The quantum Monty Hall problem

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Abstract: We consider a variation of the Monty Hall problem and we present a quantum version of it in which the quiz show master does not know where the prize is and can accidentally reveal it. We prove that if the quiz show master is allowed to play quantum and the player is not the game becomes a fair game in contrast with the classical version of the game.

I. INTRODUCTION

Inspired by the work of von-Neumann and Oskar Morgesten [10] many mathematicians and economists have used simple games to understand concepts of game theory, well-known examples are the prisoner's dilemma [9] and "the battle of sexes" [7], among others. Moreover the study of games of chance has been used for classical information theorists since 1950s. Meyer [6] in 1999 introduces a way of generalizing the classical game theory into the quantum world, creating quantum games. In quantum games we assume some of the parts (or all of them) of the classic game have been quantized, we can quantize the players' strategies or the objects used in the game.

Some games have already been brought into the quantum world, in relation to prisoner's dilemma, Eisert, Wilkens, and Lewenstein [2] show that this game ceases to pose a dilemma if quantum strategies are allowed. They also construct a particular quantum strategy which always gives reward if played against any classical strategy. Another well-known dilemma in classical information is the Monty Hall problem and the revision of this problem into the quantum world can be of interest in the study of quantum strategies of quantum measurements.

Our paper is organized as follows, we will explain the classical Monty Hall problem and a variation of it in section II. In section III we expound a genuine way to quantize the variation of the problem previously explained and we solve the problem giving the best choice for the player in any case. Finally in section IV we discuss some other versions of the quantum Monty Hall problem which have been already published.

II. THE CLASSIC MONTY HALL PROBLEM

The Monty Hall problem [8] is a well-known and a frustrating brainteaser in all of mathematics. On the surface this problem appears to be simple, however it can be challenging. This problem is due to a TV quiz show where are shown three doors, two of them hiding a goat and the other one hiding a car. The player, lets call him Bob, makes a guess and picks one of the three doors. Then, the quiz show master, let's call her Alice, who knows where the prize is hiding, opens a door in accordance with the following rules:

I Alice must not open the door picked by Bob.

II Alice must not open the door containing the prize.

III If Alice can open more than a door (i.e., Bob made the right choice) without violating rules I and II, then she opens one of these doors randomly.

Given this extra information, Bob is now asked if he wants to stay with his initial choice or otherwise change his choice to the unopened door. It can be shown that Bob has $\frac{1}{3}$ probability of winning the car if he sticks to his initial choice and $\frac{2}{3}$ probability of winning if he switches to the unopened door. For further information see [8]. We want to present a variation of this game when some of the rules concerning Alice's choice of a door have been relaxed.

A. A variation of the classic problem

Suppose that the prize has been hidden by Clarice, whose only function in this game is to prepare the game, therefore the quiz show master, Alice, does not know where the prize is hidden. In this case after Bob's initial choice, Alice will open one of the two remaining doors randomly, therefore there is a chance of her opening the door containing the car. In that way we eliminated condition II.

Notice the main differences between the original game and this version, Alice can open the door with the prize and Bob has no more the two strategies after Alice opens a door (stay, switch) and has three strategies (pick door one, pick door two, pick door three).

As the reader can check, this is a simple problem of probabilities, and Bob expects to win in $\frac{2}{3}$ of the times no matter what strategy he follows as long as he is a rational player, in other words this means Bob is not switching to the opened door if Alice reveals no prize in it, the result is not sensible to Bob's strategy in this case. And if Alice reveals the prize when she opens a door, then Bob is picking that door as his final choice. Thus this variation of the game does not modify the classical probabilities even though impoverishes the game.

III. THE QUANTUM MONTY HALL PROBLEM

A. Analysis of the game under limited information

In this section we will set forth the quantum version of the variation of the classic Monty hall problem described in the previous section. The main quantum variable will be the position of the prize, which lies in a 3-dimensional Hilbert space $\mathcal{H}$. We will also quantize the Alice's choice of a box in contrast with Bob's first and second choice of a Box that will remain classical.

Before the game starts the system is prepared quantum mechanically (by Clarice) so that a quantum particle is in a superposition of the states $|0\rangle, |1\rangle, |2\rangle$; representing the doors of the classic problem. Let $\{|0\rangle, |1\rangle, |2\rangle\}$ be the orthonormal basis of $\mathcal{H}$. The prize will be in any point of the upper hemisphere of the $\mathbb{S}^2$ as don't take into account the imaginary parts, in this work we will stay real. Then, the initial state can be described as:

$$|\psi_0\rangle = \sin\theta_0 \cos\varphi_0 |0\rangle + \sin\theta_0 \sin\varphi_0 |1\rangle + \cos\theta_0 |2\rangle. \quad (1)$$

Where $\varphi_0 \in [0, 2\pi]$ and $\theta_0 \in [0, \frac{\pi}{2}]$. We consider only the upper hemisphere since the states (θ_0, φ_0) and $(\pi - \theta_0, \varphi_0 + \pi)$ are the same, with this restriction, from a quantum point of view we remove any redundancy. We can use different probability distributions for the parameters θ_0 and φ_0. We will assume that the prize is any point of the upper hemisphere with equal probability so the probability distribution we will use:

$$P(\theta_0, \varphi_0) d\theta_0 d\varphi_0 = \frac{1}{2\pi} \sin\theta_0 d\theta_0 d\varphi_0. \quad (2)$$

The game proceeds in the following stages:

I Bob chooses a projection $p = |\phi\rangle\langle\phi|$ along one of the axis on $\mathcal{H}$, i.e., Bob is restricted to choose either $|\phi\rangle = |0\rangle, |1\rangle, |2\rangle$.

II Alice's choice of a door is $q = |\chi\rangle\langle\chi|$, by rule I, it must be another door so $q \perp p$. During this step Alice could show the "location" of the prize as we do not assume $q \perp |\psi_0\rangle\langle\psi_0|$, that is due to the fact that Alice does not know where the car is. After Alice performs a measurement the quantum system will have collapsed in the two-dimensional space $(\mathbb{1} - q)\mathcal{H}$ if Alice don't hit the prize and will collapse in $q\mathcal{H}$ if she hits the prize.

III After seeing Alice's move and with all the information but the parameters θ_0 and φ_0, Bob can pick a direction again as in step I.

We want to study how Alice can decide whether help Bob to win the car or make a fair game for her with her choice of a "door", which in this case will be a projection on $\mathcal{H}$.

Let the initial state of the prize be as equation (1) and without loss of generality $|\phi_0\rangle = |2\rangle$ the initial choice of Bob. Alice must pick a direction orthogonal to Bob's choice so she picks any direction in the plane $\{|0\rangle, |1\rangle\}$:

$$|\chi\rangle = \sin\theta_1 |0\rangle + \cos\theta_1 |1\rangle. \quad (3)$$

With $\theta_1 \in [0, \frac{\pi}{2}]$. Alice can control the θ_1 parameter and it is public information so Bob knows exactly Alice's move. We can compute now the expected value of the probability of Alice hitting the prize:

$$\int_S |\langle\psi_0|\chi\rangle|^2 P(\theta_0, \varphi_0) d\theta_0 d\varphi_0 = \frac{1}{2\pi} \int_S |\langle\psi_0|\chi\rangle|^2 \sin\theta_0 d\theta_0 d\varphi_0$$

$$= \frac{1}{2} \sin^2\theta_1 \int_0^{\frac{\pi}{2}} \sin^3\theta_0 d\theta_0 + \frac{1}{2} \cos^2\theta_1 \int_0^{\frac{\pi}{2}} \sin^3\theta_0 d\theta_0$$

$$= \frac{1}{2} \cdot \frac{2}{3} \sin^2\theta_1 + \frac{1}{2} \cdot \frac{2}{3} \cos^2\theta_1 = \frac{1}{3}. \quad (4)$$

So we expect that in one third of the games Alice reveals the prize, however this does not ensure Bob a easy win. In this case the position of the prize will have collapsed in the state $|\psi'\rangle = |\chi\rangle = \sin\theta_1 |0\rangle + \cos\theta_1 |1\rangle$, but Bob is still restricted to pick a direction along one of the axis, so if he picks the door $|0\rangle$ his probability of hitting the prize is $\sin^2\theta_1$, and $\cos^2\theta_1$ if he picks the door $|1\rangle$. Therefore if Bob is a rational player his expected payoff is:

$$\max_{\theta_1 \in [0, \frac{\pi}{2}]} \{\sin^2\theta_1, \cos^2\theta_1\}. \quad (5)$$

In the other two thirds of the games Alice won't hit the prize so after her measurement the position of the prize will have collapsed to the space $(\mathbb{1} - q)\mathcal{H}$. Now the position of the prize can be given by:

$$|\psi'\rangle = (\mathbb{1} - |\chi\rangle\langle\chi|)|\chi\rangle$$

$$= g(\theta_0, \varphi_0) \Big(\cos\theta_1 |0\rangle + \sin\theta_1 |1\rangle \Big) + \cos\theta_0 |2\rangle. \quad (6)$$

Where $g(\theta_0, \varphi_0) = \sin\theta_0 \cos(\varphi_0 + \theta_1)$. As the reader may see $|\psi'\rangle$ is not normalized so we have to normalize it. Let $N = \cos^2\theta_0 + g^2(\theta_0, \varphi_0)$ be the normalizing constant, then the state of the prize is:

$$|\psi_1\rangle = \frac{1}{\sqrt{N}} \left[g(\theta_0, \varphi_0) \Big(\cos\theta_1 |0\rangle + \sin\theta_1 |1\rangle \Big) + \cos\theta_0 |2\rangle \right]. \quad (7)$$

At this moment, Bob knows that the state of the prize is the one described in equation (7). Now he is asked to decide whether stick to his initial choice $|\phi_0\rangle$ or switch to another door. He can compute the quantum probabilities $P^i(\theta_0,\varphi_0;\theta_1) = |\langle\psi_1|i\rangle|^2$ of hitting the prize when the state (θ_0,φ_0) is known if Bob picks the door $|i\rangle$. Then if he picks the $|0\rangle$:

$$P^0(\theta_0,\varphi_0;\theta_1) = \frac{\cos^2\theta_1}{1+\cot^2\theta_0\sec^2(\varphi_0+\theta_1)}. \tag{8}$$

If he picks $|1\rangle$:

$$P^1(\theta_0,\varphi_0;\theta_1) = \frac{\sin^2\theta_1}{1+\cot^2\theta_0\sec^2(\varphi_0+\theta_1)}. \tag{9}$$

And if he picks $|2\rangle$:

$$P^2(\theta_0,\varphi_0;\theta_1) = \frac{1}{1+\tan^2\theta_0\cos^2(\varphi_0+\theta_1)}. \tag{10}$$

Now we can compute Bob's expected value of the probability of winning when he picks any of the doors by integrating these probabilities on the upper hemisphere of $\mathbb{S}^2$ because (θ_0,φ_0) are unknown to Bob. Remember that the distribution of the probability is given by equation (2). Then:

$$\langle\$_0\rangle = \int_S P^0(\theta_0,\varphi_0;\theta_1)P(\theta_0,\varphi_0)d\theta_0 d\varphi_0$$

$$= \frac{\cos^2\theta_1}{2\pi}\int_0^{\frac{\pi}{2}}\sin\theta_0\left[2\pi\Big(1-\sqrt{\frac{\cot^2\theta_0}{1+\cot^2\theta_0}}\Big)\right]d\theta_0$$

$$= \cos^2\theta_1\int_0^{\frac{\pi}{2}}\Big(\sin\theta_0-\sin\theta_0\cos\theta_0\Big)d\theta_0 = \frac{1}{2}\cos^2\theta_1. \tag{11}$$

In the same way we can find:

$$\langle\$_1\rangle = \int_S P^1(\theta_0,\varphi_0;\theta_1)P(\theta_0,\varphi_0)d\theta_0 d\varphi_0 = \frac{1}{2}\sin^2\theta_1. \tag{12}$$

And finally:

$$\langle\$_2\rangle = \int_S P^2(\theta_0,\varphi_0;\theta_1)P(\theta_0,\varphi_0)d\theta_0 d\varphi_0$$

$$= \frac{1}{2\pi}\int_S \sin\theta_0\frac{d\theta_0 d\varphi_0}{1+\tan^2\theta_0\cos^2(\varphi_0+\theta_1)}$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sin\theta_0}{\sqrt{1+\tan^2\theta_0}}d\theta_0 = \frac{1}{2}. \tag{13}$$

Whatever Alice's choice of θ_1, Bob expects more or equal if he picks $|2\rangle$ because $\frac{1}{2} \geq \frac{1}{2}\sin\theta_1, \frac{1}{2}\cos\theta_1$, so at least, if Bob is a rational player, his expected payoff equals $\frac{1}{2}$ by picking $|\phi_1\rangle = |2\rangle$ as his final choice in this case. Now we can compute the total expected payoff for Bob in this quantum version of the game, which will take into account the probabilities of Bob winning the game when Alice hits the prize when she picks a door and when she doesn't hit the prize.

$$\langle\$\rangle = \frac{1}{3}\max_{\theta_1\in[0,\frac{\pi}{2}]}\left\{\sin^2\theta_1,\cos^2\theta_1\right\} + \frac{2}{3}\cdot\frac{1}{2}$$

$$= \frac{1}{3}\left(1+\max_{\theta_1\in[0,\frac{\pi}{2}]}\left\{\sin^2\theta_1,\cos^2\theta_1\right\}\right). \tag{14}$$

It is fulfilled that $\frac{1}{2} \leq \langle\$\rangle \leq \frac{2}{3}$. The lower bound is reached when Alice plays $\theta_1 = \frac{\pi}{4}$, which corresponds with a move $|\chi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|0\rangle$. Playing like this, Alice plays the "the most quantum possible", taking advantage of the fact that Bob is restricted to playing classical, and makes a fair game for her. On the other hand the upper bound is reached when Alice plays $\theta_1 = 0$ or $\theta_1 = \frac{\pi}{2}$, which corresponds with the moves $|\chi\rangle = |0\rangle$ or $|\chi\rangle = |1\rangle$. These plays do not take profit of the possible quantum moves which Alice can make as she remains classical, so Bob can reach his expected payoff of $\langle\$\rangle = \frac{2}{3}$ like in the classical game.

B. The influence of complete information

In this section we will analyze how often does Clarice agree with Bob's decision. We will do this by plotting the probabilities $P^i(\theta_0,\varphi_0;\theta_1)$ (which Clarice knows because she knows the state (θ_0,φ_0)) as a functions of θ_0 and φ_0 which are unknown to Bob. Let $u = \frac{\varphi_0}{2\pi}$ and $\omega = \cos\theta_0$. As $\theta_0 \in [0,\frac{\pi}{2}]$ and $\varphi_0 \in [0,2\pi]$ then $(u,\omega^2) \in [0,1]\times[0,1]$. Therefore we can rewrite equations (8) to (10) as:

$$P^0(u,\omega;\theta_1) = \frac{1}{N}\cos^2\theta_1(1-\omega^2)\cos^2(2\pi u+\theta_1). \tag{15}$$

$$P^1(u,\omega;\theta_1) = \frac{1}{N}\sin^2\theta_1(1-\omega^2)\cos^2(2\pi u+\theta_1). \tag{16}$$

$$P^2(u,\omega;\theta_1) = \frac{1}{N}\omega^2. \tag{17}$$

It is obvious that $\max\{P^0(u,\omega;\theta_1),P^1(u,\omega;\theta_1)\} = \max_{\theta_1\in[0,\frac{\pi}{2}]}\{\sin^2\theta_1,\cos^2\theta_1\}$, so we want to study when the probability of "door" 2 is higher or lower than "doors" 0 and 1. i.e., we want to study when it is preferable for Bob to change his initial choice or to stick to his initial choice. Let $\beta = \max\{\sin^2\theta_1,\cos^2\theta_1\}$, combining equations (15),(16) and (17), we reach the following conclusion: as Bob, picking as final decision $|2\rangle$ has higher probability than picking $|0\rangle$ or $|1\rangle$ when:

$$\omega^2 > \frac{\beta^2\cos^2(2\pi u+\theta_1)}{1+\beta^2\cos^2(2\pi u+\theta_1)}, \tag{18}$$

and it is worthier to pick $|0\rangle$ or $|1\rangle$ when:

$$\omega^2 < \frac{\beta^2 \cos^2(2\pi u + \theta_1)}{1 + \beta^2 \cos^2(2\pi u + \theta_1)}. \tag{19}$$

In this last case Bob would have to pick $|0\rangle$ or $|1\rangle$ depending on the parameter θ_1 chosen by Alice which he already knows.

We can plot these probabilities to further comprehension of Bob's better choice and what does Clarice think about Bob's decision. First of all we present a graph showing the behavior of inequation (19) when $\theta_1 = 0$ and $\theta_1 = \frac{\pi}{4}$:

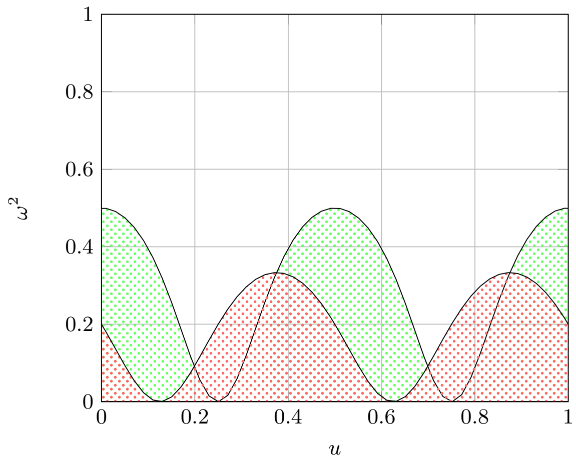

FIG. 1: Inquation (19) for $\theta_1 = 0$ (green) and $\theta_1 = \frac{\pi}{4}$ (red).

The filled area shows us all the possible pairs (θ_0, φ_0) where is would be preferable to choose $|0\rangle$ or $|1\rangle$ as final decision, so it shows when Clarice and Bob would disagree (remember that Bob best strategy is to stick to his initial choice $|2\rangle$ as we mentioned in the previous section). The following graphs put on display how the curve changes when the parameter θ_1 vary.

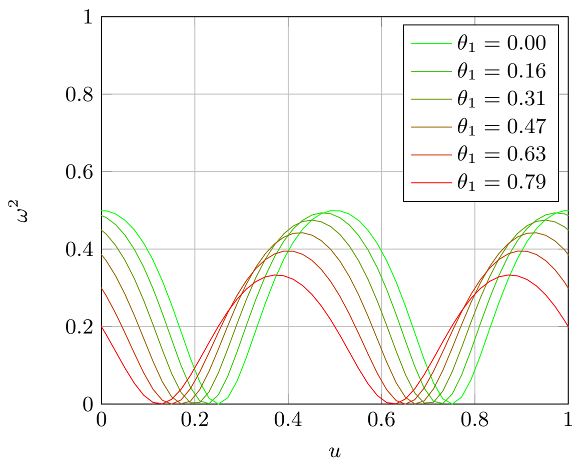

FIG. 2: Curves of probability for $0 \leq \theta_1 \leq \frac{\pi}{4}$.

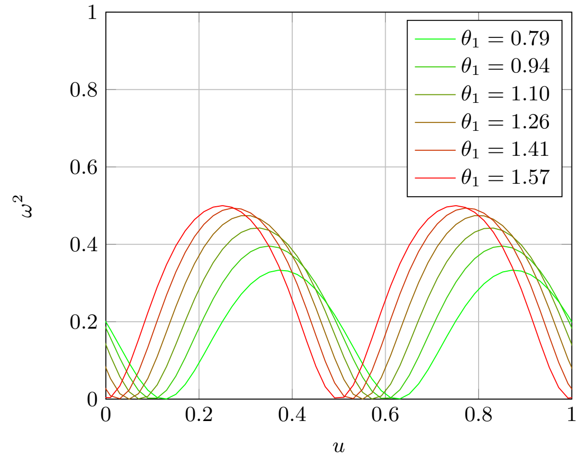

FIG. 3: Curves of probability for $\frac{\pi}{4} \leq \theta_1 \leq \frac{\pi}{2}$.

The conclusion is that picking $|0\rangle$ or $|1\rangle$ than $|2\rangle$, i.e., it would be preferable switching than sticking to your initial choice has maximum worth when Alice plays $\theta_1 = 0$ or $\theta_1 = \frac{\pi}{2}$, and it has minimum worth when Alice plays $\theta_1 = \frac{\pi}{4}$.

IV. OTHER VERSIONS OF THE PROBLEM

In this section we want to summarize three other versions of the problem that have been published. This three other versions are based in the original Monty Hall problem instead of a variation of the problem as we do. Nevertheless, these examples show that the quantization of a classical game is non-unique.

- The quantization proposed in [4] is completely different from our approach. The parts that have been quantized from the original problem are not measurements as we do, Flitney and Abbott suggest that the moves Alice and Bob can do are operations on a tripartite system. Alice's and Bob's choices are represented by qutrits and they start in some initial state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_O$. These Hilbert spaces represent the door where Alice hides the prize, the door that Bob chooses and the door that has been opened by Alice respectively. If the initial state is $|\psi_i\rangle$, the final state of the system is:

$$|\psi_f\rangle = \left(\hat{S}\cos\gamma + \hat{N}\sin\gamma\right)\hat{O}\left(\mathbb{1}\otimes\hat{B}\otimes\hat{A}\right)|\psi_i\rangle. \tag{20}$$

  Where $\hat{A}$ is Alice's choice of operator, $\hat{B}$ is Bob's choice of operator, $\hat{O}$ the opening box operator, $\hat{S}$ and $\hat{N}$ are Bob's switching and not switching operators. Bob also controls the parameter γ to play a mixed strategy of switching or not switching. They reach the following result: if Alice has access to quantum strategies and Bob doesn't, she can make the game fair with a expected payoff of $\frac{1}{2}$ for each player. Otherwise, if Bob has access to quantum strategies and Alice does not, then he can win the game all the times.

- A completely different different way to implement quantum strategies is proposed in [5]. In this version, Alice puts the quantum particle in a box, maybe in a superposition state like $|\psi\rangle_p = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$. After Bob picks one box, for instance $|0\rangle$, and Alice reveals no particle in $|2\rangle$, the state of the particle may be described by the density matrix:

$$\rho_p = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|. \tag{21}$$

  Now Alice is allowed to perform a von-Neumann measurement on the particle, shuffling it around and the game is reduced to a coin tossing game.

- The most closely related published version to ours is [1]. Although the idea for this paper came up in a bar during a workshop as the authors confess, they came up with some interesting ideas to face the quantum Monty Hall problem. The most important one and the one we have exploited in this work is to see the position of the prize and all the strategies of the players as directions in the space.

V. CONCLUSIONS

We have presented a variation of the classical Monty Hall problem where the quiz show master does not know where the prize is. In contrast with the original version where the player expected a payoff of $\frac{1}{3}$ if sticking to the initial choice and $\frac{2}{3}$ if switching, in our version, if the player is rational, he expects a payoff of $\frac{2}{3}$, see section II A.

In the quantum version of our problem the prize is quantum particle which is in a random superposition (1) on the upper hemisphere of the $\mathbb{S}^2$. Alice is allowed to perform quantum measurements and Bob is restricted to classical moves. We prove that the expected payoff for Bob is $\frac{1}{2} \leq \langle \$ \rangle \leq \frac{2}{3}$. Playing classical, Bob can never overcome his expected payoff of the classical game. In fact, Alice decides the value of Bob's payoff when she makes her move (3). That's due to the fact that Alice is allowed to play quantum and Bob isn't, and this is a clear advantage for Alice who can make the game fair for both players by playing quantum.

This game offers more research than we have done, it may be interesting to study how the players' payoff is affected when the probability distribution of the prize does not verify equation (2), or how it is affected when Bob has to pay or cede part of the prize if he wants to change his initial choice after Alice's move.

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