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Zero sets of Gaussian analytic functions

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Abstract

We study point processes given as zero sets of Gaussian analytic functions and prove that these point processes show local repulsion. We define Gaussian analytic functions and introduce its covariance kernel, which determines its probabilistic properties, and its first intensity which can be computed using the Edelman-Kostlan formula.

Finally, we also study rigidness of some model examples -by computing the variance of the counting random variable of the zeros of the GAF- and we compare it with the independence of the Poisson point process -shown in an introductory section of this project- for the same model cases.

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Contents

1 Introduction

Point processes are discrete sets of points randomly located, according to some probability law, on some underlying space. They are used in different fields of science to describe many objects representable as points. We first recall some properties of the best known point process, the Poisson process. Here the number of points falling in a given region is a random variable following a Poisson law. Its other main characteristic is that these counting random variables are independent if the corresponding regions are disjoint. The Poisson process has found many applications since it was first published in the work *Recherches sure la probabilité des jugements en matière criminelle et en matière civile* in 1837 by Siméon Denis Poisson.

In this memory we deal with a different point process, in which the independence of the Poisson process is replaced by a local repulsion between points. This is natural in many physical phenomena, for instance when modelling electrically charged particles. The point processes we study are obtained as zero sets of the so-called Gaussian analytic functions, that is, of power series with independent standard complex Gaussians. The underlying spaces we consider are mainly the complex plane (planar case), the unit disk (hyperbolic case) and the Riemann sphere (parabolic case).

The main focus of this work is to prove that the process constructed in this way is more rigid than the Poisson process, in the sense that the fluctuations -the variance of the counting functions- are of smaller order and that it shows the aforementioned repulsion. We shall see that the main tool to achieve these goals is a good control of the covariance kernel of the Gaussian analytic function.

2 Poisson process

Point processes are used to describe different physical phenomena that can be modeled by random discrete sets. The most widely known is the Poisson point process, which is a classical model. Its main characteristic is independence between the number of points of disjoint measurable subsets.

Definition 2.1. We consider a random sequence S in a domain X (usually $X = \mathbb{C}, \mathbb{D}, \mathbb{S}^2, \ldots$). Let A be a subset of X and let $n(A) = \#(A \cap S)$ be the counting function associated to A and S. S is a Poisson process if it satisfies:

1. $n(A)$ follows a Poisson distribution. Denoting its mean by $\mu(A)$, the probability function is:
$$P[n(A) = m] = \frac{(\mu(A))^m e^{-\mu(A)}}{m!}, \quad m = 0, 1, 2, ... \tag{1}$$
2. If $A \cap B = \emptyset$, then $n(A)$ and $n(B)$ are independent random variables.

It is important to point out that the independence property (b) is not always natural, so the Poisson process will not be a good model for all situations. For instance, if we are dealing with electrical charged particles, knowing that there is one charge in a concrete position makes it more unlikely that there is another charged particle in a neighbourhood of that position, due to Coulomb's force.

Another example, in the opposite sense, could be the case in which our set of points represents people who have a contagious illness. Then, it is more likely to find another case near the location of a known one.

Given a Poisson process with parameter $\mu(A)$ of the random variable $n(A)$, it can be seen that the parameter μ(A) defines a measure in the underlying space X.

Reciprocally, for every positive measure μ defined on X, there exists a Poisson process S such that the random variable $n(A) = \#(A \cap S)$ follows a Poisson distribution with parameter $\mu(A)$. For instance, if $X = \mathbb{C}$, then we can take $\mu(A) = Area(A)$ and produce a Poisson random point process for which the average number of points in A coincides with its area.

Sometimes it is useful to deal with the measure, $L > 0$, $L\mu$ instead of μ and, for instance, see what happens when L tends to infinity. The parameter L is called *intensity* of the process and it represents the average number of points per unit area (with respect to μ).

Examples: Let us see some standard Poisson processes. Later, using Gaussian analytic functions, we will produce different processes having in average the same number of points, but with other characteristics.

1. **The planar case**. Consider $X = \mathbb{C}$ and the *invariant (under translations)* measure

$$d\nu_L = L\frac{dm(z)}{\pi}.$$

Here $dm(z)$ represents the Lebesgue measure on the complex plane. We take the Poisson process associated to the measure ν_L. Then, for a disk $A = D(z,r)$, $r > 0$ $z \in \mathbb{C}$, we have $\nu_L(A) = \mathbb{E}[n_{\nu_L}(A)] = Lr^2$ and the law of the random variable $n_{\nu_L}(A) = n_{\nu_L}(D(z,r))$, we obtain:

$$P[n_{\nu_L}(D(z,r)) = m] = e^{-Lr^2}\frac{(Lr^2)^m}{m!}; \qquad m = 0, 1, 2, ... \tag{2}$$

In particular, we can compute the "hole probability", i.e., the probability that a disk of centre z and radius r is empty:

$$P[n_{\nu_L}(D(z,r)) = 0] = e^{-Lr^2}. \tag{3}$$

We are also able to give the variance of the random variable $n_{\nu_L}(A)$, knowing that follows the Poisson distribution:

$$\mathrm{Var}[n_{\nu_L}(A)] = \mathbb{E}[n_{\nu_L}(A)] = Lr^2. \tag{4}$$

In Chapter 3, we will compute the variance of the random variable associated to the zero set of a Gaussian analytic function and we will compare it with these results.

2. **The hyperbolic case**. Here, we consider $X = \mathbb{D}$ and the following measure:

$$d\nu_L(z) = L\,\frac{dm(z)}{\pi(1-|z|^2)^2}. \tag{5}$$

The measure $d\nu(z) = \frac{dm(z)}{\pi(1-|z|^2)^2}$ is the volume measure for the hyperbolic metric in $\mathbb{D}$. In consequence $d\nu_L(z)$ is invariant by the automorphisms of the unit disk

$$\varphi_{a,\theta}(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}, \quad a, z \in \mathbb{D}, \theta \in [0, 2\pi). \tag{6}$$

A computation shows that for $r \in (0,1)$, $a \in \mathbb{D}$,

$$L\nu(D(a,r)) = L\nu(D(0,r)) = \frac{Lr^2}{1-r^2}.$$

In effect:

$$L\nu(D(0,r)) = \frac{L}{\pi}\int_{D(0,r)} \frac{dm(z)}{(1-|z|^2)^2} = \frac{L}{\pi}\int_0^r \int_0^{2\pi} \frac{\rho d\theta d\rho}{(1-\rho^2)^2}$$
$$= L\int_0^{r^2} \frac{dt}{(1-t)^2} = L\Big(1-\frac{1}{1-r^2}\Big) = \frac{Lr^2}{1-r^2}.$$

As we did in the planar case, for $a \in \mathbb{D}$ and $r > 0$, the law of the random variable $n_{\nu_L}(D(a,r))$ is given by:

$$P[n_{\nu_L}(D(a,r)) = m] = e^{\frac{-Lr^2}{1-r^2}} \frac{(Lr^2)^m}{m!(1-r^2)^m}; \qquad m = 0, 1, 2, ... \tag{7}$$

In particular,

$$\mathbb{E}[n_{\nu_L}(D(a,r))] = \frac{Lr^2}{1-r^2}.$$

Similar computation to the planar case gives

$$\mathrm{Var}[n_{\nu_L}(D(a,r))] = \frac{Lr^2}{1-r^2} \tag{8}$$

and the *hole probability* is

$$P[n_{\nu_L}(D(a,r)) = 0] = e^{\frac{-Lr^2}{1-r^2}}; \quad a \in \mathbb{D} \quad r > 0.$$

3. **The parabolic case**. Let $X = \mathbb{S}^2$ be the unit sphere. We represent $\mathbb{S}^2 \setminus \{\infty\}$ on $\mathbb{C}$ using the stereographic projection. In this case, the following measure is the area form associated to the parabolic geometry:

$$d\nu_L(z) = \frac{Ldm(z)}{\pi(1+|z|^2)^2}. \tag{9}$$

The measure $d\nu(z) = \frac{dm(z)}{\pi(1+|z|^2)^2}$ is the volume measure for the chordal metric on the Riemann sphere and, in consequence, $d\nu_L$ is invariant under the rotations of $\mathbb{S}^2$, which in $\mathbb{C}$ have the form

$$\varphi_{a,\theta}(z) = e^{i\theta}\frac{z+a}{1+\bar{a}z}, \quad a, z \in \mathbb{C}, \theta \in [0, 2\pi).$$

As in the hyperbolic case, considering a disk $D(a,r)$, $a \in \mathbb{C}$ $r > 0$;

$$\nu_L(D(a,r)) = \frac{Lr^2}{1+r^2}.$$

Therefore:

$$P[n_{\nu_L}(D(a,r)) = m] = e^{\frac{-Lr^2}{1+r^2}} \frac{(Lr^2)^m}{m!(1+r^2)^m}; \qquad m = 0, 1, 2, ... \tag{10}$$

In particular, the hole probability for the parabolic case is

$$P[n_{\nu_L}(D(a,r)) = 0] = e^{\frac{-Lr^2}{1+r^2}}$$

and the variance

$$\mathrm{Var}[n_{\nu_L}(D(a,r)] = \frac{Lr^2}{1+r^2}; \quad a \in \mathbb{C}, L > 0,\ r > 0.$$

Originally, the Boltzmann model gave sense to the concept of intensity L that we are using in this project. Concretely, the inverse of the temperature T is proportional to the intensity L defined above. That model gives us the distribution of molecules confined in a microscopic region in a determinate energy state depending on temperature. Let $n(E)$ be the random variable that represents the number of molecules in a determinate energy state E, depending on temperature T. Let K_B be the Boltzmann's constant and let n_0 be the number of molecules in the ground state (lowest energy state). Then we have:

$$n(E) = n_0 e^{\frac{-E}{K_B T}}. \tag{11}$$

From this perspective, letting L go to infinity, is equivalent to letting T tend to absolute zero, something that is called *transition to the liquid phase*. In Figure 1 we can see the dependence between the quotient $\frac{n(E_2)}{n(E_1)}$ and the temperature T.

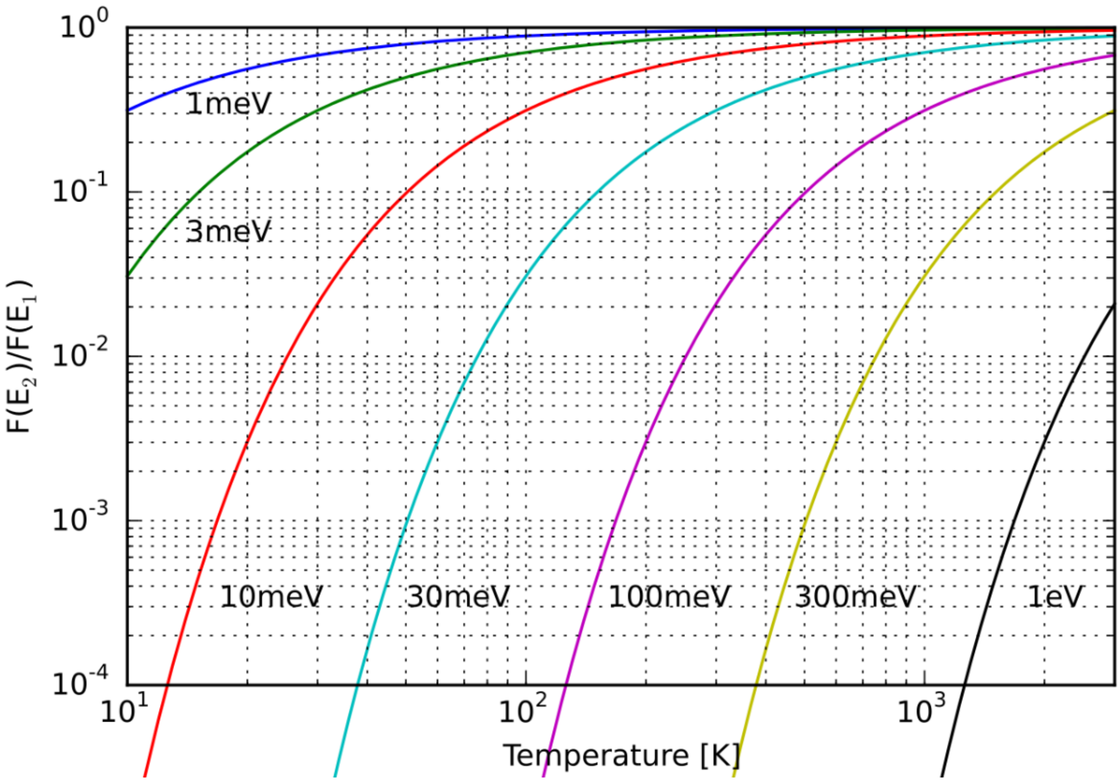

Figure 1: Boltzmann distribution([Wiki19]).

2.1 Zeros of random analytic functions.

A point process with local repulsion.

In this work we will deal with point processes given as zeros sets of random analytic functions. We shall see that these processes are more ”rigid” than the Poisson process seen above (in a sense to be made precise later) and that there is a local repulsion between points. Just to give an idea why there is such local repulsion we consider the following simplified, informal setting:

Let’s consider $X = \mathbb{C}$ and a polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0; \qquad a_0, ..., a_{n-1} \in \mathbb{C}. \tag{12}$$

The fundamental theorem of algebra allows us to write $p(z)$ in the following form:

$$p(z) = \prod_{i=1}^{n}(z - z_i); \qquad z_1, \ldots, z_n \in \mathbb{C}. \tag{13}$$

From this identity it is not difficult to see that the determinant of the transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(z_1, \ldots, z_n) = (a_{n-1}, \ldots, a_0)$ is the Vandermonde determinant:

$$\det(T) = \prod_{i<j} |z_i - z_j|^2. \tag{14}$$

Hence, if the coefficients are chosen randomly uniformly in $\mathbb{C}$ then the roots of $p(z)$ are distributed with density proportional to $(\prod_{i<j} |z_i - z_j|^2) dm(z)$. This shows **local repulsion** between roots of the polynomial (near a z_i the density is small). That is a fundamental difference with respect to the Poisson process (Figure 2.).

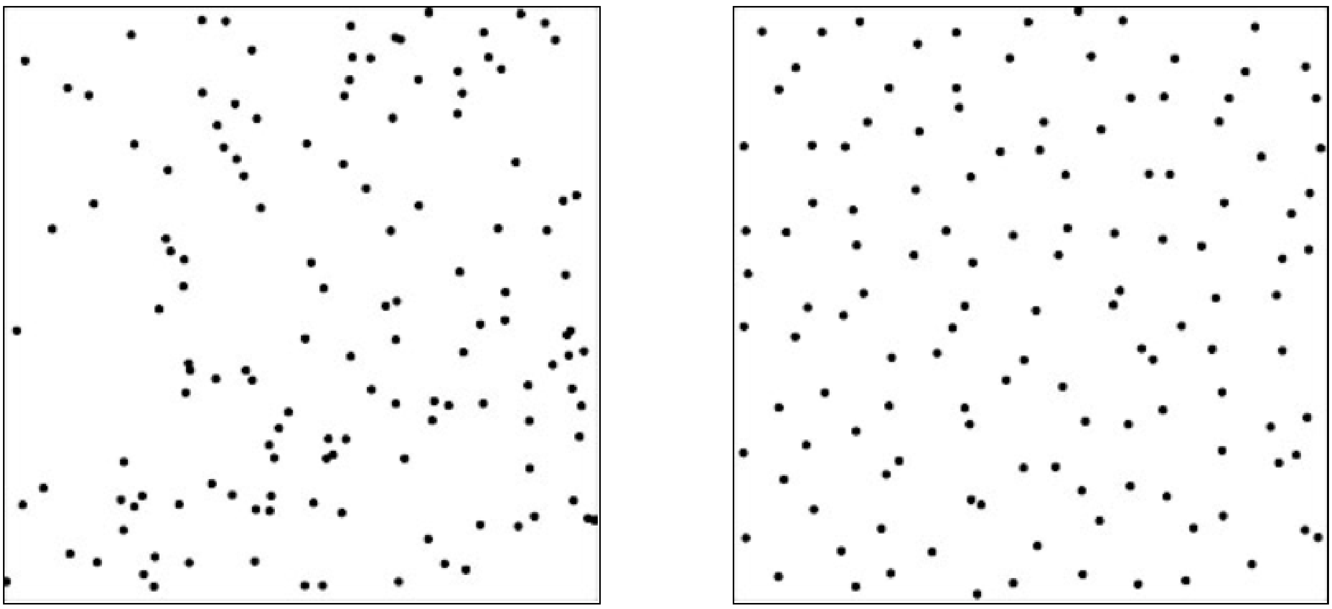

Figure 2: Left picture is a Poisson point process and the right picture is a GAF point process. Point processes with the same $\mathbb{E}(n(A))$.

3 Gaussian Analytic Functions

As said before, our aim is to study point processes defined as zero sets of certain random analytic functions. Here we will introduce such functions and some of their main properties.

3.1 Complex Gaussians

Recall that a random variable X follows a standard real normal distribution ($X \sim N_{\mathbb{R}}(\mu, \sigma)$, where μ represents the mean of X and σ^2 its variance) if its density function is:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R}, \tag{15}$$

Definition 3.1. A random complex variable Z follows a standard complex Gaussian ($Z \sim N_{\mathbb{C}}(0,1)$) if its density function (with respect to the Lebesgue measure on the complex plane) is:

$$f_Z(z) = \frac{1}{\pi} e^{-|z|^2} \qquad z \in \mathbb{C}. \tag{16}$$

Remark. If $Z \sim N_{\mathbb{C}}(0,1)$, then $Re(Z), Im(Z) \sim N_{\mathbb{R}}(0, \frac{1}{\sqrt{2}})$. In fact, the reciprocal is also true if X and Y are independent. Writing $Z = Re(Z) + iIm(Z) = X + iY$ and denoting the density function of X and Y by f_X and f_Y respectively:

$$f_Z(z) = \frac{1}{\pi} e^{-(x^2+y^2)} = \left(\frac{1}{\sqrt{\pi}} e^{-x^2}\right)\left(\frac{1}{\sqrt{\pi}} e^{-y^2}\right) = f_X(x) f_Y(y)$$

Thus, $X = Re(Z) \sim N_{\mathbb{R}}(0, \frac{1}{2})$ and $Y = Im(Z) \sim N_{\mathbb{R}}(0, \frac{1}{2})$.
Now, we state three properties of complex gaussians that we will use along this project.

Lemma 3.1.

(a) Let $Z \sim N_{\mathbb{C}}(0,1)$. Then $|Z|^2$ follows an exponential distribution of parameter 1, i.e., for $t \geq 0$

$$F(t) = P(|Z|^2 \leq t) = 1 - e^{-t}. \tag{17}$$

(b) Let $\{a_n(\omega)\}_n$ be a sequence of independent standard complex Gaussians ($a_n(\omega) \sim N_{\mathbb{C}}(0,1)$). Then:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(\omega)|} = 1 \quad a.s. \tag{18}$$

(c) If a and b are independent complex standard Gaussians, then $\mathbb{E}[a\bar{b}] = 0$.

Proof. (a). By definition, and using polar coordinates and the change of variables $s = r^2$ we see that the distribution function of $|Z|^2$ is

$$\begin{aligned} F_{|Z|^2}(t) = P(|Z|^2 \leq t) &= \int_{|z|^2 \leq t} \frac{1}{\pi} e^{-|z|^2} dm(z) = \int_{r^2 \leq t} \int_0^{2\pi} \frac{1}{\pi} e^{-r^2} r d\theta dr \\ &= 2\int_{r^2 \leq t} e^{-r^2} r dr = \int_0^t e^{-s} ds = [-e^{-s}]_0^t = 1 - e^{-t}. \end{aligned}$$

(b). First we shall use the Borel-Cantelli lemma to see that

$$P\Big(\limsup_{n\to\infty} \sqrt[n]{|a_n(\omega)|} < 1\Big) = 0.$$

By definition:

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{|a_n(\omega)|} < 1 &\Leftrightarrow \exists \epsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad \sqrt[n]{|a_n(\omega)|} < 1 - \epsilon \\ &\Leftrightarrow \exists \epsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad \sqrt[n]{|a_n(\omega)|} < 1 - \epsilon \\ &\Leftrightarrow \exists \epsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad |a_n(\omega)| < (1-\epsilon)^n.\end{aligned}$$

Let's consider the sequence of events:

$$A_n = \{\omega : |a_n|^2 < (1-\epsilon)^{2n}\} \qquad n = 0, 1, 2, \ldots$$

and let

$$A = \limsup_{n\to\infty} A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k.$$

It is clear that

$$P\Big(\limsup_{n\to\infty} \sqrt[n]{|a_n(\omega)|} < 1\Big) = 0$$

is equivalent to $P(A) = 0$. We need to prove that $\sum_n P(A_n) < +\infty$ to be able to apply the Borel-Cantelli's lemma. Here, by (a), $P(A_n) = 1 - e^{-(1-\epsilon)^{2n}}$, and using the Taylor's approximation $1 - e^{-t} \approx t$ for $t \sim 0$, we obtain:

$$\sum_n P(A_n) = \sum_n 1 - e^{-(1-\epsilon)^{2n}} \approx \sum_k (1-\epsilon)^{2n} < +\infty.$$

The proof that

$$P\Big(\limsup_{n\to\infty} \sqrt[n]{|a_n(\omega)|} > 1\Big) = 0$$

goes along the same lines.

(c). By definition, since the density function of a and b are $f_a(z) = \frac{1}{\pi}e^{-|z|^2}$, $f_b(w) = \frac{1}{\pi}e^{-|w|^2}$;

$$\begin{aligned}\mathbb{E}[a\bar{b}] &= \int_{\mathbb{C}}\int_{\mathbb{C}} z\bar{w}\frac{1}{\pi}e^{-|z|^2}\frac{1}{\pi}e^{-|w|^2}dm(z)dm(w) \\ &= \int_{\mathbb{C}} z\frac{1}{\pi}e^{-|z|^2}dm(z)\int_{\mathbb{C}} \bar{w}\frac{1}{\pi}e^{-|w|^2}dm(w) = \mathbb{E}[a]\mathbb{E}[\bar{b}] = 0. \quad \square\end{aligned}$$

Remark. From (a) we deduce, in particular, that $P(|Z|^2 > t) = e^{-t}$, $t > 0$, and $\mathbb{E}[|Z|^2] = 1$. Indeed:

$$\begin{aligned}E[|Z|^2] &= \int_{\mathbb{C}} |z|^2\frac{1}{\pi}e^{-|z|^2}dm(z) = \int_0^\infty r^2e^{-r^2}2rdr = \int_0^\infty te^{-t}dt \\ &= [-te^{-t}]_0^\infty + \int_0^\infty e^{-t}dt = [-e^{-t}]_0^\infty = 1.\end{aligned}$$

3.2 Gaussian Analytic Functions

Here we define the functions that produce the point process we are interested in.

Definition 3.2. Let $A \subset X$ and $\{e_n(z)\}_n$ be a sequence of the subspace of holomorphic functions in A, denoted by *Hol*(A) such that $\sum_{n=0}^{\infty} |e_n(z)|^2$ converges uniformly on compact subsets of A . Then a Gaussian Analytic Function (GAF) is a function of the form:

$$f_\omega(z) = \sum_{n=0}^{\infty} a_n(\omega) e_n(z) \quad z \in A, \tag{19}$$

where $a_n(\omega)$ are independent standard complex Gaussians ($a_n \sim N_{\mathbb{C}}(0,1)$, i.i.d.).

Remarks.

1. $f \in$ *Hol*(A) with probability 1. It is not difficult to see that for each $z \in A$ the series in (19) converges almost surely. But it might happen that the exceptional set where the series diverges depends on z in a way that $f_\omega(z)$ is not even a function. This can be ruled out using a version of Kolmogorov's inequality for Hilbert spaces (see [HKPV09, Lemma 2.2.3]).

2. As a random variable, f follows a normal distribution with mean 0. That is a direct consequence from the fact that f is a lineal combination of mean zero Gaussians ($a_n(\omega) \sim N_{\mathbb{C}}(0,1)$).

3. In many cases, the functions $\{e_n(z)\}_n$ are an orthonormal system in a given Hilbert space of holomorphic functions $\mathcal{H}$. Quite often, as we will see in the examples, these $e_n(z)$ are just normalizations of the monomials $\{z^n\}_{n\geq 0}$. Therefore, Lemma 2.1. (b) helps to determine where the series (19) converges.

4. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_n\}_n$. The function defined by (17) is not in $\mathcal{H}$ almost surely. Notice that

$$||f||^2 =< f, f >= \sum_{n,m} a_n \bar{a}_m < e_n, \bar{e}_m >= \sum_n |a_n|^2.$$

   To see that this is finite with probability 0, fix any $\epsilon > 0$ and $n_0 \in \mathbb{N}$. Then, by the independence of the coefficients a_n:

$$P(|a_n|^2 < \epsilon, \forall n \geq n_0) = \prod_{n \geq n_0} P(|a_n|^2 < \epsilon^2) = \prod_{n \geq n_0} (1 - e^{-\epsilon^2}) = 0.$$

From here on we shall introduce three model cases of GAF, but first it will be necessary to make a brief overview of the gamma function and the Riemann zeta function.

3.3 The gamma function.

The gamma function, represented by Γ, is defined, for $y > 0$, as

$$\Gamma(y) = \int_0^{\infty} x^{y-1} e^{-x} dx.$$

A well-known value of the gamma function at a non-integer argument is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
For our work, the most relevant properties of this function are:

1. $\Gamma(1) = 1$. This follows directly from the definition.
2. $\Gamma(y+1) = y\Gamma(y)$. This follows by integration by parts:
$$\Gamma(y+1) = \int_0^{\infty} x^y e^{-x} dx = [-x^y e^{-x}]_0^{\infty} + y \int_0^{\infty} x^{y-1} e^{-x} dx = y\Gamma(y).$$
3. $\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}$. This is a consequence of the first two properties.
4. Consider the beta function, defined for $a, b > 0$ by
$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$
Then:
$$\int_0^1 x^{a-1}(1-x)^{b-1} dx = B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (20)$$
See the proof on [Wc18]).

We will consider, for $s > 1$, the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Examples. Let us introduce three model cases of GAF.

1. **The plane case.** Let $X = \mathbb{C}$ and let $L > 0$. Consider the so-called Bargmann-Fock space of weight L
$$\mathcal{H}_L = \left\{ f \in Hol(\mathbb{C}) : ||f||^2_{\mathcal{H}_L} = \frac{L}{\pi} \int_{\mathcal{C}} |f(z)|^2 \, e^{-L|z|^2} \, dm(z) < +\infty \right\}. \quad (21)$$

Thus is a Hilbert space, with inner product given by

$$< f, g >_L = \frac{L}{\pi} \int_{\mathbb{C}} f(z)\bar{g}(z) e^{-L|z|^2} dm(z), \quad f, g \in \mathcal{H}_L.$$

Lemma 3.2. The functions

$$e_n(z) = \frac{L^{n/2}}{\sqrt{n!}} z^n, \quad n \geq 0, \tag{22}$$

form an orthonormal basis.

Proof. Let us see first these functions are orthonormal. Taking polar coordinates:

$$\begin{aligned} < e_n, e_m >_{\mathcal{H}_L} &= \frac{L}{\pi} \int_{\mathbb{C}} \frac{L^{n/2}}{\sqrt{n!}} \frac{L^{m/2}}{\sqrt{m!}} z^n \bar{z}^m e^{-L|z|^2} dm(z) \\ &= \frac{L}{\pi} \frac{L^{n/2}}{\sqrt{n!}} \frac{L^{m/2}}{\sqrt{m!}} \int_0^{2\pi} \int_0^{+\infty} r^{n+m+1} e^{i\theta(n-m)} e^{-Lr^2} dr d\theta. \end{aligned} \tag{23}$$

If $n \neq m$,

$$\int_0^{2\pi} e^{i\theta(n-m)} d\theta = \left[\frac{e^{i\theta(n-m)}}{i(n-m)} \right]_{\theta=0}^{\theta=2\pi} = 0.$$

Then:

$$< e_n, e_m >_{\mathcal{H}_L} = 0 \quad \forall n \neq m.$$

If $n = m$, from (23), we get

$$\begin{aligned} < e_n, e_n >_{\mathcal{H}_L} &= \frac{L}{\pi} \int_0^{+\infty} \frac{L^n}{n!} r^{2n} e^{-Lr^2} 2\pi r dr = \frac{2L}{n!} \int_0^{+\infty} L^n r^{2n+1} e^{-Lr^2} dr \\ &= \frac{1}{n!} \int_0^{+\infty} t^n e^{-t} dt = \frac{\Gamma(n+1)}{n!} = 1. \end{aligned}$$

We used a change of variables $t = Lr^2$ and the third property of gamma functions. □

Remark. With this we see that the expression of a GAF in the plane is

$$f_L(z) = \sum_{n=0}^{\infty} a_n \sqrt{\frac{L^n}{n!}} z^n. \tag{24}$$

As said before, we can compute the radius of convergence of a Gaussian analytic function $f \in \mathcal{H}_L$ in the plane seen on Remark above. Let us denote $c_n = \frac{L^{n/2}}{\sqrt{n!}}$ (normalization factor). Then,

$$f(z) = \sum_{n=0}^{\infty} a_n c_n z^n$$

and the convergence radius of the series is:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n(\omega)| c_n|}} = \limsup_{n\to\infty} \sqrt[2n]{\frac{n!}{L^n}} = +\infty.$$

2. **The hyperbolic case.** Now, consider $X = \mathbb{D}$ and, for $L > 1$, the weighted Bergman space defined by:

$$\mathbb{B}_L = \left\{ f \in Hol(\mathbb{D}) : ||f||_L^2 = \frac{L-1}{\pi} \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{L-2} dm(z) < +\infty \right\}. \quad (25)$$

The constant $\frac{L-1}{\pi}$ is chosen so that

$$\int_{\mathbb{D}} \frac{L-1}{\pi} (1-|z|^2)^{L-2} dm(z) = 1.$$

As before, the monomials are dense, and using the orthogonality of $\{z^n\}_n$ we can compute the normalization factor:

$$\begin{aligned} < z^n, z^n >_{\mathcal{B}_L} &= \frac{L-1}{\pi} \int_{\mathbb{D}} |z|^{2n} (1-|z|^2)^{L-2} dm(z) \\ &= \frac{L-1}{\pi} \int_0^1 \int_0^{2\pi} r^{2n} (1-r^2)^{L-2} e^{i(n-n)\theta} r d\theta dr \\ &= (L-1) \int_0^1 r^{2n} (1-r^2)^{L-2} 2r dr = (L-1) \int_0^1 t^n (1-t)^{L-2} dt \\ &= (L-1) \frac{\Gamma(n+1)\,\Gamma(L-1)}{\Gamma(n+L)} = \frac{\Gamma(n+1)\,\Gamma(L)}{\Gamma(n+L)} = \frac{n!\,\Gamma(L)}{\Gamma(n+L)}. \end{aligned}$$

Thus,

$$||z^n||_L = \sqrt{\frac{\Gamma(n+1)\Gamma(L)}{\Gamma(n+L)}} \quad \text{and} \quad e_n(z) = \sqrt{\frac{\Gamma(L+n)}{\Gamma(n+1)\Gamma(L)}} z^n. \quad (26)$$

The expression of the hyperbolic GAF is thus

$$f_L(z) = \sum_{n=1}^{\infty} a_n \left(\frac{\Gamma(L+n)}{n!\,\Gamma(L)} \right)^{\frac{1}{2}} z. \quad (27)$$

This sum can be analytically counted to $L > 0$, which we will assume from now on.

3. **The parabolic case.** Here, we deal with $X = \mathbb{S}^2$ and with the Hilbert space of polynomials of degree at most $L \in \mathbb{N}$, denoted by $\mathbb{P}_L[\mathbb{C}]$, with the following norm, given by the invariant measure on $\mathbb{C}$:

$$\mathcal{P}_L = \left\{ p \in \mathbb{P}_L[\mathbb{C}] : ||p||_L^2 = (L+1) \int_{\mathbb{C}} \frac{|p(z)|^2}{(1+|z|^2)^L} \frac{dm(z)}{\pi(1+|z|^2)^2} < +\infty \right\}. \quad (28)$$

In this case, as the previous examples, the orthonormal basis is:

$$e_n(z) = \sqrt{\frac{\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}} z^n = \binom{L}{n}^{\frac{1}{2}} z^n, \quad (29)$$

and therefore the general expression of the parabolic GAF is

$$f_L(z) = \sum_{n=0}^{L} \binom{L}{n}^{\frac{1}{2}} z^n. \quad (30)$$

3.4 Covariance kernel

Since f is a mean zero Gaussian, its probabilistic properties are determined by its variance, or more generally, its covariance kernel. It is a key concept to understand why point processes given as zero sets of Gaussian analytic functions present a local repulsion.

Lemma 3.3. Let f be a GAF as in (19). Then:

$$K(z,w) := \mathrm{Cov}(f(z), f(w)) = \sum_{n=0}^{\infty} e_n(z)\bar{e}_n(w).$$

Proof. Since $E[f(z)] = 0$, we have:

$$\begin{aligned} K(z,w) = \mathrm{Cov}(f(z), f(w)) = E[f(z)\bar{f}(w)] = E\left[\sum_{n,m} a_n \bar{a}_m e_n(z)\bar{e}_m(w)\right] \\ = \sum_{n,m} E[a_n \bar{a}_m] e_n(z)\bar{e}_m(w). \end{aligned}$$

By Lemma 2.1. (a) and (c), we have:

$$\mathbb{E}[a_n \bar{a}_m] = \delta_{n,m}\mathbb{E}[|a_n|^2] = \delta_{n,m};$$

and the result follows. □

Examples. In many cases the covariance kernel can be computed explicitly.

1. **The planar case.** For a given intensity $L > 0$ and $z, w \in \mathbb{C}$, and using (22):

$$K_L(z,w) = \sum_{n=0}^{\infty} e_n(z)\bar{e}_n(w) = \sum_{n=0}^{\infty} \frac{L^{n/2}}{\sqrt{n!}} z^n \frac{L^{n/2}}{\sqrt{n!}} \bar{w}^n = \sum_{n=0}^{\infty} \frac{(Lz\bar{w})^n}{n!} = e^{Lz\bar{w}}.$$

2. **The hyperbolic case.** Here we have; for $L > 0$ and $z, w \in \mathbb{D}$, and using (26):

$$\begin{aligned} K_L(z,w) = \sum_{n=0}^{\infty} e_n(z)\bar{e}_n(w) = \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(L+n)}{\Gamma(n+1)\,\Gamma(L)}} z^n \sqrt{\frac{\Gamma(L+n)}{\Gamma(n+1)\,\Gamma(L)}} \bar{w}^n \\ = \sum_{n=0}^{\infty} \frac{\Gamma(L+n)}{\Gamma(n+1)\,\Gamma(L)} (z\bar{w})^n = \frac{1}{(1-z\bar{w})^L}. \end{aligned}$$

3. **The parabolic case.** In this example, for every natural number L and $z, w \in \mathbb{C}$, and using (29):

$$\begin{aligned} K_L(z,w) &= \sum_{n=0}^{\infty} e_n(z)\bar{e}_n(w) \\ &= \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}} z^n \sqrt{\frac{\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}} \bar{w}^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)} (z\bar{w})^n = (1+z\bar{w})^L. \end{aligned}$$

3.5 Zeros of Gaussian analytic functions

In this section we finally introduce the point process we study in this project. Let

$$Z_f := \{z \in \Omega; f(z) = 0\} = \{\lambda_n\}_n$$

be the zero set of a given Gaussian analytic function f on Ω. We can introduce the discrete measure:

$$\mu_f = \sum_{n=1}^{\infty} \delta_{\lambda_n}. \tag{31}$$

Here δ_{λ_n} is the Dirac delta on $\lambda_n \in \Omega$. For a subset $U \subset X$, we have the counting function:

$$n_f(U) = \int_U d\mu_f = \#(Z(f) \cap U). \tag{32}$$

A crucial feature of processes given as zero sets of holomorphic functions is that this empirical measure μ_f can be computed using the Laplacian operator:

$$\mu_f = \frac{1}{2\pi}\Delta \log|f| = \frac{1}{4\pi}\Delta \log|f|^2. \tag{33}$$

Definition 3.3. The **first intensity** of a GAF f is the distribution $\mathbb{E}[\mu_f]$ defined by:

$$< \mathbb{E}[\mu_f], \varphi >= \mathbb{E}[< \mu_f, \varphi >], \qquad \varphi \in \mathcal{C}_c^{\infty}(\Omega). \tag{34}$$

Last equality can be rewritten in terms of integrals, that is:

$$\int_\Omega \varphi d\mathbb{E}[\mu_f] = E\biggl(\int_\Omega \varphi d\mu_f\biggr), \quad \varphi \in \mathcal{C}_c^{\infty}(\Omega). \tag{35}$$

A key property of Gaussians is that the first intensity can be obtained just by taking the expectation value of $|f|^2$ in the expression (33), as we shall see next.

3.5.1 The Edelman-Kostlan formula

Theorem 3.1. Let f be a GAF and μ_f its empirical measure. Then, the first intensity of f is given by:

$$\mathbb{E}[\mu_f] = \frac{1}{4\pi}\Delta \log \mathbb{E}[|f|^2] = \frac{1}{4\pi}\Delta \log K(z,z), \tag{36}$$

where $K(z,z)$ is the covariance kernel of f.

Proof. We indicate the proof for the case where $K(z,z)$ has no zeros. For the general proof see [HKPV09, Lemma 2.4.1]. By definition:

$$\begin{aligned}
\mathbb{E}[< \mu_f, \varphi >] &= \mathbb{E}\bigg[\int_\Omega \varphi d\mu_f\bigg] = \mathbb{E}\bigg[\int_\Omega \varphi \frac{1}{2\pi}\Delta \log |f(z)|\bigg] = \mathbb{E}\bigg[\int_\Omega \frac{1}{2\pi}\Delta\varphi \log |f(z)|\bigg] \\
&= \mathbb{E}\bigg[\int_\Omega \frac{1}{2\pi}\Delta\varphi\bigg(\log \frac{|f(z)|}{\sqrt{K(z,z)}} + \log \sqrt{K(z,z)}\bigg)\bigg] \\
&= \int_\Omega \frac{1}{2\pi}\Delta\varphi \log \sqrt{K(z,z)} + \mathbb{E}\bigg[\int_\Omega \frac{1}{2\pi}\Delta\varphi \log \frac{|f(z)|}{K(z,z)}\bigg].
\end{aligned}$$

If we denote

$$\xi = \frac{f(z)}{\sqrt{K(z,z)}},$$

then $\xi \sim N_{\mathbb{C}}(0,1)$ and $\mathbb{E}(\log|\xi|)$ is a constant independent of z. Then, using Fubini's theorem:

$$\begin{aligned}
\mathbb{E}[< \mu_f, \varphi >] &= \frac{1}{2\pi}\int_\Omega \varphi\Delta \log \sqrt{K(z,z)} + \frac{1}{2\pi}\int_\Omega \Delta\varphi \mathbb{E}\bigg[\log|\xi|\bigg] \\
&= \frac{1}{2\pi}\int_\Omega \varphi\Delta \log \sqrt{K(z,z)} + \frac{1}{2\pi}\int_\Omega \varphi\Delta \mathbb{E}\bigg[\log|\xi|\bigg] \\
&= \frac{1}{2\pi}\int_\Omega \varphi\Delta \log \sqrt{K(z,z)}.
\end{aligned}$$

Finally we obtain:

$$\mathbb{E}[< \mu_f, \varphi >] = \frac{1}{2\pi}\int_\Omega \varphi\Delta \log \sqrt{K(z,z)}.$$

From the defining identity

$$< \mathbb{E}[\mu_f], \varphi >= \int_\Omega \varphi d\mathbb{E}[\mu_f],$$

we get the result. □

Examples. For the model cases, the first intensity can be computed explicitly.

1. **The plane case.** Remember that $K_L(z,z) = e^{L|z|^2}$, $z \in \mathbb{C}$. Then:

$$E[\mu_{f_L}] = \frac{1}{4\pi}\Delta L|z|^2 = L\frac{1}{4\pi}4\frac{\partial}{\partial z\partial \bar{z}}(z\bar{z}) = \frac{L}{\pi}dm(z).$$

   Notice that this is the same measure we used to illustrate the Poisson process in Chapter 1.

2. **The hyperbolic case.** Here $K_L(z,z) = (1-|z|^2)^{-L}$, $z \in \mathbb{D}$. Then:

$$\begin{aligned} E[\mu_{f_L}] &= \frac{1}{4\pi}\Delta \log(1-|z|^2)^{-L} = L\frac{1}{4\pi}4\frac{\partial}{\partial z\partial \bar{z}}\log\left(\frac{1}{1-z\bar{z}}\right) \\ &= \frac{L}{\pi}\frac{1}{(1-|z|^2)^2}dm(z) = Ld\nu(z). \end{aligned}$$

3. **The parabolic case.** Here $K_L(z,z) = (1+|z|^2)^{L}$, $z \in \mathbb{S}^2$. Then:

$$\begin{aligned} E[\mu_{f_L}] &= \frac{1}{4\pi}\Delta \log(1+|z|^2)^{L} = L\frac{1}{4\pi}4\frac{\partial}{\partial z\partial \bar{z}}\log(1+z\bar{z}) \\ &= \frac{L}{\pi}\frac{1}{(1+|z|^2)^2}dm(z) = Ld\nu(z). \end{aligned}$$

   Remark. The $d\nu(z)$ appearing above in the hyperbolic case corresponds to the invariant measure associated to the hyperbolic geometry and it is not the same as the one appearing in the parabolic case, which corresponds to the invariant measure of the parabolic geometry. Context will help us to detect which invariant measure we refer.

4 Variance. General formula.

This chapter is the main goal of the project. We are going to see some results that show that point processes produced in this way are more rigid than the Poisson process seen at the beginning. A way to measure this rigidness is to evaluate the variance of the counting random variables seen at the beginning.

For an open region $U \subset X$, remember that

$$n_L(U) = \#(Z_{f_L} \cap U) = \int_U \frac{1}{4\pi} \Delta \log |f_L|^2.$$

Then, to see how fluctuates $n_L(U)$ we can compute its variance and after that, we will compare it with the variances obtained in the Poisson processes associated to each standard example (Chapter 1).

Theorem 4.1. Let $U \subseteq X$ be an open region with $\mathcal{C}^1$ boundary. Let f be a GAF as in (19) and let $K(z,w)$ denote its covariance kernel. Then:

$$Var[n_L(U)] = -\frac{1}{4\pi^2} \int_{\partial U} \int_{\partial U} \frac{1}{1-I} \frac{\partial}{\partial \bar{z}} \left(\frac{K(w,z)}{K(z,z)} \right) \frac{\partial}{\partial \bar{w}} \left(\frac{K(z,w)}{K(w,w)} \right) d\bar{z} d\bar{w}, \tag{37}$$

where

$$I(z,w) = \frac{|K(z,w)|^2}{K(z,z)K(w,w)}.$$

Notice, once more, that this depends only on the covariance kernel.

Proof. By definition:

$$\mathrm{Var}[n_L(U)] = \mathbb{E}[(n_L(U) - \mathbb{E}[n_L(U)])^2]. \tag{38}$$

Using the Edelman-Kostlan formula:

$$\begin{aligned} n_L(U) - \mathbb{E}[n_L(U)] &= \int_U \frac{1}{2\pi} \Delta \log |f_L(z)| - \int_U \frac{1}{4\pi} \Delta \log K(z,z) \\ &= \int_U \frac{1}{2\pi} \Delta \log \frac{|f_L(z)|}{\sqrt{K(z,z)}}. \end{aligned}$$

We shall use the following form of Green's identity (a particular version of the Stokes theorem). Let Ω be an open region and let U be open, bounded set such that $\bar{U} \subseteq \Omega$ and with $\mathcal{C}^1$ boundary. For $f \in \mathcal{C}^2(\Omega)$,

$$\begin{aligned} i \int_{\partial U} \frac{\partial f}{\partial \bar{z}}(\xi) d\bar{\xi} &= \int_U \frac{\partial^2 f}{\partial z \partial \bar{z}}(\xi) i d\xi \wedge d\bar{\xi} = 2 \int_U \frac{\partial^2 f}{\partial z \partial \bar{z}}(\xi) dm(\xi) \\ &= \frac{1}{2} \int_U \Delta f(\xi) dm(\xi). \end{aligned} \tag{39}$$

Using (39) we obtain:

$$n_L(U) - \mathbb{E}[n_L(U)] = \frac{i}{\pi}\int_{\partial U} \frac{\partial}{\partial \bar{z}} \log \frac{|f_L(z)|}{\sqrt{K_L(z,z)}}. \tag{40}$$

Note that the expression above could have a singularity when $K_L(z,z) = 0$, but this happens with probability 0 due to the fact that $Z_{f_L} \cap \partial U = 0$ a.s. Similarly to the proof of the Edelman-Kostlan formula, we denote the normalized GAF by $\hat{f}_L(z) = \frac{|f_L(z)|}{\sqrt{K_L(z,z)}} \sim N_{\mathbb{C}}(0,1)$. Then:

$$\begin{aligned}\mathrm{Var}[n_L(U)] &= \mathbb{E}\left[\int_{\partial U} \frac{i}{\pi}\frac{\partial}{\partial \bar{z}} \log \frac{|f(z)|}{\sqrt{K_L(z,z)}} \quad \int_{\partial U} \frac{i}{\pi}\frac{\partial}{\partial \bar{w}} \log \frac{|f(w)|}{\sqrt{K_L(w,w)}}\right] \\ &= \mathbb{E}\left[\int_{\partial U}\int_{\partial U} \left(\frac{i}{2\pi}\right)^2 \frac{\partial}{\partial \bar{z}} \log |\hat{f}_L(z)|^2 \frac{\partial}{\partial \bar{w}} \log |\hat{f}_L(w)|^2\right].\end{aligned}$$

Since $\hat{f}_L(z)$ and $\hat{f}_L(w)$ follow a $N_{\mathbb{C}}(0,1)$, then the term

$$\left(\frac{i}{2\pi}\right)^2 \frac{\partial}{\partial \bar{z}} \log |\hat{f}_L(z)|^2 \frac{\partial}{\partial \bar{w}} \log |\hat{f}_L(w)|^2$$

is integrable on ∂U. Thus, all needed conditions are fullfied so we can apply the Fubini's theorem twice. Therefore:

$$\mathrm{Var}[n_L(U)] = \int_{\partial U}\int_{\partial U} \mathbb{E}\left[\left(\frac{i}{2\pi}\right)^2 \frac{\partial}{\partial \bar{z}} \log |\hat{f}_L(z)|^2 \frac{\partial}{\partial \bar{w}} \log |\hat{f}_L(w)|^2\right].$$

Consecutively, we use the differentiation under the integral sign, which is the general form of the Leibnitz rule (see on [Wik19]) and we get:

$$\mathrm{Var}[n_L(U)] = \int_{\partial U}\int_{\partial U} \left(\frac{i}{\pi}\right)^2 \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial \bar{w}} \mathbb{E}[\log |\hat{f}_L(z)| \log |\hat{f}_L(w)|]. \tag{41}$$

It is important to remark that the expectation in this integral depends only on the standard complex Gaussian, and it is known.

Lemma 4.1. ([HKPV09, Lemma 3.5.2]) Let ξ and ν be complex Gaussians with $\mathbb{E}[\xi\bar{\nu}] = \theta$. Then:

$$\mathrm{Cov}(\log|\xi|, \log|\nu|) = \frac{1}{4} Li_2(|\theta|^2) = \frac{1}{4}\sum_{m=1}^{\infty} \frac{|\theta|^{2m}}{m^2}. \tag{42}$$

The function Li_2 is called the dilogarithm.

In our case, we have:

$$\theta(z,w) = \mathbb{E}[\hat{f}(z)\hat{f}(w)] = \frac{K(z,w)}{\sqrt{K(z,z)}\sqrt{K(w,w)}}. \tag{43}$$

For simplicity, we denote:

$$I(z,w) = |\theta(z,w)|^2. \tag{44}$$

Since

$$\mathrm{Cov}[\log \hat{f}_L(z), \log \hat{f}_L(w)] = \mathbb{E}[\log \hat{f}_L(z)\ \log \hat{f}_L(w)] - \mathbb{E}[\log \hat{f}_L(z)]\mathbb{E}[\log \hat{f}_L(w)],$$

and this second term is constant,

$$\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial \bar{w}}\mathbb{E}[\log|\hat{f}_L(z)|\log|\hat{f}_L(w)|] = \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial \bar{w}}\,\mathrm{Cov}[\log|\hat{f}_L(z)|\log|\hat{f}_L(w)|].$$

From (41) we obtain:

$$\mathrm{Var}[n_L(U)] = -\frac{1}{4\pi^2}\int_{\partial U}\int_{\partial U}\frac{\partial^2}{\partial\bar{z}\partial\bar{w}}Li_2(I(z,w))d\bar{z}d\bar{w}. \tag{45}$$

Now, we need to compute $\frac{\partial^2}{\partial\bar{z}\partial\bar{w}}Li_2(I(z,w))$:

$$\begin{aligned}
\frac{\partial^2}{\partial\bar{z}\partial\bar{w}}Li_2(I(z,w)) &= \frac{\partial}{\partial\bar{z}}\left(\sum_{m=1}^{\infty}\frac{I^{m-1}}{m}\right)\frac{\partial I}{\partial\bar{w}} \\
&= \left(\sum_{m=2}^{\infty}\frac{m-1}{m}I^{m-2}\right)\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}} + \left(\sum_{m=1}^{\infty}\frac{I^{m-1}}{m}\right)\frac{\partial^2 I}{\partial\bar{z}\partial\bar{w}} \\
&= \left(\sum_{n=0}^{\infty}\frac{n+1}{n+2}I^{n}\right)\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}} + \left(\sum_{n=0}^{\infty}\frac{I^{n}}{n+1}\right)\frac{\partial^2 I}{\partial\bar{z}\partial\bar{w}}.
\end{aligned} \tag{46}$$

Let us see next that

$$\frac{\partial^2 I}{\partial\bar{z}\partial\bar{w}} = \frac{1}{I}\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}}. \tag{47}$$

In effect, from (43) and (44) we get:

$$\frac{\partial I}{\partial\bar{z}} = \frac{\partial}{\partial\bar{z}}\left[\frac{K(w,z)}{K(z,z)}\right]\frac{K(z,w)}{K(w,w)}$$

$$\frac{\partial I}{\partial\bar{w}} = \frac{\partial}{\partial\bar{w}}\left[\frac{K(z,w)}{K(w,w)}\right]\frac{K(w,z)}{K(z,z)}.$$

Therefore:

$$\begin{aligned}
\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}} &= \frac{\partial}{\partial\bar{z}}\left[\frac{K(w,z)}{K(z,z)}\right]\frac{\partial}{\partial\bar{w}}\left[\frac{K(z,w)}{K(w,w)}\right]\frac{K(z,w)K(w,z)}{K(w,w)K(z,z)} \\
&= \frac{\partial^2 I}{\partial\bar{z}\partial\bar{w}}I.
\end{aligned} \tag{48}$$

Applying (47) to the expression (46) we obtain:

$$\begin{aligned}
\frac{\partial^2}{\partial\bar{z}\partial\bar{w}}Li_2(I(z,w)) &= \left(\sum_{n=0}^{\infty}\frac{n+1}{n+2}I^n\right)\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}} + \left(\sum_{n=0}^{\infty}\frac{I^n}{n+1}\right)\frac{1}{I}\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}} \\
&= \frac{1}{I}\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}}\left(\sum_{n=0}^{\infty}\frac{n+1}{n+2}I^{n+1} + \sum_{n=0}^{\infty}\frac{1}{n+1}I^n\right) \\
&= \frac{1}{I}\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}}\left(\sum_{n=1}^{\infty}\frac{n}{n+1}I^n + \sum_{n=0}^{\infty}\frac{1}{n+1}I^n\right) \\
&= \frac{1}{I}\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}}\left(1+\sum_{n=1}^{\infty}I^n\right) = \frac{1}{I(1-I)}\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}}. \qquad (49)
\end{aligned}$$

From (45) we obtain:

$$Var[n_L(U)] = -\frac{1}{4\pi^2}\int_{\partial U}\int_{\partial U}\frac{1}{I(1-I)}\frac{\partial I}{\partial\bar{z}}\frac{\partial I}{\partial\bar{w}}d\bar{z}d\bar{w}. \qquad (50)$$

As we did in (48),

$$\frac{1}{I}\frac{\partial I}{\bar{z}}\frac{\partial I}{\partial w} = \frac{\partial^2 I}{\partial\bar{z}\partial\bar{w}} = \frac{\partial}{\partial\bar{z}}\left(\frac{K(w,z)}{K(z,z)}\right)\frac{\partial}{\partial\bar{w}}\left(\frac{K(z,w)}{K(w,w)}\right);$$

and the result follows. □

4.1 Example: the plane case.

Now, we are able to apply the general formula (37) to **the planar case.** For simplicity, we can consider $U = D(a,r)$ to be a disc of radius $r > 0$. Due to invariance under translations of the measure $d\nu = \frac{dm(z)}{\pi}$, we can choose $a = 0$. Using that $K_L(z,w) = e^{Lz\bar{w}}$ we get:

$$I(z,w) = \frac{K(z,w)K(w,z)}{K(z,z)K(w,w)} = \frac{e^{Lz\bar{w}}e^{Lw\bar{z}}}{e^{Lz\bar{z}}e^{Lw\bar{w}}} = e^{Lz(\bar{w}-\bar{z})}e^{Lw(\bar{z}-\bar{w})} = e^{-L|z-w|^2}. \qquad (51)$$

Computing the corresponding derivatives of the expression (37), taking the polar coordinates for a given $r > 0$ and making the change of variables $t = \theta - \phi, \quad \theta,\phi \in (0,2\pi)$; we obtain:

$$\mathrm{Var}[n_L(r)] = \frac{L^2r^4}{2\pi}\int_0^{2\pi}\frac{e^{-Lr^2|1-e^{it}|^2}}{1-e^{-Lr^2|1-e^{it}|^2}}|1-e^{it}|^2dt. \qquad (52)$$

The hyperbolic case, which is more difficult, will be discussed in detail in the next chapter. From the last expression, and doing the change of variables $x = 2Lr^2(1-\cos(t))$:

$$\mathrm{Var}[n_L(r)] = \frac{\sqrt{Lr^2}}{2\pi}\int_0^{4Lr^2}\frac{e^{-x}}{1-e^{-x}}\frac{\sqrt{x}}{\sqrt{1-x/(4Lr^2)}}dx. \qquad (53)$$

Knowing that

$$\frac{e^{-x}}{1-e^{-x}} = \sum_{n=1}^{\infty} e^{-nx},$$

we obtain

$$\operatorname{Var}[n_L(r)] = \frac{\sqrt{Lr^2}}{2\pi} \sum_{n=1}^{\infty} \int_0^{4Lr^2} e^{-nx} \frac{\sqrt{x}}{\sqrt{1-x/(4Lr^2)}} dx. \tag{54}$$

We want to stress that this is an exact expression of $\operatorname{Var}[n_L(r)]$ given by a sum of positive terms, actually a sum of integrals of positive functions. A lot of information can be extracted from such objects.

For instance, let us see what happens when $\mathbb{E}[n_L(r)] = Lr^2 \to \infty$:

$$I_n = (1+o(1)) \int_0^{\infty} e^{-nx} \sqrt{x}\, dx = (1+o(1)) \frac{1}{n^{3/2}} \int_0^{\infty} e^{-y} \sqrt{y}\, dy = (1+o(1)) \frac{\Gamma(3/2)}{n^{3/2}}.$$

This is so because the singularity is integrable at $4Lr^2$. Replacing this result to (54):

$$\begin{aligned} \operatorname{Var}[n_L(r)] &= \frac{\sqrt{Lr^2}}{2\pi} \sum_{n=1}^{\infty} \frac{\Gamma(3/2)}{n^{3/2}} (1+o(1)) = \frac{\sqrt{Lr^2}}{2\pi} \Gamma(3/2)\zeta(3/2)(1+o(1)) \\ &= \frac{1}{2\sqrt{\pi}} \zeta(3/2) \sqrt{Lr^2} (1+o(1)) \sim \sqrt{Lr^2}. \end{aligned}$$

Remember that in the Poisson processes the variance of $n_L(r)$ was exactly Lr^2. Thus, we see that point process associated to the zeros of a Gaussian analytic function in the complex plane is more ”rigid”, in the sense that the variance is of lower order. This explains what we see in Figure 2, page 6.

5 The hyperbolic case.

In this last chapter of the project we will study the variance of the random variable $n_L(U)$ for the hyperbolic case. In particular, we will study the fluctuations of $n_L(D(z,r))$ as the intensity L tends to infinite. The main results follow from the following theorem, which is the specification in this context of (37). Due to invariance under automorphisms seen on (6), the $\mathrm{Var}[n_L(D(z,r)]$ does not depend on where we take the centre z of that disk, so, from now on, we will use the notation $\mathrm{Var}[n_L(r)]$.

Theorem 5.1. Let f be an hyperbolic Gaussian analytic function of intensity L as in (27). Then, for a disk $U = D(z,r)$, $z \in \mathbb{D}$, $r \in (0,1)$,

$$\mathrm{Var}[n_L(r)] = \frac{L^2 r^4}{(1-r^2)^2} \int_{-\pi}^{\pi} \frac{(1-r^2)^{2L} \quad |1-e^{it}|^2}{[|1-r^2e^{it}|^{2L} - (1-r^2)^{2L}] \quad |1-r^2e^{it}|^2} \frac{dt}{2\pi}. \tag{55}$$

We can also rewrite the last expression in the following form:

$$\mathrm{Var}[n_L(r)] = \frac{L^2 r^4}{(1-r^2)^2} \int_{-\pi}^{\pi} \frac{(1-r^2)^{2L} \quad 2(1-\cos(t))}{[|1-r^2e^{it}|^{2L} - (1-r^2)^{2L}] \quad |1-r^2e^{it}|^2} \frac{dt}{2\pi}. \tag{56}$$

Notice that the integrand of this expression is a positive function, so we can use all the tools we have to estimate such integrals.

Proof. Recall that for $L > 0$, the covariance kernel hyperbolic Gaussian analytic function of intensity L takes the form:

$$K_L(z,w) = \frac{1}{(1-z\bar{w})^L}.$$

In this case, the elements appearing in the general formula seen in the previous chapter can be computed explicitly. First:

$$I(z,w) = \frac{K(z,w)K(w,z)}{K(z,z)K(w,w)} = \left(\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-w\bar{z})}\right)^L = \left(\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}\right)^L.$$

Therefore:

$$\frac{1}{1-I} = \frac{[(1-z\bar{w})(1-w\bar{z})]^L}{[(1-z\bar{w})(1-w\bar{z})]^L - [(1-|z|^2)(1-w\bar{w})]^L} = \frac{|1-\bar{z}w|^{2L}}{|1-\bar{z}w|^{2L} - (1-|z|^2)^L(1-|w|^2)^L}.$$

From this, we deduce that:

$$\begin{aligned}
\frac{\partial I}{\partial \bar{z}} &= \left(\frac{1-|w|^2}{1-z\bar{w}}\right)^L L \left(\frac{1-|z|^2}{1-w\bar{z}}\right)^{L-1} \left(\frac{-z(1-w\bar{z}) + (1-|z|^2)w}{(1-w\bar{z})^2}\right) \\
&= L\left(\frac{1-|w|^2}{1-z\bar{w}}\right)^L \left(\frac{1-|z|^2}{1-w\bar{z}}\right)^{L-1} \frac{w-z}{(1-w\bar{z})^2}
\end{aligned}$$

$$\frac{\partial I}{\partial \bar{w}} = \left(\frac{1-|z|^2}{1-w\bar{z}}\right)^L L\left(\frac{1-|w|^2}{1-z\bar{w}}\right)^{L-1}\left(\frac{-w(1-z\bar{w})+(1-|w|^2)z}{(1-z\bar{w})^2}\right)$$
$$= L\left(\frac{1-|z|^2}{1-w\bar{z}}\right)^L\left(\frac{1-|w|^2}{1-z\bar{w}}\right)^{L-1}\frac{z-w}{(1-z\bar{w})^2}.$$

Replacing the last three results to the general expression (37) we obtain:

$$\mathrm{Var}[n_L(U)] = -\frac{1}{4\pi^2}\int_{\partial U}\int_{\partial U}\left(\frac{(1-\bar{z}w)^{2L}}{(1-|z|^2)^L(1-|w|^2)^L}\right)\left(\frac{(1-\bar{z}w)^{2L}}{(1-\bar{z}w)^{2L}-[(1-|z|^2)(1-|w|^2)]^L}\right)\times$$
$$\times L^2\left(\frac{(1-|w|^2)^{2L-1}(1-|z|^2)^{2L-1}(z-w)^2}{|1-\bar{z}w|^{2(2L+1)}}\right)d\bar{z}d\bar{w}.$$

Notice that

$$\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2} = 1-\left|\frac{z-w}{1-\bar{z}w}\right|^2,$$

so the first term of the last expression has no singularities due to the invariance by automorphisms of the unit disk, seen on Chapter 1. Thus, we take $U = D(0,r)$ and polar coordinates; $z = re^{i\theta}$ and $w = re^{i\phi}$, $r \in (0,1)$ and $\theta, \phi \in (0, 2\pi)$; we get:

$$\mathrm{Var}[n_L(r)] = \int_0^{2\pi}\int_0^{2\pi}\frac{L^2(1-r^2)^{2(L-1)}r^2(e^{i\theta}-e^{i\phi})}{[(1-r^2e^{i(\theta-\phi)})^L(1-r^2e^{i(\phi-\theta)})^L-(1-r^2)^{2L}]}\times$$
$$\times\frac{(e^{i\phi}-e^{i\theta})re^{-i\theta}re^{-i\phi}}{(1-r^2e^{i(\theta-\phi)})(1-r^2e^{i(\phi-\theta)})}\frac{d\theta}{2\pi}\frac{d\phi}{2\pi}.$$

Now, changing the variables $t = \theta - \phi$, we get finally:

$$\mathrm{Var}[n_L(r)] = \frac{L^2r^4}{(1-r^2)^2}\int_{-\pi}^{\pi}\frac{(1-r^2)^{2L} \quad |1-e^{it}|^2}{[|1-r^2e^{it}|^{2L}-(1-r^2)^{2L}] \quad |1-r^2e^{it}|^2}\frac{dt}{2\pi}$$
$$= \frac{L^2r^4}{(1-r^2)^2}\int_{-\pi}^{\pi}\frac{(1-r^2)^{2L} \quad 2(1-\cos(t))}{[|1-r^2e^{it}|^{2L}-(1-r^2)^{2L}] \quad |1-r^2e^{it}|^2}\frac{dt}{2\pi}. \qquad \square$$

For the case $L = 1$, the previous integral can be computed explicitly.

Theorem 5.2. For $L = 1$ and for any $r \in (0,1)$,

$$\mathrm{Var}[n_1(r)] = \frac{r^2}{1-r^4}.$$

Proof. Replacing $L = 1$ to the expression (55):

$$\mathrm{Var}[n_1(r)] = r^4\int_{-\pi}^{\pi}\frac{1}{|1-r^2e^{it}|^2-(1-r^2)^2}\frac{|1-e^{it}|^2}{|1-r^2e^{it}|^2}\frac{dt}{2\pi}. \tag{57}$$

In order to use the Residue theorem take $\xi = e^{it}$; we obtain:

$$\mathrm{Var}[n_1(r)] = r^4 \int_{|\xi|=1} \frac{1}{|1-r^2\xi|^2 - (1-r^2)^2} \frac{|1-\xi|^2}{|1-r^2\xi|^2} \frac{d\xi}{\xi 2\pi i}. \tag{58}$$

Notice that $\bar{\xi} = e^{-it} = \frac{1}{\xi}$. Then:

$$\begin{aligned}
\mathrm{Var}[n_1(r)] &= r^4 \int_{|\xi|=1} \frac{1}{(1-r^2\xi)(1-\frac{r^2}{\xi}) - (1-r^2)^2} \frac{(1-\xi)(1-\frac{1}{\xi})}{(1-r^2\xi)(1-\frac{r^2}{\xi})} \frac{d\xi}{\xi 2\pi i} \\
&= r^4 \int_{|\xi|=1} \frac{1}{2\pi i} \frac{1}{(1-r^2\xi)(1-\frac{r^2}{\xi}) - (1-r^2)^2} \frac{(1-\xi)(\xi-1)}{(1-r^2\xi)(\xi-r^2)} \frac{d\xi}{\xi} \\
&= r^4 \int_{|\xi|=1} \frac{1}{2\pi i} \frac{1}{(1-r^2\xi)(\xi-r^2) - (1-r^2)^2\xi} \frac{(1-\xi)(\xi-1)}{(1-r^2\xi)(\xi-r^2)} d\xi.
\end{aligned}$$

In order to find the poles of the integrand, we factorize the denominator:

$$\begin{aligned}
(1-r^2\xi)(\xi-r^2) - (1-r^2)^2\xi &= -r^2\xi^2 - [(1-r^2)^2 - r^4 - 1]\xi - r^2 \\
&= -r^2\xi^2 + 2r^2\xi + r^2 = -r^2(\xi-1)^2.
\end{aligned}$$

Replacing this result to the expression above we get:

$$\mathrm{Var}[n_1(r)] = r^2 \int_{|\xi|=1} \frac{1}{2\pi i} \frac{1}{(1-r^2\xi)(\xi-r^2)} d\xi = \int_{|\xi|=1} \frac{1}{2\pi i} \frac{1}{(\frac{1}{r^2}-\xi)(\xi-r^2)} d\xi.$$

Denote:

$$g(\xi) := \frac{1}{(\frac{1}{r^2}-\xi)(\xi-r^2)}.$$

Notice that $g(\xi)$ is an meromorphic function with simple poles at r^2 and r^{-2}. Since $r \in (0,1)$, r^{-2} is not in the interior of the unit circle $|\xi| = 1$. Applying the residue theorem we obtain:

$$\mathrm{Var}[n_1(r)] = \mathrm{Res}(g(\xi), r^2) = \lim_{\xi \to r^2} \frac{1}{\frac{1}{r^2} - \xi} = \frac{r^2}{1-r^4}. \quad \square$$

Note that in principle, we could follow the same procedure for any $L \in \mathbb{N}$. However, for general L, we get in the denominator a polynomial of degree $2L$, which in general we do not know how to factorize.

We shall see next that Theorem 4.1 is still useful to study the asymptotic behaviour of $\mathrm{Var}[n_L(r)]$ as $L \to +\infty$.

Theorem 5.3. For any $r \in (0,1)$ fixed, as $L \to +\infty$,

$$\mathrm{Var}[n_L(r)] = \frac{r}{4\sqrt{\pi}} \frac{\zeta(3/2)}{1-r^2} \sqrt{L}\,(1+o(1)).$$

Proof. Denote by I_L the integral in (56), that is:

$$I_L(r) = \int_{-\pi}^{\pi} \frac{(1-r^2)^{2L}}{[|1-r^2e^{it}|^{2L} - (1-r^2)^{2L}]} \frac{2(1-\cos(t))}{|1-r^2e^{it}|^2} \frac{dt}{2\pi}.$$

We can simplify this integral dividing both the numerator and the denominator by $(1-r^2)^{2L}$:

$$I_L(r) = \int_{-\pi}^{\pi} \frac{1}{\left(\frac{|1-r^2e^{it}|}{1-r^2}\right)^{2L} - 1} \frac{2(1-\cos(t))}{|1-r^2e^{it}|^2} dt. \tag{59}$$

Notice that:

$$|1-r^2e^{it}|^2 = 1 + r^4 - 2r^2\cos(t) = (1-r^2)^2 + 2r^2(1-\cos(t)).$$

Therefore:

$$\frac{|1-r^2e^{it}|^2}{(1-r^2)^2} = 1 + \frac{2r^2(1-\cos(t))}{(1-r^2)^2}.$$

Replacing the last two results to (59) we obtain:

$$\begin{aligned} I_L(r) &= \int_{-\pi}^{\pi} \frac{1}{\left[\left(1+\frac{2r^2(1-\cos(t))}{(1-r^2)^2}\right)^L - 1\right]} \frac{2(1-\cos(t))}{(1-r^2)^2 + 2r^2(1-\cos(t))} dt \\ &= \frac{2}{r^2} \int_0^{\pi} \frac{1}{\left[\left(1+\frac{2r^2(1-\cos(t))}{(1-r^2)^2}\right)^L - 1\right]} \frac{\frac{2r^2(1-\cos(t))}{(1-r^2)^2}}{1+\frac{2r^2(1-\cos(t))}{(1-r^2)^2}} dt. \end{aligned} \tag{60}$$

Now, we do the following change of variables:

$$x = \frac{2r^2(1-\cos(t))}{(1-r^2)^2}.$$

Thus, we have:

$$t = \arccos\left(1 - \frac{(1-r^2)^2}{2r^2}x\right),$$

and

$$dt = \frac{(1-r^2)^2}{2r^2} \frac{dx}{\sqrt{\frac{(1-r^2)^2}{r^2}x - \frac{(1-r^2)^4}{4r^4}x^2}} = \frac{(1-r^2)}{2r\sqrt{x}} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}},$$

Going back to (60) we obtain:

$$I_L(r) = \frac{1-r^2}{r^3} \int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L - 1} \frac{\sqrt{x}}{1+x} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}}. \tag{61}$$

Denote

$$J_L(r) = \int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L - 1} \frac{\sqrt{x}}{1+x} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}}.$$

For any x fixed the integrand tends to 0 as L goes to ∞. As L grows, the main contribution to the integral comes from the x near 0, where the term $\frac{1}{(1+x)^L-1}$ is "big". For those x we have $\frac{1}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} \sim 1$. Thus, we write:

$$J_L(r) = \int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}dx \left\{ 1 + \frac{\int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\left(\frac{1}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}}-1\right)dx}{\int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}dx} \right\}. \tag{62}$$

First we want to see that

$$\lim_{L\to+\infty} \frac{\int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\left(\frac{1}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}}-1\right)dx}{\int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}dx} = 0.$$

Take $\epsilon(L) \in (0,1)$ such that $\lim_{L\to+\infty} \epsilon(L) = 0$; the numerator of the last expression can be split into two parts (denoted by J_{1_L} and J_{2_L} respectively). The first one is where x is small:

$$J_{1_L} = \int_0^{\epsilon(L)\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\left(\frac{1}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}}-1\right)dx. \tag{63}$$

Define $t = \frac{(1-r^2)^2}{4r^2}x$. Then, by Taylor's formula, for $t \in (0, \epsilon(L))$,

$$\frac{1}{\sqrt{1-t}} = 1 + \frac{1}{2}t + O(t^2) \le 1 + t.$$

Thus:

$$\frac{1}{\sqrt{1-t}} - 1 \le t \le \epsilon(L).$$

Therefore, from (63), we obtain

$$J_{1_L} \le \epsilon(L)\left(\int_0^{\epsilon(L)\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}dx\right),$$

and

$$\lim_{L\to+\infty} \frac{J_{1_L}}{\int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}dx} = 0.$$

The second part of the integral is

$$J_{2_L} = \int_{\epsilon(L)\frac{4r^2}{(1-r^2)^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\left(\frac{1}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}}-1\right)dx. \tag{64}$$

Doing the change of variables proposed above, $t = \frac{(1-r^2)^2}{4r^2}x$, we get

$$J_{2_L} = \frac{4r^2}{(1-r^2)^2}\int_{\epsilon(L)}^{1} \frac{1}{(1+\frac{4r^2}{(1-r^2)^2}t)^L - 1}\frac{\frac{2r}{1-r^2}\sqrt{t}}{1+\frac{4r^2}{(1-r^2)^2}t}\Big(\frac{1}{\sqrt{1-t}} - 1\Big)dt.$$

We split the integral into two parts again. It is clear that the t near 1 are not a problem:

$$\lim_{L\to+\infty}\int_{1/2}^{1} \frac{1}{(1+\frac{4r^2}{(1-r^2)^2}t)^L - 1}\frac{\frac{2r}{1-r^2}\sqrt{t}}{1+\frac{4r^2}{(1-r^2)^2}t}\Big(\frac{1}{\sqrt{1-t}} - 1\Big)dt = 0.$$

On the other hand,

$$\begin{aligned}\int_{\epsilon(L)}^{1/2} &\frac{1}{(1+\frac{4r^2}{(1-r^2)^2}t)^L - 1}\frac{\frac{2r}{1-r^2}\sqrt{t}}{1+\frac{4r^2}{(1-r^2)^2}t}\Big(\frac{1}{\sqrt{1-t}} - 1\Big)dt \leq \\ \leq \int_{\epsilon(L)}^{1/2} &\frac{1}{(1+\frac{4r^2}{(1-r^2)^2}t)^L - 1}\frac{\frac{2r}{1-r^2}\sqrt{t}}{1+\frac{4r^2}{(1-r^2)^2}t}\frac{1}{\sqrt{1-t}}dt. \end{aligned} \tag{65}$$

Take

$$\epsilon(L) = \frac{(1-r^2)^2}{4r^2}\frac{1}{\log(L)}.$$

For L large enough, $\epsilon(L) \in (0,1)$ and following the equation (65):

$$\begin{aligned} &\leq \int_{\epsilon(L)}^{1/2} \frac{1}{(1+\frac{1}{\log L})^L - 1}\frac{1}{1+\frac{1}{\log L}}\frac{1}{\sqrt{1-\frac{4r^2}{(1-r^2)^2}\frac{1}{\log L}}}dt \\ &\leq \frac{1}{(1+\frac{1}{\log L})^{\frac{L}{\log L}\log L} - 1}\frac{1}{\sqrt{\log L}}\frac{1}{1+\frac{1}{\log L}} \\ &\leq \lim_{L\to+\infty}\frac{1}{(1+\frac{1}{\log L})^{\log L\frac{L}{\log L}} - 1} \approx \frac{1}{e^{\frac{L}{\log L}}}. \end{aligned} \tag{66}$$

We shall see later that

$$\int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L - 1}\frac{\sqrt{x}}{1+x}dx = O(L^{-3/2}) \tag{67}$$

and therefore

$$\lim_{L\to+\infty}\frac{J_{2_L}}{\int_0^{\frac{4r^2}{(1-r^2)^2}}\frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}dx} = 0. \tag{68}$$

At this point, we have seen that the expression in (62) is:

$$J_L(r) = \int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L - 1}\frac{\sqrt{x}}{1+x}dx(1+o(1)).$$

Going back to (61), we have:

$$I_L(r) = \frac{1-r^2}{r^3}\int_0^\infty \frac{1}{(1+x)^L - 1}\frac{\sqrt{x}}{1+x}dx(1+o(1))$$
$$= \frac{1-r^2}{r^3}\int_0^\infty \frac{1}{(1+x)^L\Big(1-\frac{1}{(1+x)^L}\Big)}\frac{\sqrt{x}}{1+x}dx(1+o(1)). \quad (69)$$

Making a new change of variables $y = \frac{1}{1+x}$ we obtain:

$$I_L(r) = \frac{1-r^2}{r^3}\int_0^1 \frac{y^L}{1-y^L}\sqrt{\frac{1-y}{y}}\frac{dy}{y}(1+o(1)) = \frac{1-r^2}{r^3}\int_0^1 \frac{y^{L-3/2}}{1-y^L}(1-y)^{1/2}dy(1+o(1))$$
$$= \frac{1-r^2}{r^3}\sum_{n=0}^{\infty}\int_0^1 y^{Ln+L-3/2}(1-y)^{1/2}dy(1+o(1)). \quad (70)$$

Using the second and the fourth property of the gamma function (see section 3.3), the non-integer value $\Gamma(1/2) = \sqrt{\pi}$ and the Riemann zeta function's definition, we get:

$$I_L(r) = \frac{1-r^2}{r^3}\sum_{n=0}^{\infty}\frac{\Gamma(Ln+L-1/2)\Gamma(3/2)}{\Gamma(Ln+L+1)}(1+o(1))$$
$$= \frac{(1-r^2)\sqrt{\pi}}{2r^3}\sum_{n=0}^{\infty}\frac{\Gamma(Ln+L-1/2)}{\Gamma(Ln+L+1)}(1+o(1)) = \frac{(1-r^2)\sqrt{\pi}}{2r^3}\sum_{n=0}^{\infty}\frac{1}{(Ln+L)^{3/2}}(1+o(1))$$
$$= \frac{(1-r^2)\sqrt{\pi}}{2r^3L^{3/2}}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}(1+o(1)) = \frac{(1-r^2)\sqrt{\pi}}{2r^3L^{3/2}}\zeta(3/2)(1+o(1)). \quad (71)$$

From (5), we can see:

$$\mathrm{Var}[n_L(r)] = \frac{L^2r^4}{2\pi(1-r^2)^2}I_L(r).$$

Then, replacing (70) above we get the result. □

Remarks.

1. Remember that in the Poisson processes, for the hyperbolic case,

$$\mathrm{Var}(n_L(r)) = \frac{Lr^2}{1-r^2}.$$

We can compare it to the result just obtained. In this case,

$$\mathrm{Var}(n_L(r)) = c\,\frac{\sqrt{L}r}{1-r^2} + o(\sqrt{L}).$$

We verify the same that happened to the planar case seen in the final of Chapter 3. Points corresponding to a Poisson process are more dispersed than the corresponding to the zeros of a hyperbolic Gaussian analytic function.

2. We can also study the expression (56) from different points of view. For example, we can fix the parameter L and see the behaviour of $n_L(r)$ when $r \to 1^-$. In this way, three different results are obtained depending on the value of L with $r \to 1^-$.

Theorem 5.4. [Buc13b, Proposition 7] Let f be the hyperbolic Gaussian analytic function as in (27). Then:

(a) For each fixed $L < 1/2$,

$$\mathrm{Var}[n_L(r)] = \frac{c_L}{(1-r)^{2-2L}}(1+o(1)), \quad \text{as } r \to 1^-,$$

where

$$c_L = \frac{L^2}{4\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}-L)}{\Gamma(1-L)}.$$

(b) For $L = 1/2$,

$$\mathrm{Var}[n_{1/2}(r)] = \frac{1}{8\pi}\frac{1}{1-r}\log\frac{1}{1-r}(1+o(1)), \quad \text{as } r \to 1^-.$$

(c) For each fixed $L > 1/2$,

$$\mathrm{Var}[n_L(r)] = \frac{c_L}{1-r}(1+o(1)), \quad \text{as } r \to 1^-,$$

where

$$c_L = \frac{L^2}{2\pi}\int_0^\infty \frac{1}{(1+x^2)^L - 1}\frac{x^2}{1+x^2}dx = \frac{L^2}{8\sqrt{\pi}}\sum_{n=1}^\infty \frac{\Gamma(Ln-1/2)}{\Gamma(Ln+1)}.$$

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