}(Y_{Fl^X_t(e)}) - Y_e] = \lim_{t\to 0}\frac{1}{t}[d\mu^{\exp(-tX)}(Y_{\exp(tX)}) - Y_e] \\ &= \lim_{t\to 0}\frac{1}{t}[d\mu^{\exp(-tX)}d\mu_{\exp(tX)}(Y_e) = \lim_{t\to 0}\frac{1}{t}[d(\mathrm{ad}_{exp(tX)})(Y_e) - Y_e].\end{aligned}$$

□

Lemma 2.33. *Let G be a connected Lie group and let U be an open neighborhood of e. Then, $G = \bigcup_{n=1}^{\infty} U^n$, where $U^n = \{ x_1x_2\cdots x_n \mid x_i \in U \ \forall i = 1,...,n \}$. We say that U* generates *G.*

Proof. Let $V = U \cap U^{-1}$, where $U^{-1} = \{ x^{-1} \mid x \in U \}$. V is an open subset of U which contains e and fulfills $V = V^{-1}$. Since $V \subseteq U$, $H := \bigcup_{n=1}^{\infty} V^n \subseteq \bigcup_{n=1}^{\infty} U^n$. Then H is an abstract subgroup of G, as it also fulfills $H = H^{-1}$. Moreover, it is an open subset of G, since for every $x \in H$, $xV \subseteq H$. Therefore, each coset mod H is also open in G. Now, H is the complement in G of the union of all the open cosets mod H different from H itself. Hence, H is also closed, as it is the complement of an open set. As G is connected, the only subsets of G which are both open and closed are the empty set and G itself. But H is non-empty because $e \in H$. Therefore $H = G$, and since $H = \bigcup_{n=1}^{\infty} V^n \subseteq \bigcup_{n=1}^{\infty} U^n \subseteq G$, $G = \bigcup_{n=1}^{\infty} U^n$. □

Corollary 2.34. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let V be a neighborhood of 0 in $\mathfrak{g}$. Then the group generated by $\exp(V)$ equals G.*

Proof. Immediate from Lemma 2.33 and (4.) of Theorem 2.28 above. □

Remark 2.35. If G is not connected, then the subgroup generated by $\exp(V)$ is the connected component of the identity in G.

Remark 2.36. One could mistakenly believe that Corollary 2.34 extends the diffeomorphisms described in Theorem 2.28(4.) from a neighborhood of e to all the connected component of the identity, but that is not the case. For example, the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = \exp\begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}\exp\begin{pmatrix} 0 & 0 \\ 0 & \ln 2 \end{pmatrix} \in Gl_0(2,\mathbb{R}),$$

where $Gl_0(n,\mathbb{R}) = \{B \in Gl(n,\mathbb{R}) \mid \det B > 0\}$ is connected component of the identity of $Gl(n,\mathbb{R})$. By Cor. 2.34, $Gl_0(2,\mathbb{R}) = \bigcup_{n=1}^{\infty}(\exp(V))^n$ for some neighborhood V of 0 in $\mathfrak{gl}(2,\mathbb{R})$, as indeed A suggests. However, there is no $\mathfrak{A} \in \mathfrak{gl}(2,\mathbb{R})$ such that $A = \exp(\mathfrak{A})$.

We have seen that every Lie group G has a Lie algebra $\mathfrak{g}$ associated, and that in fact $\mathfrak{g}$ generates G through the exponential map if G is connected, thus emphasizing the fact that Lie algebras can be interpreted as the infinitesimal versions of Lie groups. Moreover, we have seen a one-to-one correspondence between connected subgroups of a Lie group and Lie subalgebras of its Lie algebra (Theorem 2.21). Is this correspondence true in general? Obviously every connected Lie group G has its Lie algebra $\mathfrak{g}$, but is there also a connected Lie group associated to every Lie algebra? The answer is affirmative, though not easy to prove at all. Its first version was proved locally by Lie, and is known as Lie's 3rd theorem. The global version states that for every finite dimensional Lie algebra $\tilde{\mathfrak{g}}$ there is a Lie group G whose Lie algebra $\mathfrak{g}$ is isomorphic to $\tilde{\mathfrak{g}}$. It was not until 1924 that É. Cartan proved the global case, thus it is also called the Cartan-Lie theorem. Several proofs exist, but all are long and require deeper concepts.

Modern stronger versions of this theorem have been developed, such as Ado's theorem, which states that every finite dimensional Lie algebra has a faithful representation in $\mathfrak{gl}(n, \mathbb{C})$.

One of its useful consequences is the theorem stated below. For a discussion on the pathway to such result, see Warner [8], although no proof is given either. For a short proof, though not easy, of the result below, see Gorbatsevich [18].

Theorem 2.37. *Let $\mathfrak{g}$ be a Lie algebra. Then there exists a connected, simply connected Lie group G with Lie algebra $\mathfrak{g}$.*

2.4 Lie Transformation Groups

Throughout this section the elements of a Lie group G will be denoted by the letters g, h; and the points of a manifold M by the letters p or x, y, z.

Definition 2.38. *A* Lie transformation group *of a manifold M is a pair (G, M) where G is a Lie group and where to each $g \in G$ there is a given diffeomorphism of M, $l_g \in \mathrm{Diff}(M)$, such that*

$$\begin{aligned} l : G \times M &\longrightarrow M \\ (g, p) &\longmapsto l(g, p) = l_g(p) \end{aligned}$$

is smooth and fulfills $l_g \circ l_h = l_{gh}$ and $l_e = Id_M$. Such a map l is called a left action *of the Lie group G on M. The* partial mappings *of l are denoted by $l_g : M \to M$ and $l^p : G \to M$, given by $l(g, p) = l_g(p) = l^p(g) = g \cdot p$ for all $g \in G$ and $p \in M$.*

Remark 2.39. Again, as in the previous section, one can also define Lie transformation groups through right actions rather than left actions.

We jump directly to the main result of this chapter. The following theorem, due to Palais [1], is the main stone on which we will build the proof that the group of isometries of a semi-Riemannian manifold is a Lie transformation group. It provides a necessary condition for an abstract group of automorphisms of a smooth manifold M to be a Lie transformation group. The proof given is from Chu-Kobayashi [5], adapted from Kobayashi [6].

Theorem 2.40. *Let G be a group of diffeomorphisms of a manifold M onto itself. Let S be the set of all vector fields $\tilde{X}$ on M which generate global 1-parameter groups $\phi_t = Fl_t^{\tilde{X}}$ of transformations of M such that $\phi_t \in G$. If the set S generates a finite-dimensional Lie algebra of vector fields on M, then (G, M) is a Lie transformation group and S is the Lie algebra of G.*

Proof. Let $\mathfrak{g}^*$ be the Lie algebra of vector fields on M generated by S. Let $\tilde{G}$ be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}^*$ (Theorem 2.37). It is an abstract Lie group, not a transformation group yet. Now, for each $X \in \mathfrak{g}^*$, we denote by $\exp(tX)$ the 1-parameter subgroup of $\tilde{G}$ generated by X (Def. 2.27); while we denote by Fl_t^X the 1-parameter local group of local transformations of M generated by the flow of the vector field X. Then, thanks to these identifications, there exist a neighborhood U of $\{e\} \times M$ in $\tilde{G} \times M$ and a mapping $f : U \to M$ such that $f(\exp(tX), p) = Fl_t^X(p)$ for all $(\exp(tX), p) \in U \subseteq \tilde{G} \times M$. We say that the group $\tilde{G}$ acts locally on M.

Lemma A. Given $X, Y \in \mathfrak{g}^*$, let $Z \in \mathfrak{g}^*$ be $Z = \mathrm{ad}'_{\exp(X)}(Y)$. If $X, Y \in S$ then also $Z \in S$.

Proof of Lemma A. As in the example following Prop. 2.30, we have

$$\exp(tZ) = \exp(\mathrm{ad}'_{\exp(X)}(tY)) = \mathrm{ad}_{\exp(X)}(\exp(tY)) = (\exp(X))(\exp(tY))(\exp(-X)).$$

Therefore, when we evaluate f at $(\exp(tZ), p)$ we obtain

$$f(\exp(tZ), p) = Fl_t^Z(p) = Fl_1^X Fl_t^Y Fl_{-1}^X(p).$$

If $X, Y \in S$, then the right hand side of the equation is defined for all $p \in M$ and for all $t \in \mathbb{R}$, as X and Y are complete vector fields by the definition of S. Then also the the flow of Z is defined for all p and t, meaning that Z is complete, hence $Z \in S$.

Lemma B. S spans $\mathfrak{g}^*$ as a vector space.

Proof of Lemma B. Let V be the vector subspace of $\mathfrak{g}^*$ spanned by S. By Lemma A, we have $(\mathrm{ad}'_{\exp(S)})S \subseteq S$ and therefore $(\mathrm{ad}'_{\exp(S)})V \subseteq V$, as every element in V can be written as a linear combination of elements in S and tangent maps are linear. Since S generates the Lie algebra $\mathfrak{g}^*$, taking the exponential map, this translates to (exp S) generating $\tilde{G}$ (using an adaptation of Corollary 2.34), as we have defined $\tilde{G}$ as the connected Lie group with Lie algebra $\mathfrak{g}^*$. Hence $(\mathrm{ad}'_{\tilde{G}})V \subseteq V$. In particular, as $S \subseteq V \subseteq \mathfrak{g}^*$ so $\exp(S) \subseteq \exp(V) \subseteq \tilde{G}$, $(\mathrm{ad}'_{\exp(V)})V \subseteq V$. In turn, this can be shown to imply $[V, V] \subseteq V$, using Prop. 2.32. Therefore, $V \subseteq \mathfrak{g}^*$ as a subalgebra. Since V contains S, and $\mathfrak{g}$ is the Lie algebra generated by S, it follows that V generates $\mathfrak{g}^*$, thus $V = \mathfrak{g}^*$.

Lemma C. $S = \mathfrak{g}^*$.

Proof of Lemma C. Using the previous lemma, let $X_1, ...X_r \in S$ be a basis for $\mathfrak{g}^*$. Then the mapping

$$\sum a_i X_i \in \mathfrak{g}^* \longrightarrow (\exp(a_1 X_1)) \cdots (\exp(a_r X_r)) \in \tilde{G}.$$

gives a diffeomorphism of a neighborhood N of 0 in $\mathfrak{g}^*$ onto a neighborhood U of the identity element in $\tilde{G}$ (Theorem 2.28, 4.). Let $Y \in \mathfrak{g}^*$. Let $\delta > 0$ such that $\exp(tY) \in U$

for $|t| < \delta$. Therefore, for each t with $|t| < \delta$, there exists a unique element $\sum a_i(t)X_i \in N$ such that

$$\exp(tY) = \exp(a_1(t)X_1)\cdots\exp(a_r(t)X_r) \in \tilde{G}.$$

The action of exp tY on M is therefore given, for $p \in M$ and $|t| < \delta$, by the map f, yielding

$$f(\exp(tY), p) = Fl_t^Y(p) = Fl_{a_1(t)}^{X_1}\cdots Fl_{a_r(t)}^{X_r}(p).$$

This shows that every element Y of $\mathfrak{g}^*$ generates a global 1-parameter group of transformations of M: as in the proof of Lemma A, the right hand side of the equality is well defined globally because $X_1, ..., X_r \in S$. Therefore also, $Y \in S$. So we see that every $Y \in \mathfrak{g}*$ is also an element of S, so $\mathfrak{g}^* \subseteq S$. But obviously $S \subseteq \mathfrak{g}^*$, as $\mathfrak{g}^*$ is the Lie algebra generated by S. Hence, $S = \mathfrak{g}^*$ and Lemma C is proved.

Now, let G^* be the connected Lie transformation group acting on M generated by $\mathfrak{g}^*$. G^* exists since every element of $\mathfrak{g}^*$ generates a global 1-parameter group of transformations of M, as we have seen through the previous Lemmas, concluding that $\mathfrak{g}^* = S$. Since G^* is connected, the assumption on the statement of the theorem implies that $G^* \subseteq G$. Let $\phi \in G$ and let ψ_t be a 1-parameter subgroup of G^*. Then, $\phi\psi_t\phi^{-1}$ is a 1-parameter group of transformations of M contained in G. From the construction of G^*, it follows that this 1-parameter group is also a subgroup of G^*, since G^* is the Lie group generated by $\mathfrak{g}^* = S$. This implies that G^* is a normal subgroup of G and each $\phi \in G$ defines an automorphism $A_\phi : G^* \to G^*$ by $A_\phi(\psi) = \phi\psi\phi^{-1}$. Since the automorphism A_ϕ sends every global 1-parameter subgroup of G^* to a global 1-parameter subgroup of G^*, A_ϕ is continuous. (Chevalley [15] p. 128).

Lemma D. Let G be a group and G^* a topological group contained in G as a normal subgroup. If $A_\phi : G^* \to G^*$ is continuous for each $\phi \in G$, then there exists a unique topology on G which makes G^* open in G.

Proof of Lemma D. Let $\{W\}$ be the system of open neighborhoods of the identity element in G^*, that is, let $\{W\}$ be the collection of all open neighborhoods in G^* about e. Then, we define $\{\phi(W)\}$ as the system of open neighborhoods of $\phi \in G$ in G. It is straightforward to see that G^* is open in G with respect to the topology thus defined in G, and that this topology is also unique.

Applying Lemma D to our case, we see that G is a topological space in addition to its group structure (given by the compact-open topology defined in G^* (Def. A.13)), with now the group operation (the composition of diffeomorphisms) being continuous. Moreover, the identity component of G is G^*, as G^* is connected and $e \in G^*$. And as G^* is a Lie group, its differentiable structure can be translated to the other connected components of G. Therefore G is also a Lie group. This, combined with the differentiability of the left action $l^* : G^* \times M \to M$ implies the differentiability of $l : G \times M \to M$. Hence, G is a Lie transformation group of M.

□

Chapter 3

Semi-Riemannian Geometry

Semi-Riemannian Geometry is the branch of Differential Geometry devoted to the study of manifolds equipped with a particular tensor field: the metric tensor. The metric tensor is a bilinear non-degenerate symmetric (0,2) tensor field which provides a scalar product to all tangent spaces of a manifold, allowing one to properly define notions of angles between vectors and length of curves, generalizing these concepts from $\mathbb{R}^n$ to manifolds.

Two special cases are worth mentioning: the Riemannian metric, the first one of such metrics to be described, which is positive-definite; and the Lorentz metric, which has index one and describes space-time in modern physics through Einstein's theories of Relativity.

The study of both metrics was initially done in different styles. Riemannian Geometry, as the name suggests, was first described by the German mathematician Bernhard Riemann in mid-XIXth century, sparked by previous results of his doctoral thesis advisor, Carl F. Gauss. It was developed using coordinate-free methods and mainly devoted to the study of global problems. In contrast, Lorentz Geometry, although commented by Killing, Poincaré and others at the end of the XIXth century, was greatly boosted by theoretical physicists and mathematicians interested in Relativity after the publication of Einstein's theories (Special Relativity, 1905; General Relativity 1915). In 1907, it was again a German mathematician, Hermann Minkowski, who noticed that space-time in Special Relativity could be described as a 4-dimensional manifold. Unluckily for science, Minkowski died from appendicitis on 1909 and could not work along his dear friend David Hilbert on developing General Relativity, which he did independently of Einstein. From thereon the subject was mainly developed in classical tensor notation. It was not until the 1960s, when General Relativity finally entered mainstream physics, that leading theoretical physicists and mathematicians such as S. Hawking, R. Penrose and many others turned their attention to invariant methods for General Relativity in order to solve causality issues and further understand singularities. We refer the interested reader to [22] for a beautiful account on the historical development of General Relativity.

In this work we shall define concepts in a coordinate invariant manner, and then show their coordinate expression. Some important proofs will need coordinates, so they are not to be neglected.

3.1 The Metric: Basic Definitions

Definition 3.1. *A* metric tensor $\mathbf{g}$ *on a smooth manifold M is a symmetric non-degenerate (0,2) smooth tensor field on M which has constant index ν.*

That is, $\mathbf{g}$ assigns to each point $p \in M$ a scalar product g_p for its tangent space T_pM; and the index of g_p is $\nu_p = \nu$ for all $p \in M$. (See Defs. A.17 - A.20). The metric tensor is smooth in the following sense: for all X, Y smooth vector fields on M, the function $\mathbf{g}(X,Y) : M \to \mathbb{R}$ defined by $\mathbf{g}(X,Y)(p) = g_p(X_p, Y_p)$ is smooth.

Definition 3.2. *A* semi-Riemannian manifold *is a pair $(M, \mathbf{g})$ of a smooth manifold M and a metric tensor $\mathbf{g}$ on M. We say that $(M, \mathbf{g})$ is a*

- Riemannian manifold *if $\nu = 0$, i.e., if g_p is a positive-definite inner product for all $p \in M$.*
- Lorentz manifold *if $\nu = 1$ and* $\dim M \geq 2$.

Although we will prove that the isometry group of a semi-Riemannian manifold is a Lie group regardless of the index of its metric tensor in the next chapter, it is useful sometimes to make this distinction in order to present more familiar examples or notions that the reader may know. Moreover, there are some characteristic traits of Riemannian manifolds that do not hold for other semi-Riemannian manifolds. For example, one can turn Riemannian manifolds into metric spaces by defining a distance on them. Also, the existence of partitions of unity on smooth manifolds guarantees the existence of Riemannian metrics on any manifold, but that is not true for Lorentz metrics. Let us first, however, comment on the coordinate expression of $\mathbf{g}$. Afterwards we will define some concepts shared by all semi-Riemannian manifolds, paying special attention to the ones we shall need in the next chapter on isometries. We will end this chapter showing the existence theorems for the Riemannian and Lorentz case.

Remark 3.3. Let (U, φ) be a chart on a semi-Riemannian manifold $(M, \mathbf{g})$. Let $x^1, ..., x^n$ be the coordinate functions for φ. Recall that the vector fields $\partial_1, ..., \partial_n$ form a basis for each T_pM when evaluated at $p \in U$. So for smooth vector fields defined on U, $X = X^i\partial_i$ and $Y = Y^j\partial_j$, we have

$$\mathbf{g}(X,Y) = \mathbf{g}(X^i\partial_i, Y^j\partial_j) = X^iY^j\mathbf{g}(\partial_i, \partial_j).$$

Thus, it is only natural to define $\mathbf{g} = g_{ij}\, dx^i \otimes dx^j$ with $g_{ij} = \mathbf{g}(\partial_i, \partial_j)$. Also, $\mathbf{g}$ is symmetric, hence $g_{ij} = g_{ji}$. Moreover, as $\mathbf{g}$ is non-degenerate, the components g_{ij} form a non-singular matrix. Therefore we can define an *inverse metric* as $\mathbf{g}^{-1} := g^{ij}\ \partial_i \otimes \partial_j$, where $(g^{ij}) = (g_{ij})^{-1}$.

Example 3.4.

1. $\mathbb{R}^n$ with the usual Euclidean metric given by $g_{ij} = \delta_{ij}$ in canonical coordinates is a Riemannian manifold, identifying $T_p\mathbb{R}^n \cong \mathbb{R}^n$ for all $p \in \mathbb{R}^n$.

2. $\mathbb{R}^n$ with a metric tensor $g_{ij} = \mathrm{diag}(-1, \overset{\nu}{\ldots}, -1, 1, \overset{n-\nu}{\ldots}, 1)$ in canonical coordinates is the easiest example of a semi-Riemannian manifold of index ν, which we shall denote by $\mathbb{R}^n_\nu$, again identifying $T_p\mathbb{R}^n_\nu \cong \mathbb{R}^n_\nu$ for all $p \in \mathbb{R}^n_\nu$. If $n = 4$ and $\nu = 1$, it is called the *Minkowski space-time*, and it is the space-time of Special Relativity. It is a Lorentz manifold.

Remark 3.5. These two examples are of utmost importance. It is a well know result from Euclidean geometry that all vector spaces with an Euclidean inner product are isomorph to $\mathbb{R}^n$ with an orthogonal basis for which $g_{ij} = \delta_{ij}$, through a Gram-Schmidt process. Equivalently, any given vector space with an inner product with index $\nu \neq 0$ is isomorph to $\mathbb{R}^n_\nu$ with $g_{ij} = \mathrm{diag}(-1, \overset{\nu}{\ldots}, -1, 1, \overset{n-\nu}{\ldots}, 1)$, by a process similar to Gram-Schmidt but taking into account the negative signs that might appear. (See O'Neill p. 47-52).
What is not so obvious is that given a semi-Riemannian manifold $(M, \mathbf{g})$, one can locally find smooth vector fields $E_1, ..., E_n$ so that for all points p in some open subset $U \subseteq M$, $E_1(p), ..., E_n(p)$ form an orthonormal basis of T_pM. We shall go back to this matter in a few pages.

In order to distinguish between the metric tensor as a tensor field or as scalar product at T_pM for some point $p \in M$, we have used $\mathbf{g}$ and g_p respectively. From now on we shall simply write g for both cases whenever its use is understood from the context.

Remark 3.6. A remarkable feature of semi-Riemannian geometry is the one-to-one correspondence between vector fields and one-forms that arises through the metric. Let X be a smooth vector field on a semi-Riemannian manifold (M, g). Then, $X^\flat := g(X, \cdot) : \mathfrak{X}(M) \to \mathcal{F}(M)$ is a one-form on M. On a chart with coordinate functions $x^1, ..., x^n$ where $X = X^i \partial_i$, $X^\flat = X^\flat_i dx^i$, with $X^\flat_i = g_{ij} X^j$. Similarly we can define the vector field $\omega^\sharp = \omega^{\sharp i} \partial_i = g^{ij} \omega_j \partial_i$ for any one-form $\omega = \omega_i dx^i$, where $g^{ij} = (g_{ij})^{-1}$. The reason behind this notation is that the metric tensor *lowers* and *raises* indices respectively for $\flat$ and $\sharp$, similarly to their effect in musical notation.

Definition 3.7. *Let (N, g) be a semi-Riemannian manifold. Let $\phi : M \to N$ be a smooth mapping. We define the* pullback *$(\phi^* g)$ of the metric tensor $\mathbf{g}$ by ϕ as*

$$(\phi^* g)_p(u, v) = g_{\phi(p)}(d\phi(u), d\phi(v)),$$

for all $u, v \in T_pM$ and all $p \in M$.

It is easy to check that indeed $(\phi^* g)$ is a (0,2) tensor field on M. However, it need not be a metric tensor if the metric on N has index $\nu \neq 0$.

Definition 3.8. *Let (M, ϕ) be a submanifold of a semi-Riemannian $(N, \mathbf{g})$. If the pullback $(\phi^* \mathbf{g})$ is a metric tensor on M, then we say that $(M, (\phi * \mathbf{g}))$ is a semi-Riemannian* submanifold *of N.*

Example 3.9. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Given an atlas of polar charts on S^2, with coordinate functions (θ, ϕ), the induced metric tensor resulting from the pullback of the usual Euclidean metric on $\mathbb{R}^3$ is given by $g_{\theta\theta} = 1, g_{\phi\phi} = \sin^2 \theta$, and $g_{\theta\phi} = g_{\phi\theta} = 0$.

3.2 Levi-Civita Connection

Informally speaking, a connection is a map which tells us *how* to transport a tangent vector from the tangent space of a point in M to the tangent space of another neighboring point. There are many ways to do so, but only one is interesting from the point of view of semi-Riemannian geometry: the Levi-Civita connection. It is the unique connection which "behaves nicely" on a semi-Riemannian manifold, and in fact it can be derived from the metric itself. We will focus exclusively on the Levi-Civita connection. For more information on general connections see chapter II and III of Kobayashi and Nomizu [9].

Definition 3.10. *Let M be a manifold. A* connection ∇ *is a map*

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

such that

1. $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$, *for all $a, b \in \mathbb{R}$ and for all $X, Y_1, Y_2 \in \mathfrak{X}(M)$.*
2. $\nabla_{fX_1+hX_2} Y = f\nabla_{X_1} Y + h\nabla_{X_2} Y$, *for all $f, h \in \mathcal{F}(M)$ and for all $X_1, X_2, Y \in \mathfrak{X}(M)$.*
3. $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, *for all $f \in \mathcal{F}(M)$ and for all $X, Y \in \mathfrak{X}(M)$.*

We say that $\nabla_X Y$ is the covariant derivative *of Y with respect to X for the connection ∇.*

We can also define the covariant derivative of a vector field Y at a point $p \in M$ with respect to a tangent vector $v \in T_pM$. One need only find a vector field $X \in \mathfrak{X}(M)$ such that $X_p = v$ and then set $\nabla_v Y = (\nabla_X Y)_p \in T_pM$.

Also, ∇ is called *symmetric* or *torsion-free* if, in addition,

4. $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \mathfrak{X}(M)$.

And we say that ∇ is *compatible with the semi-Riemannian metric tensor* if

5. $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all $X, Y, Z \in \mathfrak{X}(M)$.

Theorem 3.11. *Let (M, g) be a semi-Riemannian manifold. Then, there exists a unique connection ∇ which is also torsion-free and compatible with the metric tensor g. It is called the* Levi-Civita connection.

Proof. We will first prove uniqueness and then existence. Let us write all cyclic permutations of property (5.) for $X, Y, Z \in \mathfrak{X}(M)$:

$$\begin{aligned} X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \\ Y(g(Z, X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X). \\ Z(g(X, Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{aligned}$$

Adding the first two equalities and subtracting the last one, we obtain

$$\begin{aligned} &X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\qquad = g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X) \\ &\qquad\qquad = g(2\nabla_X Y - [X, Y], Z) + g([X, Z], Y) + g([Y, Z], X), \end{aligned}$$

where we have used property (4.) on the last equality. Rearranging terms,

$$\begin{aligned} 2g(\nabla_X Y, Z) = &X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X), \end{aligned} \tag{3.1}$$

which proves the uniqueness of the map $(X, Y) \mapsto \nabla_X Y$, as it defines it with an implicit equation. Notice also that eq. (3.1) above depends solely on the choice of metric.

To prove the existence of ∇ one need only prove that the covariant derivative defined through the equation above indeed fulfills conditions (1.), (2.), and (3.). It is easy to check if one keeps in mind bilinearity of both the metric tensor and the bracket operation, and the product rule for Lie brackets, $[X, fY] = X(f)Y + f[X, Y]$, although they are long calculations so we will not include them here. $\square$

Definition 3.12. *Let (U, φ) be a chart with coordinate functions $x^1, ..., x^n$ on a semi-Riemannian manifold (M, g). The functions $\Gamma^i_{jk} : U \to \mathbb{R}$ defined by*

$$\nabla_{\partial_j} \partial_k = \Gamma^i_{jk} \partial_i$$

are called the Christoffel symbols *in the chart (U, φ).*

Proposition 3.13. *Let (M, g) be a Riemannian manifold, and let X, Y be two vector fields on M. Let (U, φ) be a chart with coordinate functions $x^1, ..., x^n$. Then, given $g = g_{ij} dx^i \otimes dx^j, (g^{ij}) = (g_{ij})^{-1}, X = X^i \partial_i$, and $Y = Y^i \partial_i$, the coordinate expressions for Γ and $\nabla_X Y$ are*

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}). \tag{3.2}$$

$$\nabla_X Y = (X^j \partial_j Y^i + \Gamma^i_{jk} X^j Y^k) \partial_i. \tag{3.3}$$

Proof. We will only sketch the proof as it is again an easy but tedious calculation. The coordinate expression of the Christoffel symbols can be obtained by subsituting their definition into equation (3.1), with $X = \partial_j$, $Y = \partial_k$ and $Z = \partial_l$. Now, since $[\partial_i, \partial_j] = 0$ for all $i, j = 1, ..., n$, we obtain

$$2g(\Gamma^i_{jk} \partial_i, \partial_l) = 2g(\nabla_{\partial_j} \partial_k, \partial_l) = \partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}, \tag{3.4}$$

from which one can easily derive expression (3.2), since $2g(\Gamma^i_{jk} \partial_i, \partial_l) = 2\Gamma^i_{jk} g_{il}$. Regarding the coordinate expression of the covariant derivative of two vector fields $\nabla_X Y$, again it is a straightforward calculation after substituting each term for its coordinate expression on property (3.) of the definition of ∇ (Def. 3.10). $\square$

Remark 3.14. From the formula (3.2) it can easily be noticed that the Christoffel symbols are symmetric on the lower indices ($\Gamma^i_{jk} = \Gamma^i_{kj}$), because the metric tensor is also symmetric.

Remark 3.15. It may be sometimes useful to write down the partial derivatives of the coordinates of the metric tensor in terms of the Christoffel symbols. Adding eq. (3.4) to its equivalent expression switching the j and l indices, we obtain

$$\partial_k g_{lj} = g_{ij} \Gamma^i_{kl} + g_{il} \Gamma^i_{kj}. \tag{3.5}$$

We have seen that the Levi-Civita connection ∇ can be understood as a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ with $(X, Y) \mapsto \nabla_X Y$, but also that for a given X it can act as a derivation $\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ with $Y \mapsto \nabla_X Y$. However, we could consider the map $(\nabla Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $X \mapsto \nabla_X Y$.

Definition 3.16. *Let $Y \in \mathfrak{X}(M)$. We define the* covariant differential ∇Y *as the (1,1) tensor field on M such that $(\nabla Y)(X) = \nabla_X Y$ for all $X \in \mathfrak{X}(M)$.*

Proposition 3.17. *Given a chart with coordinate functions $x^1, ..., x^n$ wherein $Y = Y^i \partial_i$, the local coordinate expression of ∇Y is*

$$\nabla Y = (\nabla Y)^i_j \, \partial_i \otimes dx^j, \qquad (\nabla Y)^i_j = \partial_j Y^i + \Gamma^i_{jk} Y^k. \tag{3.6}$$

Proof. Immediate from eq. (3.3) taking $X = \partial_j$. □

3.3 Parallel Transport and Geodesics

We shall now focus on a special kind of curves. Geodesics are the generalization of straight lines from Euclidean spaces to semi-Riemannian manifolds. They are smooth curves $\gamma : [a, b] \to M$, with the defining property that their velocity vector γ' is parallel transported along them, that is, informally, $\gamma'' = \nabla_{\gamma'} \gamma' = 0$. Let us first properly define covariant derivatives on a curve and the concept of parallel transport.

Parallel Translation

Definition 3.18. *A vector field V on a curve $\alpha : [a, b] \to M$ is a mapping $[a, b] \to TM$ such that $\pi \circ V = \alpha$, where π is the projection $TM \to M$. The set of all smooth vector fields on a curve α is denoted by $\mathfrak{X}(\alpha)$.*

Proposition 3.19. *Let $\alpha : [a, b] \to M$ be a smooth curve on a semi-Riemannian manifold (M, g) with Levi-Civita connection ∇. Then, there is a unique map*

$$\begin{aligned} \frac{D}{dt} : \mathfrak{X}(\alpha) &\longrightarrow \mathfrak{X}(\alpha) \\ V &\longmapsto \frac{D}{dt} V \end{aligned}$$

such that for all $c, d \in \mathbb{R}$, $f \in \mathcal{F}([a, b])$, $t \in [a, b]$, and $X \in \mathfrak{X}(M)$,

1. $\frac{D}{dt}(cV_1 + dV_2) = c\frac{D}{dt}V_1 + d\frac{D}{dt}V_2$,
2. $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{D}{dt}V$,
3. $\frac{D}{dt}(X|_{\alpha(t)}) = \nabla_{\alpha'(t)} X$.

It is called the induced covariant derivative *on α.*

Proof. As in previous results from this section, we shall prove uniqueness assuming existence by finding a coordinate equation defining $\frac{D}{dt}$. Then, existence will follow by straightforwardly verifying that the defining properties above are fulfilled by the formula found. Let there be an induced covariant derivative $\frac{D}{dt}$ on a smooth curve $\alpha : [a, b] \to M$. Without loss of generality we can assume that $\alpha([a, b]) \subseteq U$, where (U, φ) is a chart of M with coordinate functions $x^1, ..., x^n$. Then, for any $V \in \mathfrak{X}(\alpha)$, $V(t) = V|_{\alpha(t)} = V^i(t)\partial_i$, where

$V^i(t) := V(x^i(\alpha(t))) \in \mathcal{F}([a,b])$ are the coordinates of V. Now, by the properties defined above,

$$\frac{D}{dt}V = \frac{D}{dt}(V^i\partial_i) = \frac{dV^i}{dt}\partial_i + V^i\frac{D}{dt}\partial_i = \frac{dV^i}{dt}\partial_i + V^i\nabla_{\alpha'}\partial_i\ , \tag{3.7}$$

which proves uniqueness, as it defines $\frac{D}{dt}$ uniquely from ∇. $\square$

Before introducing parallel translation, let us now see a useful result about the rate of change of the scalar product of two vector fields on a curve. It is the analogue to property (5.) of the definition of the Levi-Civita covariant derivative (Def. 3.10)

Proposition 3.20. *Let (M,g) be a semi-Riemannian manifold. Let $\alpha : [a,b] \to M$ be a smooth curve on M, and let $\frac{D}{dt}$ be the induced covariant derivative on α. Then, for all X,Y smooth vector fields on α,*

$$\frac{d}{dt}\big(g(X,Y)\big) = g(\frac{D}{dt}X,Y) + g(X,\frac{D}{dt}Y).$$

Proof. We can work locally in the domain of a chart (U,φ) without loss of generality. Let $x^1,...,x^n$ be the coordinates functions of φ. Then, we can write

$$\frac{d}{dt}\big(g(X,Y)\big) = \frac{d}{dt}\big(g_{ij}X^iY^j\big) = \frac{dg_{ij}}{dt}X^iY^j + g_{ij}\frac{dX^i}{dt}Y^j + g_{ij}X^i\frac{dY^j}{dt}. \tag{3.8}$$

The first term of the right-hand side expression can be expanded by the chain rule of $\frac{dg_{ij}}{dt} = \partial_k g_{ij}\,\frac{d(x^k\circ\alpha)}{dt}$ and using equation (3.5) for the partial derivatives of the metric:

$$\frac{dg_{ij}}{dt}X^iY^j = \partial_k g_{ij}\frac{d(x^k\circ\alpha)}{dt}X^iY^j = \left(g_{lj}\Gamma^l_{ki} + g_{li}\Gamma^i_{kj}\right)\frac{d(x^k\circ\alpha)}{dt}X^iY^j. \tag{3.9}$$

Substituting eq. (3.9) to eq. (3.8) above, rearranging terms and relabelling dummy indices whenever necessary, we can see that it matches the expression below, obtained by expressing the induced covariant derivative of X and Y in coordinates (eq. (3.10) derived in the proof of the next Proposition):

$$g(\frac{D}{dt}X,Y) + g(X,\frac{D}{dt}Y) = g_{ij}\frac{dX^i}{dt}Y^j + g_{ij}\Gamma^i_{kl}\frac{d(x^k\circ\alpha)}{dt}X^lY^j + g_{ij}X^i\frac{dY^j}{dt} + g_{ij}\Gamma^i_{kl}\frac{d(x^k\circ\alpha)}{dt}X^jY^l.$$

$\square$

Definition 3.21. *We say that a vector field V on a curve α is* parallel *if $\frac{D}{dt}V = 0$.*

Proposition 3.22. *Let $\alpha : [a,b] \to M$ be a smooth curve on a semi-Riemannian manifold (M,g). Let $c \in [a,b]$ and let $v = v^i\partial_i|_p \in T_pM$, where $p = \alpha(c)$. Then, there is a unique parallel vector field V on α such that $V_p = v$. We say that V is the* parallel transportation *of v along α.*

Proof. Again, we can assume without loss of generality that $\alpha([a,b]) \subseteq U$, where (U,φ) is a chart of M with coordinate functions $x^1,...,x^n$. If we substitute ∇ by its coordinate expression as Christoffel symbols (eq. 3.3) in equation 3.7 for the induced covariant derivative, we obtain

$$\frac{D}{dt}V = \left(\frac{dV^k}{dt} + \Gamma^k_{ij}\frac{d(x^i\circ\alpha)}{dt}V^j\right)\partial_k. \tag{3.10}$$

Then, the condition $\frac{D}{dt}V = 0$ for parallel vector fields can be seen as a system of linear ordinary differential equations for V^i with initial condition $V^i(c) = v^i$. Existence and uniqueness of solutions of linear systems of ODE's gives us the desired result. □

Remark 3.23. In the notation of the previous proposition, let $d \in [a, b]$ and $q = \alpha(d)$. We will denote the *parallel translation* of v from T_pM to T_qM along α by

$$\begin{aligned} P = P_c^d(\alpha) : T_pM &\longrightarrow T_qM \\ v &\longmapsto P(v) = V_q. \end{aligned}$$

Proposition 3.24. *Parallel translation is a linear isometry of vector spaces.*

Proof. Let $v, w \in T_pM$ correspond to parallel vector fields V, W. Let $k \in \mathbb{R}$. Obviously $V + W$ and kV are also parallel vector fields: the (induced) covariant derivative is $\mathbb{R}$-linear. Therefore parallel translation is a linear morphism of vector spaces, since $P(v + w) = (V + W)_q = V_q + W_q = P(v) + P(w)$ and $P(kv) = (kV)_q = kV_q = kP(v)$. Uniqueness of solutions of Prop. 3.22 and dim $T_pM =$ dim T_qM yield bijectivity. Hence P is a linear isomorphism. We only have to see that the scalar product is preserved. It is straightforward from 3.20 and the fact that V, W have null covariant derivative along α due to being parallel.

$$\frac{d}{dt}g(V, W) = g(\frac{D}{dt}V, W) + g(V, \frac{D}{dt}W) = 0.$$

Therefore $g(V, W)$ is constant along α, so

$$g_q(P(v), P(w)) = g_q(V_q, V_q) = g_p(V_p, V_p) = g_p(v, w).$$ □

Geodesics

Definition 3.25. *A curve $\gamma : [a, b] \to M$ is called a* geodesic *if its velocity vector is parallel. In other words, γ is a geodesic if it has null* acceleration*: $\gamma'' := \frac{D}{dt}\gamma' = 0$.*

Corollary 3.26. *Let $\gamma : [a, b] \to M$ be a geodesic on a Riemannian manifold (M, g). Let $\frac{D}{dt}$ be its induced covariant derivative. Then, for all $X \in \mathfrak{X}(\gamma)$,*

$$\frac{d}{dt}(g(X, \gamma')) = g(\frac{D}{dt}X, \gamma') \tag{3.11}$$

Proof. Immediate from Prop. 3.20 and the fact that $\frac{D}{dt}\gamma' = 0$. □

The following results present the existence and uniqueness of geodesics starting at a point with a given initial velocity. They are an immediate consequence of the existence and uniqueness of parallel translation.

Consider the particular case of $V = \alpha'$ of Proposition 3.22:

Corollary 3.27. *A curve $\gamma : [a,b] \to M$ is a geodesic if and only if, for any given chart (U,φ), $\varphi = (x^1, ..., x^n)$, its local coordinate functions $(x^k \circ \gamma)$, $k = 1, ..., n$, fulfill*

$$\frac{d^2(x^k \circ \gamma)}{dt^2} + \Gamma^k_{ij} \frac{d(x^i \circ \gamma)}{dt} \frac{d(x^j \circ \gamma)}{dt} = 0. \tag{3.12}$$

Now, existence and uniqueness of solutions of systems of linear ODE's can be used to easily prove the following six results, starting from the Corollary above. For the detailed proofs, see O'Neill [11] p. 68-72.

Lemma 3.28. *For any $p \in M$ and any $v \in T_pM$ there is an interval $0 \in [a,b]$ and a unique geodesic $\gamma : [a,b] \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.*

Lemma 3.29. *Let $\gamma, \tau : [a,b] \to M$ be geodesics. If $\gamma(c) = \tau(c)$ for some $c \in [a,b]$, then $\gamma = \tau$.*

Proposition 3.30. *For any $p \in M$ and any $v \in T_pM$ there is a unique geodesic $\gamma_v : I_v \to M$ such that $\gamma_v(0) = p$, $\gamma'_v(0) = v$, and I_v is maximal. We say that γ_v is a* maximal geodesic.

In a similar fashion as we did for left-invariant flows on Lie groups, we can define a map which collects all geodesics starting at a point in M. We will also call it the exponential map. As opposed to the exponential map on Lie groups, which is defined on the identity (although then it can be left-transported), we can define the geodesic exponential map on any point $p \in M$. This map will too give us a diffeomorphism between a neighborhood of $0 \in T_pM$ and a neighborhood of p.

Definition 3.31. *Let p be a point on a semi-Riemannian manifold (M,g). Let $\mathcal{D}_p$ be the set of vectors v in T_pM such that their maximal geodesic γ_v is defined at least on $[0,1]$. The* exponential map *of M at p is defined as*

$$\begin{aligned} \exp_p : \mathcal{D}_p &\longrightarrow M \\ v &\longmapsto \exp_p(v) = \gamma_v(1). \end{aligned}$$

As we did with Lie groups, the identification of a vector space with its own tangent vector space, $T_pM \cong T_0(T_pM)$ allows us to easily see that $d\exp_p = Id_{T_pM}$. Therefore, by the Inverse Function Theorem we obtain the following corollary.

Corollary 3.32. *For each point $p \in M$ there exists a neighborhood V of 0 in T_pM on which the exponential map $\exp_p$ is a diffeomorphism onto a neighborhood U of p in M.*

We say that such a neighborhood $U \subseteq M$ is *normal* if its corresponding $V \subseteq T_pM$ is *starshaped*, i.e., if for all $v \in V$ also $\rho(t) := tv \in V$, with $0 \geq t \geq 1$.

Proposition 3.33. *If U is a normal neighborhood of $p \in M$, then for each $q \in U$ there is a unique geodesic $\tau = \exp_p \circ \rho : [0,1] \to U$ such that $\tau(0) = p, \tau(1) = q$ and $\tau'(0) = \exp_p^{-1}(q) \in V$.*

Such a geodesic is called a *radial geodesic*.

Remark 3.34. We are now in position to retake the local existence of orthonormal vector fields presented in Remark 3.5. We saw that on any tangent space T_pM of a semi-Riemannian manifold $(M, \mathbf{g})$, we can find a basis $e_1, ..., e_n$ through a Gram-Schmidt process so that the metric tensor on p has coordinates $g_{ij} = \operatorname{diag}(-1, \overset{\nu}{\ldots}, -1, 1, \overset{n-\nu}{\ldots}, 1)$. Let α be a curve on M going through p. By Proposition 3.22, there exist parallel vector fields $E_1, ..., E_n$ along α so that their evaluation at p is $e_1, ..., e_n$. And since parallel translation is a linear isometry of vector spaces (Prop. 3.24), we have that $E_1, ..., E_n$ become orthonormal basis when evaluated at all points along the curve α. Finally, to see that this also can be applied to small subsets of M, one need only extend this process to all radial geodesics on a normal neighborhood U of p. (See O'Neill [11], p.84-85 for a complete proof).

Riemannian Metric Spaces

We end this section presenting the well-known metrization of Riemannian manifolds.

Definition 3.35. *Let $\alpha : [a, b] \to M$ be a smooth curve on a Riemann manifold (M, g^R). We define the* arc-length *of α as*

$$L[\alpha] := \int_a^b \sqrt{g_{\alpha(t)}\big(\alpha'(t), \alpha'(t)\big)}dt$$

This definition can be straightforwardly generalized to piecewise smooth curves.

Definition 3.36. *Let (M, g^R) be a connected Riemannian manifold. Then, we define the* distance *$d : M \times M \to \mathbb{R}$ as*

$$d(p, q) := \inf \{L[\alpha] \mid \alpha : [0, 1] \to M \text{ piecewise smooth},\ \alpha(0) = p,\ \alpha(1) = q\}.$$

The previous definition indeed defines a *distance* in the topological sense (see Gallot, Hulin and Lafontaine, Section 2.C.3 (p.89-94)). If M is not connected it also holds if we define the distance of two points in different connected components to be ∞. Therefore,

Theorem 3.37. *All Riemannian manifolds (M, g^R) are metrizable.*

3.4 Curvature

Definition 3.38. *Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection ∇. The* Riemann curvature tensor *R is the (1,3) tensor field on M defined by*

$$\begin{aligned} R : (\mathfrak{X}(M))^3 &\longrightarrow \mathfrak{X}(M) \\ (X, Y, Z) &\longmapsto R(X, Y)(Z) = R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \end{aligned}$$

where $[\nabla_X, \nabla_Y] = \nabla_X\nabla_Y - \nabla_Y\nabla_X$, even though these terms are not vector fields.

Remark 3.39. The fact that the definition above gives indeed a tensor field is not obvious. Recall that neither the bracket operation nor the covariant derivative are tensor fields. However, due to some beautiful cancellations from both terms on the definition, the resulting R is indeed a tensor field. It is again an easy but tedious exercise to prove the $\mathcal{F}(M)$-linearity for X, Y, and Z.

Informally, the Riemann curvature tensor describes the difference between a tangent vector at a point Z_p and the tangent vector $R(X,Y)(Z)_p$ resulting of parallel transporting the initial vector Z_p around an infinitesimal loop following the flow lines of X and Y.

Proposition 3.40. *Let $X, Y, Z, V, W \in \mathfrak{X}(M)$. Then,*

1. $R_{XY}Z = -R_{YX}Z$.
2. $g(R_{XY}V, W) = g(R_{VW}X, Y)$.

Proof. Property (1.) is immediate from the definition of the curvature tensor and the fact that the bracket is antisymmetric. The proof of (2.) is rather long, as it involves summing over all cyclic permutations of X, Y, V and W, so we leave it out for conciseness. The interested reader can find a proof in O'Neill ([11], p.75). □

Proposition 3.41. *Let (U, φ) be a chart on M with coordinate functions $x^1, ...x^n$. Then, the local coordinate expression of R on U is given by $R_{\partial_k \partial_l}\partial_j = R^i_{jkl}\partial_i$, where*

$$R^i_{jkl} = \frac{\partial \Gamma^i_{kj}}{\partial x^l} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} + \Gamma^i_{lr}\Gamma^r_{kj} - \Gamma^i_{kr}\Gamma^r_{lj}. \tag{3.13}$$

Proof. For coordinate vector fields, $R_{\partial_k \partial_l}\partial_j = \nabla_{[\partial_k, \partial_l]}\partial_j - [\nabla_{\partial_k}, \nabla_{\partial_l}]\partial_j = \nabla_{\partial_l}(\nabla_{\partial_k}\partial_j) - \nabla_{\partial_k}(\nabla_{\partial_l}\partial_j)$, as the commutator $[\partial_k, \partial_l] = 0$. Now, substituting the coordinate expression for the covariant derivative (eq. (3.3)) into the first term, and using property 3. of the covariant derivative (Def. 3.11),

$$\nabla_{\partial_l}(\nabla_{\partial_k}\partial_j) = \nabla_{\partial_l}(\Gamma^r_{kj}\partial_r) = \frac{\partial \Gamma^r_{kj}}{\partial x^l}\partial_r + \Gamma^r_{kj}\nabla_{\partial_l}\partial_r = \frac{\partial \Gamma^r_{kj}}{\partial x^l}\partial_r + \Gamma^r_{kj}\Gamma^s_{lr}\partial_s = \left(\frac{\partial \Gamma^i_{kj}}{\partial x^l} + \Gamma^r_{kj}\Gamma^i_{lr}\right)\partial_i,$$

where we have relabeled the dummy indices in the last step. Finally, subtracting the equivalent expression for the second term yields the desired result. □

3.5 Variations and Jacobi Fields

Informally speaking, Jacobi fields are vector fields on a geodesic which describe the difference between the geodesic and neighboring geodesics, controlled by the Riemann curvature tensor of the manifold. Therefore we must first define the concept of variation, a 1-parameter family of smooth curves around a given smooth curve on M.

Definition 3.42. *Let $\alpha : [a, b] \to M$ be a smooth curve on a semi-Riemannian manifold (M, g). A* variation *of the curve α is a smooth map $\theta : [a, b] \times (-\delta, \delta) \to M$ with $(t, s) \mapsto \theta(t, s)$ such that $\alpha(t) = \theta(t, 0)$ for all $t \in [a, b]$.*

Notice that for a fixed $s_0 \in (-\delta, \delta)$ the maps $\theta(\cdot, s_0) : [a, b] \to M$ are also smooth curves on M, called the *longitudinal curves* of θ. Therefore we can also see the variation θ as a 1-parameter family of curves on M parametrized by s. We say that θ is a *geodesic variation* if $\theta(t, s_0)$ is a geodesic for all $s_0 \in (-\delta, \delta)$.

However, we could also consider the curves $\theta(t_0, \cdot) : (-\delta, \delta) \to M$ for a fixed $t_0 \in [a, b]$. We will call them the *transverse curves* of the variation. The vector fields defined as

$$\theta_t = d\theta\left(\frac{\partial}{\partial t}\right) \qquad \theta_s = d\theta\left(\frac{\partial}{\partial s}\right)$$

are called the *partial velocities* of longitudinal and transverse curves, respectively. For a given chart (U, φ) with coordinate functions $x^1, ..., x^n$, their coordinate expressions are

$$\theta_t = \frac{\partial(x^i \circ \theta)}{\partial t}\partial_i = \frac{\partial \theta^i}{\partial t}\partial_i \qquad \theta_s = \frac{\partial(x^i \circ \theta)}{\partial s}\partial_i = \frac{\partial \theta^i}{\partial s}\partial_i, \tag{3.14}$$

where we have defined the coordinates of θ as $\theta^i := x^i \circ \theta$ for simplicity.

We can now define a vector field V on α by restricting θ_s on it: $V(t) = V_{\alpha(t)} = \theta_s(t, 0)$. This vector field can be regarded as the infinitesimal model of the variation θ of α, and therefore it is called the *variation vector field* of θ.

We will want to know how this vector field behaves, so first we have to properly define the derivatives on θ. If X is a smooth vector field on θ, we define its *partial covariant derivatives* the following way: the longitudinal partial covariant derivative $DX/\partial t$ is given by $\frac{DX}{\partial t}|_{(t_0,s_0)}$ being the induced covariant derivative at t_0 of the vector field $\tilde{X}(t) = X(\theta(t, s_0))$ on the longitudinal curve $\theta(t, s_0)$; equivalently the transverse partial covariant derivative $DX/\partial s$ is defined by $\frac{DX}{\partial s}|_{(t_0,s_0)}$ being the induced covariant derivative at s_0 of the vector field $\bar{X}(s) = X(\theta(t_0, s))$ on the transverse curve $\theta(t_0, s)$. Their coordinate expressions, given $X = X^i\partial_i$ where $X^i = X(\partial_i)$, can be found using equation (3.10) deduced along the proof of Proposition 3.22:

$$\frac{D}{\partial t}X = \left(\frac{\partial X^k}{\partial t} + \Gamma^k_{ij}\frac{\partial \theta^i}{\partial t}X^j\right)\partial_k \quad , \quad \frac{D}{\partial s}X = \left(\frac{\partial X^k}{\partial s} + \Gamma^k_{ij}\frac{\partial \theta^i}{\partial s}X^j\right)\partial_k. \tag{3.15}$$

Remark 3.43. Notice that $\frac{D}{\partial t}\theta_t$ is the acceleration of the longitudinal curves, and $\frac{D}{\partial s}\theta_s$ is the acceleration of the transverse curves. As for the cross terms, they actually fulfill $\frac{D}{\partial t}\theta_s = \left(\frac{\partial^2\theta^k}{\partial t\,\partial s} + \Gamma^k_{ij}\frac{\partial\theta^i}{\partial t}\frac{\partial\theta^j}{\partial s}\right)\partial_k = \frac{D}{\partial s}\theta_t$. The proof is trivial using coordinates, because $\Gamma^k_{ij} = \Gamma^k_{ji}$ (eq. (3.2)), and second partial derivatives of smooth functions on $\mathbb{R}^n$ commute (Schwarz's theorem).

Before properly defining Jacobi fields, let us prove a useful property of the partial covariant derivatives on a variation.

Lemma 3.44. *Let θ be a variation on a semi-Riemannian manifold (M, g), and let X be a vector field on θ. Then,*

$$\frac{D}{\partial t}\frac{D}{\partial s}X - \frac{D}{\partial s}\frac{D}{\partial t}X = R(\theta_t, \theta_s)X.$$

Proof. Again, we shall use coordinates to prove this Lemma. Using eq. (3.15) and Prop. 3.7, and relabelling the dummy indices whenever necessary,

$$
\begin{aligned}
\frac{D}{\partial t}\frac{D}{\partial s}X &= \left(\frac{\partial^2 X^k}{\partial t\,\partial s} + \frac{\partial}{\partial t}\left[\Gamma^k_{ij}\frac{\partial\theta^i}{\partial s}X^j\right]\right)\partial_k + \left(\frac{\partial X^k}{\partial s} + \Gamma^k_{ij}\frac{\partial\theta^i}{\partial s}X^j\right)\frac{D}{\partial t}\partial_k \\
&= \left(\frac{\partial^2 X^r}{\partial t\,\partial s} + \frac{\partial\Gamma^r_{ij}}{\partial t}\frac{\partial\theta^i}{\partial s}X^j + \Gamma^r_{ij}\frac{\partial^2\theta^i}{\partial t\,\partial s}X^j + \Gamma^r_{ij}\frac{\partial\theta^i}{\partial s}\frac{\partial X^j}{\partial t} + \Gamma^r_{lk}\frac{\partial\theta^l}{\partial t}\frac{\partial X^k}{\partial s} + \Gamma^k_{ij}\Gamma^r_{lk}\frac{\partial\theta^i}{\partial s}\frac{\partial\theta^l}{\partial t}X^j\right)\partial_r
\end{aligned}
$$

Therefore, by eq. (3.14) and Prop. 3.41,

$$
\begin{aligned}
\frac{D}{\partial t}\frac{D}{\partial s}X - \frac{D}{\partial s}\frac{D}{\partial t}X &= \left(\frac{\partial\Gamma^r_{ij}}{\partial t}\frac{\partial\theta^i}{\partial s}X^j + \Gamma^k_{ij}\Gamma^r_{lk}\frac{\partial\theta^i}{\partial s}\frac{\partial\theta^l}{\partial t}X^j - \frac{\partial\Gamma^r_{ij}}{\partial s}\frac{\partial\theta^i}{\partial t}X^j - \Gamma^k_{ij}\Gamma^r_{lk}\frac{\partial\theta^i}{\partial t}\frac{\partial\theta^l}{\partial s}X^j\right)\partial_r \\
&= \frac{\partial\theta^i}{\partial t}\frac{\partial\theta^l}{\partial s}X^j\left(\frac{\partial\Gamma^r_{ij}}{\partial x^l} - \frac{\partial\Gamma^r_{lj}}{\partial x^i} + \Gamma^k_{lj}\Gamma^r_{ik} - \Gamma^k_{ij}\Gamma^r_{lk}\right)\partial_r \\
&= \frac{\partial\theta^i}{\partial t}\frac{\partial\theta^l}{\partial s}X^j R(\partial_i,\partial_l)\partial_j = R(\theta_t,\theta_s)X. \qquad \square
\end{aligned}
$$

So we see that the behaviour of a vector field on a variation is related to the Riemann curvature tensor, which is again expressing the non-commutativity of the covariant derivative.

Definition 3.45. *Let γ be a geodesic on a semi-Riemannian manifold (M, g). A vector field Y on γ which satisfies the* Jacobi differential equation $\frac{D^2Y}{dt^2} + R(\gamma', Y)\gamma' = 0$ *is called a* Jacobi vector field.

The Jacobi equation is a linear second order ordinary differential equation, so a solution is uniquely determined by the values of Y and $\frac{DY}{dt}$ at a point of the geodesic.

Theorem 3.46. *Let $\theta : [a, b] \times (-\delta, \delta) \to M$ be a geodesic variation of a geodesic $\gamma : [a, b] \to M$ on a semi-Riemannian manifold (M, g). Then, the variation vector field $V(t) = \theta_s(t, 0)$ of θ is a Jacobi vector field.*

Proof. Since $t \to \theta(t, s_0)$ is a geodesic for all $s_0 \in (-\delta, \delta)$, we have (Def. 3.25) that $\frac{D}{\partial t}(\theta_t(t, s_0)) = 0$ for all $s_0 \in (-\delta, \delta)$. Now, by Lemma 3.44,

$$
\frac{D^2V(t)}{\partial t^2} = \frac{D}{\partial t}\frac{D}{\partial t}\theta_s(t,0) = \frac{D}{\partial t}\frac{D}{\partial s}\theta_t(t,0) = \frac{D}{\partial s}\cancel{\frac{D}{\partial t}\theta_t(t,0)} + R(\theta_s,\theta_t)\theta_t = R(\theta_s,\theta_t)\theta_t\ .
$$

Hence, $V = \theta_s$ satisfies the Jacobi equation, as $R(\theta_s, \theta_t)\theta_t = -R(\theta_t, \theta_s)\theta_t$ (Prop. 3.40). $\square$

For a more complete description of Jacobi fields see Michor [2] Chapter 26, or Gallot [13] Chapter 3.C. Their applications in General Relativity are a fundamental tool to describe the gravitational *tidal force* associated to the curvature of space-time, as they describe the relative kinematics of neighboring freely falling particles (See O'Neill [11] chapter 12 for a more detailed account of such applications). It is useful in this case to decompose vector fields as a sum of a tangent and a perpendicular vector field, as the relative acceleration of such neighboring falling particles produced by tidal force depends only on the perpendicular component of this force.

Definition 3.47. *Let (M, g) be a semi-Riemannian manifold. A smooth vector field X on a curve $\alpha : [a, b] \to M$ is called*

- tangent *to α if $X = f\alpha'$ for some smooth real function f. We denote this by $X \parallel \alpha$.*
- perpendicular *to α if $g(X, \alpha')|_t = 0$ for all $t \in [a, b]$. We denote this by $X \perp \alpha$.*

Remark 3.48. It is straightforward to see that if $\alpha'(t) \neq 0$ for all $t \in [a, b]$ then X has a unique expression $X = X^{\parallel} + X^{\perp}$, where $X^{\parallel}$ and $X^{\perp}$ are respectively tangent and perpendicular to α.

Proposition 3.49. *Let X be a vector field on a geodesic $\gamma : [a, b] \to M$. Then,*

1. *$X \parallel \gamma \Rightarrow \frac{D}{dt} X \parallel \gamma$.*
2. *$X \perp \gamma \Rightarrow \frac{D}{dt} X \perp \gamma$.*

Proof.

1. If $X \parallel \gamma$, then $X = f\gamma'$ for some smooth real function f, so $\frac{D}{dt} X = \frac{D}{dt}(f\gamma') = \frac{df}{dt}\gamma' + f\cancel{\frac{D}{dt}\gamma'}$, where the last term is 0 because γ is a geodesic. Hence also $\frac{D}{dt} X \parallel \gamma$.
2. By Corollary 3.26, $\frac{d}{dt}\big(g(X, \gamma')\big) = g(\frac{D}{dt} X, \gamma')$. So if $g(X, \gamma') = 0$ obviously its derivative is null too. Therefore $g(\frac{D}{dt} X, \gamma') = 0$ and $X \perp \gamma$.

$\square$

Let us now see how these concepts relate to Jacobi fields.

Proposition 3.50. *Let X be a vector field on a geodesic $\gamma : [a, b] \to M$. Then,*

1. *If $X \parallel \gamma$, then X is a Jacobi field $\iff \frac{D^2}{dt^2} X = 0 \iff X(t) = (mt + n)\gamma'(t)$ for all $t \in [a, b]$, for some $m, n \in \mathbb{R}$.*
2. *If X is a Jacobi field, then $X \perp \gamma \iff$ there exist $t_0 \neq t_1$ such that $g(X, \gamma')|_{t_0} = g(X, \gamma')|_{t_1} = 0 \iff$ there exists t_0 such that $g(X, \gamma')|_{t_0} = g(\frac{D}{dt} X, \gamma')|_{t_0} = 0$.*

Proof.

1. If $X = f\gamma'$, since $R(X, \gamma') = fR(\gamma', \gamma') = 0$ by antisymmetry, the Jacobi equation is $\frac{D^2}{dt^2} X = 0$. Solving the second order ODE yields the second equivalence.
2. Since γ is a geodesic, $\frac{d^2}{dt^2}\big(g(X, \gamma')\big) = \frac{d}{dt}\big(g(\frac{D}{dt} X, \gamma')\big) = g(\frac{D^2}{dt^2} X, \gamma')$. Also, X fulfills the Jacobi equation: $\frac{D^2}{dt^2} X = R(X, \gamma')\gamma'$. Therefore, recalling Prop 3.40, $\frac{d^2}{dt^2}\big(g(X, \gamma')\big) = g\big(R(X, \gamma')\gamma', \gamma'\big) = g\big(R(\gamma', \gamma')X, \gamma'\big) = 0$ again by antisymmetry. Solving the second order differential equation we obtain $g(X, \gamma') = As + B$. The rest of the proof is straightforward.

$\square$

3.6 Existence theorems

We end this chapter with the aforementioned existence theorems of Riemannian and Lorentz metrics on a smooth manifold.

Theorem 3.51. *Every smooth manifold M admits a Riemannian metric tensor.*

Proof. Let $\{(U_a, \varphi_a)\}$ be an atlas on M. Let $\{f_a\}$ be a partition of unity subordinate to the covering $\{U_a\}$. Then, for each α the coordinate functions for φ_α are $x_\alpha^1, ..., x_\alpha^n$, and we define $\mathbf{g}_\alpha = dx_\alpha^i \otimes dx_\alpha^i$ on U_α. A linear combination of positive-definite inner products with positive coefficients is again a positive-definite inner product. Therefore $\mathbf{g} := \sum f_\alpha \mathbf{g}_\alpha$ is a Riemannian metric on M.

□

Theorem 3.52. *Let M be a smooth manifold. Then, the following are equivalent:*

1. *The manifold M admits a Lorentz metric tensor.*
2. *There exists a non-vanishing smooth vector field on M.*
3. *Either M is not compact; or M is compact and has Euler number $\chi(M) = 0$.*

Proof. (2.) $\Rightarrow$ (1.) : By the previous Theorem 3.51, there is a Riemannian metric tensor g^R on M. Let $\tilde{X} \in \mathfrak{X}(M)$ be a non-vanishing smooth vector field on M. Then, if we define $X = \tilde{X}/g^R(\tilde{X}, \tilde{X})$, it fulfills $g^R(X_p, X_p) = 1$ for all $p \in M$. We can now define a Lorentz metric tensor g^L setting, for all $p \in M$ and for all $Y, Z \in \mathfrak{X}(M)$,

$$g^L(Y_p, Z_p) = g^R(Y_p, Z_p) - 2g^R(X_p, Y_p) \cdot g^R(X_p, Z_p). \tag{3.16}$$

To see that this is indeed a Lorentz metric, choose a basis $X_p, e_2, ..., e_n$ at each T_pM such that $g^R(e_i, e_j) = \delta_{ij}$ and $g^R(X_p, e_i) = 0$. (In fact, we can locally choose vector fields $E_2, ..., E_n$ in U such that together with X they form a basis for each T_pM with $p \in U$, with $g(E_i, E_j) = \delta_{ij}$ and $g^R(X, X) = -1$, as seen in Remark 3.34). Therefore, by eq. (3.16),

$$g^L(X_p, X_p) = -g^R(X_p, X_p) \qquad g^L(X_p, e_i) = 0 \qquad g^L(e_i, e_j) = g^R(e_i, e_j) = \delta_{ij}.$$

(1.) $\Longrightarrow$ (2.) : (*Sketch.* See Curtis and Miller [12] p. 88-89). Let g^L be the Lorentz metric on M, and let g^R be a Riemannian metric on M by Theorem 3.51. Let $\sharp^R : T_pM^* \to T_pM$ and $\flat^L : T_pM \to T_pM^*$ be the lowering and raising isomorphisms defined on Remark 3.6, for g^R and g^L respectively. It is clear that $\flat^L \circ \sharp^R : T_pM \to T_pM$ is a linear isomorphism for all $p \in M$, on which it depends smoothly. The sought non-vanishing vector field will be given by the unique eigenvector with negative eigenvalue of $\flat^L \circ \sharp^R$.

(2.) $\Longleftrightarrow$ (3.) : See Milnor [17] p. 33-41. □

A particular case of the equivalence between (2.) and (3.) is the well known Hairy ball Theorem, which informally states that you cannot comb a hairy ball without leaving one hair up. It was first proven by Brower in 1912.

However, theoretical physicists should not be worried about the non-existence of Lorentz metrics on compact manifolds with $\chi(M) \neq 0$, as they will not usually work with compact manifolds. This is due to the following result. (See Hawking and Ellis [16] p. 189-190).

Definition 3.53. *Let (M, g) be a semi-Riemannian manifold, and let $p \in M$. A tangent vector $v \in T_pM$ is called*

1. spacelike *if $g_p(v, v) > 0$ or $v = 0$.*
2. lightlike *if $g_p(v, v) = 0$ and $v = 0$.*
3. timelike *if $g_p(v, v) < 0$.*

The set of all lightlike vectors of T_pM is called the lightcone *at p.*

We also can extend the previous classification to curves: we say that a smooth curve $\alpha : [a, b] \to M$ is *spacelike*, *lightlike*, or *timelike* if respectively, their tangent vector α' has positive, null, or negative norm for all $t \in [a, b]$.

Theorem 3.54. *Let (M, g^L) be a Lorentz manifold. If M is compact, then there exists a closed time-like curve on M.*

The result above tells us that on a Lorentz compact manifold you can have regions of space-time whose evolution does not depend on its past, thus violating causality.

Chapter 4

Isometry

In this final chapter all previous results are combined to reach our main goal: to prove that the isometry group of a semi-Riemannian manifold is a Lie group. First, we will have a look at the linear isometries of the scalar product vector space $\mathbb{R}^n_\nu$. Then, we will formally define the concepts of *isometry* and *Killing vector field*, and we will show some of their properties, giving the necessary tools to show that Palais' Theorem (Theorem 2.40) can be applied to the group of isometries of a semi-Riemannian manifold, proving that they form a Lie transformation group. We end this work presenting some further results and open questions on isometry groups.

4.1 Linear isometries of $\mathbb{R}^n_\nu$

Since each tangent space T_pM of a semi-Riemannian manifold $(M, \mathbf{g})$ is linearly isometric to $\mathbb{R}^n_\nu$ through a choice of an orthonormal basis, we shall first study the linear isometry group of this scalar product vector space before tackling the general case.

Recall that a linear isometry between two scalar product vector spaces (V, g_V) and (W, g_W) is a linear isomorphism $A : V \to W$ such that for all $u, v \in V$,

$$g_W(A(u), A(u)) = g_V(u, v).$$

In column vector convention, $u = (u^i)$, $G = (g_{ij})$ and $v = (v^i)$, the scalar product is expressed as $g(u, v) = u^t G v$. Hence, if we impose that $A = (A_{ij})$ is a linear isometry,

$$g(Au, Av) = g(u, v) \iff (Au)^t G A v = u^t G v \iff u^t (A^t G A) v = u^t G v \iff A^t G A = G.$$

Now, choosing orthonormal coordinates, $G = \eta := \operatorname{diag}(-1, \overset{\nu}{\ldots}, -1, 1, \overset{n-\nu}{\ldots}, 1) = \eta^t = \eta^{-1}$, so A is a linear isometry if and only if $A^t \eta A = \eta \iff A^t = \eta A^{-1} \eta$.

Hence, we recover the well-known result from Euclidean geometry that the linear isometry group of $\mathbb{R}^n$ seen as a vector field with an inner product is isomorph to the orthogonal group $O(n) = O(n, \mathbb{R}) = \{A \in Gl(n, \mathbb{R}) \mid A^t = A^{-1}\}$, since $\eta = Id$ if $\nu = 0$.

Let us now state some properties of the orthogonal group. The corresponding proofs and discussions can be found in Sepanski [14]. The orthogonal group $O(n) = O(n, \mathbb{R}) =$

$\{A \in Gl(n,\mathbb{R}) \mid A^t = A^{-1}\}$ is a closed Lie subgroup of the Lie group $Gl(n,\mathbb{R})$. Moreover, it is compact. It has two connected components: those matrices with determinant +1 and those with determinant -1. Its Lie algebra is $\mathfrak{o}(n) = \{S \in \mathfrak{gl}(n,\mathbb{R}) \mid S^t = -S\}$. It has dimension $n(n-1)/2$, since all $S \in \mathfrak{o}(n)$ have $S_{ii} = 0$ and $S_{ij} = -S_{ji}$ for all $i,j = 1,...,n$.

Going back to the general result we found for $\mathbb{R}^n_\nu$,

Proposition 4.1. *The linear isometry group of* $\mathbb{R}^n_\nu$ *is the* semi-orthogonal group, $O(n, n-\nu) = O_\nu(n) = \{A \in Gl(n,\mathbb{R}) \mid A^t = \eta A^{-1}\eta\}$.

The semi-orthogonal group is also a closed group of the Lie group $Gl(n,\mathbb{R})$, and hence it is also a Lie group under matrix multiplication. If $\nu \neq 0,n$, then $O_\nu(n)$ is not compact. Another difference with $O(n)$ if $\nu \neq 0,n$ is that $O_\nu(n)$ has four connected components instead of two. Its Lie algebra is the set of all skew-adjoint matrices $\mathfrak{o}_\nu(n) = \{S \in \mathfrak{gl}(n,\mathbb{R}) \mid S^t = -\eta S\eta\}$.

Lemma 4.2. *The Lie algebra* $\mathfrak{o}_\nu(n) = \{S \in \mathfrak{gl}(n,\mathbb{R}) \mid S^t = -\eta S \eta\}$ *as dimension* $n(n-1)/2$ *regardless of* ν.

Proof. Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with a a $\nu \times \nu$ matrix and d a $(n-\nu)\times(n-\nu)$ matrix. Since $\eta = \mathrm{diag}(-1,\overset{\nu}{\ldots},-1,1,\overset{n-\nu}{\ldots},1) = \eta^t = \eta^{-1}$, we have

$$S^t = \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix}, \qquad \text{and} \qquad -\eta S \eta = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}.$$

Hence, $a^t = -a$, so $a \in \mathfrak{o}(\nu)$; $d^t = -d$, so $d \in \mathfrak{o}(n-\nu)$; and $c^t = b$. Therefore $S \in \mathfrak{o}_\nu(n)$ if and only if $S = \begin{pmatrix} a & b \\ b^t & d \end{pmatrix}$, with $a \in \mathfrak{o}(\nu)$, $d \in \mathfrak{o}(n-\nu)$, and b an arbitrary $\nu \times (n-\nu)$ matrix. Thus,

$$\begin{aligned} \dim \mathfrak{o}_\nu(n) &= \dim \mathfrak{o}(\nu) + \dim \mathfrak{o}(n-\nu) + \nu(n-\nu) \\ &= \nu(\nu-1)/2 + (n-\nu)(n-\nu-1)/2 + \nu(n-\nu) = n(n-1)/2. \end{aligned}$$ □

4.2 Isometry and Killing vector fields

Definition 4.3. *Let* (M, g^M) *and* (N, g^N) *be semi-Riemannian manifolds. An* isometry *from M to N is a diffeomorphism* $\phi : M \to N$ *such that it preserves the metric tensor, that is,* $\phi^*(g^N) = g^M$. *We then say that M and N are* isometric.

Explicitly, for an isometry $\phi : M \to N$, we have $g^N_{\phi(p)}(d\phi_p(u), d\phi_p(v)) = g^M_p(u,v)$ for all $p \in M$ and $u, v \in T_pM$. Also, as ϕ is a diffeomorphism, each differential map $d\phi_p$ is a linear isomorphism; therefore the metric condition means that each differential map $d\phi_p : T_pM \to T_{\phi(p)}N$ is a linear isometry between vector spaces $\cong \mathbb{R}^n_\nu$.

So isometric manifolds are diffeomorphic and preserve the metric structure. Therefore, besides all concepts preserved through diffeomorphisms, all notions derived from the metric tensor are also preserved, such as the covariant derivative, geodesics, curvature,... Hence, from the point of view of semi-Riemannian geometry, isometric manifolds are *the same*, much as in the same way as homeomorphic spaces are topologically the same or diffeomorphic manifolds are the same from the point of view of manifold theory.

Proposition 4.4. *The set of all isometries of a semi-Riemannian manifold (M, g) onto itself form a group.*

Proof. The identity map is obviously an isometry. We need to prove that a composition of isometries is an isometry and that the inverse map of an isometry is an isometry. Let $\phi, \psi : M \to M$ be isometries. Then, $\phi \circ \psi$ is also an isometry, as $(\phi \circ \psi)^*(\mathbf{g}) = \phi^*(\psi^*(\mathbf{g})) = \phi^*(\mathbf{g}) = \mathbf{g}$. More explicitly, for all $p \in M$ and for all $u, v \in T_pM$,

$$g_{(\phi\circ\psi)(p)}\Big(d(\phi \circ \psi)_{\psi(p)}(u), d(\phi \circ \psi)_{\psi(p)}(v)\Big)$$
$$= g_{\phi\left(\psi(p)\right)}\Big(d\phi_{\psi(p)}\big(d\psi_p(u)\big), d\phi_{\psi(p)}\big(d\psi_p(v)\big)\Big) = g_{\psi(p)}\big(d\psi_p(u), d\psi_p(v)\big) = g_p(u, v).$$

Similarly, ϕ^{-1} is also an isometry, as $\mathbf{g} = (\phi^{-1} \circ \phi)^*(\mathbf{g}) = (\phi^{-1})^*(\phi^*(\mathbf{g})) = (\phi^{-1})^*(\mathbf{g})$. □

Definition 4.5. *The group of all isometries of a semi-Riemannian manifold (M, g) onto itself is denoted $I(M, \mathbf{g})$ and called the* isometry group of M.

We can also define a *softer* version of isometry, its local counterpart. Many concepts preserved through isometries are also preserved through local isometries, such as the covariant derivative, as they are local concepts.

Definition 4.6. *Let M, N be semi-Riemannian manifolds. A* local isometry *is a smooth map $\phi : M \to N$ such that for all $p \in M$, the differential map $d\phi_p : T_pM \to T_{\phi(p)N}$ is a linear isometry of vector spaces.*

Due to the Inverse Function Theorem, and the fact that linear isometries are vector space isomorphisms, the definition above is equivalent to the following: ϕ is a local isometry if for all $p \in M$ there exist a neighborhood U of p and a neighborhood V of $\phi(p) \in N$ such that $\phi|_U : U \to V$ is an isometry. Notice that all isometries are obviously local isometries, but only local isometries which are also diffeomorphisms are global isometries.

Proposition 4.7. *Let $X \in \mathfrak{X}(M)$. Then $\pounds_X g = \lim_{t\to 0} \frac{1}{t}[(Fl_t^X)^*(g) - g]$.*

Proof. Since $\pounds_X$ is a tensor derivation, and recalling that it is defined by $\pounds_X Y = [X, Y]$ and $\pounds_X f = X(f)$, by Prop. 1.55 we have that

$$(\pounds_X \mathbf{g})(Y, Z) = X\big(\mathbf{g}(Y, Z)\big) - \mathbf{g}([X, Y], Z) - \mathbf{g}(Y, [X, Z]), \tag{4.1}$$

for all $Y, Z \in \mathfrak{X}(M)$.
We will work the second half of the equation we want to prove to see that it matches (4.1) above. As $((Fl_t^X)^*g)_p(Y_p, Z_p) = g_{Fl_t^X(p)}(dFl_t^X Y_p, dFl_t^X Z_p)$ for a fixed point $p \in M$,

$$\lim_{t\to 0} \frac{1}{t}\Big[(Fl_t^X)^*g - g\Big]_p(Y_p, Z_p) = \lim_{t\to 0} \frac{1}{t}\Big[g_{Fl_t^X(p)}(dFl_t^X(Y_p), dFl_t^X(Z_p)) - g_p(Y_p, Z_p)\Big]$$
$$= \lim_{t\to 0} \frac{1}{t}\Big[g_{Fl_t^X(p)}(dFl_t^X(Y_p), dFl_t^X(Z_p)) - g_{Fl_t^X(p)}(Y_{Fl_t^X(p)}, Z_{Fl_t^X(p)})\Big]$$
$$+ \lim_{t\to 0} \frac{1}{t}\Big[g_{Fl_t^X(p)}(Y_{Fl_t^X(p)}, Z_{Fl_t^X(p)}) - g_p(Y_p, Z_p)\Big] = L1 + L2. \tag{4.2}$$

For the second limit of (4.2) above, $L2$, let us call $\alpha(t) = Fl^X(t,p)$ the integral curve of X with $\alpha(0) = p$. We have

$$\begin{aligned} L2 = \lim_{t\to 0} \frac{1}{t}\Big[g_{Fl_t^X(p)}(Y_{Fl_t^X(p)}, Z_{Fl_t^X(p)} - g_p(Y_p, Z_p)\Big] &= \lim_{t\to 0}\frac{1}{t}\Big[g_{\alpha(t)}(Y_{\alpha(t)}, Z_{\alpha(t)} - g_p(Y_p,Z_p)\Big] \\ &= \frac{d}{dt} g_{\alpha(t)}(Y_{\alpha(t)}, Z_{\alpha(t)})|_0 = \alpha'(0)\big(g_p(Y_p,Z_p)\big) = X_p\big(g_p(Y_p,Z_p)\big). \quad (4.3) \end{aligned}$$

Now, since g is bilinear,

$$g(u,v) - g(u',v') = g(u,v) - g(u',v) + g(u',v) - g(u',v') = g(u-u',v) - g(u', v-v'),$$

so for the first limit $L1$ of (4.2) we can write

$$\begin{aligned} L1 = \lim_{t\to 0}\frac{1}{t}\Big[g_{Fl_t^X(p)}\big(dFl_t^X(Y_p), dFl_t^X(Z_p)\big) - g_{Fl_t^X(p)}\big(Y_{Fl_t^X(p)}, Z_{Fl_t^X(p)}\big)\Big] \\ = \lim_{t\to 0}\frac{1}{t}\Big[g_{Fl_t^X(p)}\big(dFl_t^X(Y_p) - Y_{Fl_t^X(p)}, dFl_t^X(Z_p)\big)\Big] \\ + \lim_{t\to 0}\frac{1}{t}\Big[g_{Fl_t^X(p)}\big(Y_{Fl_t^X(p)}, dFl_t^X(Z_p) - Z_{Fl_t^X(p)}\big)\Big] = L1.1 + L1.2. \quad (4.4) \end{aligned}$$

Again, this expression can be split into two terms. On both of them we will again use that g is bilinear, and that if t small enough so that the flow is well defined we have $id = Fl_0^X = (Fl_t^X \circ Fl_{-t}^X)$. Then, the first term of (4.4) can be rewritten as

$$\begin{aligned} L1.1 &= \lim_{t\to 0}\frac{1}{t}\Big[g_{Fl_t^X(p)}\big(dFl_t^X(Y_p) - Y_{Fl_t^X(p)}, dFl_t^X(Z_p)\big)\Big] \\ &= \lim_{t\to 0}\frac{1}{t}\Big[g_{Fl_t^X(p)}\Big(dFl_t^X\big(Y_p - dFl_{-t}^X(Y_{Fl_t^X(p)})\big), dFl_t^X(Z_p)\Big)\Big] \\ &= -g_p\Big(\lim_{t\to 0} dFl_t^X\big(\frac{1}{t}[dFl_{-t}^X(Y_{Fl_t^X(p)}) - Y_p]\big), \lim_{t\to 0} dFl_t^X(Z_p)\Big) = -g_p\Big([X,Y]_p, Z_p\Big), \end{aligned}$$

where in the last equality we have used Proposition 1.45, which allows one to write the Lie bracket of vector fields in terms of flows, $\lim_{t\to 0}\frac{1}{t}\big[dFl_{-t}^X(Y_{Fl_t^X(p)}) - Y_p\big]$. Similarly,

$$\begin{aligned} L1.2 &= \lim_{t\to 0}\frac{1}{t}\Big[g_{Fl_t^X(p)}\big(Y_{Fl_t^X(p)}, dFl_t^X(Z_p) - Z_{Fl_t^X(p)}\big)\Big] \\ &= \lim_{t\to 0}\frac{1}{t}\Big[g_{Fl_t^X(p)}\Big(Y_{Fl_t^X(p)}, dFl_t^X\big((Z_p) - dFl_{-t}^X(Z_{Fl_t^X(p)})\big)\Big)\Big] \\ &= -g_p\Big(\lim_{t\to 0} Y_{Fl_t^X(p)}, \lim_{t\to 0} dFl_t^X\big(\frac{1}{t}[dFl_{-t}^X(Z_{Fl_t^X(p)}) - Z_p]\big)\Big) = -g_p\Big(Y_p, [X,Z]_p\Big). \end{aligned}$$

Wrapping up,

$$\begin{aligned} \lim_{t\to 0}\frac{1}{t}\Big[(Fl_t^X)^* g - g\Big]_p (Y_p, Z_p) &= L1 + L2 = L2 + L1.1 + L1.2 \\ &= X_p\big(g_p(Y_p,Z_p)\big) - g_p\big([X,Y]_p, Z_p\big) - g_p\big(Y_p, [X,Z]_p\big) = (\pounds_X g)_p(Y_p, Z_p), \end{aligned}$$

where the last equality is precisely what we wanted to prove, as it is (4.1) evaluated at $p \in M$, thus concluding the proof. $\square$

Definition 4.8. *A* Killing vector field *on a semi-Riemannian manifold M is a vector field $\xi \in \mathfrak{X}(M)$ relative to which the Lie derivative of the metric tensor field vanishes:*

$$£_\xi g = \lim_{t\to 0} \frac{1}{t}[(Fl_t^\xi)^*(g) - g] = 0$$

Therefore, the metric tensor $\mathbf{g}$ remains invariant along the flow of ξ. This is why Killing vector fields are sometimes also called *infinitesimal isometries*. In fact:

Proposition 4.9. *A vector field ξ is Killing if and only if for all t in which the local flow Fl^ξ of ξ is properly defined, the transformation $Fl_t^\xi : M \to M$ is an isometry.*

Proof. $(\Leftarrow)$: If Fl_t^ξ is an isometry, then $(Fl_t^\xi)^*(\mathbf{g}) = \mathbf{g}$. Therefore $£_\xi \mathbf{g} = \lim_{t\to 0} \frac{1}{t}[(Fl_t^\xi)^*(\mathbf{g}) - \mathbf{g}] = \lim_{t\to 0} \frac{1}{t}[\mathbf{g} - \mathbf{g}] = 0$.

$(\Rightarrow)$: Now, let ξ be a Killing vector field, $£_\xi \mathbf{g} = 0$, and let Fl^ξ be the local flow of ξ. Let $p \in M$ and $v \in T_pM$. For s tending to 0, $u = dFl_s^\xi(v)$ can also be considered a vector of T_pM. Hence, using Proposition 4.7, and bearing in mind that $Fl_s^\xi \circ Fl_t^\xi = Fl_{s+t}^\xi$,

$$\begin{aligned} 0 = (£_\xi g)(u,u) &= \lim_{t\to 0} \frac{1}{t}\Big[g\big(dFl_t^\xi(u), dFl_t^\xi(u)\big) - g\big(u,u\big)\Big] \\ &= \lim_{t\to 0} \frac{1}{t}\Big[g\big(dFl_{t+s}^\xi(v), dFl_{t+s}^\xi(u)\big) - g\big(dFl_s^\xi(v), dFl_s^\xi(v)\big)\Big] = \frac{d}{ds}\Big[g\big(dFl_s^\xi(v), dFl_s^\xi(v)\big)\Big] \end{aligned}$$

So the function $s \mapsto g\big(dFl_s^\xi(v), dFl_s^\xi(v)\big)$ has derivative 0 for all s. Hence it is constant, so $g\big(dFl_s^\xi(v), dFl_s^\xi(v)\big) = g\big(v,v\big)$ for all s, and also for all $p \in M$ and for all $v \in T_pM$. Therefore we can conclude that $(Fl_s^\xi)^*(g) = g$ so Fl_s^ξ is an isometry for all s where the flow of ξ is properly defined. □

Thus, given a Killing vector field ξ, its flow Fl^ξ defines a 1-parameter group of isometries. And *vice versa*, every 1-parameter group of isometries is the flow of some Killing vector field on M. Therefore a practical way of finding isometries of a semi-Riemannian manifold would be finding Killing vector fields. This is usually done by solving the equation below.

Remark 4.10. Let (U, φ) be a chart on a semi-Riemannian manifold $(M, \mathbf{g})$, with coordinate functions $x^1, ..., x^n$. Since $\mathbf{g}$ is a (0,2) tensor field on M, then also $£_X \mathbf{g}$ must be a (0,2) tensor field on M for any smooth tensor field $X \in \mathfrak{X}(M)$. Its coordinate expression can be found subsituting $Y = \partial_i$ and $Z = \partial_j$ on eq. (4.1) within the proof of Prop 4.7. One obtains

$$(£_X \mathbf{g})_{ij} = (£_X \mathbf{g})(\partial_i, \partial_j) = X^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial X^k}{\partial x^j} + g_{kj}\frac{\partial X^k}{\partial x^i}.$$

Therefore the following proposition is immediate.

Proposition 4.11. *Let (M, g) be a semi-Riemannian manifold. A smooth vector field $\xi \in \mathfrak{X}(M)$ on M is a Killing vector field is and only if its coordinate expression $\xi = \xi^k \partial_k$ for any chart (U, φ) fulfills*

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{kj}\frac{\partial \xi^k}{\partial x^i} = 0.$$

The proposition above defines a system of partial first order differential equations for the coordinates of ξ. Usually, it has no solutions, since is is an overdetermined system: there are n coordinates of ξ to solve for ($k = 1, ..., n$), but there are $n(n+1)/2$ independent equations ($i, j = 1, ..., n$ but the equations are symmetric under $i \leftrightarrow j$). Hence, generic manifolds in general are not symmetric enough to have Killing vector fields. However, when the manifold considered does have Killing vector fields, Proposition 4.11 is a useful way to find them.

Proposition 4.12. *Let $\xi \in \mathfrak{X}(M)$. Then, the following are equivalent:*

1. *ξ is a Killing vector field.*
2. *$\xi\big(g(Y,Z)\big) = g\big([\xi,Y],Z\big) + g\big(Y,[\xi,Z]\big)$ for all $Y, Z \in \mathfrak{X}(M)$.*
3. *$g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) = 0$ for all $Y, Z \in \mathfrak{X}(M)$.*

Proof.

1. $\iff$ 2. : Immediate from $£_\xi$ being tensor derivation, so $(£_\xi g)(Y,Z) = \xi(g(Y,Z)) - g([\xi,Y],Z) - g(Y,[\xi,Z])$ for all $Y, Z \in \mathfrak{X}(M)$, and ξ is Killing iff $£_\xi g = 0$.
2. $\iff$ 3. : Recall that the Levi-Civita covariant derivative is torsion-free, so $\nabla_\xi Y - \nabla_Y \xi = [\xi, Y]$ for all $\xi, Y \in \mathfrak{X}(M)$; and also compatible with the metric, so $\xi\big(g(Y,Z)\big) = g(\nabla_\xi Y, Z) + g(\nabla_\xi Z, Y)$ for all $\xi, Y, Z \in \mathfrak{X}(M)$. Therefore, we have
$$\begin{aligned} g\big(\nabla_Y \xi, Z\big) + g\big(\nabla_Z \xi, Y\big) &= g\big(\nabla_\xi Y, Z\big) - g\big([\xi,Y],Z\big) + g\big(\nabla_\xi Z, Y\big) - g\big([\xi,Z],Y\big) \\ &= \xi\big(g(Y,Z)\big) - g\big([\xi,Y],Z\big) - g\big(Y,[\xi,Z]\big). \end{aligned}$$
□

Proposition 4.13. *Let ξ be a Killing vector field, and let γ be a geodesic. Then $\xi|_\gamma$ is a Jacobi field on γ and $g(\gamma', \xi)$ is constant along γ.*

The second part of the proposition is a result known by physicists as the conservation lemma. It translates to the fact that for each Killing vector field there is a physical quantity which is conserved along geodesics. And as geodesics are the trajectories followed by free particles, and more importantly, by inertial reference frames, this result is of great convenience when studying physical systems in General Relativity.

Proof. Let Fl_s^ξ be the local flow of ξ near a point $\gamma(t)$ on the geodesic. Now, the two-parameter map $(t,s) \to \theta(t,s) := Fl_s^\xi(\gamma(t))$ is a geodesic variation of γ. For a fixed t_0, the curve $s \to \theta(t_0, s) = Fl_s^\xi(\gamma(t_0))$ is an integral curve of ξ, starting at $\gamma(t_0)$ with initial velocity $\xi_{\gamma(t_0)}$. Therefore the variation vector field of the geodesic variation is $\theta_s(t,0) = \xi|_\gamma$. Hence, by Theorem 3.46, $\xi|_\gamma$ is a Jacobi vector field on γ.
Now, as ξ is Killing, by (3.) from the previous Proposition 4.12, we have $g(\nabla_{\gamma'}\xi, \gamma') = 0$. But also, γ is a geodesic, so by Corollary 3.26,
$$\frac{d}{dt} g(\xi|_\gamma, \gamma') = g(\frac{D}{dt}\xi|_\gamma, \gamma') = g(\nabla_{\gamma'}\xi, \gamma') = 0.$$
Thus, $g(\gamma', \xi)$ is constant along γ as it has null derivative. □

Lemma 4.14. *Let ξ be a Killing vector field on a connected semi-Riemannian manifold (M, g). If there exists a point $p \in M$ such that $\xi_p = 0$ and $(\nabla\xi)_p = 0$, then ξ is identically 0.*

Proof. Let $A \subseteq M$ be the set of points at which both ξ and $\nabla\xi$ are null. A is the intersection of two sets which are the preimage of a closed set ($\{0\}$) by a continuous map (the evaluation map), therefore A is closed. Obviously A is nonempty, as $p \in A$. Thus, if we prove that A is also open, we will have that $A = M$ and therefore that ξ is identically 0 on all of M. Let U be a normal neighborhood of p, and let τ be a radial geodesic starting from p. The previous Proposition tells us that the restriction $\xi|_\tau$ of ξ on τ is a Jacobi field. Moreover, $\xi_{\tau(0)} = \xi_p = 0$ and $\frac{D}{dt}(\xi|_\tau)(0) = \nabla_{\tau'(0)}\xi = (\nabla\xi)_p(\tau'(0)) = 0$, so again by the previous Proposition both $g(\xi|_\tau, \tau')$ and $g(\frac{D}{dt}\xi|_\tau, \tau')$ are identically zero. Using Proposition 3.50(2.) we can now see that $\xi|_\tau$ is zero on all of the geodesic, and therefore ξ is identically 0 on U and so $\nabla\xi$ is null too. □

4.3 The Lie Algebra $\mathfrak{i}(M, \mathbf{g})$

Proposition 4.15. *Let X, Y be Killing vector fields on a semi-Riemannian manifold $(M, \mathbf{g})$. Then,*

1. *$aX + bY$ is also a Killing vector field, for all $a, b \in \mathbb{R}$.*
2. *$[X, Y]$ is also a Killing vector field.*

Therefore the set of all Killing vector fields on a semi-Riemannian manifold (M, g) is a Lie algebra.

Proof. Immediate from $\mathbb{R}$-linearity of tensor derivations and the remark following 1.58. If X and Y are Killing vector fields, $\pounds_X g = 0$ and $\pounds_Y g = 0$, so

1. $\pounds_{aX+bY} g = a\pounds_X g + b\pounds_Y g = 0$.
2. $\pounds_{[X,Y]} g = [\pounds_X, \pounds_Y] g = \pounds_X(\pounds_Y g) - \pounds_Y(\pounds_X g) = 0$.

□

Definition 4.16. *The set of all Killing vector fields on a semi-Riemannian manifold $(M, \mathbf{g})$ is a Lie algebra, called the* Lie algebra of Killing vector fields *and denoted by* $\mathfrak{i}(M, \mathbf{g})$.

Corollary 4.17. *Let X and Y be Killing vector fields on a semi-Riemannian manifold (M, g). If there exists a point $p \in M$ such that $X_p = Y_p$ and $(\nabla X)_p = (\nabla Y)_p$, then $X = Y$.*

Proof. As a linear combination of Killing vector fields is a Killing vector field, the proof is immediate from applying Lemma 4.14 to $\xi = X - Y$. □

Lemma 4.18. *Let (M, g) be a connected semi-Riemannian manifold of dimension n. Then, its Lie algebra of killing vector fields $\mathfrak{i}(M, \mathbf{g})$ has dimension* $\dim \mathfrak{i}(M, \mathbf{g}) \leq n(n+1)/2$.

Proof. Fix a point $p \in M$. The set of all linear operators A on T_pM which fulfill $g_p(Au, v) = -g_p(u, Av)$ form a Lie algebra, which we denote by $\mathfrak{o}(T_pM)$. The fact that a linear combination of operators in $\mathfrak{o}(T_pM)$ is still in $\mathfrak{o}(T_pM)$ is immediate from the linearity of such operators, and $\mathfrak{o}(T_pM)$ is also closed under bracket operation, since

$$g_p([A, B]u, v) = g_p(ABu, v) - g_p(BAu, v) = -g_p(u, ABv) + g_p(u, BAv) = -g_p(u, [A, B]v).$$

Therefore $\mathfrak{o}(T_pM)$ is a Lie algebra, which we call the Lie algebra of skew-adjoint operators on T_pM. Using an orthonormal basis and the column vector convention, these operators become matrices in $\mathfrak{o}_\nu(n)$, since then $g_p = \eta = \text{diag}(-1, \overset{\nu}{\ldots}, -1, 1, \overset{n-\nu}{\ldots}, 1)$:

$$\begin{aligned} g_p(Au, v) = -g_p(u, Av) &\iff (Au)^t \eta v = -u^t \eta A v \iff u^t A^t \eta v = -u^t \eta A v \\ &\iff A^t \eta = -\eta A \iff A^t = -\eta A \eta \iff A \in \mathfrak{o}_\nu(n) \end{aligned}$$

so we could have also seen that $\mathfrak{o}(T_pM)$ is a Lie algebra because it is isomorphic to the Lie albegra $\mathfrak{o}_v(n)$. Now, let the map $E : \mathfrak{i}(M, \mathbf{g}) \to T_pM \times \mathfrak{o}(T_pM)$ be defined by $\xi \mapsto (\xi_p, (\nabla\xi)_p)$. The fact that $(\nabla\xi)_p$ is indeed a skew-adjoint operator on T_pM is immediate by Prop. 4.12, (3.). By Corollary 4.17 E is one-to-one. Therefore,

$$\dim \mathfrak{i}(M, g) = \dim E(\mathfrak{i}(M, g)) \leq \dim T_pM + \dim \mathfrak{o}_v(n) = n + n(n-1)/2 = n(n+1)/2.$$

□

4.4 The Lie Group $I(M, \mathbf{g})$

We are very close to our main goal. Let us recall Theorem 2.40: *Let G be a group of diffeomorphisms of a manifold M onto itself. Let S be the set of all vector fields $\tilde{X}$ on M which generate global 1-parameter groups $\phi_t = Fl_t^{\tilde{X}}$ of transformations of M such that $\phi_t \in G$. If the set S generates a finite-dimensional Lie algebra of vector fields on M, then (G, M) is a Lie transformation group and S is the Lie algebra of G.*

We have already proved that the set of all isometries of a manifold onto itself is a group, so we set $G = I(M, g)$. Let $\mathfrak{ci}(M, g) = \{\xi \in \mathfrak{i}(M, g) \mid \xi \text{ is complete}\}$ be the set of all complete Killing vector fields on M. We want to see that $\mathfrak{ci}(M, g)$ is also the set of all vector fields on M which generate global 1-parameter groups of isometries, that is, we want to see $S = \mathfrak{ci}(M, g)$.

Lemma 4.19. *Let (M, g) be a semi-Riemannian manifold. The set of all complete Killing vector fields on M, $\mathfrak{ci}(M, g)$, is the set of all vector fields on M which generate global 1-parameter groups ϕ_t of isometries of M.*

Proof. First, let ξ be a complete Killing vector field. As it is complete, its flow Fl^ξ is globally defined. So, by Proposition 4.9, $\phi_t = Fl_t^X$ is an isometry for all $t \in \mathbb{R}$. Therefore ξ generates a global 1-parameter group which is a subgroup of $I(M, g)$, as $Fl_s^\xi \circ Fl_t^\xi = Fl_{s+t}^\xi$ and $Id = Fl_0^\xi = Fl_t^\xi \circ Fl_{-t}^\xi$, for all $s, t \in \mathbb{R}$.

Conversely, let ξ be a vector field on M which generates a global 1-parameter group $\phi_t = Fl_t^\xi$ of isometries of M. Again, by Proposition 4.9, ξ is a Killing vector field. And it is obviously complete because its flow is a global 1-parameter group. □

So, if we prove that $\mathfrak{ci}(M, g)$ generates a finite-dimensional Lie algebra, we will have all hypothesis for Theorem 2.40, thus proving that $I(M, g)$ is a Lie group.

Lemma 4.20. *$\mathfrak{ci}(M, g)$ generates a finite-dimensional Lie algebra.*

Proof. It is a direct consequence of Proposition 4.18, which says that the Lie algebra $\mathfrak{i}(M, g)$ of all Killing vector fields (not necessarily complete) has dimension at most $\frac{1}{2}n(n+1)$. One easy case would be if M were compact, then Proposition 1.39 would imply that $\mathfrak{i}(M, g) = \mathfrak{ci}(M, g)$. But the general case is also straightforward. Let $\mathfrak{g}$ be the Lie algebra generated by $\mathfrak{ci}(M, g)$. As $\mathfrak{ci}(M, g) \subseteq \mathfrak{i}(M, g)$, we have that $aX + bY \in \mathfrak{i}(M, g)$ and $[X, Y] \in \mathfrak{i}(M, g)$ for all $X, Y \in \mathfrak{ci}(M, g)$ (Prop. 4.15). So the Lie algebra $\mathfrak{g}$ generated by $\mathfrak{ci}(M, g)$ is a Lie subalgebra of $\mathfrak{i}(M, g)$, and therefore it is also finite-dimensional, as it has dim $\mathfrak{g} \leq$ dim $\mathfrak{i}(M, g) \leq n(n+1)/2$. □

Hence, all hypothesis of Theorem 2.40 are fulfilled:

Theorem 4.21. *Let (M, g) be a semi-Riemannian manifold. Let $I(M, g)$ be the set of all isometries of M onto itself, and let $\mathfrak{ci}(M, g)$ be the set of all complete Killing vector fields on M. Then, the pair $\big(I(M, g), M\big)$ is a Lie transformation group, and $\mathfrak{ci}(M, g)$ is the Lie algebra of the Lie group $I(M, g)$.*

More explicitly, we have

1. $I(M, g)$ is a Lie group.
2. The map $I(M, g) \times M \to M$ naturally defined by $(\phi, p) \mapsto \phi(p)$ is smooth.
3. A homomorphism $\alpha : \mathbb{R} \to I(M, g)$ is smooth if the map $\mathbb{R} \times M \to M$ which sends (t, p) to $\alpha(t)(p)$ is smooth.

Remark 4.22. One could get from the construction above the wrong idea that *all* isometries are generated as flows of Killing vector fields. That is not true. What we have proven is that all "smooth" isometries, that is, those which generate global 1-parameter subgroups of $I(M, g)$, are indeed generated by Killing vector fields. However, we could have "discrete" isometries in $I(M, g)$, such as reflections. These kind of isometries do not have a Killing vector field related to them. Informally speaking, these are the ones that allow you to "jump from one connected component of $I(M, g)$ to another".

Example 4.23. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$ be the n-dimensional sphere, where $|| \cdot ||$ denotes the Euclidean norm, endowed with the usual Riemannian metric g it gets as a submanifold of $\mathbb{R}^{n+1}$. Then its isometry group is $I(S^n, g) = O(n+1) = \{x \in Gl(n+1, \mathbb{R}) \mid AA^t = A^tA = \mathbb{1}\}$, the orthogonal group, as the linear isometries of $\mathbb{R}^{n+1}$ carry S^n into itself. The Killing vector fields on S^n generate the group of rotations of the sphere $SO(n+1, \mathbb{R})$ which is the group of matrices with $det = +1$. Is that all there is? No! There are also reflections with respect to a hyperplane, which are not generated as the flow of any Killing vector field. The group of rotations is only one of the two connected components of $O(n+1, \mathbb{R})$, the other one being the isometries formed as a composition of a rotation and a reflection which are the matrices with $det = -1$.

Example 4.24. Let $E = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ be a triaxial ellipsoid with the usual Riemannian metric it gets as a submanifold of $\mathbb{R}^3$. Its isometry group is discrete, so it is a 0-dimensional Lie group. It only has 8 elements: the identity, the three reflections along the planes perpendicular to the axes, and the compositions of such reflections.

Let us end presenting some further results:

Theorem 4.25. *Let (M, g) be a Riemannian manifold. If M is compact then $I(M, g)$ is compact.*

Proof. See Kobayashi and Nomizu [9]. For the general results on metric spaces, see Theorems 4.6-4.10 in Chapter 1 (p. 45-50). Their application to Riemannian manifolds is Theorem 3.4 in Chapter 6 (p. 239). □

This result is proved thanks to the fact that Riemannian manifolds are also metric spaces (Theorem 3.37), and does not hold for other kinds of metrics.

See Gallot, Hulin and Lafontaine [13] sections 2.22 and 2.135 for an example of a Lorentzian flat torus with non-compact isometry group.

However, there are also some beautiful results on the positive. For example, this theorem by G. d'Ambra:

Theorem 4.26. *([19]). The isometry group of a compact simply connected analytic Lorentzian manifold is compact.*

By an analytic manifold we mean that the charts on the manifold are related analytically instead of smoothly. The interested reader will find many more results on isometry groups of compact Lorentz manifolds in this article by Adams and Stuck [20].

We conclude this work with an open question. We have proven that the isometry group of any semi-Riemannian manifold is a Lie group. But, what about the inverse question? Is every Lie group the isometry group of some semi-Riemannian manifold?

Theorem 4.27. *([21]). Every compact Lie group can be realized as the isometry group of a compact Riemannian manifold.*

The theorem above was proved by R. Saerens and W. R. Zame in 1987. Up to this date, however, the question remains unanswered for non-compact Lie groups.

Appendix: Basic Notions

This appendix contains some basic notions on Topology, Algebra, Multivariable Calculus and Geometry needed to understand the previous chapters. Most of the definitions and results should already be known to the average senior undergraduate, so we will skip most proofs. For the proofs related to Topology and Geometry, we refer the reader to Singer and Thorpe [7], except the ones explicitly stated on the text. For the results on Calculus, we recommend Fleming [23]. For the ones related to Algebra, the interested reader should find more information on Lang [24].

Topology

Definition A.1. Given a set X and a collection of subsets of X, $\tau \subseteq \mathcal{P}(X)$, we say that the pair (X, τ) is a *topological space* if:

- Both $\emptyset \in \tau$ and $X \in \tau$
- $\{U_i\}_{i\in I} \subseteq \tau \Rightarrow \bigcup\limits_{i\in I} U_i \in \tau$
- $U_1, ..., U_r \in \tau \Rightarrow U_1 \cap \cdots \cap U_r \in \tau$

We call the elements in τ the *open sets* of X. Therefore, the definition above says that both the empty set and the whole X are open, that any arbitrary union of open sets is an open set, and that any finite intersection of open sets is also an open set.

Definition A.2. *We say that a subset $T \subseteq X$ is* closed *if its complement $X \setminus T$ is open.*

Definition A.3. *A* basis *β for a topological space X with topology τ is a subcollection $\beta \subseteq \tau$ such that for all $U \in \tau$, U is the union of elements in β. We say that a space X is a* second countable *space if its topology τ allows a countable basis β.*

Definition A.4. *Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces. We say that a map $f : X \to Y$ is* continuous *if for all $U \in \tau_Y$, $f^{-1}(U) \in \tau_X$. We say that a map $f : X \to Y$ is an* homeomorphism *if it is continuous, bijective, and has inverse map $f^{-1} : Y \to X$ continuous. We say that two topological spaces are* homeomorphic *if there exists a homeomorphism between them, and we denote it by $X \cong Y$.*

From the topological point of view two homeomorphic spaces are equivalent, as there is a one-to-one correspondence between their open sets, and hence their topology. Therefore all topological concepts defined in this section are preserved through homeomorphisms.

Definition A.5. Let X be a topological space. A *cover* of X is a collection of subsets of X, $\{A_i\}_{i\in I}$, $A_i \subseteq X$, such that $\bigcup_{i\in I} A_i = X$. We say that it is an *open cover* if A_i is open for all $i \in I$. A *subcover* is a subcollection of a cover which still covers X. A *refinement* of the cover $\{A_i\}_{i\in I}$ is a collection $\{B_j\}_{j\in J}$ such that for all $j \in J$ there exists an $i \in I$ such that $B_j \subseteq A_i$.

Definition A.6. A topological space X is *compact* if for any open cover $\{U_i\}_{i\in I}$ there exists a finite subset $J \subseteq I$ such that $\bigcup_{i\in J} U_i = X$.
A topological space X is *locally compact* if every point $x \in X$ has a compact neighborhood.

Proposition A.7. *Let X be a compact topological space. Let $T \subseteq X$ be closed. Then T is also compact.*

Proposition A.8. Let X, Y be topological spaces, X compact. Let $f : X \to Y$ be a continuous and surjective function. Then, Y is compact.

Notice that Proposition A.8 tells us that for any continuous function $f : X \to Y$, if X is compact, then $f(X) \subseteq Y$ is compact.

Definition A.9. *A collection $\{A_i\}_{i\in I}$ of subsets of X is called* locally finite *if every $x \in X$ has a neighborhood U_x such that there is only a finite number of $i \in I$ such that $U_x \bigcap A_i \neq \emptyset$.*

Definition A.10. *A topological space X is said to be* paracompact *if every open cover has a locally finite subcover.*

Definition A.11. *A topological space X is a* Hausdorff space *if for all $x, y \in X$ such that $x \neq y$ there exist U, V disjoint open sets in X such that $x \in U$ and $y \in V$.*

Lemma A.12. *Let X be a locally compact, Hausdorff and second countable topological space. Then X is also paracompact.*

Proof. See Warner [8] Lemma 1.9, p.9. □

Definition A.13. *Let G be a group of homeomorphisms of a topological space X onto itself. The* compact-open *topology for G is the topology having as a basis all sets of the form*

$$W(K_1, ...K_n; U_1, ..., U_n) = g \in G \mid g(K_i) \subseteq U_i \; i = 1, ..., n$$

where all $K_i \subseteq X$ are compact and all $U_i \subseteq X$ are open.

Proposition A.14. *If X is locally compact (which is the case for manifolds), then the compact-open topology is the weakest topology for making the map $G \times X \to X$ of G acting on X, $(g, p) \mapsto g(p)$, continuous.*

Proposition A.15. *If X is locally compact and locally connected (which again is the case for manifolds), then G becomes a topological group with the compact-open topology.*

Definition A.16. *A* topological group *G is a topological space endowed with a continuous group operation.*

Euclidean Geometry

Definition A.17. *Let V be a real vector space. A* symmetric bilinear form *on V is an $\mathbb{R}$-bilinear function $g : V \times V \to \mathbb{R}$ such that for all $v, w \in V$ we have $g(v, w) = g(w, v)$.*
We say that g is

- positive-definite *if for all $v \neq 0$, $g(v, v) > 0$.*
- negative-definite *if for all $v \neq 0$, $g(v, v) < 0$.*
- non-degenerate *if $g(v, w) = 0$ for all $w \in V$ implies that $v = 0$.*

Let W be a vector subspace of V. It is trivial to prove that the restriction of g on $W \times W$, denoted by $g|_W$, is also a symmetric bilinear form on W. With that in mind, we can define the index of a symmetric bilinear form on a vector space (sometimes also called the signature).

Definition A.18. *Let V be a real vector space and let g be a symmetric bilinear form on V. The* index *ν of g is the largest dimension of a subspace $W \subseteq V$ on which $g|_W$ is negative-definite. Thus $0 \leq \nu \leq dimV$.*

Definition A.19. *A* scalar product *g on a vector space V is a non-degenerate symmetric bilinear form on V. We say that a vector space V equipped with a scalar product g is an* scalar product space *(V, g).*

The most well-known scalar product on a real vector space is the Euclidean one, which is positive-definite. Such a scalar product allows one to define a distance on the vector field V by setting $d(v, w) = \sqrt{||v - w||}$, where $||.||$ denotes the *module* defined by $||u|| = g(u, u)$ for all $u \in V$.

Definition A.20. *A* linear isometry *between to scalar product vector spaces (V, g_V) and (W, g_W) is a linear isomorphism $\phi : V \to W$ such that for all $u, v \in V$,*

$$g_W(\phi(u), \phi(v)) = g_V(u, v).$$

We say that such V and W are linearly isometric.

Calculus

Theorem A.21. ***The Inverse Function Theorem.*** *Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^n$ be a smooth map, with $f = (f_1, ..., f_n)$, i.e. $f_i = r_i \circ f$. Let $r \in U$. If the Jacobian matrix $\left\{ \frac{\partial f_i}{\partial r_j}\Big|_r \right\}$ $i, j = 1, ..., n$ is non-singular, then there exists an open set $V \subseteq U$ about r and an open set $W \subset \mathbb{R}^n$ about $f(r)$ such that $f|V : V \to W$ is a diffeomorphism.*

Algebra

Definition A.22. *A* group *is a set G together with an operation $\cdot$ such that, for all $x, y, z \in G$,*

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
2. *there exists* $e \in G$ *such that* $e \cdot x = x \cdot e = x$.
3. *there exists* $x^{-1} \in G$ *such that* $x^{-1} \cdot x = x \cdot x^{-1} = e$.

Definition A.23. *A* homomorphism *between groups is a map* $\phi : G_1 \to G_2$ *such that* $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ *for all* $x, y \in G_1$.

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