Master in Actuarial and Financial Sciences

MASTER THESIS

Title: Systemic risk in financial systems: an axiomatic approach

Author: Ana Martínez Alcaraz

Advisor: Pedro Calleja Cortés

Academic year: 2018-2019

UNIVERSITAT DE BARCELONA
Facultat d'Economia i Empresa

Màster de Ciències Actuarials i Financeres

Faculty of Economics and Business
Universitat de Barcelona

Master thesis
Master in Actuarial and Financial Sciences

Systemic risk in financial systems: an axiomatic approach

Author: Ana Martínez Alcaraz

Advisor: Pedro Calleja Cortés

The content of this document is the sole responsibility of the author, who declares that she has not incurred in plagiarism and that all reference to other authors have been expressed in the text.

Abstract: Eisenberg and Noe (2001) define a financial network where the players have claims against each other. In this system it is possible that one or several players do not have enough money to pay all their debts and default, being their total payment smaller than the total amount of their claims. Under the properties of Limited liability and Absolute priority and the bankruptcy rule of Proportionality, they prove that there exists a unique payment matrix if the system is regular. The aim of this paper to study whether these three properties are compatible or not with non-manipulability properties. In particular, we show that although agents may have incentives to split, they do not have incentives to merge.

Key words: financial systems, default, proportionality, splitting, merging.

Index

1. Introduction

On September 15th, 2008, the financial institution Lehman Brothers declared officially its bankruptcy. This had a cost of 22 trillion of dollars for the US economy and was considered by the analysts one of the most important events that triggered the global crisis, provoking a sort of tsunami whose effects we still suffer nowadays.

Although it is common knowledge, the Lehman Brothers' bankruptcy brought to light that institutions are not isolated, in fact they are connected and the financial health of one institution does not depend only of itself, but to a greater or lesser extent, depends of the other institutions of the system, with which it makes transactions.

Counterparty risk, the likelihood that one of those involved in a transaction might default on its contractual obligation, is a hazard that firms and institutions face every day. This risk can exist in investment, trading transactions and credit and its treatment and coverage are part of a good risk management. In fact, Basel III (Basel Committee on Banking Supervision, 2017), the internationally agreed set of measures to apply in the banking system, include specific capital requirements to mitigate the counterparty credit risk. Similar requirements are included in Solvency II (see Directive 2009/138/EC of the European Parament), the European Directive that harmonizes the insurance regulation in Europe.

As a regulator the most important concern must be to ensure that the default of one or more agents of the system does not trigger a chain of defaults that concludes in the failure of the whole system, but also it is important to define the way in which the creditors of a defaulting institution will be compensated. There exist several bankruptcy rules: all the creditors will be payed equally (constrained equal awards), all the creditors will not perceive the same amount of their debt (constrained equal losses) or all the creditors will perceive a proportional part of their debt (proportionality) are some examples.

From this point of view, it could be also interesting to know which desirable properties hold under certain bankruptcy rules. In fact, Eisenberg and Noe (2001), proved that under the proportional rule, if it is imposed that the firms of the system are not able to pay more money than they have (Limited liability) and also that if any of them is not able to cover entirely all its liabilities then it will destinate all its resources to the payment to its creditors (Absolute priority), there only exists one way to clear the debts in the system, a unique clearing payment matrix (there are certain characteristics that the system must be satisfied, though).

Based on this paper, we try to study other desirable properties while holding Limited liability and Absolute priority. In fact, we are interested in the agents of the system not having incentives to merge nor to split. The results are interesting since, as far as we know, there are only two other works that study financial systems from an axiomatic point of view: one study the situation with indivisibilities (Csóka and Herings, 2018) while the other works with perfectly divisible goods, as money (Csóka and Herings, 2017). Additionally, we check if the agents have incentives to compensate the bilateral liabilities between them before initiate the process of the payment of the debts.

Also, a simpler version of the algorithm developed by Eisenberg and Noe to find the unique payment matrix is presented.

The structure of the work goes as follows: in section 2, the model and its characteristics are introduced, with some examples. In section 3, the algorithm to find the unique payment matrix is presented and implemented in an example. In section 4, new properties are imposed and it is proved whether they hold under the properties defined by Eisenberg and Noe. Finally, in section 5, we present some conclusions. In the Appendix, we include the R code of the algorithm as well as the code used to solve the example in Section 3.

2. The model

In a closed financial system several economic entities, firms, institutions, participants or simply agents participate. These words will be used interchangeably throughout this work. $\mathbb{N}$ denotes the set of all potential agents and $\mathcal{N} = \{N \subseteq \mathbb{N}: |N| < \infty\}$ denotes the set of possible groups of agents. By $N = \{1,2,\dots,n\} \in \mathcal{N}$ we obtain the set of agents in the system. In this first section, we will explain the model presented in Eisenberg and Noe (2001).

Due to their economic interactions or their past transactions, the agents may have mutual liabilities or claims. We can represent all liabilities in the system by a $n \times n$ matrix L, where $\forall i, j \in N$ $l_{ij} \geq 0$ represents the liability of agent i to agent j or equivalently the claim of agent j against agent i, and $l_{ii} = 0$ $\forall i \in N$.

Finally, by $z_i \geq 0$ we will denote the exogenous operating cash flow or cash level of agent $i \in N$ that might be used to pay its debts to its creditors. By $z = (z_1, z_2, \dots, z_n)$ we denote the vector of operating cash flows.

Although operating cash flows are assumed to be nonnegative, this condition is not restrictive. Due to operational costs, as paying salaries, the costs of $i \in N$ could exceed its revenues. In order to represent this costs, Eisenberg and Noe proposed the inclusion of a "sink node", an agent 0, whose $z_0 = 0$ and $l_{0j} = 0$ $\forall j \in N$, being the claims of the sink node, l_{i0}, the representation of operating cost $\forall i \in N$. Since nothing in their setup precludes a node with these characteristics, the assumption of nonnegative z_i is made without loss of generality.

So, a financial system is represented by the triplet (N, z, L) with L of the form:

$$L = \begin{pmatrix} 0 & l_{12} & \cdots & l_{1n} \\ l_{21} & 0 & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & 0 \end{pmatrix}.$$

By Γ we define the set of all financial systems.

A financial system it is indeed a network that can be represented by a directed graph whose nodes are the institutions of the financial systems and there is a directed edge from i to j $(i \to j)$ whenever $l_{ij} > 0$.

For any given financial system (N, z, L) we can define the value of the equity of agent $i \in N$ by

$$E_i(N, z, L) = z_i + \sum_{j \in N} l_{ji} - \sum_{j \in N} l_{ij}.$$

$E_i(N, z, L)$ represents the available amount of money of the institution $i \in N$ if all the liabilities in matrix L are fully satisfied.

If $E_i(N, z, L) \geq 0 \quad \forall i \in N$ then the system is healthy, not in risk, since satisfying all its obligations does not lead to a negative equity value for any firm and, thus, all debts can be fully cancelled.

However, if there exists $i \in N$ such that $E_i(N, z, L) < 0$ it means that the debts of institution i exceed its assets, making impossible to compensate entirely to all creditors, provoking the default of firm i.

Example 1. Consider the financial system $(N, z, L) \in \Gamma$ with three firms $N = \{1,2,3\}$ and operating cash flows and liabilities presented in z and L.

$$z = (8,0,10),$$
$$L = \begin{pmatrix} 0 & 15 & 30 \\ 0 & 0 & 20 \\ 5 & 5 & 0 \end{pmatrix}.$$

In this example we can see that firms 1 and 3 have liabilities with the other two firms while firm 2 only has a liability with firm 3. The network can be represented by the next graph in figure 1:

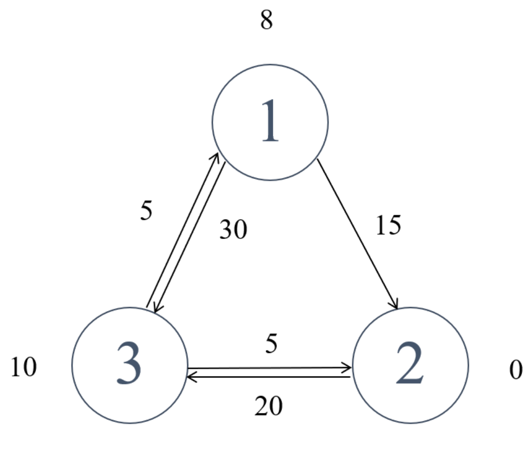

Figure 1: graph Example 1

Source: own elaboration

Then,

$$E_1(N,z,L) = 8 + 5 - (15 + 30) = -32$$
$$E_2(N,z,L) = 0 + (15 + 5) - (20) = 0$$
$$E_3(N,z,L) = 10 + (30 + 20) - (5 + 5) = 50.$$

Institution 1 defaults. Given this situation, a question arises: could the default of firm 1 lead to other firms' default? In fact, although firm 1 dedicate all its resources $(8 + 5)$ to cover its debt with firm 2, this last firm will also default, since $0 + (13 + 5) < 20$. By this example, we can see that an institution defaulting can provoke the default of other institutions in the financial system, incurring in systemic risk.

The main and fundamental question is about how much of the liabilities should be satisfied to clear the system and avoid its collapse.

We will search for ways of selecting some payoff proposal, a specification of how much of a liability l_{ij} should be paid for all $i, j \in N$. We will impose natural lower and upper bounds requirements. Those payoffs should be seen as a recommendation for the problem.

Formally, a clearing payment matrix assigns to every financial system $(N,z,L) \in \Gamma$ a $n \times n$ clearing matrix of payoffs. That clearing matrix is a function $P: \Gamma \to \mathcal{M}_{n\times n}$ where $\mathcal{M}_{n\times n}$ is the space of $n \times n$ matrix with $0 \le p_{ij}(N,z,L) \le l_{ij}\ \forall i,j \in N$. This condition simply imposes natural lower and upper bounds requirements. Note that implies $p_{ii}(N,z,L) = 0\ \forall i \in N$. Let $(N,z,L) \in \Gamma$ and $P: \Gamma \to \mathcal{M}_{n\times n}$, for simplicity of notation we will use $P(N,z,L) = (p_{ij})_{\substack{i\in N\\ j\in N}} = P$.

$P(N,z,L) = P$ is of the form

$$P = \begin{pmatrix} 0 & p_{12} & \cdots & p_{1n} \\ p_{21} & 0 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & 0 \end{pmatrix}.$$

Following Eisenberg and Noe, we impose that such solution or clearing matrix should be consistent with several principles or properties that at the same time should capture the ideas underlying the legal rules in case of bankruptcy. To define those principles or axioms let us introduce some notation. An important value is the value of equity of the agents under the clearing payment matrix $P(N,z,L) = P$:

$$E_i(N,z,P) = z_i + \sum_{j\in N} p_{ji} - \sum_{j\in N} p_{ij}.$$

The first property that Eisenberg and Noe impose is ***Limited liability*** (*LL*), which requires that the total payments made by an institution must never exceed the assets available of the institution. Formally, a clearing matrix proposal, $P(N,z,L) = P$, satisfies *LL* if for all $(N,z,L) \in \Gamma$ and all $i \in N$

$$z_i + \sum_{j\in N} p_{ji} \geq \sum_{j\in N} p_{ij}.$$

Note that this condition is equivalent to, $E_i(N,z,P) \geq 0\ \forall i \in N$.

The second property also defined by Eisenberg and Noe is ***Absolute priority*** (*AP*) of debts over equity which requires that stakeholders of an institution receives no value until the institution have payed entirely its outstanding liabilities. A clearing matrix $P(N,z,L) = P$ satisfies *AP* if for all (N,z,L) and for all $i \in N$ either $p_{ij} = l_{ij}\ \forall j \in N$ or

$$z_i + \sum_{j\in N} p_{ji} = \sum_{j\in N} p_{ij}.$$

Then, under *LL* and *AP*, $E_i(N,z,P) \geq 0\ \forall i \in N$ and it is strictly positive if and only if institution i have honored fully its obligations with the other firms in the financial system.

Finally, Eisenberg and Noe impose the principle of ***Proportionality*** (*PROP*) that requires that, if default occurs for $i \in N$, then all claimants of i are paid proportionally to their claims. Proportionality is a principle that is mostly applied in bankruptcy rules nowadays (see Aristotle, 1985). Egalitarianism and some forms of priorities are also principles with long roots in the past. However, we leave the study of these ideas for future work.

For simple claim problems (for a survey see Thomson, 2002) in which a single firm bankrupts having liabilities to others that in total exceed its estate (or operating cash flows), the proportional rule has been extensively studied (see for instance de Frutos, 1999 and Moreno-Ternero, 2006).

Formally, a clearing matrix or solution $P(N,z,l) = P$ satisfies *PROP* if for all (N,z,L) and for all $i \in N$

$$p_{ij} = \frac{l_{ij}}{\bar{l}_{i.}}\left(\sum_{j\in N} p_{ij}\right) = \Pi_{ij}(L)\left(\sum_{j\in N} p_{ij}\right) \quad \forall j \in N$$

Each $\Pi_{ij}(L)$ stands for the proportion of the total debt $\bar{l}_{i.} = \sum_{j \in N} l_{ij}$ that represents l_{ij} $\forall i \in N$, in case $\bar{l}_{i.} = 0$ we simply define $\Pi_{ij} = 0$ $\forall i, j \in N$, since firm i has no debts with others. Hence$\sum_{j \in N} \Pi_{ij}$ $\forall i \in N$, except if $\bar{l}_{i.} = 0$. Moreover, we can construct the matrix $\Pi(L)$ by setting

$$\Pi(\mathrm{L}) = \begin{pmatrix} 0 & \Pi_{12} & \cdots & \Pi_{1n} \\ \Pi_{21} & 0 & \cdots & \Pi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_{n1} & \Pi_{n2} & \ldots & 0 \end{pmatrix}.$$

For simplicity, we define for all $(N, z, L) \in \Gamma$ and a clearing matrix $P(N, z, L) = P$ the clearing vector $\bar{p} = (\bar{p}_{1.}, \bar{p}_{2}., \ldots, \bar{p}_{n.})$ where

$$\bar{p}_{i.} = \sum_{j \in N} p_{ij} \quad \forall i \in N$$

represents the total amount payed by i to the rest of agents according to matrix P.

3. The clearing payment matrix

3.1. Existence and uniqueness

For a financial system $(N, z, L) \in \Gamma$, Eisenberg and Noe and Groote, Reijnierse and Borm (2018) reached the same conclusion on the existence of a clearing payment matrix satisfying *LL*, *AP* and *PROP*, following different approaches.

Theorem 1 (Eisenberg and Noe, 2011): Let $(N, z, L) \in \Gamma$. There exists a clearing payment matrix *P* that satisfies *LL*, *AP* and *PROP*. Moreover, if $P'(N, z, L) = P'$ and $P''(N, z, L) = P''$ are two different clearing payment matrices proposals then the value of equity under P' and P'' is the same, $\forall\, i \in N$. That is

$$E_i(N, z, P') = E_i(N, z, P'') \;\; \forall i \in N$$

Eisenberg and Noe show that under some mild conditions the clearing payment matrix *P* that satisfies *LL*, *AP* and *PROP* is moreover unique. We now introduce the necessary definitions to present these conditions.

The conditions are imposed on the graph. $S \subseteq N$ is a surplus set of N if no agent in S has any liability to agents outside S, i.e. $l_{ij} = 0 \ \forall i \in S$ and $\forall j \notin S$ and agents in S together has positive operating cash flows, i.e.$\sum_{i \in S} z_i > 0$.

For each node of the system $i \in N$ the risk orbit of i, denoted by $o(i)$, is:

$$o(i) = \{j \in N\text{: there exist a directed path from i to j}\}$$

$o(i)$ can be interpreted as the set of institutions that may suffer the effects of i not facing all its obligations.

A financial system is *regular* if $\forall i \in N$, it holds that $o(i)$ is a surplus set.

To illustrate the necessity to study regular systems if we want uniqueness we present an example:

Example 2. Suppose a financial system with $N = \{1,2\}$ and $z = (0,0)$ and firms have liabilities of 1 to each other. Hence,

$$\mathrm{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The risk orbits of the system are $o(1) = o(2) = \{1,2\}$. Clearly $\{1,2\}$ is not a surplus set because $z_1 + z_2 = 0$. This implies that the financial system is not regular and, moreover, there is not a unique clearing matrix that satisfies *LL*, *AP* and *PROP*. In fact, there are infinite solutions that clear the system of the form

$$P^t = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix},$$

where $t \in [0,1]$ and the value of equity is for all different alternatives:

$$E_1(N, z, P^t) = E_2(N, z, P^t) = 0 \quad \forall t \in [0,1].$$

<u>**Theorem 2**</u>: There is a unique clearing matrix P satisfying *LL*, *AP* and *PROP*.

Next, we introduce and explain an algorithm to compute such unique clearing matrix P. This algorithm is very similar, to the sequence of defaults presented in Eisenberg and Noe. Their proof (Lemma 3) that their algorithm provides the unique clearing matrix P under *LL*, *AP* and *PROP* can also be used for the algorithm we present here.

3.2. The algorithm

In the following, on the class of regular financial systems we denote by $P^*(N,z,L) = P^*$ the clearing matrix proposal satisfying *LL*, *AP* and *PROP* and $\bar{p}^*$ will denote the associated clearing payment vector.

Our algorithm is similar, but simpler and slightly different from the Eisenberg and Noe's algorithm. Additionally, we have developed an R code implementing our algorithm. The code is included in the Appendix.

The algorithm works as follows:

Let $(N,z,L) \in \Gamma$. First, we will determine the set of firms that default if we assume *LL*, *AP* and *PROP*. By $D(L) \subset N$ we will denote the firms that default. To this aim, let us first introduce the proportion of liabilities that a firm $i \in N$ can pay in case it receives all its claims against the rest of firms, that is $\forall i \in N$ define:

$$\delta_i(L) = \begin{cases} \dfrac{z_i + \underline{l}_i}{\bar{l}_{i.}} & if\ E_i(N,z,L) < 0 \quad (z_i + \underline{l}_{\cdot i} < \bar{l}_{i\cdot}) \\ 1 & \text{otherwise} \quad (z_i + \underline{l}_{\cdot i} \geq \bar{l}_{i\cdot}) \end{cases}.$$

where $\underline{l}_i = \sum_{j \in N} l_{ji}$. Note that $1 \geq \delta_i(L) \geq 0\ \forall i \in N$.

We first assume that every $i \in N$ fully pays its debts to other firms according to L. Under this assumption either (a) no firm defaults, that is, $E_i(N,z,L) \geq 0\ \forall i \in N$ or (b) some firms default, $\exists\ i \in N$ such that $E_i(N,z,L) < 0$. If (a) by *AP*, all firms completely satisfy its obligations with other firms, thus, the unique clearing payment matrix is determined by setting $P^*(N,z,L) = L$ and the unique clearing payment vector is $\bar{p}^* = (\bar{l}_{1.}, \bar{l}_{2.}, \dots, \bar{l}_{n.})$ and our financial system is not in risk. If (b), define $D_1(L) = \{i \in N : E_i(N,z,L) < 0\}$ to be the first-order defaulting firms, such firms will be unable to satisfy all its obligations with the rest of firms, although the other firms pay entirely their debts. By *LL*, *AP* and *PROP* a firm $i \in D_1(L)$ will pay the proportion $\delta_i(L)$ of l_{ij} to all $j \in N$ which will be possible only if l_{ji} are all fully covered $\forall j \in N$.

Accordingly define the $n \times n$ matrix L^1 of liabilities by

$$l_{ij}^1 = \begin{cases} l_{ij} & \forall j \in N \quad if\ i \notin D_1(L) \\ \delta_i(L) \cdot l_{ij} & \forall j \in N \quad if\ i \in D_1(L) \end{cases}.$$

Note that for all $i \in D_1(L)$,

$$E_i(N,z,L^1) = z_i + \sum_{j\in N} l_{ji}^1 - \sum_{j\in N} l_{ij}^1 = z_i + \sum_{j\in D_1} \delta_i(L)\cdot l_{ji} + \sum_{j\in N\setminus D_1} l_{ij} - \sum_{j\in N} \delta_i(L)\cdot l_{ij} \leq$$
$$\leq z_i + \sum_{j\in N} l_{ji} - \sum_{j\in N} \delta_i(L)\cdot l_{ij} = 0.$$

where the last equality follows from the definition of $\delta_i(L)$.

On the other hand, for any $i \in N\setminus D_1(L)$

$$E_i(N,z,L^1) = z_i + \sum_{j\in N} l_{ji}^1 - \sum_{j\in N} l_{ij}^1 = z_i + \sum_{j\in D_1(L)} \delta_i(L)\cdot l_{ji} + \sum_{j\in N\setminus D_1(L)} l_{ji} - \sum_{j\in N} l_{ij}$$
$$\leq E_i(N,z,L).$$

Consequently, there might be a firm $i \in N\setminus D_1(L)$ that does not default in first-order that can default in succesive orders. Define $D_2(L) = \{i \in N\setminus D_1(L) : E_i(N,z,L^1) < 0\}$ to be the second-order defaulting firms, that do not default according to matrix L, but they do according to matrix L^1.

If $D_2 = \emptyset$ stop. There are not defaulting firms in second-order and the set of firms that default under *LL*, *AP* and *PROP* is $D(L) = D_1(L)$. If $D_2(L) \neq \emptyset$, by *LL*, *AP* and *PROP* every $i \in D_2(L)$ will only pay $\delta_i^1(L)$ of their liabilities according to L if it receives all its claims also according to L^1. $\delta_i^1(L)$ is of the form

$$\delta_i^1(L) = \begin{cases} \dfrac{z_i + \underline{l}_i^1}{\bar{l}_{i.}} & if\ i \in D_1(L) \cup D_2(L) \quad (z_i + \underline{l}^1{\cdot}_i < \bar{l}_i.) \\ 1 & if\ i \notin D_1(L) \cup D_2(L) \quad (z_i + \underline{l}^1{\cdot}_i \geq \bar{l}_i.) \end{cases}.$$

We will then define L^2 accordingly and proceed as before, defining l_{ij}^2:

$$l_{ij}^2 = \begin{cases} l_{ij} & \forall j \in N \quad if\ i \notin D_1(L) \cup D_2(L) \\ \delta_i^1(L)\cdot l_{ij} & \forall j \in N \quad if\ i \in D_1(L) \cup D_2(L) \end{cases}.$$

Must be noted that $\forall i \in D_1(L)$, $\delta_i(L) \geq \delta_i^1(L)$ because $\underline{l}_i \geq \underline{l}_i{}^1$.

This process is iterated k times until $D_k(L) = \emptyset$. Since there are n firms this will occur in $k \leq n$ steps. Then, the firms that default under *LL*, *AP* and *PROP* are

$$D(L) = \bigcup_{t=1}^{k-1} D_t(L).$$

Note that the matrix L^k, if proposed as final payoffs, will satisfy *AP* and *PROP* but may not satisfy LL.

Secondly, and knowing $D(L)$, we construct a linear system of $|D(L)|$ equations, one for each defaulting firm in $D(L)$ with variables $\bar{p}_{i.}^{*}\ \forall i \in D(L)$.

By LL, *AP* and *PROP* $\forall i \in D(L)$ we construct an equation of the form

$$E_i(N, z, P^*) = z_i + \sum_{j \in N \setminus D(L)} l_{ji} + \sum_{j \in D(L)} \Pi_{ji}(L)\, \bar{p}_{j.}^{*} - \bar{p}_{i.}^{*} = 0.$$

By Eisenberg and Noe (2001), under regularity, this system will have a unique solution that will determine $\bar{p}_{i.}^{*}\ \forall i \in D(L)$. For every $i \in N \setminus D(L)$ we set $\bar{p}_{i.}^{*} = \bar{l}_{i.}^{*}$. It is worth to notice that if the financial system is not regular, and due to Theorem 1, then any solution of the system of equations will satisfy LL, *AP* and *PROP* but the equity value of all firms will be the same for every different solution. Once all $\bar{p}_{i.}^{*}\ \forall i \in N$ are computed, the algorithm finishes. Observe that $p_{ij}^{*}(N, z, L) = \bar{p}_{i.}^{*} \cdot \Pi_{ij}(L)$.

Next, to illustrate the proper functioning of the algorithm, we present and solve a simple example. Additionally, this same example is solved with R code and it is included in the Appendix.

Example 2. Consider the financial system $(N, z, L) \in \Gamma$ with four firms $N = \{1,2,3,4\}$ with operating cash flows and liabilities presented in z and L:

$$z = (0, 0, 20, 20),$$

$$L = \begin{pmatrix} 0 & 20 & 15 & 15 \\ 10 & 0 & 10 & 20 \\ 10 & 10 & 0 & 20 \\ 20 & 10 & 20 & 0 \end{pmatrix}.$$

We construct $\bar{l}_{i.}$, $\underline{l}_{i}$ and the matrix $\Pi(\mathrm{L})$:

$$\bar{l}_{i.} = (50, 40, 40, 50),$$

$$\underline{l}_{i} = (40, 40, 45, 55),$$

$$\Pi(\mathrm{L}) = \begin{pmatrix} 0 & 0.4 & 0.3 & 0.3 \\ 0.25 & 0 & 0.25 & 0.5 \\ 0.25 & 0.25 & 0 & 0.5 \\ 0.4 & 0.2 & 0.4 & 0 \end{pmatrix}.$$

We assume fully payment of all debts and compute the vector of equities:

$$E(N, z, L) = (-10, 0, 25, 25).$$

As we can see, firm 1 is the only firm that has negative value of equity and, thus, is the only defaulting firm in first order, so $D_1(L) = \{1\}$. Then,

$$\delta(L) = (0.8, 1, 1, 1).$$

To check if there are any second-order defaults, L^1 matrix is constructed

$$L^1 = \begin{pmatrix} 0 & 16 & 12 & 12 \\ 10 & 0 & 10 & 20 \\ 10 & 10 & 0 & 20 \\ 20 & 10 & 20 & 0 \end{pmatrix}.$$

Once again, we assume that the obligations according to this matrix are completely satisfied and construct the vector of values of equity:

$$E(N, z, L^1) = (0, -4, 22, 22).$$

We can see that the only second-order default is by firm 2, then $D_2(L) = \{2\}$. We construct $\delta^1(L)$ and with this vector, L^2 is defined

$$\delta^1(L) = (0.8, 0.9, 1, 1),$$
$$L^2 = \begin{pmatrix} 0 & 16 & 12 & 12 \\ 9 & 0 & 9 & 18 \\ 10 & 10 & 0 & 20 \\ 20 & 10 & 20 & 0 \end{pmatrix}.$$

Then

$$E(N, z, L^2) = (-1, 0, 21, 20).$$

We can observe that neither 3 or 4 see their value of equity reduced enough to not being able to pay all their debts, so they do not default in third-order, so $D_3(L) = \emptyset$ and $D(L) = D_1(L) \cup D_2(L) = \{1,2\}$. Now, we stop iterating and proceed with the next step: we have to solve the linear equation system below to obtain the unique an unknown value of $\bar{p}_{1.}^*$ and $\bar{p}_{2.}^*$. The system of equations to solve in this example is

$$(10+20)+0.25\bar{p}_{2.}^{*}-\bar{p}_{1.}^{*}=0,$$
$$(10+10)+0.4\bar{p}_{1.}^{*}-\bar{p}_{2.}^{*}=0,$$

and its solutions are $\bar{p}_{1}^{*}=38.89$ and $\bar{p}_{2}^{*}=35.56$. Additionally, since 3 and 4 do not default, $\bar{p}_{3.}^{*}=\bar{l}_{3.}=40$ and $\bar{p}_{4.}^{*}=\bar{l}_{4.}=50$. Therefore,

$$\bar{p}_{i.}{}^{*}=(38.89,35.56,40,50).$$

With the help of matrix $\Pi(\mathrm{L})$ we can construct the clearing payment matrix, $P^{*}(N,z,L)=P^{*}$:

$$P^{*}=\begin{pmatrix} 0 & 15.5556 & 11.6667 & 11,6667 \\ 8.8889 & 0 & 8.8889 & 17.7778 \\ 10 & 10 & 0 & 20 \\ 20 & 10 & 20 & 0 \end{pmatrix}.$$

Finally, we can obtain the equity value vector:

$$E(N,z,P^{*})=(0,0,20.56,19.44).$$

By looking at $E(N,z,P^{*})$, we check that the properties established by Eisenberg and Noe are hold by the clearing payment vector that we have obtained: 1 and 2 are defaulting firms, so by *AP*, their value of equity must be 0 while in the case of 3 and 4 can be (and they are) positive. Moreover, none of the values of equity are negative, respecting *LL* and no institution receives a higher payment than their debt, as we can see in the clearing payment matrix P^{*}. Furthermore, the payments defined by this matrix confirm that *PROP* is hold, since each p_{ij} is proportional to the claim that firm i owe to firm j. Finally, this financial system is regular ($o(1)=o(2)=o(3)=o(4)=\{1,2,3,4\}$), so we ensure that the clearing payment matrix obtained computing the algorithm is the only matrix that hold *LL*, *AP* and *PROP*.

4. Manipulation by merging and splitting

In the traditional claims' problems (see Thompson, 2002, for a survey) a single firm goes bankrupt and faces a number of claims of n creditors $(l_1,l_2,\dots,l_n)$ among whom an insufficient state E has to be distributed. This problem is first introduced by O'Neill (1982) and motivated by numerous fragments of the Talmud.

The proportional, to claims, rule, distributes E in proportion to the list of liabilities and plays the same role than the property of Proportionality in our context. Proportionality is indeed, the most common way to distribute E in case of bankruptcy, according to law. Moreover, is the most intuitive interpretation of justice or equity in justice (see Aristotle, 1985).

The Proportional rule for claims problems has been characterized by non-manipulability axioms as splitting-proofness (non-manipulability via splitting) and merging-proofness (non-manipulability via merging) in different works as de Frutos (1999) and Ju (2003), among others.

Other works, like Moreno-Ternero (2006) impose a stronger axiom of non-manipulability that puts together non-splitting and non-merging incentives. Csoka and Herings (2017) follows this last approach in the setting of financial systems. Here we pretend to study splitting-proofness and merging-proofness separately when combining it with *LL*, *AP* and the central idea of Proportionality.

Definition: We say that a clearing matrix $P: \Gamma \to \mathcal{M}_{n\times n}$ satisfies ***splitting-proofness*** (*SP)* if $\forall N', N \in \mathcal{N}$ and for all (N, z, L) and (N', z', L'), with $N' \subset N$, if there is $m \in N'$ such that

(i) $z'_m = z_m + \sum_{k\in N\setminus N'} z_k$,
(ii) $l'_{mj} = l_{mj} + \sum_{k\in N\setminus N'} l_{kj}$ for all $j \in N'$,
(iii) $l'_{jm} = l_{jm} + \sum_{k\in N\setminus N'} l_{jk}$ for all $j \in N'$,

while $\forall i, j \in N'\setminus\{m\}$, $l'_{ij} = l_{ij}$ and $z'_i = z_i$.

Then,

$$E_m(N', z', P') \geq E_m(N, z, P) + \sum_{k\in N\setminus N'} E_k(N, z, P)$$

where $P' = P(N', z', L')$ and $P = P(N, z, L)$.

For simplicity, in case of splitting, we call $M = (N\setminus N') \cup \{m\}$ the set of firms in (N, z, L) resulting of the splitting of m in (N', z', L'), including m. Then $M = |N| - |N'| + 1$.

In words, suppose a firm m splits in $|M|$ different firms satisfying

(i) the aggregate cash flows of the firms in M sum up the operating cash flow of m.
(ii) the liabilities to another firm j of all the firms in M sum up the same liability to j of firm m.
(iii) the debts of a firm j with m equal the sum of debts of j to the firms in M.

Then, the value of the equity of m should be larger or equal to the sum of the values of the equities of the splitting firms. If not, m would have incentives to split.

Contrary to the classical claims problems with a unique firm that bankrupts, we show that in a financial system *PROP* together with *AP* and *LL*, is no longer compatible with *SP*.

Surprisingly, the incompatibility is even stronger. In the next theorem we show that *LL*, *AP* and *SP* are incompatible.

Theorem 3: There is not clearing matrix that satisfies *LL*, *AP* and *SP*.

Proof. Let *P* be a clearing matrix satisfying *LL*, *AP* and *SP*.

Consider the financial system (N', z', L') with three firms $N' = \{1,2,3\}$ and operating cash flows and liabilities:

$$z' = (30, 10, 20),$$
$$L' = \begin{pmatrix} 0 & 10 & 30 \\ 10 & 0 & 10 \\ 5 & 25 & 0 \end{pmatrix}.$$

Since no firm defaults, by *LL* and *AP* we obtain the clearing payment matrix $P' = P(N', z', L')$:

$$P' = \begin{pmatrix} 0 & 10 & 30 \\ 10 & 0 & 10 \\ 5 & 25 & 0 \end{pmatrix}.$$

Let $m = 1$, so firm 1 will split. In concrete in two new firms: one will inherit all the obligations of m to others, the other one will, on the contrary, inherit all the obligations of the other firms with m.

Consider now the financial system (N, z, L) with $N = \{1,2,3,4\}$,

$$z = (30, 10, 20, 0),$$
$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 10 & 0 & 10 & 0 \\ 5 & 25 & 0 & 0 \\ 0 & 10 & 30 & 0 \end{pmatrix}.$$

It is easy to observe that firm 1 in (N', z', L') has splitted into firms 1 and 4 in (N, z, L), so $M = \{1,4\}$.

Note that new firm 4 defaults, and it is the only firm defaulting. Moreover, note that $z_4 + \underline{l}_4 = 0$. This is because, when firm 1 splits, it keeps the collection rights and the operating cash flow while it leaves all the debts in firm 4.

Again, by *LL* and *AP* we obtain that $P = P(N, z, L)$ is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 10 & 0 & 10 & 0 \\ 5 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that for firm 4 only *LL* and *AP* are needed to obtain the payments made by the firm $p_{4j} = 0\ \forall j \in N$ since $z_4 + \underline{l}_4 = 0$.

Easy computations lead to

$$E_1(N', z', P') = 30 + 15 - 40 = 5$$

$$E_1(N, z, P) = 30 + 15 - 0 = 45$$

$$E_4(N, z, P) = 0 + 0 - 0 = 0$$

Then $E_1(N', z', P') < E_1(N, z, P) + E_4(N, z, P)$, which is a contradiction with *SP*. ∎

This result is particularly interesting because it implies that if *LL* and *AP* are imposed, independently of which principle is employed when a firm bankrupt and has to pay its obligations to others, there might be institutions with incentives to split.

The result, indeed, captures the intrinsic idea that under *LL*, it is always better to split in two firms: the good one, the one that inherits the claims against others and the operating cash flow, and the bad one, the one that inherits all the debts to the other firms.

However, and this is more surprising, *LL*, *AP* and *PROP* are compatible with merging-proofness.

Definition: We say that a clearing matrix $P: \Gamma \to \mathcal{M}_{n\times n}$ satisfies ***merging-proofness*** (*MP*) if $\forall N', N \in \mathcal{N}$ that

(i) $z'_m = z_m + \sum_{k\in N\setminus N'} z_k$,

(ii) $l'_{mj} = l_{mj} + \sum_{k\in N\setminus N'} l_{kj}$ for all $j \in N'$,

(iii) $l'_{jm} = l_{jm} + \sum_{k\in N\setminus N'} l_{jk}$ for all $j \in N'$,

while $\forall i, j \in N'\setminus\{m\}$, $l'_{ij} = l_{ij}$ and $z'_i = z_i$.

Then,

$$E_m(N', z', P') \leq E_m(N, z, P) + \sum_{k\in N\setminus N'} E_k(N, z, P)$$

where $P' = P(N', z', L')$ and $P = P(N, z, L)$.

For simplicity, in case of merging, we call $M = (N \backslash N') \cup \{m\}$ the set of firms in (N', z', L') resulting of the mergin of m in (N, z, L), including m. Then $M = |N| - |N'| + 1$.

In words, suppose m merges with all firms in $M \backslash \{m\}$ satisfying

(i) the operating cash flow of m equal the sum of all the cash flows of the firms in M.

(ii) the liabilities of the new firm m to any j equals the sum of obligations to j of all firms in M.

(iii) the debts of j with m equals the sum of debts of j with all firms in M.

Then, the value of the equity of m should be smaller or equal to the sum of the values of equities of the merging firms. If not, the firms in M would have incentives to merge.

Next theorem shows that *LL*, *AP*, *PROP* and *MP* are compatible.

Theorem 4: Every clearing matrix $P: \Gamma \to \mathcal{M}_{n \times n}$ satisfying *LL*, *AP* and *PROP* also satisfies *MP*.

Lemma: Let $N', N \in \mathcal{N}$ and (N, z, L), (N', z', L') be such that $N' \subset N$ and there is $m \in N'$ such that

(i) $z'_m = z_m + \sum_{k \in N \backslash N'} z_k$,

(ii) $l'_{mj} = l_{mj} + \sum_{k \in N \backslash N'} l_{kj}$ for all $j \in N'$,

(iii) $l'_{jm} = l_{jm} + \sum_{k \in N \backslash N'} l_{jk}$ for all $j \in N'$,

and $l'_{ij} = l_{ij}$, $z'_i = z_i$ $\forall i, j \in N' \backslash \{m\}$.

Moreover, let $E_m(N', z', L') > 0$, being $P' = P'(N', z', L')$ under *LL*, *AP* and *PROP*. Then, by *LL*, *AP* and *PROP*, when applying the algorithm we have (i) $\delta_i^t(L) \leq \delta_i^t(L')$ for all $i \in N' \backslash \{m\}$ and all $t = \{1, \dots, k\}$ being k such that $D_k(L) = \emptyset$ and (ii) $D(L') \subseteq D(L)$.

Proof. For simplicity we will denote $M = N \backslash N' \cup \{m\}$ the set of merging firms. So, $N' \backslash \{m\} = N \backslash M$ is the set of firms that do not merge.

Let $i \in N' \backslash m$. We have:

(i) $\underline{l}'_i = \sum_{j \in N'} l'_{ji} = l'_{mi} + \sum_{j \in N' \backslash \{m\}} l'_{ji} = \sum_{j \in M} l_{ji} + \sum_{j \in N \backslash M} l_{ji} = \underline{l}_i$,

(ii) $\bar{l}'_{i.} = \sum_{j \in N'} l'_{ij} = l'_{im} + \sum_{j \in N' \backslash \{m\}} l'_{ij} = \sum_{j \in M} l_{ij} + \sum_{j \in N \backslash M} l_{ij} = \bar{l}_{i.}$,

(iii) $z'_i = z_i$

Consequently:

$E_i(N', z', L') = z'_i + \underline{l}'_i - \bar{l}'_{i.} = E_i(N, z, L)$ and $\delta_i(L') = \delta_i(L)$. Hence, $i \in D_1(L) \Leftrightarrow i \in D_1(L')$ for all $i \in N' \backslash \{m\}$ and $D_1(L') \subseteq D_1(L)$. On the other hand, note that $m \notin D_1(L')$ by hypothesis.

With the aim to compare $\delta_i^1(L')$ with $\delta_i^1(L)$ we first look at the relation between $\underline{l}_{.i}^1$ and $(\underline{l}')_{.i}^1$.

$$\left(\underline{l}'\right)_{.i}^1 = \sum_{j\in N'} (l')_{ji}^1 = (l')_{mi}^1 + \sum_{j\in N'\setminus\{m\}\cup D_1(L)} (l')_{ji}^1 + \sum_{j\in D_1(L')} (l')_{ji}^1 =$$

$$= l'_{mi} + \sum_{j\in N'\setminus\{m\}\cup D_1(L)} l'_{ji} + \sum_{j\in D_1(L')} \delta_j(L')\cdot l'_{ji} =$$

$$= \sum_{j\in M} l_{ji} + \sum_{j\in N\setminus M\cup D_1(L')} l_{ji} + \sum_{j\in D_1(L')} \delta_j(L)\cdot l_{ji} \geq$$

$$\geq \sum_{j\in M} \delta_j(L)\cdot l_{ji} + \sum_{j\in N\setminus M\cup D_1(L)} l_{ji} + \sum_{j\in D_1(L)} \delta_j(L)\cdot l_{ji} = \underline{l}_i^1 .$$

Here, the second equality follows from $m \notin D_1(L')$. The third equality follows from the definition of $(L')^1$. The fourth equality, from the fact that $\delta_j(L') = \delta_j(L)$ for all $j \in N'\setminus\{m\}$.

The inequality follows since $D_1(L') \leq D_1(L)$. In particular note that there might be players in M that belong to $D_1(L)$ but not to $D_1(L')$.

As consequence $\delta_i^1(L) \leq \delta_i^1(L') \forall i \in N'\setminus\{m\}$.

And, moreover, let $i \in N\setminus\{m\}$

$$E_i(N', z', (L')^1) = z'_i + (\underline{l}')_{.i}^1 - \left(\bar{l}'\right)_{.i}^1 \geq z_i + \underline{l}_i^1 - \bar{l}_{i.}^1 .$$

If $i \in D_1(L')$ then $i \in D_1(L)$ and so $i \in D_1(L') \cup D_2(L')$ and $i \in D_1(L) \cup D_2(L)$. If $i \notin D_1(L')$ then $i \notin D_1(L)$ and then $\left(\bar{l}'\right)_{.i}^1 = \bar{l}_{i.}^1 = \bar{l}_{i.}$ and, hence, $E_i(N', z', (L')^1) \geq E_i(N, z, L^1)$. And if $i \in D_2(L')$, then $i \in D_1(L) \cup D_2(L)$ (it might have defaulted in first-order in (N, z, L) and have defaulted in second-order in (N', z', L')).

Summarizing, $\delta_i^1(L) \leq \delta_i^1(L')\ \ \forall i \in N'\setminus\{m\}$ and

$$\bigcup_{r=1}^{2} D_r(L') \subseteq \bigcup_{r=1}^{2} D_r(L) .$$

<u>Induction hypothesis</u>: Let $t = \{2, \dots, k\}$ with $\delta_i^{t-1}(L) \leq \delta_i^{t-1}(L')$ for all $i \in N'\setminus\{m\}$ and $\bigcup_{r=1}^{t} D_r(L') \subseteq \bigcup_{r=1}^{t} D_r(L)$.

We shall show that $\delta_i^t(L) \leq \delta_i^t(L') \; \forall i \in N'\backslash m$ and $\bigcup_{r=1}^{t+1} D_r(L') \subseteq \bigcup_{r=1}^{t+1} D_r(L)$. We first look at the relation between $\left(\underline{l}'\right)_{.i}^{t}$ and $\underline{l}_i^t$ following similar argument to those to show $\left(\underline{l}'\right)_{.i}^{1} \geq \underline{l}_i^1$.

$$\left(\underline{l}'\right)_{.i}^{t} = l'_{mi} + \sum_{j \in N'\backslash\{m\} \bigcup_{r=1}^{t} D_r(L')} l'_{ji} + \sum_{j \in \bigcup_{r=1}^{t} D_r(L')} \delta_j^{t-1}(L') \cdot l'_{ji} \geq$$

$$\geq \sum_{j \in M} l_{ji} + \sum_{j \in N\backslash M \bigcup_{r=1}^{t} D_r(L')} l_{ij} + \sum_{j \in \bigcup_{r=1}^{t} D_r(L')} \delta_j^{t-1}(L) \cdot l_{ji} \geq$$

$$\geq \sum_{j \in M} \delta_j^{t-1}(L) \cdot l_{ji} + \sum_{j \in N\backslash M \bigcup_{r=1}^{t} D_r(L)} l_{ji} + \sum_{j \in \bigcup_{r=1}^{t} D_r(L)} \delta_j^{t-1}(L) \cdot l_{ji} = \underline{l}_i^t \,.$$

The second equality follows from $m \notin \bigcup_{r=1}^{t} D_r(L')$. The first inequality follows from the induction hypothesis $\delta_i^{t-1}(L) \leq \delta_i^{t-1}(L') \; \forall i \in N\backslash\{m\}$ and the second inequality holds by the induction hypothesis $\bigcup_{r=1}^{t} D_r(L') \subseteq \bigcup_{r=1}^{t} D_r(L)$. As a direct consequence $\delta_i^t(L) \leq \delta_i^t(L')$ for all $i \in N'\backslash\{m\}$.

Moreover, let $i \in N'\backslash\{m\}$

$$E_i(N', z', (L')^t) = z'_i + \left(\underline{l}'\right)_{.i}^{t} - \left(\bar{l}'\right)_{.i}^{t} \geq z_i + \underline{l}_i^t - \bar{l}_{i.}^t \,.$$

If $i \in \bigcup_{r=1}^{t} D_r(L')$ then $i \in \bigcup_{r=1}^{t} D_r(L)$ by induction hypothesis.

If $i \notin \bigcup_{r=1}^{t} D_r(L')$ and $i \notin \bigcup_{r=1}^{t} D_r(L')$ then $\left(\bar{l}'\right)_{.i}^{t} = \bar{l}_{i.}^t = \bar{l}_{i.}$. Hence, $E_i(N', z', (L')^t) \geq E_i(N, z, L^t)$.

Summarizing, for all $i \in N'\backslash\{m\}$, $\delta_i^t(L) \leq \delta_i^t(L') \;\; \forall i \in N'\backslash\{m\}$ and

$$\bigcup_{r=1}^{t+1} D_r(L') \subseteq \bigcup_{r=1}^{t+1} D_r(L) \,.$$

In consequence, $D(L') \subseteq D(L)$, which finishes the proof. ■

Using the Lemma as a tool we can now prove the next Theorem.

Theorem 5: Every clearing matrix $P: \Gamma \to \mathcal{M}_{n \times n}$ satisfying *LL*, *AP* and *PROP* also satisfies *MP*.

Proof. Let $N', N \in \mathcal{N}$ and (N, z, L), (N', z', L') be such that $N' \subset N$ and there is $m \in N'$ such that

(i) $z'_m = z_m + \sum_{k\in N\setminus N'} z_k$,
(ii) $l'_{mj} = l_{mj} + \sum_{k\in N\setminus N'} l_{kj}$ for all $j \in N'$,
(iii) $l'_{jm} = l_{jm} + \sum_{k\in N\setminus N'} l_{jk}$ for all $j \in N'$,

and $l'_{ij} = l_{ij}$, $z'_i = z_i$ $\forall i, j \in N'\setminus\{m\}$.

We shall show that if the clearing matrix satisfies *LL*, *AP* and *PROP* then if $P = P(N, z, L)$ and $P' = P'(N', z', L')$, it holds that

$$E_m(N', z', P') \leq E_m(N, z, P) + \sum_{k\in N\setminus N'} E_k(N, z, P)\ .\ (1)$$

Observe first that under *LL*, *AP* and *PROP* and by Theorem 1, we do not have to impose the systems (N, z, L) and (N', z', L') are regular.

Assume, on the contrary that $E_m(N', z', P') > E_m(N, z, P) + \sum_{k\in N\setminus N'} E_k(N, z, P)$. By *LL*, *AP* and *PROP*, all equity values are non-negative. Hence (1) will hold if and only if $E_m(N', z', P') > 0$ which can only happen if $m \notin D'(L)$.

It is easy to see that for (N, z, L) we have

$$\sum_{i\in N'} E_i(N, z, P) = \sum_{i\in N}\left(z_i + \sum_{j\in N} p_{ji} - \sum_{j\in N} p_{ij}\right) = \sum_{i\in N} z_i + \sum_{i\in N}\sum_{j\in N} p_{ji} - \sum_{i\in N}\sum_{j\in N} p_{ij} = \\ = \sum_{i\in N} z_i\ .$$

The same reasoning can be applied for (N', z', L').

Hence,

$$\sum_{k\in M} E_k(N, z, P) + \sum_{k\in N'\setminus\{m\}} E_k(N, z, P) = E_m(N', z', P') + \sum_{k\in N'\setminus\{m\}} E_k(N', z', P')$$

Then (1) holds if and only if

$$\sum_{k\in N'\setminus\{m\}} E_k(N, z, P) > \sum_{k\in N'\setminus\{m\}} E_k(N', z', P')\ .\ (2)$$

Let $k \in N'\backslash m$. In view of Lemma, $D_1(L') \subseteq D_1(L)$. We distinguish three cases:

(a) $k \in D(L')$, then $k \in D(L)$ and $E_k(N,z,P) = E_k(N',z',P') = 0$,
(b) $k \notin D(L')$ and $k \in D(L)$, then $E_k(N,z,P) = 0 < E_k(N',z',P')$,
(c) $k \notin D(L')$ and $k \notin D(L)$, then by *AP* and the definition of z' and L'

$$E_k(N',z',P') = z'_k + \underline{p}'_{.i} - \bar{l}'_{i.} = z_k + \underline{p}'_{.i} - \bar{l}_{i.}\ .$$

Moreover,

$$E_k(N,z,P) = z_k + \underline{p}_{.i} - \bar{l}_{i.}\ .$$

We obtain the values $\underline{p}'_{.i}$ and $\underline{p}_{.i}$ solving the system of equations for $D(L')$ and $D(L)$ as presented in the algorithm. Some easy algebra on the linear system of equations leads to

$$\underline{p}'_{.i} \geq \underline{p}_{.i}$$

and consequently $E_k(N',z',P') \geq E_k(N,z,P)$. Hence, from (a), (b) and (c) it follows that

$$\sum_{k \in N'\backslash m} E_k(N,z,P) \leq \sum_{k \in N'\backslash m} E_k(N',z',P')$$

Which contradicts (2) and consequently contradicts (1), which finishes the proof. ■

This shows that although the firms in M do not find incentives to merge, the non-merging firms are in equal or better situation if the merging occurs.

To finish this section, we would like to study whether or not *LL*, *AP* and *PROP* are with a very natural invariance property that states that a clearing matrix should be invariant in front of bilateral compensations.

We say that a clearing matrix satisfies ***invariant under net compensations*** *(INC)* if $\forall N \in \mathcal{N}$

(i) $z_i^C = z_i$ for all $i \in N$

(ii) $l_{ij}^C = \begin{cases} l_{ij} - l_{ji} & if\ l_{ij} \geq l_{ji} \\ 0 & if\ l_{ij} < l_{ji} \end{cases}$

Then

$$E_i(N,z,P^C) = E_i(N,z,P) \;\; \forall i \in N$$

In words, the net compensation of payments consists of, previously to pay the debts of matrix L, transform matrix L in matrix L^C, in which all firms have compensated bilaterally its debts with the other firms.

<u>Theorem 6</u>: There is not clearing matrix P that satisfies *LL*, *AP*, *PROP* and *INC*.

Proof. Assume that P is a clearing matrix that satisfies *LL*, *AP*, *PROP* and *INC*. Consider the financial system (N,z,L) with four firms $N = \{1,2,3,4\}$ and operating cash flows and liabilities.

$$z = (0,20,0,20),$$
$$L = \begin{pmatrix} 0 & 10 & 20 & 10 \\ 10 & 0 & 10 & 30 \\ 0 & 10 & 0 & 0 \\ 10 & 20 & 10 & 0 \end{pmatrix}.$$

It is easy to check that under *LL*, *AP* and *PROP*, $D(L) = \{1\}$ and $\bar{p}_{i.}, P$ and the vector of values of equities $E(N,z,P)$ are

$$\bar{p}_{i.} = (20,50,10,40),$$
$$P = \begin{pmatrix} 0 & 5 & 10 & 5 \\ 10 & 0 & 10 & 30 \\ 0 & 10 & 0 & 0 \\ 10 & 20 & 10 & 0 \end{pmatrix},$$
$$E(N,z,P) = (0,5,20,15).$$

If the mutual debts are compensated before the payment, the liabilities matrix L^C is

$$L^C = \begin{pmatrix} 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{pmatrix}.$$

In this case, under *LL*, *AP* and *PROP*, again $D(L^C) = \{1\}$ and $\bar{p}^C, P^C$ and the vector $E(N,z,P^C)$ are

$$z = (0,20,0,20),$$

$$\bar{p}^C = (0, 10, 0, 10),$$

$$P^C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{pmatrix},$$

$$E(N, z, P^C) = (0, 10, 10, 20).$$

Hence, P does not satisfy INC ■.

Since the compensation of payments does not imply a change in the operating cash flow and in both sides of the inequality it is considered the value of equity of the whole firm set, the aggregate value of equity is equal to the aggregate operating cash flow of all firms, with or without compensation of payments.

$$\sum_{i \in N} E_i(N, z, L^C) = \sum_{i \in N} z_i = \sum_{i \in N} E_i(N, z, L)$$

Nevertheless, in order to be *INC*, the value of equity $\forall i \in N$ must be equal with or without compensation of the mutual debts, which is not true if *LL*, *AP* and *PROP* are hold, how the example proves.

5. Conclusions

As has been demonstrated through this work, Limited liability, Absolute priority and Proportionality are such strong properties that make very difficult that the only clearing matrix can hold other properties.

We have proved that the clearing matrix is splitting-proof under Limited liability and Absolute priority, independently of the bankruptcy rule specified. Also, the clearing matrix is not invariant to bilateral compensations under the properties defined by Eisenberg and Noe. Meanwhile, and surprisingly, we have found that the clearing matrix is merging-proof. This result is specially shocking because it contradicts the findings of de Frutos (1999), who proves that in the classical bankruptcy problems the only bankruptcy rule that provides immunity to manipulation via merging and splitting is the proportionality rule.

In our opinion, the investigation of the financial systems from an axiomatic point of view have still room to more research. In fact, the proportionality rule can be substituted for other bankruptcy rules such as constrained equal awards or constrained equal losses among others. If we had had more available time, we would have repeated the analysis with other bankruptcy rules and would have tried to find if the results are robust to the change of the bankruptcy rule.

6. Bibliography

Aristotle (1985) *Ethics*. J.A.K. Thompson, tr. Harmondsworth, UK: Penguin.

Basel Committee on Banking Supervision (2017) 'Basel Committee on Banking Supervision Basel III: Finalising post-crisis reforms'. *Bank for International Settlements*.

Csóka, P. and Herings, P. J.-J. (2017) 'An Axiomatization of the Proportional Rule in Financial Networks'. Discussion paper, MT-DP-2017-1.

Csóka, P. and Herings, P. J.-J. (2018) 'Decentralized Clearing in Financial Networks'. *Management Science*, 64, 10, pp. 4681-4699.

Eisenberg, L. and Noe, T. H. (2001) 'Systemic Risk in Financial Systems', *Management Science*, 47, pp. 236-249.

European Parliament (2009) 'Directive 2009/138/EC of the European Parliament and of the council of 25 November 2009', *Official Journal of the European Union*.

de Frutos, M. A. (1999) 'Coalitional manipulations in a bankruptcy problem', *Review of Economic Design*, 4, pp. 255-272.

Groote Schaarsberg, M., Reijnierse, H. and Borm, P. (2018) 'On solving mutual liability problems', *Mathematical Methods of Operations Research*, 87, pp. 383-409.

Ju, B.-G. (2003) 'Manipulation via merging and splitting in claims problems', *Review of Economic Design*, 8, pp. 127-139.

Moreno-Ternero, J. D. (2006) 'Proportionality and Non-Manipulability in Bankruptcy Problems', *International Game Theory Review*, 8, pp. 127-139.

O'Neill, B. (1982) 'A problem of rights arbitration from the Talmud', *Mathematical Social Sciences*, 2, 345-371.

Thomson, W. (2002) 'Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: A survey', *Mathematical Social Sciences*, 45,249-297.

7. Appendix

7.1. Appendix 1: R code

```
z<-c(0,0,20,20)

L1<-c(0,20,15,15)

L2<-c(10,0,10,20)

L3<-c(10,10,0,20)

L4<-c(20,10,20,0)

L<-rbind(L1,L2,L3,L4)        #matrix of liabilities


l<-rowSums (L)              #total amount of debts of i

pi<-L/l;pi[is.na(pi)]<-0     #matrix Pi

deltat<-rep(1,nrow(L))       #vector delta

Delta<-rep(1,nrow(L))        #matrix Delta. Each row is the delta vector of t-order
t=1,...,k

D<-rep(0,nrow(L))            #vector of defaulting firms in t+1-order or any previous
orders

D0<-rep(1,nrow(L))           #vector of defaulting firms in t-order or any previous orders

Dmat<-D                      #matrix of defaults. Each row is the vector of defaulting firms of
t-order t=1,...,k  (1 if default)


while(sum(D0)!=sum(D)){

  D0<-D

  Lt<-deltat*L               #matrix L^t

  lt<-rowSums (Lt)           #vector of sum of debts (L^t)

  ct<-colSums (Lt)           #vector of sum of claims (L^t)

  At<-z+ct                   #vector of assets or capacity to pay in t-order

  Et<-(At-l)                 #E(N,z,L^t) value of equity in t-order

  deltat<-replace(deltat,(Et<=0|deltat<1),(At[Et<=0])/l[Et<=0])

  Delta<-rbind(Delta,deltat)

  D<-replace(D,deltat<1,1) #defaulting firms set in t-order or any previous order

  Dmat<-rbind(Dmat,D)
```

```
}
if (sum(D)==1|sum(D)==0) {
  p<-lt} else {if(nrow(t(as.matrix(Lt[lt==l,lt<l])))==1){
    c<--(z[lt<l]+L[lt==l,lt<l])}else{
      c<--(z[lt<l]+colSums(L[lt==l,lt<l]))}
    def<-t(pi)[D==1,D==1]
    deff<-replace(def,diag(1,sum(D))==1,-1)
    solve(deff,c)
    d<-replace(D,D==1,solve(deff,c))
    p<-replace(D,c(D==0,D==1),c(l[D==0],d[D==1]))
    p<-replace(replace(D,D==0,l[D==0]),D==1,d[D==1]) #clearing vector p
  }
Dmat<-Dmat[-1,]
P<-p*pi                    #clearing matrix P
Ep<-z+colSums(P)-rowSums(P) #E(N,z,P)
colnames(L)<-rownames(P); colnames(P)<-rownames(P)
colnames(Dmat)<-rownames(P);rownames(Dmat)<-seq(1,nrow(Dmat))
colnames(Delta)<-rownames(P);rownames(Delta)<-seq(1,nrow(Delta))
zn<-z; Ln<-L; Deltan<-Delta;Dmatn<-Dmat; pn<-p; Pn<-P; Epn<-Ep

##SPLITTING-PROOFNESS##

#the code only represents the case where the splitting firm in (N',z',L')
#keeps all the assets and gives all the liabilites to the new firm created in (N,z,L)

s<-1      #splitting firm
L<-rbind(L,L[s,]);L<-cbind(L,rep(0,nrow(L)));L[s,]<-rep(0,nrow(L))
z<-c(z,0) #the new firm is {n+1}

l<-rowSums (L)
pi<-L/l;pi[is.na(pi)]<-0
deltat<-rep(1,nrow(L))
```

```
DeltaS<-rep(1,nrow(L))
D<-rep(0,nrow(L))
D0<-rep(1,nrow(L))
Dmat<-D

while(sum(D0)!=sum(D)){
  D0<-D
  Lt<-deltat*L
  lt<-rowSums (Lt)
  ct<-colSums (Lt)
  At<-z+ct
  Et<-(At-l)
  deltat<-replace(deltat,(Et<=0|deltat<1),(At[Et<=0])/l[Et<=0])
  DeltaS<-rbind(DeltaS,deltat)
  D<-replace(D,deltat<1,1)
  Dmat<-rbind(Dmat,D)
}
if (sum(D)==1|sum(D)==0) {
  p<-lt} else {if(nrow(t(as.matrix(Lt[lt==l,lt<l])))==1){
    c<--(z[lt<l]+L[lt==l,lt<l])}else{
      c<--(z[lt<l]+colSums(L[lt==l,lt<l]))}
    def<-t(pi)[D==1,D==1]
    deff<-replace(def,diag(1,sum(D))==1,-1)
    solve(deff,c)
    d<-replace(D,D==1,solve(deff,c))
    p<-replace(D,c(D==0,D==1),c(l[D==0],d[D==1]))
    p<-replace(replace(D,D==0,l[D==0]),D==1,d[D==1]) #clearing vector p
  }
Dmat<-Dmat[-1,]
P<-p*pi                    #clearing matrix P
Ep<-z+colSums(P)-rowSums(P) #E(N,z,P)
colnames(L)<-rownames(P); colnames(P)<-rownames(P)
```

```
colnames(Dmat)<-rownames(P);rownames(Dmat)<-seq(1,nrow(Dmat))
colnames(DeltaS)<-rownames(P);rownames(DeltaS)<-seq(1,nrow(DeltaS))
zS<-z; LS<-L; DS<-Delta; DmatS<-Dmat; pS<-p; PS<-P; EpS<-Ep

#Does s have incentives to split?

if(Epn[s]>=(EpS[s]+EpS[nrow(Ln)+1])){"NO"
}else{"YES"}

##MERGING-PROOFNESS##

m<-c(1,4)  #firms which merge (M)

z<-c(z[-m],sum(z[m]))
if(nrow(t(as.matrix(L[-m,m])))==1&nrow(t(as.matrix(L[m,-m])))==1&length(z)-
length(m)==1){
L<-rbind(c(0,sum(L[m,-m])),c(sum(L[-m,m]),0))}else{
L<-rbind(cbind(L[-m,-m],rowSums(L[-m,m])),c(colSums(L[m,-m]),0))}

l<-rowSums (L)
pi<-L/l;pi[is.na(pi)]<-0
deltat<-rep(1,nrow(L))
DeltaM<-rep(1,nrow(L))
D<-rep(0,nrow(L))
D0<-rep(1,nrow(L))
Dmat<-D

while(sum(D0)!=sum(D)){
  D0<-D
  Lt<-deltat*L
  lt<-rowSums (Lt)
  ct<-colSums (Lt)
```

```
  At<-z+ct
  Et<-(At-l)
  deltat<-replace(deltat,(Et<=0|deltat<1),(At[Et<=0])/l[Et<=0])
  DeltaM<-rbind(DeltaM,deltat)
  D<-replace(D,deltat<1,1)
  Dmat<-rbind(Dmat,D)
}
if (sum(D)==1|sum(D)==0) {
  p<-lt} else {if(nrow(t(as.matrix(Lt[lt==l,lt<l])))==1){
    c<--(z[lt<l]+L[lt==l,lt<l])}else{
      c<--(z[lt<l]+colSums(L[lt==l,lt<l]))}
    def<-t(pi)[D==1,D==1]
    deff<-replace(def,diag(1,sum(D))==1,-1)
    solve(deff,c)
    d<-replace(D,D==1,solve(deff,c))
    p<-replace(D,c(D==0,D==1),c(l[D==0],d[D==1]))
    p<-replace(replace(D,D==0,l[D==0]),D==1,d[D==1]) #clearing vector p
  }
Dmat<-Dmat[-1,]
P<-p*pi
Ep<-z+colSums(P)-rowSums(P)
colnames(L)<-rownames(P); colnames(P)<-rownames(P)
colnames(Dmat)<-rownames(P);rownames(Dmat)<-seq(1,nrow(Dmat))
colnames(DeltaM)<-rownames(P);rownames(DeltaM)<-seq(1,nrow(DeltaM))
zM<-z; LM<-L; DeltaM<-Delta; DmatM<-Dmat; pM<-p; PM<-P; EpM<-Ep

#Do have M incentives to merge?

if(EpM[nrow(LM)]<=sum(Epn[m])){"NO"
}else{"YES"}

##INVARIANT UNDER NET COMPENSATIONS##
```

```
L<-L-t(L)
L<-replace(L,L<0,0)

l<-rowSums (L)
pi<-L/l;pi[is.na(pi)]<-0
deltat<-rep(1,nrow(L))
DeltaC<-rep(1,nrow(L))
D<-rep(0,nrow(L))
D0<-rep(1,nrow(L))
Dmat<-D

while(sum(D0)!=sum(D)){
  D0<-D
  Lt<-deltat*L
  lt<-rowSums (Lt)
  ct<-colSums (Lt)
  At<-z+ct
  Et<-(At-l)
  deltat<-replace(deltat,(Et<=0|deltat<1),(At[Et<=0])/l[Et<=0])
  DeltaC<-rbind(DeltaC,deltat)
  D<-replace(D,deltat<1,1)
  Dmat<-rbind(Dmat,D)
}
if (sum(D)==1|sum(D)==0) {
  p<-lt} else {if(nrow(t(as.matrix(Lt[lt==l,lt<l])))==1){
    c<--(z[lt<l]+L[lt==l,lt<l])}else{
      c<--(z[lt<l]+colSums(L[lt==l,lt<l]))}
    def<-t(pi)[D==1,D==1]
    deff<-replace(def,diag(1,sum(D))==1,-1)
    solve(deff,c)
    d<-replace(D,D==1,solve(deff,c))
```

```
    p<-replace(D,c(D==0,D==1),c(l[D==0],d[D==1]))
    p<-replace(replace(D,D==0,l[D==0]),D==1,d[D==1]) #clearing vector p
  }
Dmat<-Dmat[-1,]
P<-p*pi
Ep<-z+colSums(P)-rowSums(P)
colnames(L)<-rownames(P); colnames(P)<-rownames(P)
colnames(Dmat)<-rownames(P);rownames(Dmat)<-seq(1,nrow(Dmat))
colnames(DeltaC)<-rownames(P);rownames(DeltaC)<-seq(1,nrow(DeltaC))
zinc<-z; Linc<-L; Deltainc<-Delta; Dmatinc<-Dmat; pinc<-p; Pinc<-P; Epinc<-Ep

#Is the financial system invariant under net compensations?

if(FALSE %in% (Epn==Epinc)){"NO"
}else{"YES"}
```

7.2. Appendix 2: Example 3

To show accurately each step of Example 3, the code has been slightly modified:

```
> # Consider the financial system (N,z,L) in Gamma with four firms N={1,2,3,4} with operating cash flows and liabilities presented in z and L:
>
> z<-c(0,0,20,20)
> L1<-c(0,20,15,15)
> L2<-c(10,0,10,20)
> L3<-c(10,10,0,20)
> L4<-c(20,10,20,0)
> L<-rbind(L1,L2,L3,L4)
>
> z
[1]  0  0 20 20
> L
   [,1] [,2] [,3] [,4]
L1    0   20   15   15
L2   10    0   10   20
L3   10   10    0   20
L4   20   10   20    0
>
> # We construct l, c and matrix Π(L):
```

```
>
> l<-rowSums (L)
> pi<-L/l;pi[is.na(pi)]<-0
> c<-colSums (L)
> l
L1 L2 L3 L4
50 40 40 50
> c
[1] 40 40 45 55
> pi
   [,1] [,2] [,3] [,4]
L1 0.00 0.40 0.30  0.3
L2 0.25 0.00 0.25  0.5
L3 0.25 0.25 0.00  0.5
L4 0.40 0.20 0.40  0.0
>
> # We assume fully payment of all debts and compute the vector of equities:
>
>
> EL<-z+c-l
> EL
 L1  L2  L3  L4
-10   0  25  25
>
> # As we can see, firm 1 is the only firm that has negative value of equity and, thus, is
the only defaulting firm in first order, so D1(L)={1}. Then,
>
> delta<-rep(1,nrow(L))
> delta<-replace(delta,EL<=0,(z[EL<=0]+c[EL<=0])/l[EL<=0])
> D1<-rep(0,nrow(L))
> D1<-replace(D1,delta<1,1)
> delta
[1] 0.8 1.0 1.0 1.0
>
> # To check if there are any second-order defaults, L^1 matrix is constructed
> L1<-delta*L
> L1
   [,1] [,2] [,3] [,4]
L1    0   16   12   12
L2   10    0   10   20
L3   10   10    0   20
L4   20   10   20    0
>
> # Once again, we assume that the obligations according to this matrix are completely
satisfied and construct the vector of values of equity:
> l1<-rowSums (L1)
> c1<-colSums (L1)
> EL1<-z+c1-l1
> EL1
L1 L2 L3 L4
```

```
 0 -4 22 22
>
> # We can see that the only second-order default is by firm 2, then D2(L)={2}. We
construct δ^1(L) and with this vector, L^2 is defined
> delta1<-rep(1,nrow(L))
> delta1<-replace(delta,(EL1<=0|delta<1),(z[EL1<=0]+c1[EL1<=0])/l[EL1<=0])
> D2<-rep(0,nrow(L))
> D2<-replace(D2,delta1<1,1)-D1
> L2<-delta1*L
> delta1
[1] 0.8 0.9 1.0 1.0
> L2
   [,1] [,2] [,3] [,4]
L1    0   16   12   12
L2    9    0    9   18
L3   10   10    0   20
L4   20   10   20    0
>
> # Then
> l2<-rowSums (L2)
> c2<-colSums (L2)
> EL2<-z+c2-l2
> EL2
L1 L2 L3 L4
-1  0 21 20
>
> # We can observe that neither 3 or 4 see their value of equity reduced enough to not
being able to pay all their debts, so they do not default in third-order, so D3(L)=∅ and
D(L)=D1(L)∪D2(L)={1,2}. Now, we stop iterating and proceed with the next step: we
have to solve the linear equation system below to obtain the unique an unknown value
of p*1 and p*2. The system of equations to solve in this example is
> delta2<-rep(1,nrow(L))
> delta2<-replace(delta2,(EL2<=0|delta2<1),(z[EL2<=0]+c2[EL2<=0])/l[EL2<=0])
> D3<-rep(0,nrow(L))
> D3<-replace(D3,delta2<1,1)-D1-D2
> D3
[1] 0 0 0 0
> D<-D1+D2
> if (sum(D)==1|sum(D)==0) {
+   p<-lt} else {if(nrow(t(as.matrix(L2[l2==l,l2<l])))==1){
+     c<--(z[l2<l]+L[l2==l,l2<l])}else{
+       c<--(z[l2<l]+colSums(L[l2==l,l2<l]))}
+     def<-t(pi)[D==1,D==1]
+     deff<-replace(def,diag(1,sum(D))==1,-1)
+     solve(deff,c)
+     d<-replace(D,D==1,solve(deff,c))
+     p<-replace(D,c(D==0,D==1),c(l[D==0],d[D==1]))
+     p<-replace(replace(D,D==0,l[D==0]),D==1,d[D==1]) #clearing vector p
+   }
>
```

```
> # and its solutions are p*1=38.89 and p*2=35.56. Additionally, since 3 and 4 do not
default, p*3=l3=40 and p*4=l4=50. Therefore,
> p
[1] 38.88889 35.55556 40.00000 50.00000
>
> # With the help of matrix Π(L) we can construct the clearing payment matrix,
P*(N,z,L)=P*:
> P<-p*pi
> P
        [,1]     [,2]      [,3]     [,4]
L1  0.000000 15.55556 11.666667 11.66667
L2  8.888889  0.00000  8.888889 17.77778
L3 10.000000 10.00000  0.000000 20.00000
L4 20.000000 10.00000 20.000000  0.00000
>
> # Finally, we can obtain the equity value vector:
> Ep<-z+colSums(P)-rowSums(P)
> Ep
      L1       L2       L3       L4
 0.00000  0.00000 20.55556 19.44444
```