Master in Actuarial and Financial Sciences

MASTER THESIS

Title: Risk quantification of an option portfolio through the introduction of the fuzzy Black-Scholes formula

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ABSTRACT

The aim of this thesis is to quantify the market risk of an option portfolio under uncertainty. The fuzzy sets theory is introduced to model the parameters of the Black-Scholes option-pricing formula. Since the option price is calculated through the fuzzy Black-Scholes formula, we can compute the Value-at-Risk as a fuzzy number. By doing so, we aim to capture extra information that is lost in traditional models given the uncertainty derived from the fluctuations of financial markets. Finally, we want to conclude whether the introduction of the fuzzy sets theory is useful in order to improve the risk management.

KEYWORDS

- Market risk
- Options
- Black-Scholes-Merton model
- Uncertainty
- Fuzzy numbers

CONTENTS

I. INTRODUCTION

Financial risk management has become a fundamental issue in the development of any business, and it is especially relevant in corporations such as banks or insurance companies. The reason is double, economic and regulatory. On the one hand, it is vital to control the risk to which a business is exposed for its sustainable growth. On the other hand, organizations have to fulfill regulatory requirements that financial authorities impose.

In order to quantify the risk of an option portfolio, we have to previously valuate this type of derivative. The Black-Scholes-Merton model introduced by Black and Scholes (1973) is the most used option pricing model since its publication. Such was the importance of this new methodology that in 1997 Merton and Scholes were awarded with the Nobel price in economics. Nevertheless, this methodology is still not suitable for fuzzy environments. Fluctuations on financial markets lead to uncertain scenarios where parameters needed to compute the option price are vague and cannot be determined as specific values.

With the appearance of the uncertainty theory introduced by Zadeh (1965), new methodologies are proposed to model this kind of information that has a difficult treatment with traditional models because it is hardly measurable or not probabilizable. Thus, we can introduce useful and, a priori, non-measurable information that make models more complete.

Fuzzy numbers seem to be a good methodology to model the parameters of the Black-Scholes-Merton model. Under this transformation, the option price obtained with the Black-Scholes formula could be represented as a fuzzy number. In that way, we could choose different levels of prices according to our subjective expectation about future values of the parameters and the derivative. Thus, we might be able to capture some additional information that could be useful on the market risk quantification of an option portfolio. Since the option price is calculated through the fuzzy Black-Scholes formula, we can compute Value-at-Risk as a fuzzy number, constructing an interval delimitated by maximum and minimum possible boundaries.

As a student of actuarial and financial sciences I am really interested in the valuation of derivatives and the market risk quantification. But although we have a strong knowledge of modelling random variables, methodologies to model uncertainty are usually quite unknown. This master thesis is a great opportunity to learn more about the uncertainty theory and its models. Additionally, it is interesting to introduce this methodology on the Black-Scholes-Merton option pricing model and see what impact has on the risk management.

The paper structure is as follows: Firstly, we are going to explain useful concepts about risk, option pricing and fuzzy sets in order to understand the calculus that further on are develop. Secondly, we are going to price with the Black-Scholes formula the option portfolio that is going to be used in this paper. Then, we will introduce fuzzy sets on the previous mentioned Black-Sholes formula, representing it as a fuzzy number. After that, we will compute the Value-at-Risk using the values obtained with both methodologies, that is, the classical and the fuzzy Black-Scholes option pricing formula. Finally, we will compare and analyze the results obtained, determining if the inclusion of the fuzzy sets theory is useful to improve risk management.

II. PREVIOUS CONCEPTS AND CONSIDERATIONS

2.1. Market risk framework and risk measures

According to the Bank for International Settlements (BIS), Basel III is an internationally set of measures developed by the Basel Committee on Banking supervision in response to the financial crisis of 2007-09. The measures aim to strength the regulation, supervision and risk management of banks. This regulation stipulates the usage of two risk measures in order to compute market risk, that is, Value-at-Risk (VaR) and Expected Shortfall (ES). In this thesis, we are only going to consider Value-at-Risk, that is the most used risk measure over the financial environment. VaR represents the worst expected loss on a portfolio of instruments resulting from market movements over a given time horizon and a pre-defined confidence level. This risk measure is mathematically described below:

The Value-at-Risk of a portfolio with loss X at a given confidence level $\rho \in (0,1)$ is defined by,

$$VaR_{\rho}(X) = \inf\{x \in \mathbb{R} : P(X > x) \leq 1 - \rho\} = \inf\{x \in \mathbb{R} : F_X(x) \geq \rho\}.$$

VaR cannot be computed in an option portfolio from a risk factor distribution because of their non-linear payoff. Instead of that, we need to valuate previously the price of the derivative in order to quantify the market risk of the portfolio. After that, we can compute VaR by means of Monte-Carlo simulation or using the Theta-Delta-Gamma approximation (Partial Monte-Carlo simulation).

Further on, we are going to explain and develop the most popular methodology to compute option prices, that is, the Black-Scholes model. But previously, we have to explain what an option is and their characteristics.

2.2. Options

An option is a financial derivative that offers the holder the right, but not the obligation, to buy or sell the underlying asset at an agreed-upon price K (strike or exercise price) within a time period or on the maturity date T. Options can be negotiated in standardized or in OTC (over-the-counter) markets.

Two types of options exist: Puts and Calls. A Put gives the buyer the right to sell the underlying asset whereas the call offers the holder the right to buy the underlying asset, both at a specified price and over a time period.

We can also distinguish plain vanilla and exotic options. Plain vanilla options are the basic ones while exotic options are more complex derivatives because of their payment structures, expiration dates and strike prices. In this thesis, we are only going to consider plan vanilla options.

In this way, we can find different types of options depending on some criteria.

- *Exercising date*

a) European style: Options that only can be exercised at the maturity time T.
b) American style: Options that can be exercised at any time during the contract time period.

- *Spread between the strike and the price of the underlying asset*

a) In the money (ITM): A call is ITM when the price of the underlying asset is higher than the exercise price. On the contrary, a put is ITM when the price of the underlying asset is lower than the strike.
b) At the money (ATM): Calls and puts are ATM when the price of the underlying asset is equal to the exercise price.
c) Out the money: A call is OTM when the price of the underlying asset is lower than the strike, while the put, is OTM when the price of the underlying asset is higher than the exercise price.

Once we know what an option is, we proceed to explain the most common model used in pricing these type of derivatives.

2.3. The Black-Scholes-Merton option pricing model

The assumptions made in addition to the variables and parameters needed for the valuation of the option are described first.

- *Elements of the model*

K: Strike or exercise price.

T: Maturity date.

t: Initial date

$S(t)$: Price of the underlying asset at time t.

σ: Volatility.

r: Interest free rate

Black and Scholes (1973) assume “ideal conditions” in the market for the stock and for the option:

a) The interest free rate is constant and known.
b) The stock pays no dividends.
c) The option is “European” and can only be exercised at maturity.
d) There are no transaction costs.

e) It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the interest free rate.
f) There are no penalties to short selling.
g) The stock price follows a Wiener process (Brownian motion) with r and σ constants described by the following stochastic differential equation (SDE).

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

As follows, the original development of the Black-Scholes-Merton model is shown.

- *Derivation of the model*

The value of the option is a function of the time and the stock price, that is, $V\big(t, S(t)\big)$. In order to simplify the notation, in large equations we are going to represent the processes without the variables (e.g: $V\big(t, S(t)\big) = V$).

According to the Ito's lemma,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS(t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 .$$

Note that following the stochastic calculus, the second derivative with respect to time and the cross derivatives disappear.

Then, substituting $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$ and $dS(t)^2 = \sigma^2 S(t)^2 dt$ and regrouping terms we obtain,

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW .$$

We construct the replicating portfolio $X(t) = \Delta S(t) - V(t)$. Applying the Ito's lemma again, we obtain the following dynamic (note that the function is linear and its second derivatives are equal to 0),

$$dX = \frac{\partial X}{\partial S}dS + \frac{\partial X}{\partial V}dV .$$

Substituting $\frac{\partial X}{\partial S(t)} = \Delta, \frac{\partial X}{\partial V(t)} = -1$, $dS(t)$ and $dV(t)$ and regrouping terms we obtain,

$$dX = \left(\Delta\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\Delta\sigma S - \sigma S\frac{\partial V}{\partial S}\right)dW .$$

By choosing $\Delta = \frac{\partial V}{\partial S(t)}$ we eliminate the randomness of the portfolio. Furthermore, if there are no arbitrage opportunities, the portfolio grows at the same rate that the risk free asset with dynamic $dB(t) = rB(t)dt$. Then,

$$\left(\Delta\mu S - \frac{\partial V}{\partial t} - \frac{\partial V}{\partial S}\mu S - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt = rXdt .$$

After eliminating and regrouping terms, we obtain the Black-Scholes partial differential equation (PDE).

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV\ .$$

The payoff of the call at the maturity time T is,

$$V(T, S(T) = [S(T) - K]^+ = \max(S(T) - K, 0)\ .$$

In addition,

$$V(t, 0) = 0$$

and

$$V\big(t, S(t)\big) = S(t) - e^{-r\tau}K \qquad\qquad \text{if } S(t) \to \infty\ ,$$

where $e^{-r\tau}$ is the discount factor with r as a continuously compounded interest rate and $\tau = T - t$.

Then, solving the heat equation we obtain,

$$V(S(t), \sigma, \tau, r, K) = \ S(t)\Phi(d1) - Ke^{-r\tau}\Phi(d2)$$

where,

$$d_1 = \frac{ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{T - t}}\ , \qquad d_2 = \ d_1 - \sigma\sqrt{\tau}\ .$$

This result is known as the Black-Scholes option pricing formula.

The Black-Scholes formula could also be derived through the risk neutral approach (martingale method). Under this methodology we can obtain the terminal stock price solving the following stochastic differential equation,

$$dS(t) = rS(t) + \sigma S(t)dW(t)\ .$$

Integrating the SDE between T and t,

$$\int_t^T \frac{dS(u)}{S(u)} = \int_t^T rdu + \int_t^T \sigma dW(u)\ ,$$

we derive the stock price at the maturity time T as,

$$S(T) = S(t)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma W(\tau)\right]$$

with $\tau = T - t$.

As $S(t)$ is the current price of the stock, we may assume that is known. The random variable $W(\tau)$ is normally distributed with mean 0 and variance τ. If Z is a standard normal random variable, therefore we may represent the terminal stock price as,

$$S(T) = S(t)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Z\right).$$

This expression will be needed in order to compute the terminal value of the portfolio in Monte-Carlo simulations.

2.4. Fuzzy sets theory

We define a fuzzy subset $\bar{A}$ of B as,

$$\bar{A} = \{ (x, \mu_{\bar{A}}(x) \mid x \in B\}$$

where $A \subset B$ is defined by its membership function,

$$\mu_{\bar{A}}(x) = \alpha \,, \quad x \in \bar{A}_\alpha \,.$$

The fuzzy subset $\bar{A}$ of B is a fuzzy number when it is normal

$$max[\mu_{\bar{A}}(x)] = 1$$

and convex,

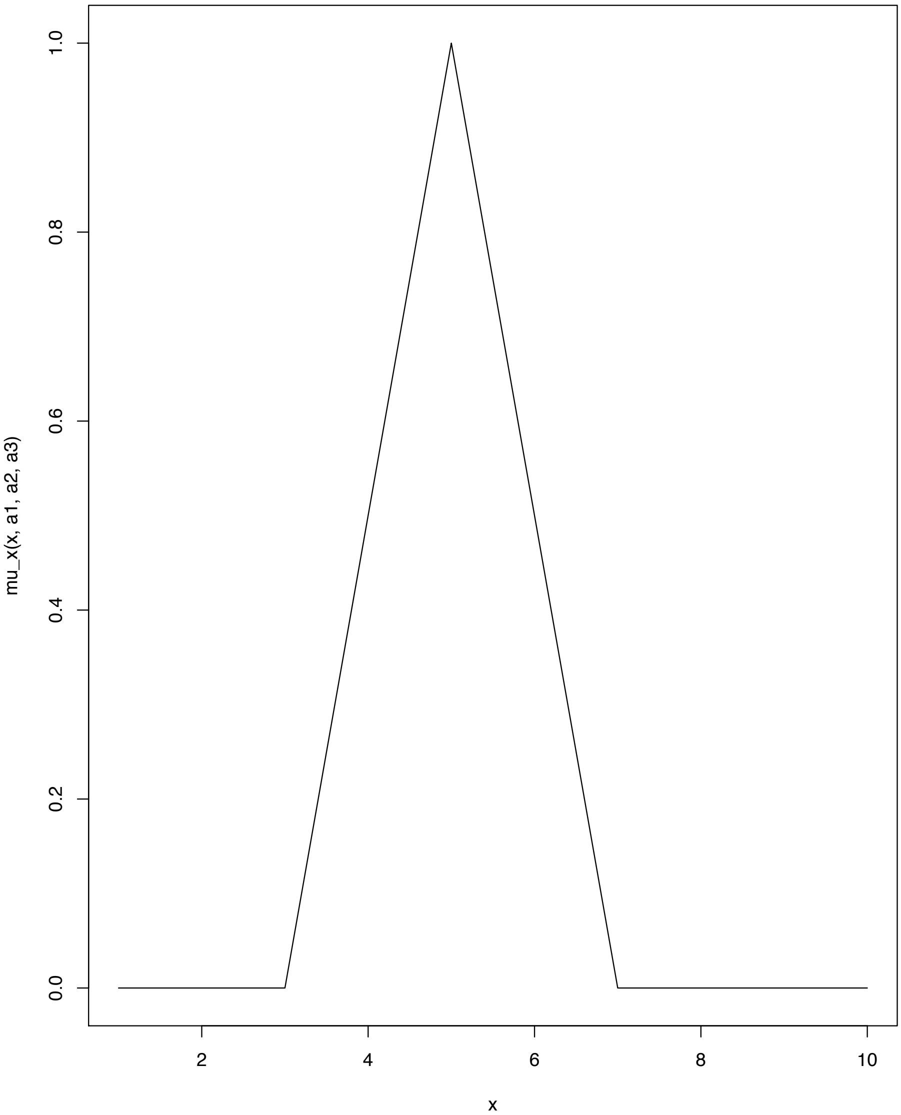
$$(\alpha' > \alpha) \Leftrightarrow ([\bar{a}_1^{\alpha'}, \bar{a}_2^{\alpha'}] \subset [\bar{a}_1^{\alpha}, \bar{a}_2^{\alpha}]), \qquad \bar{A}_\alpha = [\bar{a}_1^{\alpha}, \bar{a}_2^{\alpha}] \,.$$

A triangular fuzzy number could be defined as,

$$\bar{A} = (\bar{a}_1, \bar{a}_2, \bar{a}_3),$$

where $\bar{a}_1$ = lowest possibility, $\bar{a}_2$ = maximum level of presumption, $\bar{a}_3$ = highest possibility.

Figure 1. Graphic representation of the triangular fuzzy number $F = (3, 5, 7)$



Source: Own elaboration.

Depending on the presumption level α, we can construct subintervals of the fuzzy set. According to Buckley and Qu (1990), "*α-cuts*" are splits of a fuzzy subset producing regular non-fuzzy subsets.

$$\bar{A}_\alpha = \{x \in B \mid \mu_{\bar{A}}(x) \geq \alpha\}$$

We can also compute "*α-cuts*" using an interpolation of the function $\mu_{\tilde{A}}(x)$.

$$\bar{A}_\alpha = [\, \bar{a}_1^\alpha, \bar{a}_2^\alpha] = [\bar{a}_1 + (\bar{a}_2 - \bar{a}_1)\alpha, \bar{a}_3 - (\bar{a}_3 - \bar{a}_2)\alpha]$$

Gil Aluja & Kaufmann (1990) expose that a fuzzy number is the association of two concepts; the confidence interval linked to the uncertainty idea and the presumption level linked to the subjectivity idea.

III. OPTION PRICING

3.1. Black-Scholes option pricing formula

First of all, we have to mention that all data of this paper is extracted from Yahoo finance, and correspond to real statistics of the analysed corporation.

The portfolio that we will use in this thesis is formed by a single option. We consider a standard European call on a non-dividend-paying stock of J.P. Morgan with maturity time $T = 10/05/2019$ and exercise price $K = 106{,}000$. The underlying asset quotates on the New York Stock Exchange (NYSE) and its closing price is equal to $S(t) = 111{,}100$ at the initial time $t = 16/04/2019$. The interest free rate corresponding to the 13 week treasury bills is $r = 2{,}378\%$.

We can derive the implied volatility for the option equalizing the given market value (V_{MARKET}) with the call function, considering σ as an unknown variable. That is,

$$V_{MARKET} = V.$$

We can find iteratively a numerical solution for the equation applying the Newton-Raphson method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \,.$$

The given market price for the option is $V_{MARKET} = 5{,}500$, obtaining an implied volatility equal to $\sigma = 15{,}415\%$.

We proceed to compute the value of the option at time t through the Black-Scholes option pricing formula. Substituting the parameters that we have already defined above, we obtain the following results:

$$d_1 = 1{,}240\,,$$

$$d_2 = 1{,}201\,,$$

$$V(S(t), \sigma, \tau, r, K) = 5{,}500\,.$$

Further on, we will refer to this Black-Scholes formula as the classical Black-Scholes formula in order to distinguish it from the future transformations that we will apply on it.

3.2. "Fuzzification" process of the parameters

With the formalization of an option contract, both parts previously stipulate the exercise price (K) and the maturity time (T). However, the other necessary parameters for the option valuation remain variable depending on the evolution of the market.

The fluctuations of the market, make the price of the underlying asset $S(t)$ not possible to be determined as a specific value. Instead of that, it usually moves over an interval of possible levels. Given the continuous nature of prices, we can observe on financial markets different values for a stock over a period of time (e.g: one day). In addition, the bid/ask price plays an important role on the determination of the stock value. Following that reasoning, we think it could be interesting to model the price of the underlying asset through a fuzzy triangular number. In that way, we define the fuzzy price of the stock as $\bar{S}(t) = (S_1(t), S_2(t), S_3(t))$.

The same casuistic occurs with the interest free rate. It is not always the same value and depends on the market fluctuations. In the same way as we did above, we are going to use a triangular fuzzy number to model this parameter, that is, $\bar{r} = (r_1, r_2, r_3)$.

Given the fuzzy stock price and the fuzzy interest free rate, we can also compute the implied volatility as another triangular fuzzy number. In this way, the volatility could be expressed as $\bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

The idea behind these transformations is that in addition to the randomness, we have to consider the uncertainty associated to the prices for the computation of the parameters. Using triangular fuzzy numbers, we are trying to do a more realistic approach by considering an interval of possible values.

We take the price of the stock and the interest free rate at three different moments of the initial date. We use the highest and the lowest prices quoted on the market in addition with the closing price used before in the classical Black-Scholes formula. We obtain the fuzzy triangular parameters $\bar{S}(t) = (109{,}710, 111{,}100, 111{,}390)$ and $\bar{r}\,\% = (2{,}373, 2{,}378, 2{,}380)$. Given the previous data, we also compute the implied volatility for each moment on t obtaining $\bar{\sigma}\,\% = (15{,}294, 15{,}415, 26{,}216)$.

As we commented before, the Black-Scholes model assumes constant interest rates and volatility. These assumptions are still being viable under our fuzzy model. Even though parameters are represented by intervals, the interest free rate and the volatility are constant in a given specific point, for example, in the lowest possible values.

We proceed with the calculation of the "*α-cuts*".

Table 1. "*α-cuts*" of the triangular fuzzy parameter $S(t)$.

α	$S(t)_1^{\alpha}$	$S(t)_2^{\alpha}$
0	109,710	111,390
0,1	109,849	111,361
0,2	109,988	111,332
0,3	110,127	111,303
0,4	110,266	111,274
0,5	110,405	111,245
0,6	110,544	111,216
0,7	110,683	111,187
0,8	110,822	111,158
0,9	110,961	111,129
1	111,100	

Source: Own elaboration based on Yahoo finance data.

Table 2. "*α-cuts*" of the triangular fuzzy parameter r (%).

α	r_1^α %	r_2^α %
0	2,373	2,380
0,1	2,374	2,380
0,2	2,374	2,380
0,3	2,375	2,379
0,4	2,375	2,379
0,5	2,376	2,379
0,6	2,376	2,379
0,7	2,377	2,379
0,8	2,377	2,378
0,9	2,378	2,378
1	2,378	

Source: Own elaboration based on Yahoo finance data.

Table 3. "*α-cuts*" of the triangular fuzzy parameter σ (%).

α	σ_1^α %	σ_2^α %
0	15,294	26,216
0,1	15,306	25,136
0,2	15,318	24,056
0,3	15,330	22,976
0,4	15,342	21,895
0,5	15,354	20,815
0,6	15,366	19,735
0,7	15,378	18,655
0,8	15,391	17,575
0,9	15,403	16,495
1	15,415	

Source: Own elaboration based on Yahoo finance data.

Introducing these "*α-cuts*" cuts we add a subjective factor in our model not prior measurable with traditional methodologies. Depending on the uncertainty that an individual has on the expected value of the variable he will choose one alpha value or another. This subjectivity captures extra information that is lost with classical models and makes our estimations more realistic.

If we are absolutely sure about the value of one parameter, we will choose $\alpha = 1$ and the result will be equal to the one obtained with the classical model. But with the existence of uncertainty, it is likely that we do not know the specific value of the parameter. Depending on this degree of uncertainty and the chosen alpha, we will have a bigger or smaller interval of levels.

3.3. Fuzzy Black-Scholes option pricing formula

Once estimated the fuzzy parameters of the formula and their levels of presumption, we can compute the price of the European call as a fuzzy number.

$$\bar{V} = V(\bar{S}(t), \bar{\sigma}, \tau, \bar{r}, K) \;\Rightarrow\; \bar{V} = (V_1, V_2, V_3),$$

where $V_1(S_1(t), \sigma_1, \tau, r_1, K)$ is the lowest possible price, $V_2(S_2(t), \sigma_2, \tau, r_2, K)$ is the price with the maximum level of presumption and $V_3(S_3(t), \sigma_3, \tau, r_3, K)$ is the highest possible price.

Substituting the values of the parameters for each possible state in the analytic expression, we derive the fuzzy Black-Scholes formula,

$$V_1(S_1(t), \sigma_1, \tau, r_1, K) = 4{,}296$$

$$V_2(S_2(t), \sigma_2, \tau, r_2, K) = 5{,}500$$

$$V_3(S_3(t), \sigma_3, \tau, r_3, K) = 6{,}511$$

obtaining the triangular fuzzy number for the price of the European call.

$$\bar{V} = (4{,}296, 5{,}500, 6{,}511)\ .$$

Note that if instead of computing the fuzzy value of the call through the estimated parameters, we only take the given market values for the call, we obtain a different triangular fuzzy number for the European call price.

$$\bar{V}_{MARKET} = (5{,}250, 5{,}500, 5{,}750)\ .$$

This phenomenon occurs because we compute a specific implied volatility for every given market price according to the other fuzzy parameters. For the lowest values $S(t) = 109{,}710$, $r = 2{,}373\%$ and $V_{MARKET} = 5{,}250$, the calculated implied volatility is σ= 26,216 %. That value is the highest volatility obtained, and according to the fuzzy sets theory, it would be the upper boundary of our fuzzy volatility parameter.

This fact has an important effect on the transformed price of the option, because the lower limit and the upper limit of the fuzzy number are more extreme than the direct market values. Thus, the lower boundary is lower and the highest boundary is higher.

In addition, we can observe the deviation between the fuzzy price computed with the parameters and the fuzzy price constructed through the market values.

$$\bar{\varepsilon} = (-0{,}954, 0, 0{,}761)\ .$$

Once commented this casuistic, we precede to calculate the "*α-cuts*" for the price computed through the fuzzy Black-Scholes formula.

Table 3. "*α-cuts*" of the triangular fuzzy European call price.

α	V_1^α	V_2^α
0	4,296	6,511
0,1	4,416	6,410
0,2	4,537	6,309
0,3	4,657	6,208
0,4	4,778	6,107
0,5	4,898	6,006
0,6	5,018	5,905
0,7	5,139	5,803
0,8	5,259	5,702
0,9	5,380	5,601
1	5,500	

Source: Own elaboration based on Yahoo finance data.

IV. RISK QUANTIFICATION

Once calculated the value of the option, we can proceed to quantify the market risk of our portfolio. As we said before, we are going to compute the most used risk measure, the Value-at-Risk. This risk measure is also one of those that banking companies have to calculate according to the Basel III framework.

As the price at the maturity time is unknown, we have to simulate the values in order to derive VaR. The process that we are going to follow for computing market risk is simple. First, we are going to use the Monte Carlo simulation methodology. Secondly, we are going to apply the Partial Monte Carlo simulation (Theta-Delta-Gamma approximation) after computing the Greeks. Both methodologies will be applied for the value obtained through the classical Black-Scholes formula and for the fuzzy transformed values.

In market risk management, VaR is calculated over short time horizons given the changing nature of the markets, usually in one or ten days. The confidence level ρ is usually at the 95% or the 99%. In our case, we will compute the one-day VaR ($\Delta t = \frac{1}{360}$) at the 99% confidence level ($\rho = 0{,}99$).

4.1. Black-Scholes option pricing formula

4.1.1. Monte Carlo simulation

Given the price calculated through the classical Black-Scholes formula, we start applying the Monte-Carlo simulation in order to compute VaR. The process is described simply below.

First, we generate market moves ($\Delta S(t)$) and revaluate the portfolio at $t + \Delta t$, that is, $V(S(t) + \Delta S(t), t + \Delta t)$. Then, we calculate the loss as $L = -\Delta V(t)$. Finally, we order the sample and compute the quantile at the ρ confidence level.

Applying this methodology we obtain:

$$VaR_{0,99}(L) = 1{,}751 \, .$$

4.1.2. Partial Monte Carlo simulation (Theta-Delta-Gamma approximation)

In order to capture the non-linearity in the option payoff structure, we can model $V\big(t, S(t)\big)$ through a quadratic Taylor expansion. The theta-delta-gamma approximation is based on this idea and is a useful way to compute VaR.

$$\Delta V \approx \frac{\partial V}{\partial t}\Delta t + \delta^T \Delta S + \frac{1}{2}\Delta S^T \Gamma \Delta S \, ,$$

But before starting to compute the market risk through this method, we need to define the Greeks.

The Greeks are calculated as the partial derivatives of the Black-Scholes model, and they represent the sensitivity of the price of the option to a change in the parameters of the model. These variables are widely used in hedging and risk management. The most common Greeks are the following ones:

Theta (Θ):

$$\frac{\partial V}{\partial t} = -\frac{S(t)\phi(d1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d2) \, .$$

Delta (δ):

$$\frac{\partial V}{\partial S(t)} = \Phi(d1) \, .$$

Gamma (Γ):

$$\frac{\partial^2 V}{\partial S(t)^2} = \frac{\phi(d1)}{S(t)\sigma\sqrt{\tau}} \, .$$

Vega (ν):

$$\frac{\partial V}{\partial \sigma} = S(t)\phi(d1)\sqrt{\tau} \, .$$

Rho (ρ):

$$\frac{\partial V}{\partial r} = K\tau e^{-r\tau}\Phi(d2) \, .$$

For the theta-delta-gamma computation we only need the first three Greeks previously explained. Substituting the values of the parameters we obtain,

$$\Theta = -10{,}625$$

$$\delta = 0{,}824$$

$$\Gamma = 0{,}058\,.$$

Once the three Greeks are obtained, we calculate ΔV through the Taylor expansion described above. Then, we follow the same steps as in the Monte-Carlo method. That is, we calculate the loss as $L = -\Delta V$ and order the sample. At last, we compute the quantile at the ρ confidence level.

Applying this theta-delta-gamma approximation we obtain:

$$VaR_{0,99}(L) = 1{,}793\,.$$

4.2. "Fuzzy" Black-Scholes option pricing formula

Now we proceed to compute the fuzzy Value-at-Risk. The value of the risk measure for the maximum level of presumption coincides with the VaR calculated through the classical Black-Scholes formula. Thus, we only have to compute VaR for the boundary prices of the fuzzy call.

In order to avoid confusion between the fuzzy numbers notation used in this thesis and the typical boundaries notation ($\overline{X}$), we will represent the upper limit as X^u and the lower limit as X^l.

4.2.1. Monte Carlo simulation

As we did before, we will start computing the market risk of our portfolio with the Monte-Carlo simulation. We obtain the following results applying this methodology:

$$VaR^l_{0,99}(L) = 1{,}549$$

$$VaR^u_{0,99}(L) = 2{,}538\,.$$

These two values in addition with the VaR obtained through the classical Black-Scholes model form the fuzzy VaR. That is,

$$\overline{VaR}_{0,99}(L) = \left(VaR^l_{0,99}(L), VaR_{0,99}(L), VaR^u_{0,99}(L)\right) = (1{,}549, 1{,}751, 2{,}538\,)\,.$$

Simplifying, we could represent this fuzzy VaR as an interval with its upper and lower boundary.

$$\overline{VaR}_{0,99}(L) = \left(VaR^l_{0,99}(L), VaR^u_{0,99}(L)\right) = (1{,}549, 2{,}538\,)\,.$$

This form to represent VaR is the same that the one used in Capital allocation, where the risk measure cannot be represented as a specific value.

4.2.2. Partial Monte Carlo simulation (Theta-Delta-Gamma approximation)

Now, we continue to compute the Greeks. The results obtained by replacing the parameters in the partial derivatives are the following ones:

$$\Theta^l = -10{,}447$$

$$\delta^l = 0{,}824$$

$$\Gamma^l = 0{,}060$$

$$\Theta^u = -18{,}439$$

$$\delta^u = 0{,}785$$

$$\Gamma^u = 0{,}039$$

After constructing the loss function and ordering the sample, we can compute the quantile for both limits, obtaining the upper and the lower boundaries of VaR.

$$VaR_{0,99}^l(L) = 1{,}597$$

$$VaR_{0,99}^u(L) = 2{,}621$$

As we did with the Monte-Carlo simulation, we represent the Value-at-Risk for the fuzzy option price as a fuzzy number.

$$\overline{VaR}_{0,99}(L) = \left(VaR_{0,99}^l(L), VaR_{0,99}(L), VaR_{0,99}^u(L)\right) = (1{,}597, 1{,}793, 2{,}621\,)\,.$$

We can also simplify the fuzzy VaR representing the interval only with its upper and lower limit. Thus,

$$\overline{VaR}_{0,99}(L) = \left(VaR_{0,99}^l(L), VaR_{0,99}^u(L)\right) = (1{,}597, 2{,}621\,)\,.$$

V. RESULTS ANALYSIS

5.1. Option price

In this part, we are going to compare the main results obtained in this thesis. Although our last objective is to analyze the impact of the introduction of the fuzzy sets in the risk quantification of our option portfolio, we are going to, first, analyze the prices of the derivative obtained with the Black-Scholes formula and with its fuzzy transformation.

Black-Scholes: $V = 5{,}500$

Fuzzy Black-Scholes: $\bar{V} = (4{,}296, 5{,}500, 6{,}511)$

As we can observe, with the introduction of the fuzzy Black-Scholes formula we obtain an interval of prices instead of having a unique and specific value for the option. The length of the interval captures the uncertainty associated to the variables because it is constructed with the maximum and minimum possible values taken by the parameters. This interval range could be reduced and turned into more precise values according to the individual subjective perception about the expected value of the parameters and the option price.

It is important to mention that, although we represent the call value as a fuzzy number, we also have the price obtained by applying the classical methodology. Thus, this form to represent the prices could be useful in uncertain scenarios and for volatile markets. We add certain flexibility in the model because, depending on the self-trust about the market, we can choose a more or less risky strategy.

5.2. Value-at-Risk boundaries

Now, we continue analyzing the VaR obtained through Monte-Carlo simulation and the Theta-Delta-Gamma approximation by using the classical option price and the fuzzy transformed value.

- *Monte-Carlo simulation*

$$VaR_{0,99}(L) = 1{,}751\ .$$

$$\overline{VaR}_{0,99}(L) = \left(VaR^{l}_{0,99}(L), VaR_{0,99}(L), VaR^{u}_{0,99}(L)\right) = (1{,}549, 1{,}751, 2{,}538\ )\ .$$

- *Partial Monte-Carlo simulation*

$$VaR_{0,99}(L) = 1{,}793\ .$$

$$\overline{VaR}_{0,99}(L) = \left(VaR^{l}_{0,99}(L), VaR_{0,99}(L), VaR^{u}_{0,99}(L)\right) = (1{,}597, 1{,}793, 2{,}621\ )\ .$$

Just as it happens with the option price, by computing the VaR as fuzzy number we can obtain an interval of possible values. In this case, it is even more important because the values denote the worst expected losses of our portfolio according to the selected parameters. With this methodology, the maximum and minimum possible VaR are represented, and depending on the uncertainty and the subjective risk appetite that an individual has, we will choose a specific or a wide range of values.

We think that this way of representing uncertainty is very useful in order to have different alternatives in the risk management. It is important to emphasize that the risk measures are computed under the same assumptions and with the same confidence level. It is known that different risk measures are available and that different distributions can be used, yet, we think that representing VaR through a range of possible values is an interesting way to deal against the fluctuations of the market.

It is also interesting the fact that we could represent the VaR interval in the same form that is used in capital allocation. Thus, we only represent the upper and the lower VaR boundaries by simplifying the notation as a classic interval. In this way, we obtain for the Monte-Carlo simulation the first expression and for the partial Monte-Carlo simulation the second one.

$$\overline{VaR}_{0,99}(L) = \left(VaR^{l}_{0,99}(L), VaR^{u}_{0,99}(L)\right) = (1{,}549, 2{,}538\,)\,,$$

$$\overline{VaR}_{0,99}(L) = \left(VaR^{l}_{0,99}(L), VaR^{u}_{0,99}(L)\right) = (1{,}597, 2{,}621\,)\,.$$

VI. CONCLUSIONS

In order to quantify the market risk of an option portfolio, we have to previously valuate this type of derivative. The Black-Scholes-Merton model is the most used option pricing model, but it is still not suitable for fuzzy environments. Market fluctuations lead to uncertain scenarios where the needed parameters to valuate the option cannot be determined as specific values, and consequently, neither the price.

The introduction of fuzzy parameters to the Black-Scholes formula enables us to represent the option price as a fuzzy number by using the fuzzy Black-Scholes formula. Computing the value of the option as fuzzy numbers makes the model more flexible and realistic, because it captures extra information not considered in the traditional models. By modelling the parameters and the prices through the fuzzy sets methodology, we can capture the market uncertainty and the subjectivity of individuals in addition to the randomness considered by the original model. Also, we can obtain the option price obtained with the classical Black-Scholes formula by assuming the highest presumption level about the expected value of the derivative.

It is important to mention that, by computing the fuzzy option price using the fuzzy implied volatility, we obtain different values to the ones we obtain by constructing a fuzzy number with the option market values. By doing so, the fuzzy option price interval becomes more extreme using the fuzzy implied volatility, that is, the upper boundary is higher and the lower boundary is lower than the limits obtained through the given market values.

Once the fuzzy transformed option price is computed, we can also calculate the Value-at-Risk of the portfolio as a fuzzy number. This is a useful way to quantify risk, because we obtain the maximum and the minimum worst expected loss of the portfolio according to the option price and the selected parameters. Depending on the uncertainty and the subjective risk appetite that an individual has, we will choose a more or less risky strategy.

Even in the fuzzy transformation in the VaR, the same assumptions are considered in the calculation of the different risk levels. We can simplify the notation of the risk interval by only representing it with the highest and the lowest boundaries. This form ~~it~~ is the same that the one used in capital allocation, where risk cannot be determined as a specific value. In that way, we obtain an interval instead of a specific value.

Finally, we conclude that under uncertain scenarios, the fuzzy sets theory is a useful methodology to valuate and to quantify the market risk of an option portfolio, dealing against market fluctuations associated to the prices.

VII. APPENDIX

```
###Implied volatility###
f<-function(S,r,sigma,tau,K,Vmarket){
  d1<-(log(S/K)+(r+(1/2)*sigma^2)*tau)/(sigma*tau^(1/2))
  d2<-d1-sigma*tau^(1/2)
  V<-S*pnorm(d1)-exp(-r*tau)*K*pnorm(d2)
  f<-V-Vmarket;
  f}
vega<-function(S,r,sigma,tau,K,Vmarket){
  vega<-S*(tau^(1/2))*dnorm((log(S/K)+(r+(1/2)*sigma^2)*tau)/(sigma*tau^(1/2)))
  vega}
S<-
sigma<-
r<-
K<-
tau<-
Vmarket<-
maxlevel<-10^(-10)
repeat{
  sigmaold<-sigma
  sigma<-sigmaold-f(S,r,sigma,tau,K,Vmarket)/vega(S,r,sigma,tau,K,Vmarket)
  if(abs(sigma-sigmaold)<maxlevel){
    break
  }
}
cat(sprintf("%f\n",sigma))

###Option pricing###
##Black-Scholes formula##
```

```
Black_Scholes<-function(S,r,sigma,tau,K){
  d1<-(log(S/K)+(r+(1/2)*sigma^2)*tau)/(sigma*tau^(1/2))
  d2<-d1-sigma*tau^(1/2)
  V<-S*pnorm(d1)-exp(-r*tau)*K*pnorm(d2)
  return(V)}

S<-
sigma<-
r<-
K<-
tau<-

Black_Scholes(S,r,sigma,tau,K)

##Monte-Carlo simulation##
n<-10^5
Z<-rnorm(n)
ST<-S*exp((r-(1/2)*sigma^2)*tau+sigma*Z*(tau)^(1/2))
mcV<-rep(1,n)
for (i in 1:n){
  if (ST[i]-K>0) {mcV[i]<-exp(-r*tau)*(ST[i]-K)}
  else {mcV[i]<-0}
}
mcVn<-sum(mcV)/n;mcVn

###Fuzzy number###
mu_x<-function(x,a1,a2,a3){
  mux<-ifelse(x<=a1,0,ifelse(x>a1&x<=a2,(x-a1)/(a2-a1),ifelse(x>a2&x<=a3,(a3-
x)/(a3-a2),0)))
  return(mux)
```

```
}
x<-seq()
plot(x,mu_x(x,a1,a2,a3),type='l')

###VaR###
##Monte-Carlo simulation##
n<-10^5
deltat<-1/360
alpha<-0.99
L<-rep(1,n);length(L)
sd<-(S*sigma*(deltat)^(1/2));sd
MCd1<-rep(1,n)
MCd2<-rep(1,n)
MCV<-rep(1,n)
MCS<-rep(1,n)
deltaS<-rep(1,n)
for(i in 1:n){
  deltaS[i]<-rnorm(1,mean=0,sd=sd)
  MCS[i]<-S+deltaS[i]
  MCd1[i]<-(log(MCS[i]/K)+(r+1/2*sigma^2)*(tau))/(sigma*(tau)^(1/2))
  MCd2[i]<-MCd1[i]-sigma*(tau)^(1/2)
  MCV[i]<-MCS[i]*pnorm(MCd1[i])-exp(-r*(tau))*K*pnorm(MCd2[i])
  L[i]<-Black_Scholes(S,r,sigma,tau,K)-MCV[i]
}
L<-sort(L)
VaR<-quantile(L,alpha);VaR

###Delta-Gamma approximation##
theta<-function(S,sigma,tau,r,K){
```

```
  d1<-(log(S/K)+(r+(1/2)*sigma^2)*tau)/(sigma*tau^(1/2))
  d2<-d1-sigma*tau^(1/2)
  theta<--(S*dnorm(d1)*sigma)/(2*(tau)^(1/2))-r*K*exp(-r*tau)*pnorm(d2)
  return(theta)}
delta<-function(S,sigma,tau,r,K){
  d1<-(log(S/K)+(r+(1/2)*sigma^2)*tau)/(sigma*tau^(1/2))
  delta<-pnorm(d1)
  return(delta)}
gamma<-function(S,sigma,tau,r,K){
  d1<-(log(S/K)+(r+(1/2)*sigma^2)*tau)/(sigma*tau^(1/2))
  gamma<-dnorm(d1)/(S*sigma*(tau)^(1/2))
  return(gamma)}
theta(S,sigma,tau,r,K)
delta(S,sigma,tau,r,K)
gamma(S,sigma,tau,r,K)
deltaV<-
theta(S,sigma,tau,r,K)*deltat+delta(S,sigma,tau,r,K)*deltaS+(gamma(S,sigma,tau,r,K)*
deltaS^2)/2
Ldg<--deltaV
Ldg<-sort(Ldg)
VaRdg<-quantile(Ldg,alpha);VaRdg
```

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