SCHOTTKY VIA THE PUNCTUAL HILBERT SCHEME

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Abstract. We show that a smooth projective curve of genus g can be reconstructed from its polarized Jacobian (X, Θ) as a certain locus in the Hilbert scheme $\mathrm{Hilb}^d(X)$, for $d = 3$ and for $d = g + 2$, defined by geometric conditions in terms of the polarization Θ. The result is an application of the Gunning–Welters trisecant criterion and the Castelnuovo–Schottky theorem by Pareschi–Popa and Grushevsky, and its scheme theoretic extension by the authors.

1. Introduction

Let (X, Θ) be an indecomposable principally polarized abelian variety (ppav) of dimension g over an algebraically closed field k of characteristic different from 2. The polarization Θ is considered as a divisor class under algebraic equivalence, but for notational convenience, we shall fix a representative $\Theta \subset X$. (X, Θ) being indecomposable means that Θ is irreducible.

The geometric Schottky problem asks for geometric conditions on (X, Θ) which determine whether it is isomorphic, as a ppav, to the Jacobian of a nonsingular genus g curve C. The Torelli theorem then guarantees the uniqueness of the curve C up to isomorphism. One may ask for a constructive version: can you "write down" the curve C, starting from (X, Θ)? Even though one may embed C in its Jacobian X, there is no canonical choice of such an embedding, so one cannot reconstruct C as a curve in X without making some choices along the way. We refer to Mumford's classic [11] for various approaches and answers to the Schottky and Torelli problems, and also to Arbarello [1], Beauville [2] and Debarre [3] for more recent results.

In this note, we show that any curve C sits naturally inside the punctual Hilbert scheme of its Jacobian X. We give two versions: firstly, using the Gunning–Welters criterion [7, 14], characterizing Jacobians by having many trisecants, we reconstruct C as a locus in $\mathrm{Hilb}^3(X)$. Secondly, using the Castelnuovo–Schottky theorem, quoted below, we reconstruct C as a locus in $\mathrm{Hilb}^{g+2}(X)$. In fact, for any indecomposable ppav (X, Θ), we define a certain locus in the Hilbert scheme $\mathrm{Hilb}^d(X)$ for $d \geq 3$, and show that this locus is either empty, or one or two copies of a curve C, according to whether (X, Θ) is not a Jacobian, or the Jacobian of the hyperelliptic or nonhyperelliptic curve C. Then we characterize the locus in question for $d = 3$ in terms of trisecants, and for $d = g + 2$ in terms of being in special position with respect to 2Θ-translates.

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To state the results precisely, we introduce some notation. For any subscheme $V \subset X$, we shall write $V_x \subset X$ for the translate $V - x$ by $x \in X$. Let $\psi\colon X \to \mathbb{P}^{2^g-1}$ be the (Kummer) map given by the linear system $|2\Theta|$.

Theorem A. *Let $Y \subset \mathrm{Hilb}^3(X)$ be the subset consisting of all subschemes $\Gamma \subset X$ with support $\{0\}$, with the property that*

$$\{x \in X \,|\, \Gamma_x \subset \psi^{-1}(\ell) \text{ for some line } \ell \subset \mathbb{P}^{2^g-1}\}$$

has positive dimension. Then Y is closed and

(1) *if X is not a Jacobian, then $Y = \emptyset$;*
(2) *if $X \cong \mathrm{Jac}(C)$ for a hyperelliptic curve C, then Y is isomorphic to the curve C;*
(3) *if $X \cong \mathrm{Jac}(C)$ for a non-hyperelliptic curve C, then Y is isomorphic to a disjoint union of two copies of C.*

The proof is by reduction to the Gunning–Welters criterion; more precisely to the characterization of Jacobians by inflectional trisecants. Note that the criterion defining Y only depends on the algebraic equivalence class of Θ, and not the chosen divisor.

For the second version, we need some further terminology from [12] and [6].

Definition 1.1. A finite subscheme $\Gamma \subset X$ of degree at least $g + 1$ is *theta-general* if, for all subschemes $\Gamma_d \subset \Gamma_{d+1}$ in Γ of degree d and $d + 1$ respectively, with $d \leq g$, there exists $x \in X$ such that the translate Θ_x contains Γ_d, but not Γ_{d+1}.

Definition 1.2. A finite subscheme $\Gamma \subset X$ is *in special position* with respect to 2Θ-translates if the codimension of $H^0(X, \mathscr{I}_\Gamma(2\Theta_x))$ in $H^0(\mathscr{O}_X(2\Theta_x))$ is smaller than $\deg \Gamma$ for all $x \in X$.

Again note that these conditions depend only on the algebraic equivalence class of Θ. The term "special position" makes most sense for Γ of small degree, at least not exceeding $\dim H^0(\mathscr{O}_X(2\Theta_x)) = 2^g$.

Our second version reads:

Theorem B. *Let $Y \subset \mathrm{Hilb}^{g+2}(X)$ be the subset consisting of all subschemes $\Gamma \subset X$ with support $\{0\}$, which are theta-general and in special position with respect to 2Θ-translates. Then Y is locally closed, and*

(1) *if X is not a Jacobian, then $Y = \emptyset$;*
(2) *if $X \cong \mathrm{Jac}(C)$ for a hyperelliptic curve C, then Y is isomorphic to the curve C minus its Weierstraß points;*
(3) *if $X \cong \mathrm{Jac}(C)$ for a non-hyperelliptic curve C, then Y is isomorphic to a disjoint union of two copies of C minus its Weierstraß points.*

The proof of Theorem B is by reduction to the Castelnuovo–Schottky theorem, which is the following:

Theorem 1.3. *Let $\Gamma \subset X$ be a finite subscheme of degree $g+2$, in special position with respect to 2Θ-translates, but theta-general. Then there exist a nonsingular curve C and an isomorphism $\mathrm{Jac}(C) \cong X$ of* PPAV*s, such that Γ is contained in the image of C under an Abel–Jacobi embedding.*

Here, an *Abel–Jacobi embedding* means a map $C \to \mathrm{Jac}(C)$ of the form $p \mapsto p - p_0$ for some chosen base point $p_0 \in C$. This theorem, for reduced Γ, is due to Pareschi–Popa [12] and, under a different genericity hypothesis, Grushevsky [4, 5]. The scheme theoretic extension stated above is by the authors [6]. The scheme theoretic generality is clearly essential for the application in Theorem B.

We point out that the Gunning–Welters criterion is again the fundamental result that underpins Theorem 1.3, and thus Theorem B. More recently, Krichever [9] showed that Jacobians are in fact characterized by the presence of a single trisecant (as opposed to a positive dimensional family of translations), but we are not making use of this result.

2. Subschemes of Abel–Jacobi curves

For each integer $d \geq 1$, let

$$Y_d \subset \mathrm{Hilb}^d(X)$$

be the closed subset consisting of all degree d subschemes $\Gamma \subset X$ such that

(i) the support of Γ is the origin $0 \in X$,
(ii) there exists a smooth curve $C \subset X$ containing Γ, such that the induced map $\mathrm{Jac}(C) \to X$ is an isomorphism of PPAV's.

We give Y_d the induced reduced scheme structure.

We shall now prove analogues of (1), (2) and (3) in Theorems A and B for Y_d with $d \geq 3$:

Proposition 2.1. *With $Y_d \subset \mathrm{Hilb}^d(X)$ as defined above, we have:*

(1) *If X is not a Jacobian, then $Y_d = \emptyset$.*
(2) *If $X \cong \mathrm{Jac}(C)$ for a hyperelliptic curve C, then Y_d is isomorphic to the curve C.*
(3) *If $X \cong \mathrm{Jac}(C)$ for a non-hyperelliptic curve C, then Y_d is isomorphic to a disjoint union of two copies of C.*

As preparation for the proof, consider a Jacobian $X = \mathrm{Jac}(C)$ for some smooth curve C of genus g. It is convenient to fix an Abel–Jacobi embedding $C \hookrightarrow X$; any other curve $C' \subset X$ for which $\mathrm{Jac}(C') \to X$ is an isomorphism is of the form $\pm C_x$ for some $x \in X$. Such a curve $\pm C_x$ contains the origin $0 \in X$ if and only if $x \in C$. Hence Y_d is the image of the map

$$\phi = \phi_+ \coprod \phi_- \colon C \coprod C \to \mathrm{Hilb}^d(X)$$

that sends $x \in C$ to the unique degree d subscheme $\Gamma \subset \pm C_x$ supported at 0, with the positive sign on the first copy of C and the negative sign on the second copy.

More precisely, ϕ is defined as a morphism of schemes as follows. Let $m\colon X \times X \to X$ denote the group law, and consider

$$m^{-1}(C) \cap (C \times X)$$

as a family over C via first projection. The fibre over $p \in C$ is C_p. Let $N_d = V(\mathfrak{m}_0^d)$ be the $d-1$'st order infinitesimal neighbourhood of the origin in X. Then

$$Z = m^{-1}(C) \cap (C \times N_d) \subset C \times X$$

is a C-flat family of degree d subschemes in X; its fibre over $p \in C$ is $C_p \cap N_d$. This family defines $\phi_+\colon C \to \mathrm{Hilb}^d(X)$, and we let $\phi_- = -\phi_+$ (where the minus sign denotes the automorphism of $\mathrm{Hilb}^d(X)$ induced by the group inverse in X).

Lemma 2.2. *The map $\phi_+\colon C \to \mathrm{Hilb}^d(X)$ is a closed embedding for $d > 2$.*

In the proof of the Lemma, we shall make use of the difference map $\delta\colon C \times C \to X$, sending a pair (p, q) to the degree zero divisor $p - q$. We let $C - C \subset X$ denote its image. If C is hyperelliptic, we may and will choose the Abel–Jacobi embedding $C \subset X$ such that the involution -1 on X restricts to the hyperelliptic involution ι on C. Thus, when C is hyperelliptic, $C - C$ coincides with the distinguished surface W_2, and the difference map δ can be factored via the symmetric product $C^{(2)}$:

$$\begin{tikzcd} C \times C \arrow[r, "1\times\iota", "\cong"'] \arrow[d] & C \times C \arrow[d, "\delta"] \\ C^{(2)} \arrow[r] & X \end{tikzcd}$$

We note that the double cover $C \times C \to C^{(2)}$, that sends an ordered pair to the corresponding unordered pair, is branched along the diagonal, so that via $1 \times \iota$, the branching divisor becomes the "antidiagonal" $(1, \iota)\colon C \hookrightarrow C \times C$.

As is well known, the surface $C - C$ is singular at 0, and nonsingular everywhere else. The blowup of $C - C$ at 0 coincides with $\delta\colon C \times C \to C - C$ when C is nonhyperelliptic, and with the addition map $C^{(2)} \to W_2$ when C is hyperelliptic.

Proof of Lemma 2.2. To prove that ϕ_+ is a closed embedding, we need to show that its restriction to any finite subscheme $T \subset C$ of degree 2 is nonconstant, i.e. that the family $Z|_T$ is not a product $T \times \Gamma$. For this it suffices to prove that if Γ is a finite scheme such that

$$m^{-1}(C) \supset T \times \Gamma, \tag{1}$$

then the degree of Γ is at most 2.

Consider the following commutative diagram:

$$
\begin{tikzcd}
X \times X \arrow[r, "(m,pr_2)", "\cong"'] & X \times X \\
m^{-1}(C) \cap (X \times C) \arrow[u, hook] \arrow[r, "\cong"'] \arrow[dr, "pr_1"'] & C \times C \arrow[u, hook] \arrow[d, "\delta"] \\
& X
\end{tikzcd}
\tag{2}
$$

First suppose $T = \{p, q\}$ with $p \neq q$. The claim is then simply that $C_p \cap C_q$, or equivalently its translate $C \cap C_{q-p}$, is at most a finite scheme of degree 2. Diagram (2) identifies the fibre $\delta^{-1}(q-p)$ on the right with precisely $C \cap C_{q-p}$ on the left. But $\delta^{-1}(q-p)$ is a point when C is nonhyperelliptic, and two points if C is hyperelliptic.

Next suppose $T \subset C$ is a nonreduced degree 2 subscheme supported in p. Assuming Γ satisfies (1), we have $\Gamma \subset C_p$, so

$$m^{-1}(C) \cap (X \times C_p) \supset T \times \Gamma$$

or equivalently

$$m^{-1}(C) \cap (X \times C) \supset T_p \times \Gamma_{-p}.$$

We have $T_p \subset C - C$, and Diagram (2) identifies $\delta^{-1}(T_p)$ on the right with $m^{-1}(C) \cap (T_p \times C)$ on the left.

Suppose C is nonhyperelliptic. Then δ is the blowup of $0 \in C - C$, and $\delta^{-1}(T_p)$ is the diagonal $\Delta_C \subset C \times C$ together with an embedded point of multiplicity 1 (corresponding to the tangent direction of $T_p \subset C - C$). Diagram (2) identifies the diagonal in $C \times C$ on the right with $\{0\} \times C$ on the left. Thus $m^{-1}(C) \cap (T_p \times C)$ is $\{0\} \times C \subset X \times C$ with an embedded point. Equivalently, $m^{-1}(C) \cap (T \times C_p)$ is $\{p\} \times C_p$ with an embedded point, say at (p, q). This contains no constant family $T \times \Gamma$ except for $\Gamma = \{q\}$, so Γ has at most degree 1.

Next suppose C is hyperelliptic. We claim that $\delta^{-1}(T_p)$ is the diagonal $\Delta_C \subset C \times C$ with either two embedded points of multiplicity 1, or one embedded point of multiplicity 2. As in the previous case, this implies that $m^{-1}(C) \cap (T \times C_p)$ is $\{p\} \times C_p$ with two embedded points of multiplicity 1 or one embedded point of multiplicity 2, and the maximal constant family $T \times \Gamma$ it contains has Γ of degree 2. It remains to prove that $\delta^{-1}(T_p)$ is as claimed.

We have $W_2 = C - C$, and the blowup at 0 is $C^{(2)} \to W_2 = C - C$. The preimage of T_p is the curve $(1 + \iota) \colon C \to C^{(2)}$, together with an embedded point of multiplicity 1, say supported at $q + \iota(q)$. Now the two to one cover $C \times C \to C^{(2)}$ is branched along the diagonal $2C \subset C^{(2)}$, If $q \neq \iota(q)$, then the preimage in $C \times C$ is just $(1, \iota) \colon C \to C \times C$, together with two embedded points of multiplicity 1, supported at $(q, \iota(q))$ and $(\iota(q), q)$. If $q = \iota(q)$, i.e. q is Weierstraß, then we claim the preimage in $C \times C$ is $(1, \iota) \colon C \to C \times C$ together with an embedded point of multiplicity 2. This follows once we know that the curves $2C$ and $(1 + \iota)(C)$ in $C^{(2)}$ intersect transversally. And they do, as the tangent spaces of the two curves $(1,1)(C)$ (the diagonal) and $(1,\iota)(C)$ in $C\times C$ are invariant under the involution exchanging the two factors, with eigenvalues 1 and -1, respectively. □

Proof of Proposition 2.1. Point (1) is obvious, so we may assume $X = \operatorname{Jac} C$. By Lemma 2.2, ϕ_+ is a closed embedding and hence so is $\phi_- = -\phi_+$. If C is hyperelliptic, we have chosen the embedding $C \subset X$ such that the involution -1 on X extends the hyperelliptic involusion ι on C. It follows that $C_p = -C_{\iota(p)}$, and thus $\phi_- = \phi_+ \circ \iota$. Thus the two maps ϕ_+ and ϕ_- have coinciding image, and (2) follows.

For (3), it remains to prove that if C is nonhyperelliptic, then the images of ϕ_- and ϕ_+ are disjoint, i.e. we never have $C_p \cap N_d = (-C_q)\cap N_d$ for distinct points $p, q \in C$. In fact, $C_p \cap (-C_q)$ is at most a finite scheme of degree 2: the addition map

$$C \times C \to X$$

is a degree two branched cover of $C^{(2)} \cong W_2$ (using that C is nonhyperelliptic), and its fibre over $p+q \in W_2$ is isomorphic to $C_p \cap (-C_q)$. □

3. Proof of Theorem A

In view of Proposition 2.1, it suffices to prove that Y in Theorem A agrees with Y_3 in Proposition 2.1. This is a reformulation of the Gunning–Welters criterion: given $\Gamma \in \operatorname{Hilb}^3(X)$, consider the set

$$V_\Gamma = \{x \in X \,|\, \Gamma_X \subset \psi^{-1}(\ell) \text{ for some line } \ell \subset \mathbb{P}^{2^g-1}\}.$$

Then Gunning–Welters says that V_Γ has positive dimension if and only if (X,Θ) is a Jacobian. Moreover, when V_Γ has positive dimension, it is a smooth curve, the canonical map $\operatorname{Jac}(V_\Gamma) \to X$ is an isomorphism, and Γ is contained in V_Γ (see [15, Theorem (0.4)]). Thus Y in Theorem A agrees with Y_3 in Proposition 2.1.

4. Proof of Theorem B

Let X be the Jacobian of C. For convencience, we fix an Abel–Jacobi embedding $C \hookrightarrow X$. First, we shall analyse theta-genericity for finite subschemes of C.

Recall the notion of *theta-duality*: whenever $V \subset X$ is a closed subscheme, we let

$$T(V) = \{x \in X \,|\, V \subset \Theta_x\}.$$

It has a natural structure as a closed subscheme of X (see [13, Section 4] and [6, Section 2.2]); the definition as a (closed) subset is sufficient for our present purpose.

With this notation, theta-genericity means that for all chains of subschemes

$$\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_{g+1} \subset \Gamma, \tag{3}$$

where Γ_i has degree i, the corresponding chain of theta-duals,

$$T(\Gamma_1) \supset T(\Gamma_2) \supset \cdots \supset T(\Gamma_{g+1}),$$

consists of strict inclusions of sets.

We write $\widehat{\mathscr{F}}$ for the Fourier–Mukai transform [10, 8] of a WIT-sheaf $\mathscr{F}$ on X [10, Def. 2.3]: $\widehat{\mathscr{F}}$ is a sheaf on the dual abelian variety, which we will identify with X using the principal polarization.

Proposition 4.1. *Let $\Gamma \subset C$ be a finite subscheme of degree at least $g+1$. Then Γ is theta-general, as a subscheme of* $\mathrm{Jac}(C)$*, if and only if* $\dim H^0(\mathscr{O}_C(\Gamma_g)) = 1$ *for every degree g subscheme $\Gamma_g \subset \Gamma$. In particular, if Γ is supported at a single point $p \in C$, then Γ is theta-general if and only if p is not a Weierstraß point.*

Proof. For the last claim, note that the condition $\dim H^0(\mathscr{O}_C(gp)) > 1$ says precisely that p is a Weierstraß point.

For any effective divisor $\Gamma_g \subset C$ degree g, it is well known that $\dim H^0(\mathscr{O}_C(\Gamma_g)) = 1$ if and only if Γ_g can be written as the intersection of $C \subset \mathrm{Jac}(C)$ and a Θ-translate (this is one formulation of Jacobi inversion). If this is the case, then the point $x \in X$ satisfying $\Gamma_g = C \cap \Theta_x$ is unique.

Consider a chain (3). If there is a degree g subscheme $\Gamma_g \subset \Gamma$ not of the form $C \cap \Theta_x$, then every Θ-translate containing Γ_g also contains C, and in particular $T(\Gamma_g) = T(\Gamma_{g+1})$. Hence Γ is not theta-general.

Suppose, on the other hand, that Γ_g is of the form $C \cap \Theta_x$. Then $T(\Gamma_g) \setminus T(\Gamma_{g+1})$ consists (as a set) of exactly the point x. Thus there is a Zariski open neighbourhood $U \subset X$ of x such that $T(\Gamma_g) \cap U = \{x\}$. We claim that, for a possibly smaller neighbourhood U, there are regular functions $f_1, \ldots, f_g \in \mathscr{O}_X(U)$, such that $T(\Gamma_i) \cap U = V(f_1, \ldots, f_i)$ for all i: in fact, apply the Fourier–Mukai functor to the short exact sequence

$$0 \to \mathscr{I}_{\Gamma_i}(\Theta) \to \mathscr{O}_X(\Theta) \to \mathscr{O}_{\Gamma_i} \to 0$$

to obtain

$$0 \to \mathscr{O}_X(-\Theta) \xrightarrow{F_i} \widehat{\mathscr{O}_{\Gamma_i}} \to \widehat{\mathscr{I}_{\Gamma_i}(\Theta)} \to 0.$$

Then F_i is a section of a locally free sheaf of rank i, and its vanishing locus is exactly $T(\Gamma_i)$. Choose trivializations of $\widehat{\mathscr{O}_{\Gamma_i}}$ over U for all i compatibly, in the sense that the surjections $\widehat{\mathscr{O}_{\Gamma_{i+1}}} \to \widehat{\mathscr{O}_{\Gamma_i}}$ correspond to projection to the first i factors. Then $F_i = (f_1, \ldots, f_i)$ in these trivializations.

As $T(\Gamma_g) \cap U$ is zero dimensional, it follows that each $T(\Gamma_i) \cap U$ has codimension i in U. Hence all the inclusions $T(\Gamma_i) \supset T(\Gamma_{i+1})$ are strict, and so Γ is theta-general. □

Now we can compare the locus Y in Theorem B with Y_{g+2} in Proposition 2.1 by means of the Castelnuovo–Schottky theorem:

Corollary 4.2 (of Theorem 1.3). *Let $\Gamma \in \mathrm{Hilb}^{g+2}(X)$ be theta-general and supported at $0 \in X$. Then Γ is in the locus Y_{g+2} in Propsition 2.1 if and only if it is in special position with respect to 2Θ-translates.*

Proof. Theorem 1.3 immediately shows that if Γ is in special position with respect to 2Θ-translates, then $\Gamma \in Y_{g+2}$.

The converse is straight forward, and does not require the theta-genericity assumption. Indeed, we use that any curve $C' \subset X$ for which $\mathrm{Jac}(C') \to X$ is an isomorphism is of the form $\pm C_p$ for some $p \in X$ and we claim that if $\Gamma \subset \pm C_p$, then Γ is in special position with respect to 2Θ-translates. For ease of notation, we rename $\pm C_p$ as C, so that $\Gamma \subset C$. Then $H^0(\mathscr{I}_C(2\Theta_x)) \subset H^0(\mathscr{I}_\Gamma(2\Theta_x))$, and the exact sequence

$$0 \to H^0(\mathscr{I}_C(2\Theta_x)) \to H^0(\mathscr{O}_X(2\Theta_x)) \to H^0(\mathscr{O}_C(2\Theta_x))$$

shows that already the codimension of $H^0(\mathscr{I}_C(2\Theta_x))$ in $H^0(\mathscr{O}_X(2\Theta_x))$ is at most $\dim H^0(\mathscr{O}_C(2\Theta_x)) = g+1$. $\square$

Theorem B now follows: The set Y defined there agrees with the theta-general elements in Y_{g+2}, by the Corollary. By Proposition 4.1, $\Gamma = \phi_\pm(p)$ is theta-general if and only if the supporting point 0 of Γ is not Weierstraß in $\pm C_p$, i.e. $p \in C$ is not Weierstraß.

5. Historical remark

Assume C is not hyperelliptic. Then $C_p \cap (-C_p)$ is a finite subscheme of degree 2 supported at 0. Thus, for $d = 2$, we have $\phi_+ = \phi_-$, and the argument in Lemma 2.2 shows that ϕ_+ is an isomorphism from C onto Y_2. If C is hyperelliptic with hyperelliptic involution ι, however, we find that ϕ_+ factors through $C/\iota \cong \mathbb{P}^1$, and $Y_2 \cong \mathbb{P}^1$, and we cannot reconstruct C from Y_2 alone.

In the nonhyperelliptic situation, it is well known that the curve C can be reconstructed as the projectivized tangent cone to the surface $C - C \subset X$ at 0. This projectivized tangent cone is exactly Y_2 (when we identify the projectivized tangent space to X at 0 with the closed subset of $\mathrm{Hilb}^2(X)$ consisting of nonreduced degree 2 subschemes supported at 0). To quote Mumford [11]: "If C is hyperelliptic, other arguments are needed." In the present note, these other arguments are to increase d!

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