LEFSCHETZ PROPERTIES FOR COMPLETE INTERSECTION IDEALS GENERATED BY PRODUCTS OF LINEAR FORMS

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ABSTRACT. In this paper, we study the strong Lefschetz property of artinian complete intersection ideals generated by products of linear forms. We prove the strong Lefschetz property for a class of such ideals with binomial generators.

CONTENTS

1. INTRODUCTION

Let K be a field of characteristic zero and let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over K in n variables. A graded artinian K-algebra $A := R/I$ is said to have the *strong Lefschetz property* (*SLP* for short) if there is a linear form $\ell \in A_1$ such that the multiplication

$$\times \ell^s : A_{k-s} \to A_k$$

has maximal rank for all k and all s, i.e., $\times \ell^s$ is either injective or surjective, for all k and all s. If the multiplication $\times \ell : A_{k-1} \to A_k$ has maximal rank for all k, then $A = R/I$ is said to have the *weak Lefschetz property* (*WLP* for short). A linear form ℓ, as above, is called a *strong Lefschetz element* (resp. *weak Lefschetz element*) of A. We also say that I has the SLP (resp. WLP) if R/I has the SLP (resp. WLP). Though many algebras are expected to have the WLP or even the SLP, establishing this property is often rather difficult and even in seemingly simple cases, such as complete intersections and ideals generated by products of linear forms, much remains unknown about the presence of the Lefschetz properties.

Lefschetz properties have been studied intensively and a large toolbox, containing different approaches and methods, to check if a graded artinian K-algebra A has the WLP or the SLP has been developed. Their study is interesting not only because they put a lot of restrictions on the Hilbert function of a standard graded K-algebra but also since they have shown to be connected to a large number of problems, that appear to be unrelated at first glance. Nevertheless, all research results in this area are motivated and owe their roots to the following theorem proved by Stanley in [14], Watanabe in [15] and Reid, Roberts and Roitman in [12]: If K is a field of characteristic zero, then the artinian monomial complete intersection ideal $I = (x_1^{a_1}, \cdots, x_n^{a_n}) \subset R$ has the strong and, in particular, the weak Lefschetz property (see also [6] and [7]). As a consequence we have that a *general* complete intersection artinian ideal with fixed generator degrees has both, the WLP and the SLP. It is therefore natural to pose the following question:

Question 1.1 ([10, Question 3.1]). Do *all* artinian complete intersections have the WLP/SLP, in characteristic zero?

Some evidence that Question 1.1 has a positive answer is given by the fact that *all* artinian complete intersections in 3 variables are known to have the WLP. However, it is a challenging and still open problem to decide whether all height 3 artinian complete intersections have the SLP. Similarly, the answer to Question 1.1 for height 4 ideals is unknown.

There are plenty of results concerning Lefschetz properties of ideals generated by powers of linear forms (see e.g., [2, 9, 11, 13]), and it is natural to consider generalizations of those results to ideals generated by products of linear forms. From this point of view, it would be interesting to study Question 1.1 for artinian complete intersections generated by products of linear forms. In this short note, we make a non-trivial contribution to this problem by providing a family of artinian complete intersection ideals of arbitrary height n, which have the SLP. More precisely, the following is our main result:

Theorem 1.2. *Let $d_0, \ldots, d_{n-1}$ be positive integers and let $a \in K$ with $a \neq 1$. The algebra*

$$R/(x_n^{d_0}(x_n - ax_1), x_i^{d_i}(x_i - x_{i+1}) \mid 1 \leq i \leq n-1)$$

is an artinian complete intersection and has the SLP.

The structure of the paper is as follows. After recalling some basic results on Lefschetz properties in Section 2, we prove the main result of this paper (Theorem 1.2) and an extension of it in Section 3. In Section 4, we present some research problems concerning Lefschetz properties of ideals generated by products of linear forms.

2. Background and preparatory results

In this section, for the sake of completeness, we recall the main tools and results that will be used in the rest of the paper.

Throughout the following, we denote by K a field of characteristic zero and by $R = K[x_1, \ldots, x_n]$ the graded homogeneous polynomial ring in n variables over K. Strong respectively weak Lefschetz elements of an artinian algebra R/I are known to form a Zariski open, possibly empty, subset of $(R/I)_1$. In other words, if the artinian K-algebra R/I satisfies the strong or the weak Lefschetz property for some linear form, then it does so for a general linear form. However, for monomial ideals there is no need to consider a general linear form due to the following useful result:

Proposition 2.1 ([8, Proposition 2.2]). *Let $I \subset R$ be an artinian monomial ideal. Then R/I has the SLP if and only if $x_1 + \cdots + x_n$ is a strong Lefschetz element for R/I.*

We also recall the following well-known result, which can be seen as *the* starting point of the study of Lefschetz properties (see [12, 14, 15] and also [6, 7]).

Proposition 2.2. *Let $d_1, \ldots, d_n$ be non-negative integers and let $D = d_1 + \cdots + d_n$. Then $A = R/(x_1^{d_1+1}, \ldots, x_n^{d_n+1}) = A_0 \oplus \cdots \oplus A_D$ has the SLP. In particular, the multiplication map*

$$\times (x_1 + \cdots + x_n)^{D-2i} : A_i \to A_{D-2i}$$

is an isomorphism for $i < \frac{D}{2}$.

The following result, that will be crucial for the proof of Theorem 1.2, enables us to reduce the study of Lefschetz properties of an arbitrary artinian ideal I to the one of a monomial ideal, by passing to an initial ideal of I.

Proposition 2.3 ([16, Proposition 2.9]). *Let $I \subset R$ be an artinian ideal, τ a term order and $\mathrm{in}_\tau(I)$ the initial ideal of I with respect to τ. If $R/\mathrm{in}_\tau(I)$ has the SLP (resp. WLP), then so has R/I.*

3. A class of complete intersections

This section is dedicated to the proof of Theorem 1.2. Thereby, we provide a class of artinian complete intersection ideals, generated by products of linear forms, that satisfy the SLP.

Lemma 3.1. *Let $d_0, d_1, \ldots, d_{n-1}$ be positive integers, $a \in K$ with $a \notin \{0,1\}$, and let $I = (x_n^{d_0}(x_1 - ax_n), x_i^{d_i}(x_i - x_{i+1}) \mid 1 \le i \le n-1)$. Then the initial ideal $\mathrm{in}_{\mathrm{lex}}(I)$ of I with respect to the lexicographic order is equal to*

$$(x_1^{d_1+1}, \cdots, x_{n-1}^{d_{n-1}+1}, x_i x_n^{d_0+\cdots+d_{i-1}} \mid 1 \le i \le n).$$

In particular, R/I is an artinian complete intersection.

Proof. Let

$$f_i = x_i^{d_i}(x_i - x_{i+1}) \quad \text{for } 1 \le i \le n-1$$

and

$$g_i = x_i x_n^{d_0+\cdots+d_{i-1}} - \frac{1}{a} x_n^{d_0+\cdots+d_{i-1}+1} \quad \text{for } 1 \le i \le n.$$

We prove the first statement by showing that $G = \{f_1, \ldots, f_{n-1}, g_1, \ldots, g_n\}$ is a Gröbner basis of I with respect to the lexicographic order. In the following, we write $S(f, g)$ for the S-polynomial of f and g. A routine computation implies the following properties of the possible S-polynomials:

- $S(f_i, g_i) = -(x_{i+1}x_i^{d_i}x_n^{d_0+\cdots+d_{i-1}} - \frac{1}{a}x_i^{d_i}x_n^{d_0+\cdots+d_{i-1}})$ reduces to $-\frac{1}{a^{d_i}}g_{i+1}$ with respect to $\{f_1, \ldots, f_n, g_1, \ldots, g_i\}$ for $1 \leq i \leq n-1$.
- $S(g_i, g_j) = -\frac{1}{a}(x_j x_n^{d_0+\cdots+d_{j-1}+1} - x_i x_n^{d_0+\cdots+d_{j-1}+1})$ reduces to zero with respect to G for $1 \leq i < j \leq n-1$.
- $\mathrm{in}_{\mathrm{lex}}(f_i)$ and $\mathrm{in}_{\mathrm{lex}}(f_j)$ are relatively prime if $1 \leq i \neq j \leq n-1$.
- $\mathrm{in}_{\mathrm{lex}}(f_i)$ and $\mathrm{in}_{\mathrm{lex}}(g_j)$ are relatively prime if $1 \leq i \leq n-1$, $1 \leq j \leq n$ and $i \neq j$.
- $S(g_i, g_n) = -\frac{1}{a}x_n^{d_1+\cdots+d_n+2} = -\frac{x_n}{a-1}g_n$ for $1 \leq i \leq n-1$.

The above facts combined with Buchberger's criterion guarantee that $G \subset I$ and that G is a Gröbner basis of I with respect to the lexicographic order.

For the second statement, observe that, since $\mathrm{in}_{\mathrm{lex}}(I)$ contains pure powers of all variables, $R/\mathrm{in}_{\mathrm{lex}}(I)$ and hence also R/I is artinian. Moreover, as I is generated by n polynomials, R/I is a complete intersection. □

Theorem 1.2 will finally follow from Proposition 2.3 and the next statement:

Theorem 3.2. *Let $d_0, d_1, \ldots, d_{n-1}$ be positive integers and let*

$$J = (x_1^{d_1+1}, \cdots, x_{n-1}^{d_{n-1}+1}, x_i x_n^{d_0+\cdots+d_{i-1}} \mid 1 \leq i \leq n).$$

Then R/J has the SLP.

Proof. Let $D = d_0 + d_1 + \cdots + d_{n-1}$. It follows from Lemma 3.1, that the Hilbert series of J is the same as the one of a complete intersection ideal generated by polynomials of degrees $d_0 + 1, d_1 + 1, \ldots, d_{n-1} + 1$. Therefore, by Proposition 2.1, it suffices to prove that the multiplication

$$\times (x_1 + \cdots + x_n)^{D-2i} : (R/J)_i \to (R/J)_{D-i}$$

is an isomorphism for $i < \frac{D}{2}$.

Let $A = R/J$ and for $1 \leq i \leq n-1$ let $M^{(i)} = (x_1, \ldots, x_i)A$ be the ideal of A generated by the variables $x_1, \ldots, x_i$. Using induction on k, we first prove that the multiplication map

(1) $$\times (x_1 + \cdots + x_n)^{D-2i} : M_i^{(k)} \to M_{D-i}^{(k)}$$

is an isomorphism for all $1 \leq k \leq n-1$ and all $i < \frac{D}{2}$. For $k = 1$, we have an isomorphism

$$R/(x_1^{d_1}, x_2^{d_2+1}, \ldots, x_{n-1}^{d_{n-1}+1}, x_n^{d_0})(-1) = R/(J : x_1)(-1) \xrightarrow{\times x_1} M^{(1)}$$

and therefore, Proposition 2.2 guarantees that the map (1) is an isomorphism in this case.

Suppose that $k > 1$. Since $M^{(k)}/M^{(k-1)}$ is an ideal of $A/M^{(k-1)} = R/(J +$

$(x_1, \ldots, x_{k-1}))$ generated by the single element x_k, we have that

$$M^{(k)}/M^{(k-1)} \cong R/((J + (x_1, \ldots, x_{k-1})) : x_k)(-1)$$

$$\cong K[x_k, x_{k+1}, \ldots, x_n]/(x_k^{d_k}, x_{k+1}^{d_{k+1}+1}, \ldots, x_{n-1}^{d_{n-1}+1}, x_n^{d_0 + \cdots + d_{k-1}})(-1),$$

which, by Proposition 2.2, has the SLP. Using the induction hypothesis and the following exact sequence

$$(2) \qquad\qquad 0 \longrightarrow M^{(k-1)} \longrightarrow M^{(k)} \longrightarrow M^{(k)}/M^{(k-1)} \longrightarrow 0,$$

we conclude that (1) is an isomorphism for all $1 \leq k \leq n-1$ and all $i < \frac{D}{2}$.

Finally, it follows from the short exact sequence

$$0 \longrightarrow M^{(n-1)} \longrightarrow A \longrightarrow A/M^{(n-1)} \cong K[x_n]/(x_n^{D+1}) \longrightarrow 0,$$

A has the SLP. $\qquad\square$

Theorem 1.2 is an almost immediate consequence of Lemma 3.1 and Theorem 3.2.

Proof of Theorem 1.2. Note that, by Proposition 2.3, it is enough to show that $R/\mathrm{in}_{\mathrm{lex}}(I)$ has the SLP. If $a = 0$, then $\mathrm{in}_{\mathrm{lex}}(I) = (x_n^{d_0+1}, x_1^{d_1+1}, \ldots, x_{n-1}^{d_{n-1}+1})$ is an artinian complete intersection and it follows from Proposition 2.2 that $R/\mathrm{in}_{\mathrm{lex}}(I)$ has the SLP.

If $a \neq 0$, it follows from Lemma 3.1 and Theorem 3.2 that $R/\mathrm{in}_{\mathrm{lex}}(I)$ has the SLP. $\qquad\square$

Remark 3.3. The sum of the variables $x_1 + \cdots + x_n$ may not be a Lefschetz element of R/I in Theorem 1.2. Indeed, we have checked with Macaulay2 [3] that if $K = \mathbb{R}$, $n = 7$, $d_0 = d_1 = \cdots = d_6 = 1$ and $a = \pm\frac{4}{\sqrt{3}}$, then $x_1 + \cdots + x_n$ is not a Lefschetz element for R/I.

In the remaining part of this section, we discuss an extension of Theorem 1.2. To do so, let us first fix some notation. Let $M_{n \times n}(K)$ denote the set of all $n \times n$ matrices with entries in K. To any matrix $A \in M_{n \times n}(K)$ and any tuple $\vec{d} = (d_1, d_2, \ldots, d_n)$ of positive integers we associate the ideal

$$I_{A,\vec{d}} := \left(x_i^{d_i} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \ \Big| \ 1 \leq i \leq n \right) \subset R.$$

It is known that $I_{A,\vec{d}}$ is an artinian complete intersection ideal if and only if all principal minors of A are non-zero (see e.g., [1, Lemma 2.1], where this equivalence is proved when $\vec{d} = (1, 1, \ldots, 1)$). Since these ideals give a class of artinian complete intersection ideals, it is natural to ask if they have the SLP. Using our main result Theorem 1.2 we are able to provide a positive answer to this problem in a special case:

Theorem 3.4. *Let $A \in M_{n \times n}(K)$ be a matrix with non-zero principal minors. Assume that each row of A has exactly two non-zero entries. Then $R/I_{A,\vec{d}}$ has the SLP.*

The condition that in each row of the matrix A are precisely two non-zero entries just says that the ideal $I_{A,\vec{d}}$ is generated by binomials.

Proof. We will show that – up to a change of coordinates – $R/I_{A,\vec{d}}$ is isomorphic to an extension of a tensor product of algebras that have the SLP. Since tensor products preserve the SLP [5, Theorem 3.34], and the above extension also turns out to preserve the SLP, the claim follows.

First note that by permuting the variables x_i and x_j, the ideal $I_{A,\vec{d}}$ is changed to $I_{A',\vec{d}'}$, where A' is obtained from A by exchanging the ith and j^{th} row as well as the ith and j^{th} column.

Since all principal minors of A are non-zero, we know that the diagonal entries a_{ii} have to be non-zero. Moreover, as each row of A contains exactly two non-zero entries, for any $1 \leq i \leq n$ there exists a unique j_i such that the entry in the ith row and j_i^{th} column of A is non-zero. We denote this entry by b_{ij_i}.

We first show the following claim.

Claim: We can assume that A is of the form

$$(3) \qquad \begin{pmatrix} B & * & * & \cdots & * \\ & A_1 & 0 & \cdots & 0 \\ & & A_2 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & A_r \end{pmatrix}.$$

We associate a directed graph G_A to the matrix A in the following way: The set of vertices of G_A is defined to be the set of symbols a_{ii} and b_{ij_i} ($1 \leq i \leq n$). The set of edges is defined to be the set of all ordered tuples (a_{ii}, b_{ij_i}) and $(b_{ij_i}, a_{j_ij_i})$ ($1 \leq i \leq n$). As j_i is uniquely determined by i, we conclude that a_{ii} and b_{ij_i} both have outdegree equal to 1. Therefore, the finite graph G_A has at least one (directed) cycle, and distinct cycles are disjoint. If the vertex b_{ij_i} belongs to a cycle, then so do a_{ii} and $a_{j_ij_i}$ (as b_{ij_i} only lies in the edges (a_{ii}, b_{ij_i}) and $(b_{ij_i}, a_{j_i}j_i)$). Hence, the sets of row and column indices occurring in (vertices of) a cycle coincide. It follows from the previous discussion that we can assume that A is of the form (3), where $r \geq 1$ is the number of cycles of G_A and the A_i are $n_i \times n_i$-matrix, whose row and column indices correspond to vertices forming a cycle. The matrix B is an $n_0 \times n_0$-matrix, whose row indices correspond to vertices a_{ii} not contained in any cycle. This shows the above Claim.

Since the non-zero entries in the first n_0 rows and columns of A correspond to vertices of G_A not lying in a cycle, by permuting and scaling the variables we can assume that B is upper triangular with diagonal entries equal to 1.

Furthermore, again by permutation of variables, we can suppose that the only non-zero off-diagonal entries of A_k are the ones directly above the diagonal or the one in the bottom left corner of A. Note that, multiplying the ith column of A by $\lambda \in K \setminus \{0\}$ corresponds to a scaling of the variables ($x_i \mapsto \lambda x_i$), whereas multiplying the ith row of A by λ does not change the ideal at all. Hence, by scaling the variables,

we can even assume that A_k has the following shape:

$$A_k = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & 1 & -1 \\ a & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $a \in K \setminus \{0\}$ depends on A_k. Note that A_k gives the ideal in Theorem 1.2. We finally infer from the previous argumentation that

$$R/I_{A,\vec{d}} \cong \frac{\left(\bigotimes_{k=1}^{r} K[n_k \text{ variables}]/I_{A_k,\vec{d_k}}\right)[x_1, x_2, \ldots, x_{n_0}]}{(f_1, f_2, \ldots, f_{n_0})},$$

where f_ℓ ($1 \leq \ell \leq n_0$) is a homogeneous binomial that is monic in x_ℓ, and $\vec{d_k}$ ($1 \leq k \leq r$) contains the entries of $\vec{d}$ corresponding to A_k. The above tensor product has the SLP, since each factor has the SLP by Theorem 1.2. Then the right-hand side of the above equation can be considered as the repetition of a simple extension $S[x_\ell]/(f_\ell)$ of an algebra S that has the SLP for $1 \leq \ell \leq n_0$ (in reverse order). Finally the right-hand side has the SLP by [5, Corollary 4.17], which finishes the proof of the theorem. □

If A is an upper triangular matrix, then the initial ideal $\mathrm{in}_{\mathrm{lex}}(I_{A,\vec{d}})$ with respect to the lexicographic order is a monomial complete intersection. So, $R/\mathrm{in}_{\mathrm{lex}}(I_{A,\vec{d}})$ has the SLP and, applying Proposition 2.3, we conclude that the same holds for $R/I_{A,\vec{d}}$.

4. FINAL COMMENTS AND OPEN PROBLEMS

In this section, we will present some open questions for further research.

To simplify the notation, we will write $I_A = I_{A,(1,1,\ldots,1)}$. Our study of Theorem 1.2 and the ideals $I_{A,\vec{d}}$ is motivation by the following fact: If I a quadratic artinian complete intersection ideal generated by products of linear forms, then – by applying an appropriate change of coordinates – it follows that R/I is isomorphic to $R/I_{A,(1,\ldots,1)}$ for some A. Thus an affirmative answer to the next question will give an affirmative answer to Question 1.1 for quadratic complete intersection ideals generated by products of linear forms.

Problem 4.1. Does R/I_A (or more generally $R/I_{A,\vec{d}}$) have the WLP/SLP, if all principal minors of A are non-zero?

We also propose some special instances of Problem 4.1.

Problem 4.2. Does R/I_A (or more generally $R/I_{A,\vec{d}}$) have the WLP/SLP if A is integral and all its principal minors are equal to ± 1?

Since a positive definite symmetric matrix A always has non-zero principal minors, we suggest to consider the following problem.

Problem 4.3. Does R/I_A (or more generally $R/I_{A,\vec{d}}$) have the WLP/SLP if A is a positive definite symmetric matrix?

More generally, we suggest to study the following problem:

Problem 4.4. Let I be an artinian complete intersection ideal generated by forms f_i of degree a_i. Assume that $f_i = \ell_1^{(i)} \cdots \ell_{a_i}^{(i)}$ is a product of a_i linear forms. Does R/I have the WLP/SLP? More specifically, one can ask the same question for the case that all f_i are of the same degree, i.e., $a_i = d$ for all i.

As an extension of the previous problem, we propose the following problem:

Problem 4.5. Study the WLP/SLP for ideals generated by products of linear forms.

Concerning this last problem it is worthwhile to point out that there is a huge list of papers dealing with ideals generated by powers of linear forms. For more information on this subject the reader can see, for instance, [4], [9], [11] and [13]. The problem is really subtle since a minuscule change can alter the behavior of the WLP. Indeed, the ideals

$$I_1 = (x_1^4, x_2^4, x_3^4, x_4^4, x_1 x_2 x_3 x_4) \text{ and } I_2 = (x_1^4, x_2^4, x_3^4, x_4^4, x_1 x_2 x_3 (x_1 + x_4))$$

have the same Hilbert function: 1 4 10 20 30 36 34 24 12 4 0 but I_1 never has the WLP while I_2 does have the WLP.

Remark 4.6. One might suspect that Problem 4.1 generalizes to almost complete intersection ideals generated by quadrics, which are products of linear forms. However, in [11, Theorem 2.12], the second author proved that this is not the case. Indeed, for $n = 6$ and all $n \geq 8$, the artinian ideal $I = (\ell_1^2, \ldots, \ell_{n+1}^2) \subset K[x_1, \cdots, x_n]$ generated by the square of $n + 1$ general linear forms fails the WLP. Nevertheless, we have checked with Macaulay2 that the artinian ideal $I = (\ell_1 \ell_1', \ldots, \ell_7 \ell_7') \subset K[x_1, \cdots, x_6]$ generated by the products of general linear forms has the WLP.

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