UNIVERSITAT DE BARCELONA

Strategic Interactions in Marketing: A Dynamic Approach

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PhD in Business | Lijue Lu

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"My substitute for trying to be the first was to go off in a direction orthogonal to the mainstream and hope that I could find a small but deep backwater of my own."

—Paul R. Halmos, "How to Do Research"

献给我的父母。

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CHAPTER 1

Introduction

It is a well-known fact that a strong commitment to advertising is the key to success. This one-way communication from brands to customers is proved to be helpful in increasing the brand and product awareness, building brand images, differentiating the products from those of other companies, and so on. The importance of advertising is evidenced by the large and increasing amount of money spent by successful corporations. According to a report by AdAge (2017), Procter & Gamble, the largest global advertiser in 2016, allocated 10.5 billion U.S. dollars toward advertising activities, followed by Samsung ($9.9 billion), Nestle ($9.2 billion), Unilever ($8.6 billion) and L'Oreal ($8.3 billion).

Not surprisingly, increasing academic attention has been paid to an important research question: how should a firm decide the advertising expenditure in order to maximize the profit? One of the difficulties to answer this question might come from the inherently dynamic nature of marketing. Advertising is not a one-time announcement, instead, it is a continuous activity and requires careful inter-temporal planning. Moreover, its impact is not limited to the current period, but also in the future. As a consequence, the applications of dynamic models, which draw support from mathematical modeling and quantitative methods, to the filed of advertising have been flourishing since the early 1960s.

Academic efforts were devoted mainly, in the beginning, to one decision maker problems, where a monopolistic firm decides her optimal advertising level over time (we refer to Sethi, 1977; Feichtinger et al., 1994; Huang et al., 2012; Sethi & Thompson, 2000, for surveys of control theory models in advertising).

It was not long before the case of two or more firms competing/cooperating in advertising also attracted a great deal of attention, where the differential game approach was applied (see, e.g., Jørgensen, 1982; Erickson, 1995b; Huang et al., 2012; Erickson, 1991; Jørgensen & Zaccour, 2004; Dockner et al., 2000, for reviews on differential game models in advertising).

Due to its game theoretic foundation, apart from the dynamic perspective, this approach successfully involves another essential element of the marketing problems: strategic interactions among various agents, and has become one of the principal methodologies in marketing science.

The basic assumption of the differential game approach is that firms can, in one way or another, know or estimate the influences of advertising investment on certain state(s) which is/are relevant to the profit (for instance, sales), and summarize them in terms of differential equations. Based on different aspects of advertising's contributions, different dynamics are chosen, we can therefore identify four research lines in advertising, where the following seminal works are taken as starting points:

- Advertising market share models (Kimball, 1957)
  Originally used to depict military combat, the Lanchester model was firstly introduced by Kimball (1957) into the economic world because of the similarity between military and industrial operations. Assuming that the customers are naturally disloyal and will drift toward the firm with better advertising, and that the advertising only influences the rival firms' but not the firm's own customers, this model describes a situation where firms are competing for the market share with their advertising efforts. The Lanchester dynamics are presented as

$$\dot{x}_i(t) = f_i\left(a_i(t)\right)[1 - x_i(t)] - \sum_{\substack{j=1 \\ j \neq i}}^{N} f_j\left(a_j(t)\right) x_i(t)\,,$$

  where $x_i(t)$, $a_i(t)$ denote the market share and the advertising rate of firm i at time t, respectively. The function $f_i(a_i)$ is the advertising effectiveness function, which is assumed to be positive and increasing in a_i.

  An important contribution comes from Case (1979), where he characterized the feedback Nash equilibrium with linear advertising effectiveness functions and zero discounting. Some other influential studies include the incorporation of multiple marketing tools (Chintagunta & Vilcassim, 1994; Fruchter & Kalish, 1998), various types of advertising (Erickson, 1993), market expansion (Bass et al., 2005a,b), non zero discounting (Fruchter & Kalish, 1997; Jarrar et al., 2004; Sorger, 1989), the extension to oligopoly (Erickson, 2009), and some empirical tests run by Erickson (1992); Chintagunta & Vilcassim (1992); Chintagunta & Jain (1995); Wang & Wu (2007), and so on.

- Advertising sales model (Vidale & Wolfe, 1957)
  The Vidale-Wolfe model studies an optimal control problem, where the sales rate can be increased by advertising activities, and suffers a decay effect:
  $$\dot{S}(t) = ka(t)\,[M - S(t)] - \delta S(t)\,,$$
  where $S(t)$, $a(t)$ represent firm's sales and advertising effort, M stands for the market potential, and k and δ refer to the advertising effectiveness and sales depreciation rate. Note that this model can be expressed in terms of market share by substituting $x = S/M$.

  Sethi (1973) offered a detailed analysis of the advertising strategies in combination with bang-bang, impulse control, etc. A duopolistic differential game extension is provided by Deal (1979), where both firms can influence only the untapped market ($M - S_1(t) - S_2(t)$). However, it is also frequently combined with the Lanchester model by adding the capacity to attract rival firm's customers (e.g., Leitmann & Schmitendorf, 1978; Wang & Wu, 2001; Jørgensen et al., 2010). Some other important extensions include the case of oligopoly (Erickson, 1995a), a structural variation to incorporate the word-of-mouth communication (Sethi, 1983), and the introduction of uncertainty (Prasad & Sethi, 2009), and so forth.

- Advertising goodwill model (Nerlove & Arrow, 1962)
  This model mainly focuses on the impact of advertising in the rise in popularity and the building of the brand reputation. The goodwill of a company is considered as an intangible stock, which can be improved by advertising and depreciates over time due to the forgetting effects

of consumers, as presented in the following equation:

$$\dot{G}(t) = ka(t) - \delta G(t)\ ,$$

with $G(t)$ and $a(t)$ representing the goodwill and advertising, and k and δ being the advertising effectiveness and goodwill depreciation rate.

This model has been applied in many contexts. We refer to Tapiero (1979), Fornell et al. (1985), Chintagunta (1993), Buratto & Zaccour (2009) as some illustrations, where uncertainty, consumption experience, the sensitivity of a firm's profits with respect to deviation from equilibrium, and licensing contract are analyzed.

- New product diffusion model (Bass, 1969)
  This model studies the adoption process of a new product. Although advertising is not explicitly formulated in the original model, it is frequently incorporated as a major driver of adoption (for example, see Horsky & Simon, 1983, one of the seminal studies to introduce advertising in product diffusion). Let $S(t)$ denote cumulative sales, $a(t)$ and $f(a(t))$ denote advertising policies and the effectiveness function, M represent the potential market size, and η be the imitation coefficient, a common specification of the cumulative sales evolution is given by

$$\dot{S}(t) = f(a(t))\,[M - S(t)] + \eta\frac{S(t)}{M}\,[M - S(t)]\ .$$

  The first term on the right-hand side refer to the adoption by "innovators", who can be influenced by advertising. The second term represents the fact that "imitators" might adopt the new product after communicating with innovators.

  Some relevant studies are Dockner & Jørgensen (1988), where they suggested that advertising also affects imitators, Thompson & Teng (1984), where an oligopolistic version is provided, Kamrad et al. (2005), where word-of-mouth effect is analyzed in a stochastic environment, etc.

Note that these four research streams are not strictly separated, and some researchers have attempted to investigate multiple effects of advertising by formulating a differential game of several state variables (for example, El Ouardighi & Pasin, 2006), or to integrate different models (like the case of Lanchester and Vidale-Wolfe).

In addition to the state variables affected by advertising, different market structures also deserve special attention. The late 1990s witnessed a growing interest in the interaction among different decision makers in the supply chain (suppliers, manufacturers, distributors, retailers, customers, and so on). Leng & Parlar (2005); Cachon & Netessine (2006); He et al. (2007) offer some reviews of the applications of game theory/dynamic games in supply chain management. Among all the studies to date, static models clearly outnumber dynamic ones. Academic interest in supply chain management has been primarily paid to coordination mechanisms. Research questions such as if the cooperation is beneficial, which coordination scheme yields better outcome, how to sustain the cooperation without binding contract, and so on, are commonly addressed.

Besides the dynamics employed, another key element of any differential game model is the objective functional, which relies on the discounted utility theory proposed by Samuelson (1937). Typically, a firm's objective is to maximize the discounted profits over planning horizon to the present time (the time when the decisions are made). Due to the mathematical tractability, the discount factor is usually assumed to be an exponential function, with a constant discount rate which is independent of the time perspective. Nevertheless, this way of discounting may not be the best candidate to explain some decision making behaviors. Actually, experimental and empirical studies on intertemporal choice have demonstrated various inadequacies of the standard time discounting (Frederick et al., 2002, give a comprehensive review of time discounting). In general, decision makers exhibit declining rates of time preferences. Moreover, people apply different discount rates depending on the types of goods and categories of decisions. For example, gains are more heavily discounted than losses (Thaler, 1981; Loewenstein, 1987; MacKeigan et al., 1993), a bigger size of the reward decreases the discount rate (Thaler, 1981; Holcomb & Nelson, 1992; Kirby et al., 1999), and so forth.

In response to the limitations of constant discounting, Strotz (1955) was the first one to suggest considering alternatives to the exponential discount function. Since then various alternate discounting models are raised, among which the best documented is *hyperbolic discounting*. Its advantage in descriptive realism is that it additionally captures the phenomenon of present bias. One class of models where hyperbolic discounting is very often introduced are those related to consumption-saving behaviors (e.g., Laibson, 1994, 1997, 1998; Angeletos et al., 2001; Bernheim et al., 2015). It also frequently appears in the studies of economic growth (for example, Barro, 1999; Krusell & Smith, JR., 2003; Ekeland & Lazrak, 2010). Fischer (1999) and O'Donoghue & Rabin (1999b, 2001) explore the implications of hyperbolic discounting for procrastination, whereas those for addiction are studied by Carrillo (1998), O'Donoghue & Rabin (1999a, 2002) and Gruber & Köszegi (2001).

Diminishing discount rates are also applied in environmental economics, such as Karp (2005) and Karp & Tsur (2011), etc.

Another quite highly referenced alternative to standard discounting is *heterogeneous discounting*. It describes a situation where the decision maker discounts the utility during the planning horizon and the final function at constant but different discount rates. This single modification depicts an additional realism that is not described by standard or hyperbolic discounting: by assuming a higher discount rate at the ending point, the valuation of the final function is increasing as the time approaches the end of planning period, whereas a decreasing valuation can be formulated if a lower discount rate is connected with the final function. Marín-Solano & Patxot (2012) proposed it as a response to the standard discounting anomaly that agents discount different types of goods in different ways. This approach is also introduced in consumption-saving problems in de-Paz et al. (2013, 2014). It is worth mentioning that in a cooperative environment where two agents cooperate in order to maximize the joint utility, if each player exhibits different rate of time preferences, the grand coalition will behave like an agent with heterogeneous discounting (for more details, see de-Paz et al., 2013), and will face the trade-off between time-consistency and efficiency (Jackson & Yariv, 2015).

In spite of being a popular topic in many fields such as resource economics, behavioral economics, financial economics, and so on, general time preferences have never been, to the best of our knowledge, introduced into management science. Nonetheless, we believe that importing insights from time perspective could help us to better understand the dynamic strategic interactions in marketing. First of all, at personal level, managers and practitioners, as human beings, are bound to be influenced by the individual behavioral biases even when they are making professional decisions. Secondly, similar to public policy making, a firm might also have limited commitment which gives rise to dynamic inconsistency. The current executive can not guarantee that her successor would follow the strategies made by now. Moreover, the rate of time preferences for a firm is a reflection of multiple internal and external factors, including but not limited to firm size, uncertainty, information availability, financial health, legislative constraints, survival probabilities, state of economy, business cycles, and so on. It seems natural to consider that the discount rate can be varying across time. Besides, time inconsistency could stem from cooperation among divergent agents, even if each firm discounts the future profits at constant discount rate. Therefore, it is suggested to conduct some exploratory studies incorporating time perspectives.

Hence, one of the primary emphases in this thesis is to introduce more general time preferences into dynamic advertising strategies. To do so, we apply the market share (Lanchester) model and the goodwill (Nerlove and Arrow) model, and cover both duopolistic market and supply chain system. In this way we can have a more general and comprehensive understanding of the advertising's different functions and mechanisms in different market structures with the presence of several types of time preferences. More specifically, in Chapter 2, we have introduced individual time-inconsistent preferences into the Lanchester model, a duopolistic advertising competition model, and study the investment behaviors by computing the Markovian strategies. We have also considered, in Chapter 3, the time inconsistency derived from collective dynamic choice, and have analyzed the induced inefficiency in a cooperative setting.

After reaching a deeper understanding of advertising activities, we shift our attention to the interface between advertising and other business activities. Research questions of how firms can jointly apply advertising and other marketing tools (typically, pricing) are quite frequently studied (e.g., Thompson & Teng, 1984; Fershtman et al., 1990; Chintagunta, 1993; He et al., 2009; Bertuzzi & Lambertini, 2010; Krishnamoorthy et al., 2010, and so on). Nevertheless, as pointed out by Jørgensen (2018), although marketing plans affect and are affected by activities conducted in other functional areas, yet studies tackling these intersections using optimal control/differential game approach are scarce. Undoubtedly, firms can benefit from a highly integrated organizational structure, and researches exploring how to coordinate different business functions could offer some managerial implications.

Particularly, we confine our interest to the *operations management* (also known as *production management*). On the one hand, it is one of the most essential functional areas of a business, especially for manufacturing companies. On the other hand, the production department aims to minimize the cost, whereas the marketing department intends to maximize the revenue. The potential conflict that might arise between these two functional areas requires special attention.

The effort of modeling the interaction between these two kinds of activities in a continuous-time setting can date back to the seventies (Eliashberg & Steinberg, 1993 and Gaimon, 1998 provide two surveys of production-marketing/pricing interface). However, early studies mainly addressed pricing, and it was not until 1987 that advertising was considered in an integrated system of finance, marketing, and production in a monopolistic environment (Abad, 1987). Lambertini & Palestini (2009) investigate an oligopolistic market where a cartel is competing with various fringes. Each firm needs to decide her advertising and output level, in cooperative or non-cooperative way. The inventory management is introduced by Erickson (2011), where backlogging is allowed, and two marketing tools (advertising and pricing) are available to the managers. De Giovanni et al. (2019) also consider the interaction among adverting, pricing and inventory management, but in a supply chain where cooperative advertising is adopted.

Among all the studies undertaking the interaction between advertising and operations management, quality improvement has attracted most effort, probably because of its close connection with marketing, in that quality is often related with higher price, and the information of quality needs to be delivered via advertising.

Most of the quality-advertising interface has been placed in the context of monopoly. For instance, Ringbeck (1985) analyzes how a firm decides the quality investment, advertising and pricing strategies if quality slows down the customer attrition, and finds it recommended to enter the market with low quality and price but high advertising, and to reverse the emphasis with the passage of time. Liu et al. (2015) compare different mechanisms to coordinate the operations department, which controls design quality, and the marketing department, which makes advertising plan and sets the price. Their results show that a committed dynamic transfer price paid by the marketing department can coordinate a decentralized company. El Ouardighi et al. (2016) study the allocation of resources of a firm when conformance quality can influence the advertising attracting effectiveness via word-of-mouth effects. They conclude that the investment to improve defective items is always beneficial, whereas intense advertising should be applied only when the sales are high. Reddy et al. (2016) investigate the situation where the design quality is improved at impulse time, and a better quality can mitigate the depreciation of goodwill and sales. A higher frequency of quality improvement is proved to be able to generate higher goodwill and sales level in Vidale-Wolfe and Nerlove-Arrow models. The interaction among pricing, advertising and quality may lead to two possible optimal paths depending on the demand potential, according to Caulkins et al. (2017). If the path of higher sales and quality is more preferred, the government might be willing to subsidize quality investments. Chenavaz & Jasimuddin (2017) undertake a study to revise the linkage between advertising and quality level. A positive relationship can be found if the contribution of quality and advertising to demand is superior to the increased production cost, otherwise a negative interrelation is detected. The analysis carried out by De Giovanni (2019) demonstrates the importance of appraisal and prevention effort on non-defective products in conformance quality. Under a total quality management scheme, the need for advertising is decreasing as the brand is well recognized.

Few researchers have addressed the interface between quality and advertising in a competitive environment. Colombo & Lambertini (2003) consider a duopolistic market with product differentiation through quality, where two firms need to decide their advertising and quality investment. They have found that the firm selling low-quality products might get higher profits with more efficient advertising. In the advertising battle presented by El Ouardighi & Pasin (2006), where only customers who have experienced defective products can be attracted by the other company, the firm of smaller size would put more resources in advertising, whereas the larger competitor would concentrate on quality improvement. If quality is able to contribute directly to the goodwill accumulation, as in Nair & Narasimhan (2006), the advertising and quality policies would be inversely related.

El Ouardighi et al. (2008) and De Giovanni (2011) are the only two studies that situate this issue in a supply chain, as far as we can tell. In the former study, the manufacturer chooses the production rate, quality improvement and advertising, whereas the retailer decides how much to purchase from the manufacturer and the price. They suggest the sequence of investing firstly in quality to increase royalty, then in advertising to attract new customers. De Giovanni (2011) analyzes whether it is beneficial to employ a cooperative advertising program where the manufacturer subsidizes the retailer, and to improve the product quality. He reaches the conclusion that the cooperation will happen only when the advertising's contribution is significant, and effort will not be paid to quality unless it is highly effective.

Shapiro (1977) highlighted eight points of "necessary cooperation but potential conflict" of the coordination between marketing and production, one of which being quality assurance. Quality is also included in the list of six most important interfaces between operations management and marketing raised by Montgomery & Hausman (1986). Furthermore, in a later research (Hausman & Montgomery, 1990), quality is found among the top five production priorities, as well as the five most evaluated factors by customers. Therefore, the last study of this thesis has followed the research stream of the interaction between advertising and quality management.

Moreover, most of the research to date has considered a deterministic setting. Studies dealing with marketing decisions under uncertainty are rather sparse, and have mainly taken into account the continuous stochastic fluctuation in sales/market shares, which is caused by noncompetitive factors such as the inherent randomness of customers' purchasing behaviors, lack of product differentiation, forgetting effect, and so on. Sethi (1983) introduced such disturbances into a stochastic model of advertising sales response in a monopolistic setting. Following the same research line, Prasad & Sethi (2004) extended the previous optimization problem to a duopolistic advertising competition, where the decay term from the Vidale-Wolfe model was remained. The authors also presented an oligopolistic extension in Prasad &

Sethi (2003). He et al. (2009) located this continuous stochastic noise in a supply chain structure, where the manufacturer supports part of the retailer's advertising and decides the wholesale price.

On the contrary, another kind of uncertainty, the *black swan events*, is rarely introduced into marketing. The black swan events refer to the unexpected events that can cause broad impact and serious consequences. Such uncertain changes can be incorporated into our framework through a *piecewise deterministic game*. This approach is frequently used to model problems of regime shifts in the areas of financial economics (e.g., Ngwira & Gerrard, 2007; Josa-Fombellida & Rincón-Zapatero, 2012), and environmental economics (for example, Clarke & Reed, 1994; Harris & Vickers, 1995; Polasky et al., 2011; van der Ploeg, 2014). To the best our knowledge, Rubel et al. (2011) is the only exception which has considered this kind of uncertainty. In the paper mentioned above the authors adopted the Lanchester model to study an optimal advertising problem, where a product-harm crisis can happen at any random instant and cause a sharp decrease in sales.

To this end, in Chapter 4, we have studied a situation where firms need to consider the interface among operations management (quality improvement), marketing (advertising), and public relations management when facing potential crises.

Summarizing, the aim of the thesis is to contribute to a better understanding of the strategic and dynamic interactions in some marketing problems by using a differential game approach. We focus on the effects of the introduction of more general time preferences in an advertising competition, and in the framework of marketing channels, where we study some coordination mechanisms and the possible consequences of having asymmetric agents. Furthermore, we explore the role of quality investments when facing a potential crisis. Table 1.1 presents the general and specific objectives of the three studies we have carried out.

The current thesis is organized as follows:

In Chapter 2 we study a finite time horizon advertising dynamic game under the assumption that firms' time preferences are time-inconsistent. Specifically, we consider two types of discounting, heterogeneous discounting and hyperbolic discounting. In the case of heterogeneous discounting, the relative importance of the final function will increase/decrease as the end

Table 1.1: Objectives of the Thesis

Chapter	Objectives
2	**General Objective** O1: To introduce time-inconsistent preferences into advertising competition **Specific Objectives** O1.1: To study how firms adapt their advertising strategies in a competitive environment when they have increasing/decreasing valuations of the final functions O1.2: To analyze how companies advertise when they have diminishing time discount rates
3	**General Objective** O2: To analyze the efficiency of different time-consistent advertising coordination mechanisms in supply chain **Specific Objectives** O2.1: To extend previous supply chain models by considering that agents can differ in their time preferences, which gives rise to time-inconsistent aggregated preferences O2.2: To study situations when cooperation is inefficient O2.3: To compare different coordination mechanisms to see which can yield a better outcome
4	**General Objective** O3: To investigate the advertising and quality improvement strategies in the presence of potential crisis **Specific Objectives** O3.1: To introduce quality management into supply chain management in an inter-temporal setting O3.2: To study crisis management policies as a piecewise deterministic dynamic game O3.3: To analyze the interaction among operations management, marketing, and crisis management

of the planning horizon approaches compared with current payoffs. Whereas when agents discount future payoffs hyperbolically, their instantaneous discount rates diminish rapidly in earlier stages and then slowly in the long term. The dynamics for the market are presented in square root form as in the papers of Sethi (1983) and Sorger (1989). Subgame perfect Nash equilibria are then studied and compared with that of the standard discounting case.

In Chapter 3, we study an advertising dynamic game in supply chain management under the assumption that the agents differ in their time preference rates. We study two coordination mechanisms: the cost sharing program, where the retailer can get some reimbursement of the advertising cost from the manufacturer; and the vertical integration, where the two players aim to maximize the joint profit. We derive the time-consistent cooperative advertising strategies in each coordination setting, and we compare them with the non-cooperative case. Our results show that, the cost sharing program is Pareto superior to the non-cooperative setting, while vertical integration could be more preferred by the manufacturer and less preferred by the retailer if the initial goodwill level is sufficiently high. Besides, unlike previous results in the literature, we found that when the agents' discount rates are very different, joint profits could be lower under vertical integration than in the non-cooperative case, which yields an inefficient cooperation.

In Chapter 4 we confine our interest to the intersection of marketing and other functional activities when corporates face uncertainties. Crises can occur at random time instants (e.g., Samsung had to recall their Galaxy Note 7 which had critical failures in the batteries and can result in fires, Ford issued several recalls for millions of cars/trucks with loose steering wheels/unseated gear shift cable locking clip). Since the occurrence of such unexpected crises may damage a brand's goodwill, sales and profitability, far-sighted managers should be able to take it into account when making decisions. To address this issue, we study an advertising and quality management game with a piecewise deterministic process. We consider a supply chain consisting of a single manufacturer and a single retailer, where the manufacturer controls the global advertising and quality improvement, while the retailer focuses in the local advertising. We adopt the goodwill model proposed by Nerlove & Arrow (1962) and assume that when the crisis happens, the companies suffer a sharp decrease in the goodwill. We characterize the stationary Markovian

Nash equilibrium, and then we compare the corresponding strategies and outcomes with those of the case where the potential crises are absent. We also evaluate the effects of instantaneous crisis rate and damage rate.

Chapter 5 closes the thesis by summarizing some main findings, presenting managerial implications and suggesting future researches.

CHAPTER 2

Lanchester Duopoly Model Revisited:Advertising Competition under Time Inconsistent Preferences

2.1. Introduction

The role of advertising in marketing has been highlighted for many years. As the primary competitive marketing tool in highly competitive industries, heavy advertising expenditures are required. Coca-Cola spent \$3.342, \$3.266 and \$3.499 billion in 2012, 2013 and 2014 respectively, and the yearly commitment to advertising of its largest competitor, Pepsi, is \$2.2, \$2.4 and \$2.3 (Investopedia, 2015).

As a consequence, a great deal of academic attention has been paid to the advertising, and the tendency is still increasing. Differential game approach, drawing support from mathematical modeling and quantitative methods, successfully involves the two essential elements of the marketing problems: dynamic and strategic considerations, and has been one of the principal methodologies in marketing science.

One of the earliest and most attractive advertising market share response models is the Lanchester model introduced by Kimball (1957). It is characterized in depicting battles for market share in a simple and elegant way, and has been adopted in many researches of dynamic advertising competition.

In the earliest years academic attention was focused, probably for reasons of mathematical tractability, in the specific case where agents do not discount future payoffs. Case (1979) suggests that it can be seen as an approximation of small and positive discount rate. Following the same research line, a series of theoretical and empirical studies focusing on zero discounting are conducted. Erickson (1991, chap. 3) presents an analytical and numerical analysis of feedback equilibria, and compares it with that of open-loop form. Both Erickson (1992) and Chintagunta & Vilcassim (1992) empirically test the market share response function to advertising investment and examine which kind of strategies (open-loop or closed-loop) fits better the reality. The difference between these two researches derives from the data samples and the statistical procedures applied. Chintagunta & Vilcassim (1994) extends the previous work by considering multiple marketing tools such as advertising, detailing, sales promotion, and so on. The cases of zero discounting in a finite time horizon with salvage value are analyzed in two empirical studies. In Wang & Wu (2001), they use the Lanchester model as a benchmark case for an extended Vidale-Wolfe model (Vidale & Wolfe, 1957) in terms of model fitting and forecast accuracy. Later on, in Wang & Wu (2007) an empirical test is run for different structures of market share response function incorporating the inflation effect. However, the zero discount case may cause problems of convergence of the objective functionals (Jørgensen & Zaccour, 2004).

The first attempt of breaking the zero discounting assumption comes from Fruchter & Kalish (1997), where they study a game of infinite time horizon and propose a new approach to obtain the so-called time-varying closed-loop strategies, which are determined by time, current states and initial states. This work is later extended to an oligopolistic competition in Fruchter (1999b), Fruchter (1999a), Fruchter (2001), and Fruchter & Kalish (1998), with the incorporation of market expansion, multi-products in a growing market, and multiple marketing tools in the latter three studies. However, the equilibrium policies are not subgame perfect due to the strategy dependence on initial market share. Jarrar et al. (2004) and Breton et al. (2006) develop two numerical algorithms to compute the feedback Nash and Stackelberg equilibrium strategies, respectively. According to their numerical illustrations, when rates of time preferences are positive, in both modes of play (simultaneous and sequential) the advertising strategies are decreasing in the firm's own market share. This property differs from the results of zero discount rates. Another approach to consider positive discount rate is through modification of model structure. For instance, Sorger (1989) proposes a variant of the Lanchester model, which allows for the characterization of feedback Nash strategies. In accordance with Jarrar et al. (2004) and Breton et al. (2006), higher market share also implies a decrease in advertising. The empirical support from Chintagunta & Jain (1995) show that the specification made by Sorger is a good candidate for the market of pharmaceutical products, soft drinks, beers, and detergents.

Given that a positive discount rate can have a significant impact on the advertising strategies, it is natural to think, if the agents discount the future payoffs in another way rather than the standard way, in which discount rates are assumed to be constant and unchanged, will they behave differently? Besides, empirical and experimental studies show that how people discount the future payoffs depends on the time distance and the types of goods. The curiosity of exploring the time preferences' impact, as well as the inadequacy of standard discounting have encouraged an academic stream in the differential game literature, where alternative discounting models are applied. Although time-inconsistent preferences have proven to be important in many areas such as behavioral economics (e.g., Fischer, 1999; O'Donoghue & Rabin, 1999b, 2001; Gruber & Köszegi, 2001), environmental economics (for example, Karp, 2005; Karp & Tsur, 2011), financial economics (Laibson, 1997; de-Paz et al., 2013, 2014), and so forth, such concern has never been introduced into the business context.

Nonetheless, any other discount function but the standard exponential one would lead to time inconsistency (Strotz, 1955). If we follow the standard approach, a decision obtained at a later time does not necessarily, and in general not, coincide with that made at an earlier time. As a consequence, the agent tends to deviate from herself constantly, and the intertemporal choice, even in an optimal control problem, can be considered as a dynamic game among the "selves" of the decision maker at different instants of time.

Hence, the purpose of this chapter is to, firstly, going one step further, explore the impact of temporal discounting on advertising competition. Specifically, we confine our interest to *heterogeneous discounting* and *hyperbolic discounting*, two of the most studied alternative discount models. Secondly, we then compute different types of strategies, and compare them with the standard discounting case to analyze how firms behave under different kinds of time preferences and different commitment power.

The rest of this chapter is organized as follow. In Section 2.2 we describe a differential game model, the determination of feedback Nash equilibria follows in Section 2.3. In Section 2.4 some numerical simulations will be run to throw light on the advertising strategies and market dynamics. Finally, in Section 2.5 we summarize our results, relate them to the market observations, discuss the limitations and suggest some future studies.

2.2. Model Formulation

2.2.1 Lanchester Dynamics

The Lanchester model was originally used to model military combat. It was firstly introduced into the economic world by Kimball (1957) because of the similarity between military and industrial operations, and further advanced by Case (1979) and Little (1979). This model describes a battle for the market share where the advertising is the dominant influencing factor that only affects the customers of the rival firm.

Denote by $x_i(s)$ and $u_i(s)$ the market share and the rate of advertising expenditure of firm i ($i = 1, \ldots, N$) at time s, k_i the advertising effectiveness, the market share and advertising are originally related in a linear structure

$$\dot{x}_i(s) = k_i u_i(s)[1 - x_i(s)] - \sum_{\substack{j=1 \\ j \neq i}}^{N} k_j u_j(s) x_i(s) \; .$$

Specifically, in a duopolistic market (as in Little, 1979; Erickson, 1985; Chintagunta & Vilcassim, 1992; Jarrar et al., 2004), by letting $x = x_1$ and $x_2 = 1 - x$, the basic Lanchester dynamics can be simplified as

$$\dot{x}(s) = k_1 u_1(s)[1 - x(s)] - k_2 u_2(s) x(s) \; .$$

Sorger (1989) extended the Lanchester model adopting the square root structure in a Vidale-Wolfe extension proposed by Sethi (1983), and formulated the instantaneous variation of market share in the following way (specifically, $k_1 = k_2 = 1$ in Sorger's setting)

$$\dot{x}(s) = k_1 u_1(s)\sqrt{1 - x(s)} - k_2 u_2(s)\sqrt{x(s)} \, , \quad x(t) = x_t \; . \tag{2.1}$$

According to Sorger (1989), the two square root terms in (2.1) are approximation of $1 - x + x(1 - x)$ and $x + x(1 - x)$, respectively. Therefore, a word-of-mouth communication effect is incorporated into the market share dynamics.

Besides, (2.1) can also be explained as a joint effect of the "Lanchester-type" dynamics and the excess advertising.

This formulation is highly referenced in the literature. For instance, Prasad & Sethi (2003) extended it to an oligopoly setting with the presence of white noise. Prasad & Sethi (2004) enriched the discussion by introducing the decay effect of Vidale-Wolfe model as well as the stochastic setting. Bass et al. (2005a,b) analyzed the situation where firms invest in brand-advertising to capture rival firm's customers and in generic-advertising to increase the primary demand. Naik et al. (2008) studied the advertising competition in an oligopoly setting with market expansion and brand confusion effect, and offered some empirical evidence. He et al. (2011) considered an advertising battle where a coalition comprised of a manufacturer and a retailer is competing against another independent retailer.

In this chapter, we adopt Sorger's extension in that it offers a richer interpretation by incorporating word-of-mouth communication and excess advertising. Moreover, with the square root formulation, the equilibrium market share of firm i ($i = 1, 2$) is of S-shape, which is considered to be in better accordance with reality. Furthermore, as mentioned previously, Sorger's modification allows the computation of feedback strategies for non-zero discounting, which is critical for further discussion related to time-inconsistent discounting.

Assuming quadratic advertising costs (which give rise to diminishing effect), the two firms aim to maximize the sum of the current value of the profit stream over a finite planning interval T and the scrap value assigned to the terminal state:

$$J_1(u_1, u_2) = \int_t^T \theta_1(s-t) \left[\pi_1 x(s) - \frac{c_1}{2}(u_1(s))^2\right] ds + \theta_1(T-t) S_1 x(T), \quad (2.2)$$

$$J_2(u_1, u_2) = \int_t^T \theta_2(s-t) \left[\pi_2 \left(1-x(s)\right) - \frac{c_2}{2}(u_2(s))^2\right] ds + \theta_2(T-t) S_2 \left(1-x(T)\right), \quad (2.3)$$

where $\theta_i(s-t)$ ($i = 1, 2$) are discount functions and will be given in the next section.

The denotation of the variables and parameters is as follows:

π_i positive constant margin per unit product of firm i,

$x(t)$ market share of firm 1 at time t (state variable),

c_i positive constant cost parameter of firm i,

$u_i(t)$ rate of advertising expenditure of firm i at time t (control variable),

k_i positive constant advertising effect parameter of firm i,

S_i the valuation assigned to the final state (non-negative constant).

2.2.2 Discount Function

The pioneering researches into intertemporal choice mainly focused on the psychological motives that lead to *time preference*[1] (for example, Rae, 1834; Senior, 1836; Fisher, 1930). Samuelson (1937) put forward the discounted utility (DU) model to condense all the psychological motives underlying intertemporal decisions into a single parameter, the discount rate, which is assumed to be constant and invariant across time and for all kinds of goods. Its simplicity made it become the dominant theoretical framework to study intertemporal behaviors. However, numerous experimental and empirical studies conducted in the following years have shown that in some situations, people demonstrate diminishing discount rates. Furthermore, the rates of time preference vary in the types of goods and decisions. The findings of such inadequacy of constant discounting have encouraged the development of various alternative theoretical models (for an overview of this topic, see Frederick et al., 2002).

Marín-Solano & Patxot (2012) introduced a temporal bias where agents discount the utility during the planning horizon and the final function at constant but different rates. It is labeled as *heterogeneous discounting* and the corresponding discount function is given by:

$$\theta_i(s-t) = \begin{cases} e^{-\delta_i(s-t)} & \text{if} \quad s < T\,, \\ e^{-\rho_i(s-t)} & \text{if} \quad s = T\,,\ i = 1,2. \end{cases} \tag{2.4}$$

[1] Defined as "*the preference for immediate utility over delayed utility*" (Frederick et al., 2002).

The essence of heterogeneous discounting is to describe a situation where the valuation of the final function is changing over time. To better present this idea, we rewrite the discount factor at the ending point $e^{-\rho_i(T-t)}$ as

$$e^{-\delta_i(T-t)} \cdot e^{-(\rho_i-\delta_i)(T-t)}\ , \ i = 1, 2. \tag{2.5}$$

From (2.5) we can see that if $\rho_i > \delta_i$ $(i = 1, 2)$, the discount factor for the moment T is increasing in t, and *vice versa*. Therefore, we are able to model an increasing valuation of the final function by assuming $\rho_i > \delta_i$, and a decreasing valuation by $\rho_i < \delta_i$ $(i = 1, 2)$.

One typical application of this approach, as in Marin-Solano and Paxtot (2012), is to discount the "hard" goods, in the sense that effort has to be made prior to the enjoyment of the benefits (some examples are sports, knowledge and human capital accumulation). It has also been applied in the field of behavioral finance, such as the consumption and investment problem (de-Paz et al., 2013), and the life insurance purchase behaviors (de-Paz et al., 2014), where the final function represents the wealth at retirement or the bequest left for her descendants. In all the cases mentioned above it appears natural to assume that the agent has an increasing concern as the time t approaches to the end of the planning horizon T.

Introducing heterogeneous discounting into the business context (corporate level) could make sense due to the following concerns: 1) as discussed in Marín-Solano & Patxot (2012), the capital accumulation of a firm can be, to some degree, regarded as a "hard" good; 2) It appears restrictive to assume the discount rate to be invariant over time. The rate of time preference is affected by social factors such as regime (Pirvu & Zhang, 2014) and state of economy (Parkin, 1988), as well as by other firm-level factors like project duration, risk and fixed cost (Chen, 2012). 3) It is also of interest to consider different ways of discounting for different things. For instance (in this model), a firm could be more concerned with the cash flow if it is required to guarantee the development. Nevertheless, the emphasis might switch to the market coverage when the company reaches a steady growth.

In addition to heterogeneous discounting, a plenty of effort has been devoted to *hyperbolic discounting*. The phenomenon that decision makers exhibit declining discount rates has been supported experimentally and empirically by many studies (e.g., Thaler, 1981; Myerson & Green, 1995, and so forth) (Benzion et al., 1989; Kirby, 1997), and hyperbolic discounting is a response to such DU anomaly by relaxing the constant rate assumption.

The applications of hyperbolic discounting have been primarily located in the fields of macroeconomics (e.g., for consumption-saving behaviors, Laibson, 1994, 1997, 1998, and Barro, 1999; Krusell & Smith, JR., 2003 for economic growth), behavioral economics (for example, Fischer, 1999; O'Donoghue & Rabin, 1999b, 2001 for procrastination, and O'Donoghue & Rabin, 1999a, 2002 for addiction), and environmental economics (Karp, 2005; Karp & Tsur, 2011, and so on).

We believe that it could be meaningful to incorporate hyperbolic discounting from a company's point of view. Firstly, a manager could have limited commitment like a public policy maker, in that she is not sure if the business plans made currently would be followed by the successor. Besides, as human beings, it is likely that administrators are also influenced by the temporal bias that affect personal choice when making professional decisions. Moreover, uncertainty over the hazard rate of payoff realization or over the agents' own future discount rates would lead to hyperbolic discounting (Azfar, 1999; Dasgupta & Maskin, 2005; Farmer & Geanakoplos, 2009).

We choose a linear combination of exponential functions which is given as follows:

$$\theta_i(s-t) = \lambda e^{-\delta_i(s-t)} + (1-\lambda)e^{-\rho_i(s-t)}\,, \tag{2.6}$$

with the corresponding instantaneous discount rate

$$r_i(\tau) = -\frac{\theta_i'(\tau)}{\theta_i(\tau)} = \frac{\lambda\delta_i e^{-\delta_i\tau} + (1-\lambda)\rho_i e^{-\rho_i\tau}}{\lambda e^{-\delta_i\tau} + (1-\lambda)e^{-\rho_i\tau}}\,,\ i = 1,2, \tag{2.7}$$

where $0 < \lambda < 1$ and $\delta_i > \rho_i$ $(i = 1,2)$. The discount function (2.6) implies that the instantaneous discount rate declines relatively rapidly in the earlier stages and then more slowly in the long run. Furthermore, when the planning horizon is sufficiently large, the pure rate of time preferences will converge to ρ_i $(i = 1,2)$. This functional form is also applied in Ekeland & Pirvu (2008), Ekeland & Lazrak (2010), and Karp & Tsur (2011).

2.3. Determination of Feedback Nash Equilibria

We confine our interest to the feedback Nash equilibria for some reasons. It is theoretically desirable in that firstly, it is more robust than the open-loop equilibria; secondly, empirical studies show that the feedback strategies can better explain the real dynamic advertising competition (Chintagunta & Vilcassim, 1992; Erickson, 1992); thirdly, evaluation of different kinds of strategies has been made by means of estimating market share response model independently of the strategies, the results suggest that feedback strategies perform strategically better for profit maximization (Wang & Wu, 2007). In addition, it is managerially attractive since the feedback rules, which are time and state dependent, allow the flexibility of responding to the changing market.

For the sake of completeness we introduce the definitions of some commonly used strategy concepts in dynamic inconsistency setting. A feedback equilibrium is sub-game perfect in the standard (constant discount rate) case, however it does not necessarily, and in general it does not, hold while applying any kind of non-constant discounting. This is intuitive because a decision made at time t is (normally) not optimal for the agent herself at a future time t' due to her time-varying preferences. An individual with time-inconsistent preferences may or may not be aware of that. If the agent solves the optimization problem at the beginning of the planning horizon, and she believes that her preferences will not change in the future (and in fact they do), or she can commit herself to follow this strategy made at time 0, we call it pre-commitment solution.

Under heterogeneous discounting, the pre-commitment agents need to solve a standard game in the beginning of the planning horizon. The corresponding system of dynamic programming equations (DPEs) for feedback Nash equilibrium are given as follows (we use P to denote “pre-commitment):

$$\begin{aligned}\delta_i V_i^P - \frac{\partial V_i^P}{\partial s} \\ = \max_{\{u_i^P\}} \left\{ \pi_i x_i - \frac{c_i}{2}(u_i)^2 + \frac{\partial V_i^P}{\partial x}\left(k_1 u_1 \sqrt{1-x} - k_2 u_2 \sqrt{x}\right)\right\}, \ i = 1, 2,\end{aligned} \tag{2.8}$$

with boundary conditions $V_i^P(T, x) = e^{-(\rho_i - \delta_i)T} S_i x_i(T)$.

However, the decision maker would tend to deviate from the *ex ante* policy as time goes on. If she re-optimizes the problem in a future time t' according to her interest of that time and applies it, and repeats this procedure in a later time t''... As a consequence, she will end up solving the problem at every instant and applying the solution only in that particular point of time. This kind of strategy is defined as naive solution (denoted by superscript N).

If the decision makers under heterogeneous discounting act in a naive way, at every moment t they will solve

$$\begin{aligned} &\delta_i V_i^t - \frac{\partial V_i^t}{\partial s} \\ &= \max_{\{u_i^N\}} \left\{ \pi_i x_i - \frac{c_i}{2}(u_i^t)^2 + \frac{\partial V_i^t}{\partial x} \left[k_1 u_1^t \sqrt{1-x} - k_2 u_2^t \sqrt{x} \right] \right\} \ , \ i = 1,2, \end{aligned} \tag{2.9}$$

together with the boundary conditions $V_i^t(T,x) = e^{-(\rho_i - \delta_i)(T-t)} S_i x_i(T)$. Moreover, they will only apply the solutions obtained from (2.9) at the moment $s = t$.

Note that neither the pre-commitment nor the naive solutions are time-consistent. A solution can be time-consistent (named as sophisticated solution) if the agent can anticipate and take into account her future preferences while making decisions, which implies no reason for future selves to deviate from it. Using superscript S to represent sophisticated solutions, under heterogeneous discounting, the feedback Nash equilibrium is computed by solving

$$\rho_i V_i^S + K_i - \frac{\partial V_i^S}{\partial t} = \max_{\{u_i^S\}} \left\{ \pi_i x_i - \frac{c_i}{2}(u_i^S)^2 + \frac{\partial V_i^S}{\partial x} \left[k_1 u_1^S \sqrt{1-x} - k_2 u_2^S \sqrt{x} \right] \right\}, \tag{2.10}$$

with

$$K_i(t,x) = (\delta_i - \rho_i) \int_t^T e^{-\delta_i (s-t)} \left[\pi_i x_i(s) - \frac{c_i}{2}(u_i^*)^2 \right] ds \ , \ i = 1,2, \tag{2.11}$$

where u_i^* maximizes the right-hand side term of equation (2.10). The corresponding boundary conditions are $V_i^S(T,x) = S_i x_i(T)$, and $K_i(T,x) = 0$ ($i = 1,2$). By differentiating (2.11) with respect to t, we can get a simplified version

$$\delta_i K_i - \frac{\partial K_i}{\partial t}$$
$$= (\delta_i - \rho_i)\left[\pi_i x_i - \frac{c_i}{2}(u_i^*)^2\right] + \frac{\partial K_i}{\partial x}\left[k_1 u_1^* \sqrt{1-x} - k_2 u_2^* \sqrt{x}\right] \ , \ i = 1,2. \tag{2.12}$$

We now proceed to compute the pre-commitment, naive and sophisticated solutions under heterogeneous discounting.

By maximizing the right hand side of equations (2.8), (2.9), and (2.10), we get the optimal advertising strategies. We make the informed guess that the value functions are linear in the state variable $V_i^\sigma(t,x) = \eta_i^\sigma(t)x + \nu_i^\sigma(t)$. Then, the feedback Nash equilibrium advertising policies are given by

$$u_1^{\sigma *}(t,x) = \frac{k_1}{c_1}\eta_1^\sigma(t)\sqrt{1-x}\ , \qquad u_2^{\sigma *}(t,x) = \frac{k_2}{c_2}\eta_2^\sigma(t)\sqrt{x}\ , \qquad \sigma = P, N, S. \tag{2.13}$$

For the agent who can commit herself to following the decision taken at the beginning of planing horizon $t = 0$, she needs to solve (2.8). Substituting the advertising rules u_1^{P*}, u_2^{P*}, the value functions V_1^P, V_2^P and their partial derivatives into (2.8), after rearranging, we obtain

$$\begin{aligned}&\left[\delta_1\eta_1^P(s) - \dot\eta_1^P(s) - \pi_1 + \frac{(k_1)^2}{2c_1}\left(\eta_1^P(s)\right)^2 - \frac{(k_2)^2}{c_2}\eta_1^P(s)\eta_2^P(s)\right]x\\ &= -\delta_1\nu_1^P(s) + \dot\nu_1^P(s) + \frac{(k_1)^2}{2c_1}\left(\eta_1^P(s)\right)^2\ ,\end{aligned} \tag{2.14}$$

$$\begin{aligned}&\left[\delta_2\eta_2^P(s) - \dot\eta_2^P(s) + \pi_2 - \frac{(k_2)^2}{2c_2}\left(\eta_2^P(s)\right)^2 + \frac{(k_1)^2}{c_1}\eta_1^P(s)\eta_2^P(s)\right]x\\ &= -\delta_2\nu_2^P(s) + \dot\nu_2^P(s) + \pi_2 + \frac{(k_1)^2}{c_1}\eta_1^P(s)\eta_2^P(s)\ .\end{aligned} \tag{2.15}$$

Equations (2.14) and (2.15) hold for every x, if and only if the parameters of x are equal to zero, thus we have get a system of two Ricatti differential equations with boundary conditions

$$\eta_i^P(T) = (-1)^{i-1}e^{-(\rho_i-\delta_i)T}S_i\ ,\ i = 1,2. \tag{2.16}$$

We can find out the naive solutions following the same pattern. By substituting u_1^{N*}, u_2^{N*}, V_1^N, V_2^N and the corresponding partial derivatives into

(2.9), we get the same DPEs as (2.14) and (2.15), but with different boundary conditions

$$\eta_i^N(T) = (-1)^{i-1} e^{-(\rho_i - \delta_i)(T-t)} S_i \ , \ i = 1,2. \tag{2.17}$$

Regarding the sophisticated solutions, apart from V_i^S, we also need to make a guess of the structure of the term K_i in (2.10). We conjecture a linear structure, as for the value functions.

$$V_i^S(t,x) = \eta_i^S(t)x + \nu_i^S(t), \quad K_i(t,x) = \alpha_i(t)x + \beta_i(t), \ i = 1,2. \tag{2.18}$$

Accordingly,

$$\begin{aligned} &\frac{\partial V_i^S}{\partial x} = \eta_i^S(t) \ , \quad \frac{\partial V_i^S}{\partial t} = \dot{\eta}_i^S(t)x + \dot{\nu}_i^S(t) \ , \\ &\frac{\partial K_i}{\partial x} = \alpha_i(t) \ , \quad \frac{\partial K_i}{\partial t} = \dot{\alpha}_i(t)x + \dot{\beta}_i(t) \ , \ i = 1,2. \end{aligned} \tag{2.19}$$

We then substitute (2.13), (2.18), and (2.19) into (2.10) and (2.12). After rearrangement, we obtain

$$\begin{aligned} &\left[\rho_1 \eta_1^S(t) + \alpha_1(t) - \dot{\eta}_1^S(t) - \pi_1 + \frac{(k_1)^2}{2c_1}\left(\eta_1^S(t)\right)^2 - \frac{(k_2)^2}{c_2}\eta_1^S(t)\eta_2^S(t)\right] x \\ &= -\rho_1 \nu_1^S(t) - \beta_1(t) + \dot{\nu}_1^S(t) + \frac{(k_1)^2}{2c_1}\left(\eta_1^S(t)\right)^2 \ , \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\left[\rho_2 \eta_2^S(t) + \alpha_2(t) - \dot{\eta}_2^S(t) + \pi_2 - \frac{(k_2)^2}{2c_2}\left(\eta_1^S(t)\right)^2 + \frac{(k_1)^2)}{c_1}\eta_1^S(t)\eta_2^S(t\right] x \\ &= -\rho_2 \nu_2^S(t) - \beta_2(t) + \dot{\nu}_2^S(t) + \pi_2 + \frac{(k_1)^2}{c_1}\eta_1^S(t)\eta_2^S(t) \ , \end{aligned} \tag{2.21}$$

$$\begin{aligned} &[\delta_1 \alpha_1(t) - \dot{\alpha}_1(t) - (\delta_1 - \rho_1)\pi_1 \\ &\quad \left. - \frac{(k_1)^2(\delta_1 - \rho_1)}{2c_1}\left(\eta_1^S(t)\right)^2 + \frac{(k_1)^2}{c_1}\alpha_1(t)\eta_1^S(t) - \frac{(k_2)^2}{c_2}\alpha_1(t)\eta_2^S(t)\right] x \\ &= -\delta_1 \beta_1(t) + \dot{\beta}_1(t) - \frac{(k_1)^2(\delta_1 - \rho_1)}{2c_1}\left(\eta_1^S(t)\right)^2 + \frac{(k_1)^2}{c_1}\alpha_1(t)\eta_1^S(t) \ , \end{aligned} \tag{2.22}$$

$$\begin{aligned} &[\delta_2 \alpha_2(t) - \dot{\alpha}_2(t) + (\delta_2 - \rho_2)\pi_2 \\ &\quad \left. + \frac{(k_2)^2(\delta_2 - \rho_2)}{2c_2}\left(\eta_1^S(t)\right)^2 - \frac{(k_2)^2}{c_2}\alpha_2(t)\eta_2^S(t) + \frac{(k_1)^2}{c_1}\alpha_2(t)\eta_1^S(t)\right] x \\ &= -\delta_2 \beta_2(t) + \dot{\beta}_2(t) + (\delta_2 - \rho_2)\pi_2 + \frac{(k_1)^2}{c_1}\alpha_2(t)\eta_1^S(t) \ . \end{aligned} \tag{2.23}$$

Equations (2.20)-(2.23) hold for every x, if and only if the parameters of x are equal to zero. Therefore, we obtain a system of four differential equations of $\eta_i^S(t)$ and $\alpha_i(t)$, with boundary conditions $\eta_i^S(T) = S_i$ and $\alpha_i(T) = 0$ $(i = 1,2)$.

The equilibrium of the game under heterogeneous discounting is characterized in the following proposition.

Proposition 2.1. *The pre-commitment, naive and sophisticated feedback Nash equilibria solutions for the the advertising competition under heterogeneous discounting are determined by*

$$
\begin{aligned}
u_i^\sigma(s,x) &= (-1)^{i-1}\frac{k_i}{c_i}\eta_i^\sigma(s)\sqrt{1-x_i(s)} \\
&= (-1)^{i-1}\frac{k_i}{c_i}\eta_i^\sigma(s)\sqrt{x_j(s)}\ ,\ \{i,j\} = \{1,2\},\ \sigma = P,N,S.
\end{aligned}
\tag{2.24}
$$

- *For* ***pre-commitment*** *solutions,* $\eta_i^P(s)$ $(i = 1,2)$ *are the solutions to the system of differential equations*

$$
\begin{aligned}
\dot{\eta}_i^P(s) =&(-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^P(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^P(s)\eta_j^P(s) + \delta_i\eta_i^P(s) \\
&+ (-1)^i\pi_i\ ,\ \{i,j\} = \{1,2\},
\end{aligned}
\tag{2.25}
$$

with boundary conditions $\eta_i^P(T) = (-1)^{i-1}e^{-(\rho_i-\delta_i)T}S_i$ $(i = 1,2)$.

- *For* ***naive*** *solutions of t-agent,* $\eta_i^N(s)$ $(i = 1,2)$ *solve the system*

$$
\begin{aligned}
\dot{\eta}_i^N(s) =&(-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^N(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^N(s)\eta_j^N(s) + \delta_i\eta_i^N(s) \\
&+ (-1)^i\pi_i\ ,\ \{i,j\} = \{1,2\},
\end{aligned}
\tag{2.26}
$$

with boundary conditions $\eta_i^N(T) = (-1)^{i-1}e^{-(\rho_i-\delta_i)(T-t)}S_i$ $(i = 1,2)$.

- *For* ***sophisticated*** *solutions,* $\eta_i^S(s)$ $(i = 1,2)$ *solve the system of differ-*

ential equations

$$
\begin{aligned}
\dot{\eta}_i^S(s) =&(-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^S(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^S(s)\eta_j^S(s) + \rho_i\eta_i^S(s) \\
&+ \alpha_i(s) + (-1)^i\pi_i\ , \\
\dot{\alpha}_i(s) =&(-1)^i\frac{(k_i)^2(\delta_i-\rho_i)}{2c_i}\left(\eta_i^S(s)\right)^2 + (-1)^{i-1}\frac{(k_i)^2}{c_i}\alpha_i(s)\eta_i^S(s) \\
&+ (-1)^i\frac{(k_j)^2}{c_j}\alpha_i(s)\eta_j^S(s) + \delta_i\alpha_i(s) + (-1)^i(\delta_i-\rho_i)\pi_i\ , \\
&\{i,j\} = \{1,2\},
\end{aligned}
\tag{2.27}
$$

with boundary conditions $\eta_i^S(T) = (-1)^{i-1}S_i,\ \alpha_i(T) = 0\ (i = 1,2)$.

Next we derive the time-inconsistent (Pre-commitment and Naive) and time-consistent (Sophisticated) solutions for agents with hyperbolic discounting.

The system of DPEs for pre-commitment solutions are

$$
r_i(s)V_i^P - \frac{\partial V_i^P}{\partial s} = \max_{\{u_i^P\}}\left\{\pi_i x_i - \frac{c_i}{2}(u_i)^2 + \frac{\partial V_i^P}{\partial x}\left[k_1u_1\sqrt{1-x} - k_2u_2\sqrt{x}\right]\right\}\ , \tag{2.28}
$$

with boundary conditions $V_i^P(T,x) = S_i x_i(T)\ (i = 1,2)$.

As to the naive agents, they need to solve, at every instant t,

$$
r_i(s-t)V_i^t - \frac{\partial V_i^t}{\partial s} = \max_{\{u_i^N\}}\left\{\pi_i x_i - \frac{c_i}{2}(u_i^t)^2 + \frac{\partial V_i^t}{\partial x}\left[k_1u_1^t\sqrt{1-x} - k_2u_2^t\sqrt{x}\right]\right\}\ , \tag{2.29}
$$

together with the boundary conditions $V_i^N(T,x) = S_i x_i(T)\ (i = 1,2)$, and follow the solutions obtained only at the moment $s = t$.

The DPE for sophisticated agents in a game of finite time horizon under non-constant discounting is derived in Marín-Solano & Navas (2009). Following their approach, the time-consistent equilibrium strategies can be obtained by solving the set of DPEs

$$
\begin{aligned}
&r_i(T-t)V_i^S + K_i - \frac{\partial V_i^S}{\partial t} \\
&= \max_{\{u_i^S\}}\left\{\pi_i x_i - \frac{c_i}{2}(u_i^S)^2 + \frac{\partial V_i^S}{\partial x}\left[k_1u_1^S\sqrt{1-x} - k_2u_2^S\sqrt{x}\right]\right\}\ ,
\end{aligned}
\tag{2.30}
$$

with

$$K_i(t,x) = \int_t^T \theta_i(s-t)[r_i(s-t) - r_i(T-t)] \left[\pi_i x_i(s) - \frac{c_i}{2}(u_i^S)^2\right] ds \ , \quad (2.31)$$

and

$$V_i^S(T,x) = S_i x_i(T) \ , \quad K_i(T,x) = 0 \ , \ i = 1,2, \quad (2.32)$$

where u_i^S ($i = 1,2$) maximize the right-hand side of (2.30). Similarly, the term K_i ($i = 1,2$) can be simplified by differentiating both sides with respect to t. If the discount factor is a linear combination of exponential functions given in (2.6), by differentiating (2.31) we obtain

$$\Omega_i(t)K_i - \frac{\partial K_i}{\partial t} = \Phi_i(t)\left[\pi_i x_i - \frac{c_i}{2}(u_i^S)^2\right] + \frac{\partial K_i}{\partial x}\left[k_1 u_1^S \sqrt{1-x} - k_2 u_2^S \sqrt{x}\right] \ , \quad (2.33)$$

with

$$\begin{aligned} \Omega_i(t) &= \frac{\lambda \rho_i e^{-\delta_i(T-t)} + (1-\lambda)\delta_i e^{-\rho_i(T-t)}}{\lambda e^{-\delta_i(T-t)} + (1-\lambda)e^{-\rho_i(T-t)}} \ , \\ \Phi_i(t) &= \frac{\lambda(1-\lambda)(\rho_i - \delta_i)\left[e^{-\delta_i(T-t)} - e^{-\rho_i(T-t)}\right]}{\lambda e^{-\delta_i(T-t)} + (1-\lambda)e^{-\rho_i(T-t)}} \ . \end{aligned} \quad (2.34)$$

Following the same procedures for the case of heterogeneous discounting, by solving equations of (2.28), (2.29), (2.30) and (2.33), we characterize the feedback Nash equilibria for the case of hyperbolic discounting, which are summarized in the following proposition.

Proposition 2.2. *The pre-commitment, naive and sophisticated feedback Nash equilibria solutions for the advertising competition under hyperbolic discounting are determined by*

$$\begin{aligned} u_i^\sigma(s,x) &= (-1)^{i-1}\frac{k_i}{c_i}\eta_i^\sigma(s)\sqrt{1-x_i(s)} \\ &= (-1)^{i-1}\frac{k_i}{c_i}\eta_i^\sigma(s)\sqrt{x_j(s)} \ , \ \{i,j\} = \{1,2\}, \ \sigma = P,N,S. \end{aligned} \quad (2.35)$$

- *For* ***pre-commitment*** *solutions,* $\eta_i^P(s)$ ($i = 1,2$) *are the solutions to*

$$\begin{aligned} \dot{\eta}_i^P(s) =& (-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^P(s)\right)^2 + (-1)^i \frac{(k_j)^2}{c_j}\eta_i^P(s)\eta_j^P(s) + r_i(s)\eta_i^P(s) \\ &+ (-1)^i \pi_i \ , \ \{i,j\} = \{1,2\}, \end{aligned} \quad (2.36)$$

with boundary conditions $\eta_i^P(T) = (-1)^{i-1}S_i$, $i = 1,2$.

- *For **naive** solutions of t-agent,* $\eta_i^N(s)$ $(i = 1,2)$ *solve the system*

$$\begin{aligned}\dot{\eta}_i^N(s) =&(-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^N(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^N(s)\eta_j^N(s) \\ &+ \left[\lambda\delta_i + (1-\lambda)\rho_i\right]\eta_i^N(s) + (-1)^i\pi_i\ ,\ \{i,j\} = \{1,2\},\end{aligned} \tag{2.37}$$

with boundary conditions $\eta_i^N(T) = (-1)^{i-1}S_i$ $(i = 1,2)$.

- *For **sophisticated** solutions,* $\eta_i^S(s)$ $(i = 1,2)$ *solve the system of differential equations*

$$\begin{aligned}\dot{\eta}_i^S(s) =&(-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^S(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^S(s)\eta_j^S(s) + r_i(T-t)\eta_i^S(s) \\ &+ \alpha_i(s) + (-1)^i\pi_i\ , \\ \dot{\alpha}_i(s) =&(-1)^i\frac{(k_i)^2}{2c_i}\Phi_i(s)\left(\eta_i^S(s)\right)^2 + (-1)^{i-1}\frac{(k_i)^2}{c_i}\alpha_i(s)\eta_i^S(s) \\ &+ (-1)^i\frac{(k_j)^2}{c_j}\alpha_i(s)\eta_j^S(s) + \Omega_i(s)\alpha_i(s) + (-1)^i\pi_i\Phi_i(s)\ , \\ \{i,j\} =&\{1,2\},\end{aligned} \tag{2.38}$$

with $\Omega_i(s)$ *and* $\Phi_i(s)$ *given in* (2.34)*, and with boundary conditions* $\eta_i^S(T) = (-1)^{i-1}S_i$, $\alpha_i(T) = 0$ $(i = 1,2)$.

2.4. Numerical Illustrations

Since the system of differential equations cannot be solved explicitly, we provide some numerical illustration to throw light on the impact time preferences have on firms' behaviors and the evolution of the market. Numerical solutions are calculated using *Wolfram Mathematica v11.2*.

For reasons of research interest, the two firms are assumed to be symmetric, with the exception of their time preferences and initial market share. By controlling $\pi_1 = \pi_2$, $c_1 = c_2$, and $k_1 = k_2$, we are able to concentrate on how firms' advertising investments alter in accordance with their time preferences. Furthermore, it is not impractical to assume such symmetry.

For products satisfying some specific properties, it is likely that both firms have similar net profit ratio, have achieved excellence in cost control, and are of symmetric abilities in relation to media buying, quality control and some other capabilities, which implies the technical/economic symmetry. For instance, Chintagunta & Jain (1995) conducted some empirical tests using Sorger's specification, and found that the advertising effectiveness of the two duopolies in markets of pharmaceutical product, soft drink and beer are almost identical. In the following, we set $\pi_1 = \pi_2 = 300$, $c_1 = c_2 = 2$, $k_1 = k_2 = 0.3$, as what has been used in Jarrar et al. (2004).

The values of S_1 and S_2 should be carefully chosen. Sorger (1989), Wang & Wu (2001) and Wang & Wu (2007) show that the time dependence of advertising efforts is highly connected with S_1 and S_2, the parameters representing the importance of market share in the end of planning horizon for each firm. Specifically, let $\hat{u}_i(\hat{\eta}_i(s), x)$ denote the feedback Nash equilibrium strategies of both firms under standard discounting, and $\bar{\eta}_i$ be the values such that $\dot{\hat{u}}_i(s) = 0$ $(i = 1, 2)$. If the ending market shares are relatively important $(S_i > \bar{\eta}_i)$ for both firms, then once the market shares reach near the steady state, they will both increase the advertising budget over time when approaching time T, whereas the contrary happens if the final functions are relatively unimportant $(S_i < \bar{\eta}_i)$[2].

In order to mitigate these effects, here we let S_1 and S_2 be proportional to the shadow prices of market share (A_1 and A_2) for the game of infinite time horizon starting at time T with discount rate ρ_i $(i = 1, 2)$. Specifically, $S_i = \omega_i A_i$ $(i = 1, 2)$, where A_1 and A_2 are the solutions to the system

$$\rho_i A_i + (-1)^i q_i + (-1)^{i+1}\frac{(k_i)^2}{2c_i}(A_i)^2 + (-1)^i\frac{(k_j)^2}{c_j}A_iA_j = 0\,,\ \{i, j\} = \{1, 2\}.$$

It can be easily verified that A_i decreases in ρ_i, and that A_i coincide with $\bar{\eta}_i$ $(i = 1, 2)$. The purpose of introducing ω_i $(i = 1, 2)$ is to gain the flexibility of formulating a greater variety of situations under heterogeneous discounting, which will be explained in the next section. For standard and hyperbolic discounting, we assume that $\omega_1 = \omega_2 = 1$.

Under this setting, the current model under standard discounting and with $\omega_1 = \omega_2 = 1$ will coincide with the game of infinite time horizon.

[2]For more detailed discussion, we refer to the Section 4 of Sorger 1989.

2.4.1 Heterogeneous Discounting

We start by discussing the possible situations we can take into account by assigning different values of ω_i ($i = 1, 2$). Without loss of generality, take the symmetric case under heterogeneous discounting with $\delta_i < \rho_i$ as an example. The discounted final function is given by $e^{-\rho_i(T-t)}\omega_i A_i$ ($i = 1, 2$), which is an increasing function in t (from the previous discussion in Section 2.2.2). Depending on the values of ω_i ($i = 1, 2$), we can model the following cases:

- If $\omega_i = 1$, then
  $e^{-\rho_i(T-t)}\omega_i A_i < e^{-\delta_i(T-t)}A_i$ and $e^{-\rho_i(T-T)}\omega_i A_i < e^{-\delta_i(T-T)}A_i$.
  The values that firms ascribe to the ending market shares are relatively low. Though as time goes by, the valuations of the final states are increasing, they are always inferior to the valuations of the profits during the period t to T.

- If $1 < \omega_i < e^{(\rho_i-\delta_i)(T-t)}$, then
  $e^{-\rho_i(T-t)}\omega_i A_i < e^{-\delta_i(T-t)}A_i$ and $e^{-\rho_i(T-T)}\omega_i A_i > e^{-\delta_i(T-T)}A_i$.
  The assessment of scrap values is relatively lower in the beginning of the planning horizon t, then increases as firms move toward the ending point and eventually surpasses the importance of the profits before the end of the planning period.

- If $\omega_i > e^{(\rho_i-\delta_i)(T-t)}$, then
  $e^{-\rho_i(T-t)}\omega_i A_i > e^{-\delta_i(T-t)}A_i$ and $e^{-\rho_i(T-T)}\omega_i A_i > e^{-\delta_i(T-T)}A_i$.
  The importance of final states is higher in the beginning in comparison with the cash flow during the period, and such importance is increasing across time.

For the case of $\delta_i > \rho_i$, situations of always (relatively) higher but decreasing importance, initially higher then eventually lower importance, and always lower and even decreasing importance of final functions can also be modeled by letting $\omega_i = 1$, $e^{(\rho_i-\delta_i)(T-t)} < \omega_i < 1$, and $\omega_i < e^{(\rho_i-\delta_i)(T-t)}$ ($i = 1, 2$), respectively.

Figures 2.1 and 2.2 illustrate the advertising strategies of both firms in a symmetric case of $\delta_1 = \delta_2 = 0.05$, $\rho_1 = \rho_2 = 0.1$, $\omega_1 = \omega_2 = 1.4$,

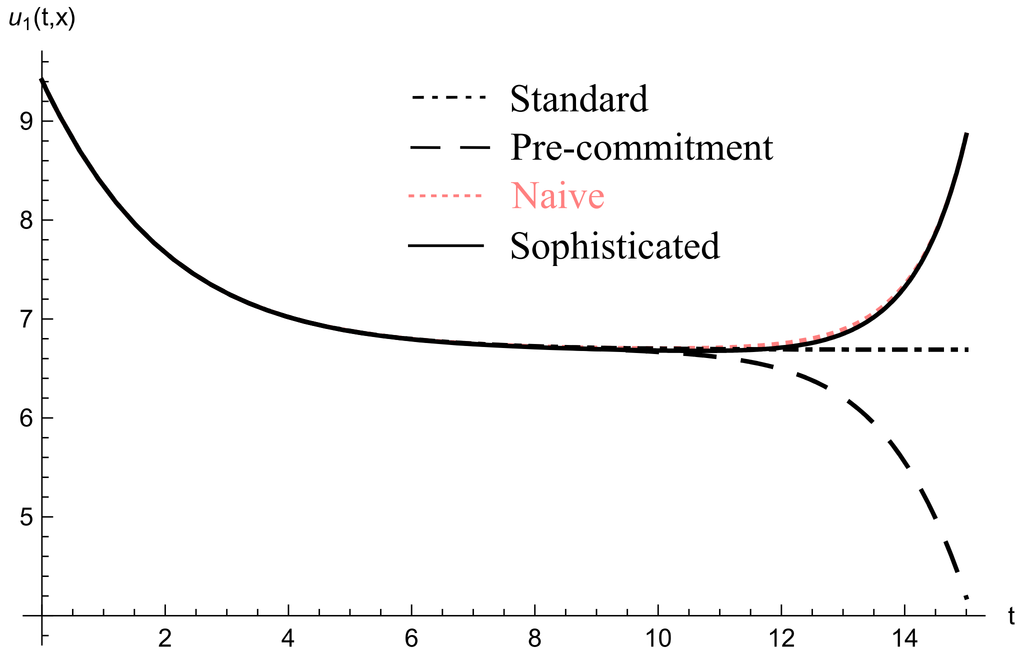

Figure 2.1: Advertising of Firm 1 (Heterogeneous Discounting)

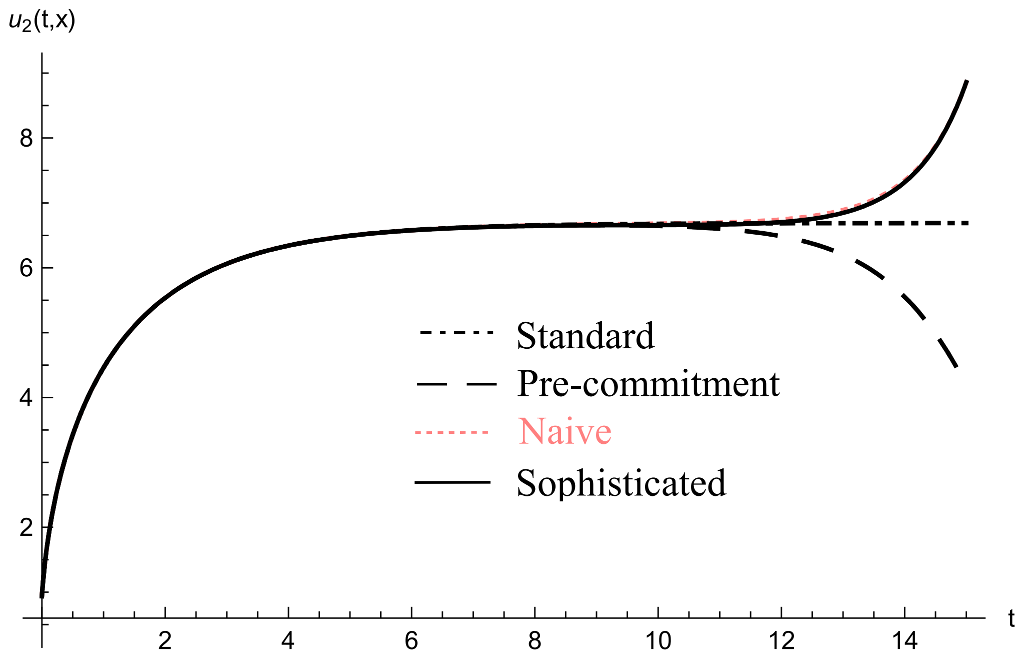

Figure 2.2: Advertising of Firm 2 (Heterogeneous Discounting)

$x_0 = 0.01$, $t = 0$, and $T = 15$. Since $\rho_i > \delta_i$ $(i = 1, 2)$, both firms have increasing valuations of the ending market shares. Furthermore, as explained previously, these valuations are initially inferior to the concerns with the profits throughout the planning horizon, but eventually become dominant. We confine our interest to this special case since it can best demonstrate the difference between time-inconsistent and time-consistent strategies. The standard case of $\delta_i = \rho_i = 0.05$ and $\omega_i = 1$ $(i = 1, 2)$ is also graphed to serve as a benchmark.

As shown in Figure 2.1, for all kinds of discounting and solution types, firm 1, which is at a disadvantage at the beginning (as $x_0 < 0.5$), pumps money into advertising in order to seize market share as soon as possible. The investment is decreasing over time, as her own market share is growing and the target market is reducing the size. On the contrary, holding a dominant market position, firm 2 invests little in the beginning and eventually increases the budget (Figure 2.2). This battle stage lasts until the market share stay in the neighborhood of steady state. From then on, both firms show almost the same advertising patterns.

If firms have standard time preferences, during the quasi-stationary period, both firms would keep the same advertising efforts until the end of planning horizon (due to the values chosen for S_1 and S_2). However, firms under heterogeneous discounting make last-minute shifts in accordance with how they discount the final market. Note that when agents commit themselves to the decision made at the beginning, they act as if they were under standard discounting, but with different boundary conditions. Here, the precommitment solutions are consistent with that of a standard discounting game with final function $e^{-(\rho_i-\delta_i)(T-t)}\omega_i S_i x_i$. Given $\rho_i > \delta_i$, the values that firms ascribe to the ending market share are relatively low, which implies a sharp decrease in advertising. However, as time goes by and firms approach the ending point, the relevance of the final states is increasing and at one point, it takes the priority. Anticipating such changing taste, sophisticated agents' last-minute accommodation is an increase in advertising, which is contrary to the behaviors of players with commitment power. It is worth mentioning that naive solutions and sophisticated solutions are almost identical, probably because when the last-minute change happens, the importance of final states is already dominant.

By anticipating future preferences, time-consistent strategies can help firms to act according to their true preferences. Nonetheless, sophisticated solutions do not necessarily increase or decrease the payoffs. The graphic presentation of market share dynamics is omitted because the patterns in all four cases are extremely similar. Intuitively, lower advertisement spending yields higher payoffs. We can see that agents are better off with pre-commitment than sophisticated solutions in this case.

Next we study some other case with asymmetric discounting.

New Entrant Game: A new entrant in the industry is competing with the incumbent. As the new entrant could have a smaller firm size, more financial constraints, higher instantaneous crisis rate and more urgent developing necessities, she would be more impatient with the financial return, thus discounting future payoffs more heavily. However, the manager believes that after some years' developing, the firm will be less constrained and relatively more far-sighted.

We can incorporate such future belief using heterogeneous discounting. For firm 1 (the new entrant) we set $\delta_1 = 0.15$ and $\rho_1 = 0.05$, whereas firm 2 (the incumbent) uses the same constant and smaller discount rate $\delta_2 = \rho_2 = 0.05$. Since the emphasis here is not the time-variant final function, we let $\omega_1 = \omega_2 = 1$. The initial market distribution is set to be $x_0 = 0.01$. For better interpretation, we also present graphically two benchmark cases of (a) $\delta_1 = \rho_1 = 0.15$, $\delta_2 = \rho_2 = 0.05$ and (b) $\delta_1 = \rho_1 = \delta_2 = \rho_2 = 0.05$. Figures 2.3 to 2.6 demonstrate the scenario described above. Here we focus on the sophisticated solutions, since they are theoretically more desirable and the corresponding equilibrium is subgame perfect.

As shown in Figure 2.3, instead of making last-minute changes as in the symmetric case, here the new entrant, the sophisticated agent under heterogeneous discounting, starts her accommodation much earlier. In the battle period both firms act similarly as in Figures 2.1 and 2.2, the initially smaller firm tries hard to steal the market share from her rival, whereas the market dominant allocates relatively little but increasing resources. In the adapting stage, the new entrant raises her advertising budget at a firstly increasing then decreasing speed. As a response to the new entrant's adjustment, firm 2 (the incumbent) chooses a lower advertising level, in comparison with the standard case, in the accommodation stage. Notice that the new entrant will

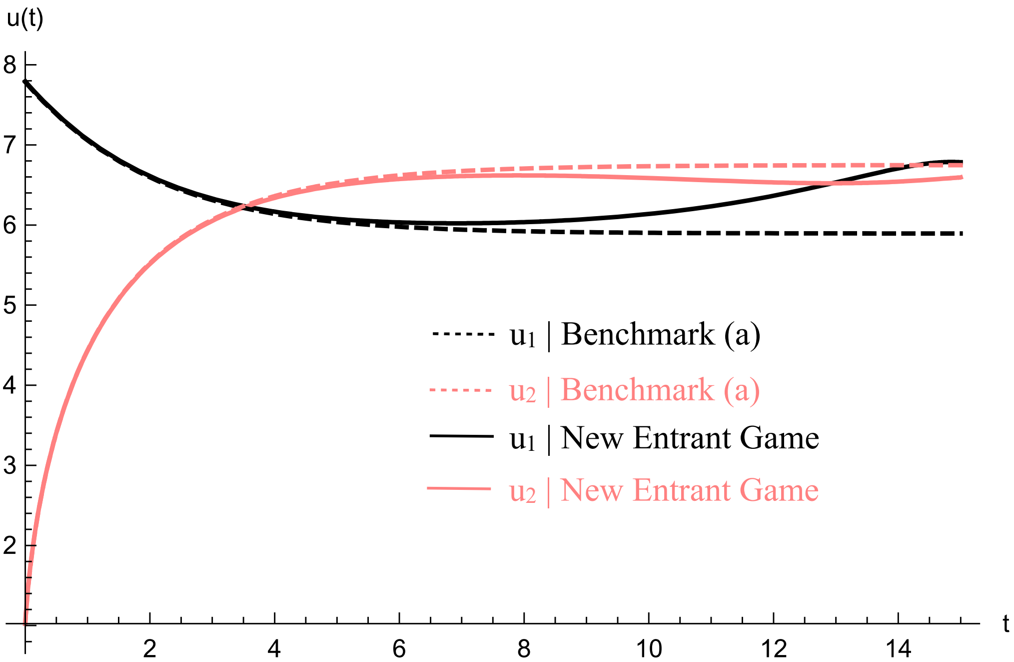

Figure 2.3: Advertising Strategies in New Entrant Game (a)

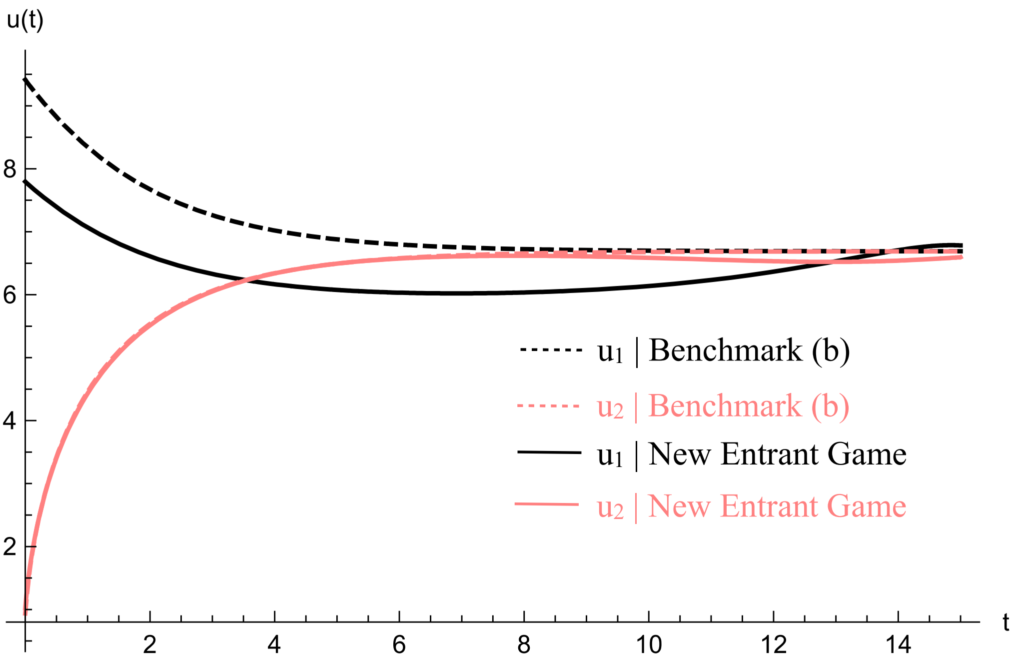

Figure 2.4: Advertising Strategies in New Entrant Game (b)

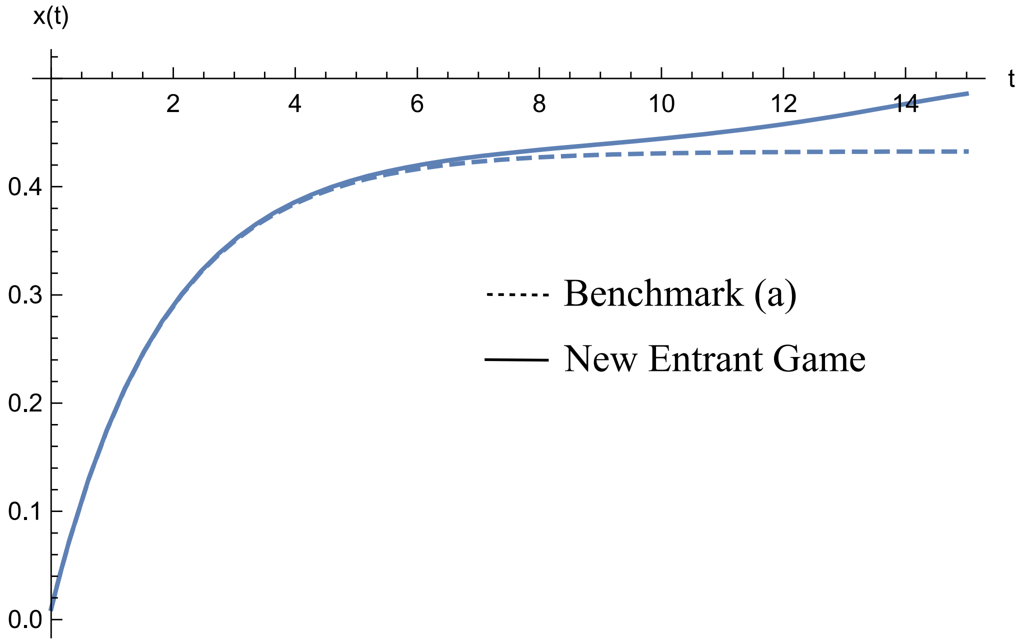

Figure 2.5: Market Share Dynamics in New Entrant Game (a)

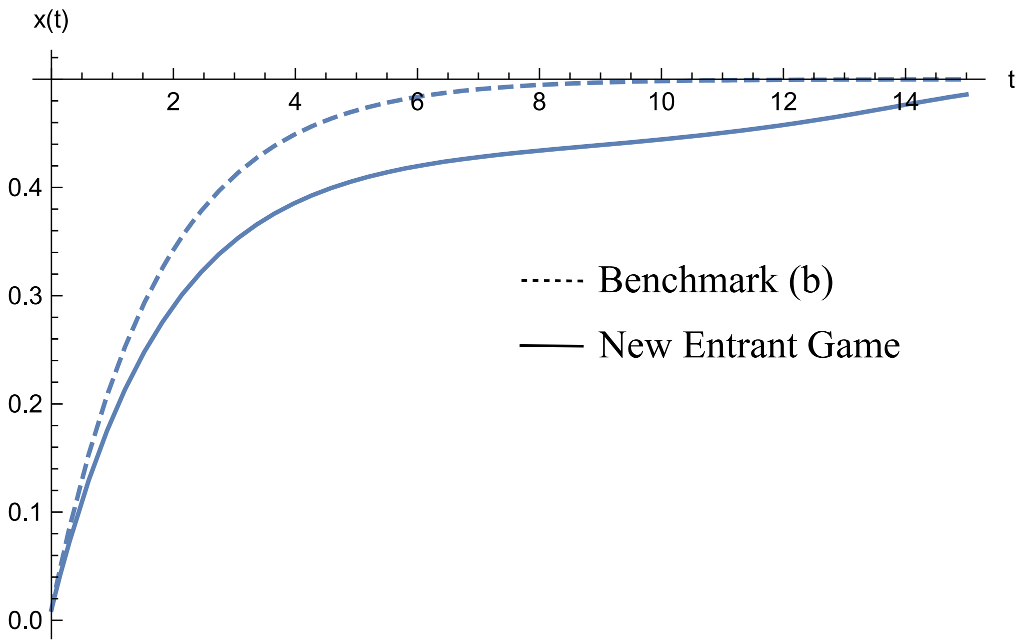

Figure 2.6: Market Share Dynamics in New Entrant Game (b)

end up with higher advertising spending than the incumbent, even when the difference between δ_1 and ρ_1 is extremely small. Figure 2.5 displays the corresponding market share evolution. If the new entrant discounts the future in a standard way with a relatively higher discount rate compared with the incumbent, she will end up with a smaller portion of the whole market. If the manager believes that the new firm can catch up with the incumbent regarding the financial achievement, crisis management, etc., which may lead to a convergence in time preferences, the two firms will share almost equally the market in the end.

As to the benchmark (b), by comparing Figures 2.3 and 2.4, and Figures 2.5 and 2.6, one can clearly see that the new entrant game is an intermediate case between these two benchmarks.

2.4.2 Hyperbolic Discounting

In the following we present some numerical illustrations of advertising competition under hyperbolic discounting.

We start, as previously, with a symmetric case of $\delta_1 = \delta_2 = 0.3$, $\rho_1 = \rho_2 = 0.05$, and $\omega_1 = \omega_2 = 1$. The benchmark here, is the standard discounting case with $\theta_i(s) = e^{-\rho_i s}$, the convergence of the discount function (2.6) when T is sufficiently large. We can also observe different stages in the advertising competition. In the battle stage, both firms' policies are qualitatively consistent with those under standard and heterogeneous discounting. There also appears an adjustment of increasing budget during the last years. To some extent the hyperbolic discounting is similar to heterogeneous discounting with $\delta_i > \rho_i$, in the sense that in both cases, the ending discount rate is smaller compared to that during the planning period. Therefore, firms in Figure 2.7 show similar behaviors as firm 1 in Figure 2.3, they all increment the advertising efforts in the final years. Nonetheless, different types of solution show divergence throughout the planning horizon and converge in the end, which is contrary to the heterogeneous discounting case. Furthermore, the sophisticated agents apply higher advertising policies than naive agents, but in general they are quite similar. Pre-commitment solutions are located between sophisticated strategies and policies made under standard discounting, and the corresponding adjustment stage starts earlier. Note that,

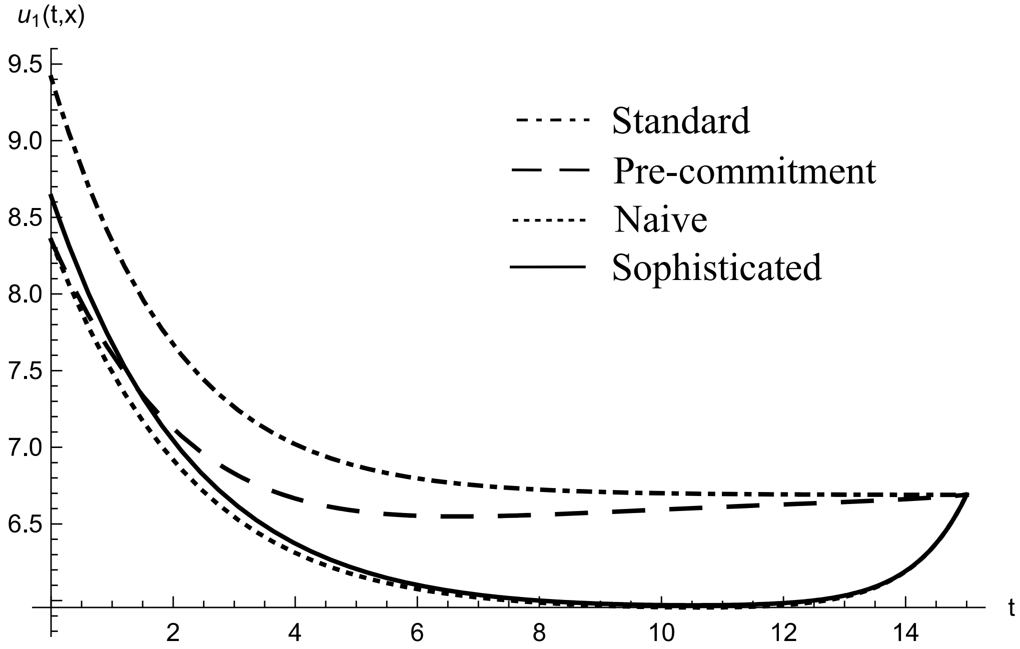

Figure 2.7: Advertising of Firm 1 (Hyperbolic Discounting)

in Figure 2.8, the market share dynamics are roughly the same under time-inconsistent and time-consistent solutions. However, if firms precommit their advertising policies, or are unaware of the change in time preferences, their spending is much higher. Therefore, under hyperbolic discounting, the lack of information about future selves preferences or strong commitment power leads to over investment.

Let us reconsider the *New Entrant Game* described in Section 2.4.1. If both new entrant and the incumbent are under hyperbolic discounting, it is likely that the new entrant has a faster decreasing discount factor due to greater uncertainty. Next we consider the case of $\delta_1 = 0.6$, $\delta_2 = 0.3$, $\rho_1 = \rho_2 = 0.05$, $\omega_1 = \omega_2 = 1$ (Figures 2.9 and 2.10). Notice that, in this setting, both firms share the same (and almost same) discount factor at time 0 and T. Firm 2 applies a lower discount rate throughout the whole planning horizon, which leads her to implement a higher advertising rate when the equilibrium stays in the neighborhood of stationary state (year 7 to 12). Firm 1, being more impatient, starts the accommodation stage earlier, and with a greater increasing rate than firm 2. However, this last-minute effort can not

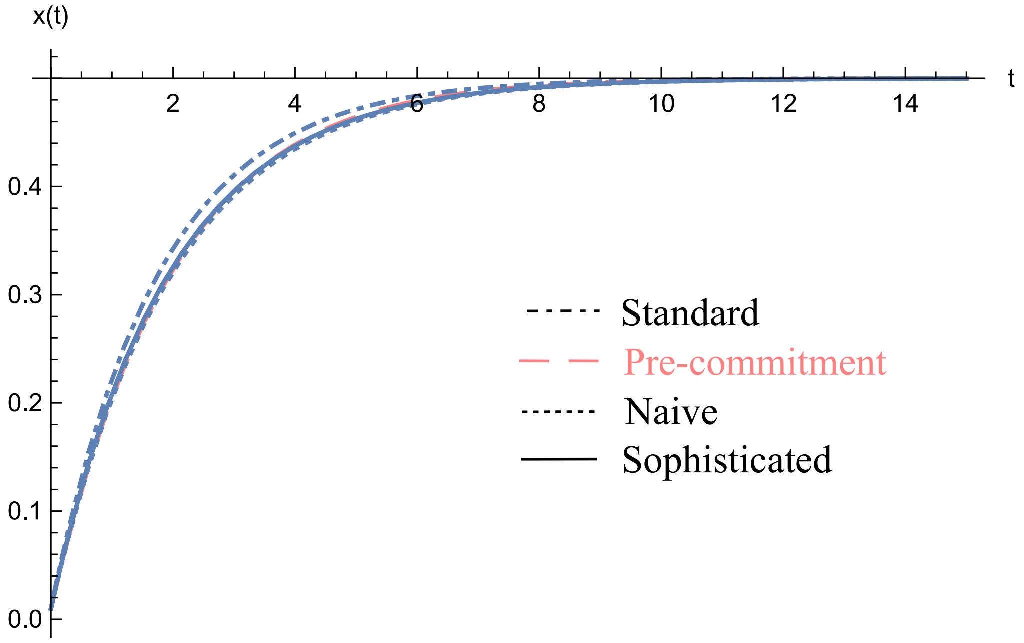

Figure 2.8: Market Share Dynamics (Hyperbolic Discounting)

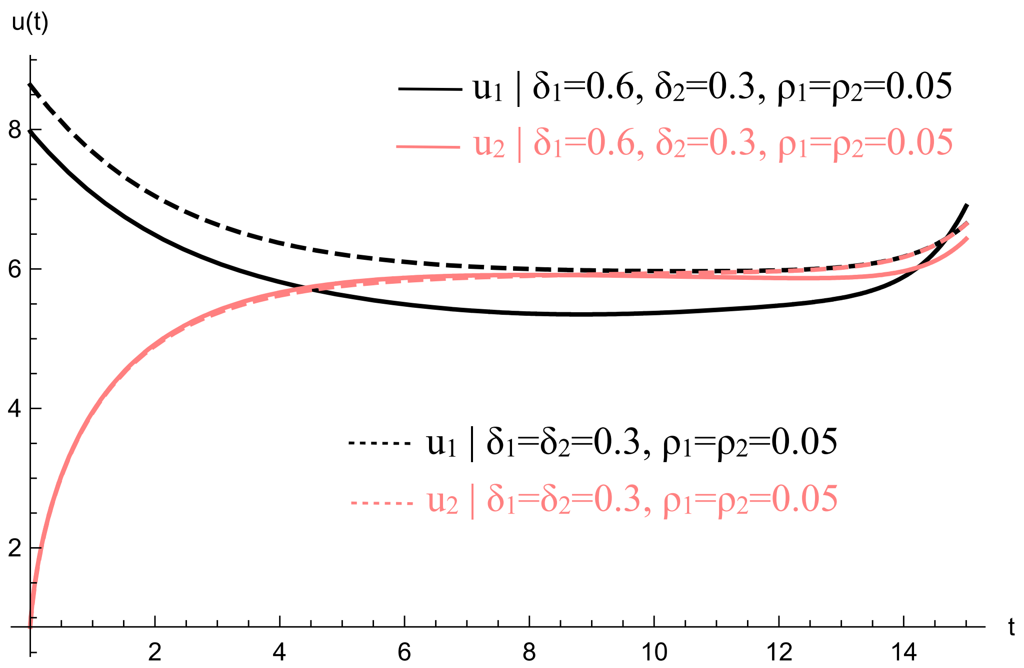

Figure 2.9: Advertising Strategies in New Entrant Game

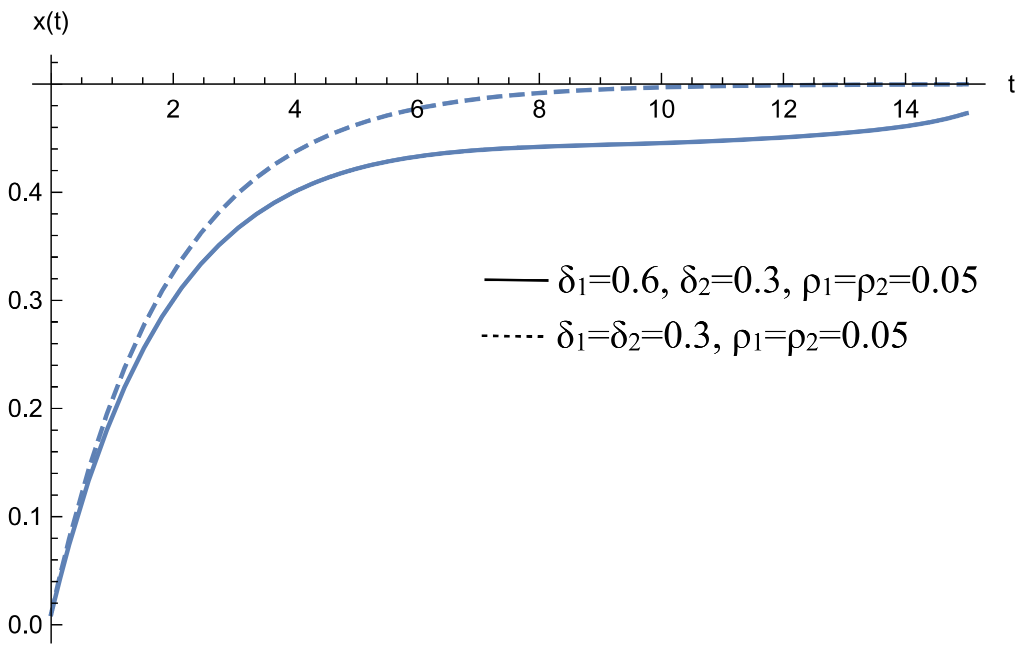

Figure 2.10: Market Share Dynamics in New Entrant Game

compensate completely the loss during the planning horizon, as shown in Figure 2.10, the market ends up with firm 1 acquiring less portion than firm 2.

2.5. Conclusions

This chapter aims to study the advertising competition by introducing some biases in the temporal preferences. Specifically, we have applied a decision making framework that allows agents to take into account the future shift by discounting heterogeneously the future payoffs during and at the end of the planning horizon. The phenomenon of diminishing discount rate is also considered. We have computed three types feedback Nash equilibria strategies for both alternative discounting models. Numerical simulations were run to illustrate some properties of the time-inconsistent and time-consistent advertising behaviors and the corresponding market dynamics. Comparison is made among different types of solutions and those of standard discounting.

The results reveal that firms under heterogeneous discounting act in a different manner compared to those with standard discounting (Sorger's setting). In general, the advertising strategies can be categorized in two phases. The first phase is the battle phase, in which the firm with a larger initial market share invests little at the beginning and increases the advertising effort in time, and the firm with a smaller initial market portion behaves the other way around. If the planning period is sufficiently long, they can arrive near the steady state and remain in its neighborhood for some time. The second phase is the accommodation stage where agents raise/cut their advertising rate according to the increasing/decreasing importance they assign to the final states when approaching to the end of the planning horizon. Our numerical illustrations have demonstrated that the pre-commitment solution can show contrary adjustment direction compared with the sophisticated solution in this stage, whereas the naive solution basically coincides with the sophisticated one.

As to the advertising policies under hyperbolic discounting, a similar battle stage is also present. Another coincidence with heterogeneous discounting is the discrepancy between pre-commitment strategies and time-consistent strategies, as well as the similarity between naive and sophisticated solutions. Due to the model structure, a higher discount rate implies lower advertising effort. Different from heterogeneous discounting, here the lack of information about declining discount rates in the future or strong commitment power would lead to over investment.

The model is built based on some simplifying assumptions. First of all, we have focused on a mature market, which implies a stable market size. We have also assumed that the advertisement cannot influence the purchasing decisions of consumers who are not participating in this industry, in this sense the model could explain the alcohol and beverage industry, but might fail in explaining those industries where outsiders can be attracted by advertisement. Besides, we have not gone into detail on the advertising efficiency, which is described by a parameter. However it would be of interest to consider the factors determining the advertising efficiency apart from technical/economic ones, such as goodwill, brand loyalty and so on.

Variable market size would deserve further research, it sounds more practical that agents have decreasing concern of the final state when the industry is going downhill. It would also be interesting to break the assumption of symmetric advertising efficiency by combining intangible asset of the firm like goodwill. Finally, this model can also be extended to the oligopoly setting under time inconsistent preferences as in Prasad & Sethi (2003).

CHAPTER 3

An Analysis of Efficiency of Time-consistent Coordination Mechanisms in a Model of Supply Chain Management

3.1. Introduction

Research interest regarding the interaction between the members of a supply chain covers various topics including inventory management, production and pricing competition, quality improvement and advertising competition, among others. Advertising coordination, which is commonly believed to be beneficial to the channel, has been highlighted in recent years.

There exist different interpretations of what is understood by advertising coordination (see, e.g., Aust & Buscher, 2014; Jørgensen & Zaccour, 2014, for reviews on advertising coordination). One prevailing setting is the cost-sharing program, also called cooperative advertising/co-op advertising, which is a binding contract on the sharing of the advertising cost: the follower of the supply chain (typically, the retailer) can get some reimbursement of advertising cost from the leader (typically, the manufacturer). According to a report by Marketing-Land (2018), the annual cooperative advertising expenditure was estimated to be 70 billion dollars in the United States. This program has been empirically tested and quite intensively studied in

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static models (for example, see Berger, 1972; Bergen & John, 1997; Dutta et al., 1995; Nagler, 2006). Adopting the goodwill dynamics to model the carry-over effect of advertising, Jørgensen et al. (2000) introduce this co-operation scheme into inter-temporal setting. They consider the case where both manufacturer and retailer can (but are not forced to) implement two types of advertising with long-term and short-term effect, which contribute to the goodwill and market demand respectively. In line with the interest on different advertising effects, Jørgensen et al. (2003) argue that the negative influence of the retailer's promotion on goodwill should be studied. Jørgensen et al. (2001) apply a more flexible functional form for the demand function, and introduce decreasing marginal returns to goodwill. The situation in which the retailer can launch a private-label product is studied in Karray & Zaccour (2005), whereas in De Giovanni (2011), the manufacturer has quality improvement as an additional operational tool. Other cases include a pre-launch advertising campaign with two customers segments to which the players' access is asymmetric (as in Buratto et al., 2007), mechanisms combining revenue sharing and advertising cost sharing (De Giovanni & Roselli, 2012), and the interaction between inventory management and cost-sharing program (De Giovanni et al., 2019).

Despite of the different elements incorporated in different models, what is clear is that when the cost-sharing program is implemented, i.e., the subsidy-rate is strictly positive (this mainly depends on the relationship between the margins of each member, with the exception of De Giovanni, 2011, where the necessary conditions are related to the effectiveness parameter), the retailer is induced, directly or indirectly (through higher goodwill level), to invest more in advertising, and the outcome is Pareto-improving.

Another common mechanism is vertical integration, also known as centralized coordination. In this setting, all members of the supply chain act in a coordinated way to maximize the joint profit. The vertical integration happens rather frequently, for instance, Amazon acquired Whole Foods Market partly for their private label products. Due to its higher total channel profit (in the case of equal discount rates), research interest has been mainly put on how to maintain the cooperation over time with the implementation of incentive strategies (Jørgensen et al., 2006; Jørgensen & Zaccour, 2003b). It is worth mentioning that this setting sometimes also serves as a benchmark in the literature of cost-sharing to make the comparison of channel efficiency (Buratto et al., 2007; Zhang et al., 2013).

Nevertheless, the vertical integration efficiency is based on an essential hypothesis that all agents share equal time preferences. However, asymmetry in time discounting may arise as the result of many different aspects, such as the distinct firm sizes, which could imply distinct financial costs or financial constraints. Besides, different firms conduct divergent economic activities, which are regulated by the corresponding (very often divergent) legislation. Moreover, the asymmetric power in the chain could be associated with different survival probabilities for the firms involved. It is known that when firms have different survival density functions to evaluate the expected utility, by assuming an exponential distribution, the survival probabilities are integrated into the discount rates. Hence, it appears appealing to generalize the time preferences setting via incorporating a possible asymmetry. Such asymmetry in time discounting implies that the joint time preferences are time-inconsistent and we face a trade-off between efficiency and time-consistency (Jackson & Yariv, 2015). If players cooperate by using time-consistent (subgame perfect) strategies, it can happen that joint profits are lower than the joint non-cooperative payoffs. This situation, that we call group inefficiency, may arise for the reason that the set of non-cooperative strategies is, in general, not included in the set of time-consistent strategies. In this chapter we concentrate our attention on the time-consistency since they can be seen as more credible, in the sense that agents have no incentives to deviate from their decisions.

The objective of the study is threefold. First, we extend previous supply chain models by considering that agents can differ in their time preferences, and analyzing how this asymmetry affects non-cooperative and cooperative outcomes. While different time preferences have been considered in other economic areas (e.g., in environmental resources models, de-Paz et al., 2013; Breton & Keoula, 2014), to the best of our knowledge these issues have not been addressed in the framework of dynamic marketing models. Second, and as a consequence of the previous extension, we study situations when cooperation does not pay off by identifying cases of group inefficiency. Third, we adopt a more general demand function that integrates the private effect of goodwill, the synergistic effect of goodwill and retailer's advertising, and the decreasing marginal returns to goodwill. We compare different cooperation programs to see which cooperation mechanism can yield a better outcome, thus offering some implementation guidelines.

We analyze three scenarios. Scenario Nash (N) describes the situation where the two firms act simultaneously in the absence of cooperation. In scenario Stackelberg (S) a cost sharing program is applied: the manufacturer, acting as the leader in a Stackelberg game (this is a prevalent assumption in the literature and is consistent with the nature of many industries such as automobile, gasoline and so on), supports part of the retailer's advertising cost. Another cooperation mechanism, the vertical integration, is modeled in scenario joint maximization (J) where the two players of the supply chain act coordinately to maximize the joint profit.

The main results show that: (1) The cost sharing mechanism implies higher advertising efforts from both manufacture and retailer, and leads to a Pareto superior outcome in comparison with non-cooperative case. This is in accordance with most of the studies. (2) Contrary to the existing marketing literature, a centralized coordination does not necessarily grant higher joint payoffs compared to scenario N. Particularly, if the retailer has a much higher discount rate than the manufacturer, low initial goodwill level and low revenue sharing rate could give rise to group inefficiency. Whereas for the opposite case, when the manufacturer discounts future payoffs much more heavily, and the revenue sharing rule does not extensively favor the retailer, the vertical integration is inefficient, no matter how the initial goodwill level is.

The rest of the chapter is organized as follows. In Section 3.2 we describe the differential game model. The determination of feedback/time-consistent equilibria follows in Section 3.3. In Section 3.4 we make a fully detailed comparison of the strategies and payoffs obtained in Section 3.3 among the three distinct scenarios. We also run some numerical simulations to throw light on the existence of group inefficiency. Finally, in Section 3.5 we present the concluding remarks and suggest some future studies.

3.2. Model Formulation

We consider a two-echelon supply chain model consisting of one manufacturer and one retailer, and where the mechanism works in the way that the manufacturer's advertising policies have more impact on goodwill and sales. In practice, the advertising activities of each member usually have different properties. The manufacturer's global advertising $A_M(t)$ is normally more general and nationwide, with the objective of creating and improving the brand image; it doesn't necessarily generate immediate sales. On the contrary, the retailer's advertising $A_R(t)$ (e.g., promotion, fliers, point-of-sale display, etc.) works more in a local scale and could typically influence directly on the consumer demand (Aust & Buscher, 2014). In this sense, the manufacturer is to some extent dependent on the local advertising, and this is an important reason justifying the usefulness of cooperation.

We depart from the goodwill model proposed by Nerlove & Arrow (1962), where the goodwill is considered as a stock with dynamics given by

$$\dot{G}(t) = k_m A_M(t) - \delta G(t), \quad G(0) = G_0 \geq 0\,, \tag{3.1}$$

where k_m and δ are positive constants representing the effectiveness of the manufacturer's advertising and the depreciation rate, respectively. From expression (3.1), goodwill only increases through manufacture's advertising. In the literature, assumptions on how the retailer's advertising effort affects the brand image are mixed. Some papers consider a positive effect (see, e.g., Jørgensen & Zaccour, 2003a; De Giovanni, 2014; De Giovanni & Roselli, 2012; Jørgensen et al., 2000; Zhang et al., 2013), some others report a negative effect (the main idea is that consumers very often relate frequent promotions to poor quality, for more details, see Jørgensen et al., 2003), and finally some assume a null effect and capture all the retailer's advertising influence in the sales function (Karray & Zaccour, 2005; Jørgensen et al., 2001, 2006).

One common way to model the market demand is to consider the sum of functions of the goodwill and the advertising rate, assuming that they influence the demand independently. One example comes from Jørgensen et al. (2003) (we refer to Jørgensen et al., 2001; Karray & Zaccour, 2005; Zhang et al., 2013, for more variations), where

$$S(t) = \mu G(t) + \gamma A_R(t)\ .$$

Alternatively, the following structure proposed by Jørgensen et al. (2006) (for more examples, see Jørgensen et al., 2001, 2000) assumes that they interact in a multiplicative way:

$$S(t) = \theta + \gamma A_R(t)\sqrt{G(t)}\ .$$

In this setting, what could be arguable is that, in the most extreme case when there is no advertising or no goodwill, the global effect is null.

In this chapter, we combine the separate effect of goodwill and the joint effect, thus sales are given by

$$S(t) = \theta + \mu G(t) + \gamma A_R(t)\sqrt{G(t)}, \tag{3.2}$$

where θ, μ and γ are positive constants representing the baseline sales, the effectiveness of goodwill and the synergy, respectively. Expression (3.2) can be considered as an extension of the additive type model by moderating the retailer's advertising's effect, or as the extension of the multiplicative type model by adding the separate effect of goodwill. This specification captures the properties of some specific markets such as car, infant food, domestic appliances etc., where goodwill plays a determinant role in consumer buying decisions. It reflects moreover the limited influence of the retailer, in that her advertising only works as a booster to the demand.

As in many other studies, advertising cost functions are assumed to be convex and given by

$$C(A_M) = \frac{c_m}{2}A_M(t)^2, \quad C(A_R) = \frac{c_r}{2}A_R(t)^2,$$

where c_m and c_r are positive constant cost parameters.

Finally, we assume that agents' time preferences can differ. As mentioned before, time discount rate can be thought as an aggregation of a series of factors (firm size, legislation, survival probabilities and so on). It is natural to think that such aggregation could result differently in different agents.

Let ρ_m and ρ_r denote the discount rate of the manufacturer and the retailer; $\pi \in (0, 1)$ the revenue sharing rate, which is given exogenously; and $\Phi(t)$ the cost sharing rate, which is the fraction of the advertising cost of the retailer that the manufacturer offers to support. The objective functional of the manufacturer is

$$J_M = \int_0^\infty e^{-\rho_m t}\left[\pi S(t) - \frac{c_m}{2}A_M(t)^2 - \frac{c_r}{2}\Phi(t)A_R(t)^2\right] dt, \tag{3.3}$$

and the objective functional of the retailer is

$$J_R = \int_0^\infty e^{-\rho_r t} \left[(1-\pi)S(t) - \frac{c_r}{2}(1-\Phi(t))A_R(t)^2 \right] dt. \tag{3.4}$$

Equations (3.1), (3.2), (3.3) and (3.4) define a two-player differential game with one state variable $G(t) \geq 0$, and with the manufacturer controlling $A_M(t) \geq 0$ and the retailer controlling $A_R(t) \geq 0$. We conduct a different treatment for the control variable $\Phi(t)$ depending on the scenario. Specifically, in the non-cooperation setting (N), $\Phi(t) = 0$. In the cost sharing scenario (S), the manufacturer can decide freely the value of $\Phi(t)$ in the interval $[0, 1]$. In the vertical integration scenario (J) we keep $\Phi(t)$ as a constant. In a standard cooperative game of joint maximization with equal discount rates, neither revenue sharing rate nor cost sharing rate have impact on the strategies or on the joint outcome, since they are ultimately side payments. However when agents discount future revenues/costs in a heterogeneous way, it may happen that their behaviors are influenced thus the dynamics of the state also evolve differently.

In the following sections the time argument is omitted for brevity unless obvious ambiguity arises.

3.3. Determination of Feedback Equilibria

In this section, we compute the feedback Nash equilibrium, stage-wise feedback Stackelberg equilibrium (for the definition and distinction of Stackelberg solution types, we refer to Long, 2010; Haurie et al., 2012; Basar et al., 2018) and the time-consistent cooperative solution for the scenarios described above.

For the computation of feedback equilibria, we initially restrict the manufacturer's strategy space to the set of linear strategies and make use of the Lemma 3.1. We prove later that there is only one solution in each scenario, which gives rise to constant strategies for the manufacturer (this property coincides with previous results in the literature, see, e.g., Jørgensen et al., 2003), and to linear value functions for both players. It is important to realize that the structure of the strategies for the retailer does not depend on the assumptions on the manufacturer's strategy space, as can be easily checked from the proofs of Propositions 3.1-3.3.

Lemma 3.1. *If the manufacturer and retailer strategies are given by* $A_M = A_m G + B_m$ *and* $A_R = A_r\sqrt{G}$*, then the corresponding value functions are quadratic in the goodwill for the manufacturer and linear in the goodwill for the retailer, i.e.,* $V_M = \lambda_M^2/2 + \alpha_M G + \beta_M$ *and* $V_R = \alpha_R G + \beta_R$*, with* λ_M*,* α_M*,* β_M*,* α_R *and* β_R *constant numbers.*

Proof. See the Appendix. □

3.3.1 Determination of the Feedback Nash Equilibrium

In this non-cooperative scenario, the manufacturer and the retailer decide simultaneously their strategies and there is no cost sharing, $\Phi = 0$. With the absence of cost sharing program, the manufacturer cannot influence the retailer's decisions due to our model structure. Accordingly, the following Nash equilibrium coincides with the Stackelberg equilibrium without cost sharing program (for a rather detailed discussion on such coincidence, we refer to Rubio, 2006). Using a superscript N to denote "Nash", Proposition 3.1 characterizes the equilibrium.

Proposition 3.1. *The feedback Nash equilibrium is given by the pair of strategies*

$$A_M^{\mathrm{N}} = \frac{k_m}{c_m}\alpha_M^{\mathrm{N}}\,, \tag{3.5}$$

$$A_R^{\mathrm{N}} = \frac{(1-\pi)\gamma}{c_r}\sqrt{G}\,, \tag{3.6}$$

and the corresponding value functions are given by

$$V_M^{\mathrm{N}} = \alpha_M^{\mathrm{N}} G + \frac{k_m^2}{2c_m\rho_m}(\alpha_M^{\mathrm{N}})^2 + \frac{\pi\theta}{\rho_m}\,, \tag{3.7}$$

$$V_R^{\mathrm{N}} = \alpha_R^{\mathrm{N}} G + \frac{k_m^2}{c_m\rho_r}\alpha_M^{\mathrm{N}}\alpha_R^{\mathrm{N}} + \frac{(1-\pi)\theta}{\rho_r}\,, \tag{3.8}$$

where

$$\alpha_M^{\mathrm{N}} = \frac{c_r\pi\mu + \pi(1-\pi)\gamma^2}{c_r(\rho_m+\delta)}\,,$$

$$\alpha_R^{\mathrm{N}} = \frac{2c_r(1-\pi)\mu + (1-\pi)^2\gamma^2}{2c_r(\rho_r+\delta)}\,.$$

Proof. See the Appendix. □

As usual, the manufacturer's advertising strategy is determined in the way that the advertising's marginal cost is equal to its marginal revenue:

$$c_m A_M^{\mathrm{N}} = \frac{\pi k_m}{(\rho_m + \delta)} \left[\mu + \frac{(1-\pi)\gamma^2}{c_r} \right] .$$

From (3.5) we can see that the manufacturer's policy is proportional to the ratio of effectiveness parameter to cost parameter, k_m/c_m, which can represent the efficiency of this investment, and is decreasing in δ, the goodwill depreciation rate. Next we study the sensitivity of A_M^{N} with respect to π. From

$$\frac{\partial A_M^{\mathrm{N}}}{\partial \pi} = \frac{k_m(c_r\mu + \gamma^2 - 2\gamma^2\pi)}{c_m c_r(\rho_m + \delta)} ,$$

it holds that $\partial A_M^{\mathrm{N}}/\partial\pi > 0$ for all $\pi \in (0,1)$ when $c_r\mu \geq \gamma^2$. This implies that the manufacturer is motivated to invest more when her corresponding revenue sharing rate π is higher, as expected. However, when γ, the effectiveness parameter of the synergy $A_R^{\mathrm{N}}\sqrt{G}$, is sufficiently high, such that $c_r\mu < \gamma^2$, A_M^{N} increases in π in the interval $(0, \tilde{\pi})$, and decreases in π in the interval $(\tilde{\pi}, 1)$ with $\tilde{\pi} = (c_r\mu + \gamma^2)/(2\gamma^2) > 1/2$. The same happens if c_r and/or μ, the retailer's advertising cost parameter and the effectiveness of goodwill, are sufficiently small. We can understand this behavior via two aspects. Firstly, both large γ and small c_r imply high efficiency of the retailer's advertising, and μ partly measures the efficiency of the manufacturer's advertising. Secondly, large π leads to low local advertising effort, and the manufacturer's advertising is somehow constrained by the retailer's level due to the synergistic effect of goodwill and promotion. As a result, in the situation where the retailer has highly efficient marketing tool but takes little part of the revenue (and low A_R^{N} follows), the manufacturer reduces her investment as π increases. This result is new in the literature and can provide some managerial insights to practitioners. It is also worth mentioning that when the manufacturer is more impatient (larger ρ_m), she invests less. In our model, the manufacturer has no direct influence on the revenue. In addition, an impatient agent discounts heavily the future rewards, which makes the immediate loss prioritized in the decision making.

Regarding the retailer's advertising strategy, it is increasing in the goodwill with a decreasing marginal effect. This is intuitive. Recall how the retailer's advertising works jointly with the goodwill level in (3.2): on the one hand, its effect gets strengthened by the goodwill level, thus a response of increase results; on the other hand, such reinforcement effect is decreasing and, as a consequence, the retailer adjusts her increasing speed. Besides, A_R^{N} is proportional to γ/c_r, the ratio of effectiveness parameter to cost parameter. A higher revenue sharing rate corresponding to her $(1-\pi)$ implies a higher advertising effort.

3.3.2 Determination of the Feedback Stackelberg Equilibrium under Cost Sharing

In this scenario, a cost sharing program is applied. The retailer, which is the follower (as it is usual in most of the studies and in many industries like automobile and gasoline; this is also a natural assumption derived from the limited influence of the retailer on the sales in our setting), can get some reimbursement of the advertising cost from the leader - the manufacturer. As seen in scenario N, the retailer slows down the increment in advertising as the goodwill level goes higher. By supporting part of the retailer's cost, the manufacturer can reach a more desirable sales level. Letting the superscript S refer to "Stackelberg", the following proposition describes the equilibrium.

Proposition 3.2. *The feedback Stackelberg equilibrium is given by the strategies*

$$A_M^{\mathrm{S}} = \begin{cases} \frac{k_m}{c_m}\alpha_M^{\mathrm{S}} & \text{if} \quad \frac{1}{3} < \pi < 1\,, \\ \frac{k_m}{c_m}\alpha_M^{\mathrm{N}} & \text{if} \quad 0 < \pi \leq \frac{1}{3}\,, \end{cases} \tag{3.9}$$

$$\Phi^{\mathrm{S}} = \begin{cases} \frac{3\pi-1}{\pi+1} & \text{if} \quad \frac{1}{3} < \pi < 1\,, \\ 0 & \text{if} \quad 0 < \pi \leq \frac{1}{3}\,, \end{cases} \tag{3.10}$$

$$A_R^{\mathrm{S}} = \begin{cases} \frac{(1+\pi)\gamma}{2c_r}\sqrt{G} & \text{if} \quad \frac{1}{3} < \pi < 1\,, \\ \frac{(1-\pi)\gamma}{c_r}\sqrt{G} & \text{if} \quad 0 < \pi \leq \frac{1}{3}\,, \end{cases} \tag{3.11}$$

and the corresponding value functions are given by

$$V_M^{\mathrm{S}} = \begin{cases} \alpha_M^{\mathrm{S}}G + \frac{k_m^2}{2c_m\rho_m}(\alpha_M^{\mathrm{S}})^2 + \frac{\pi\theta}{\rho_m} & \text{if} \quad \frac{1}{3} < \pi < 1\,, \\ \alpha_M^{\mathrm{N}}G + \frac{k_m^2}{2c_m\rho_m}(\alpha_M^{\mathrm{N}})^2 + \frac{\pi\theta}{\rho_m} & \text{if} \quad 0 < \pi \leq \frac{1}{3}\,, \end{cases} \tag{3.12}$$

$$V_R^{\mathrm{S}} = \begin{cases} \alpha_R^{\mathrm{S}} G + \frac{k_m^2}{c_m \rho_r} \alpha_M^{\mathrm{S}} \alpha_R^{\mathrm{S}} + \frac{(1-\pi)\theta}{\rho_r} & if \quad \frac{1}{3} < \pi < 1 \,, \\ \alpha_R^{\mathrm{N}} G + \frac{k_m^2}{c_m \rho_r} \alpha_M^{\mathrm{N}} \alpha_R^{\mathrm{N}} + \frac{(1-\pi)\theta}{\rho_r} & if \quad 0 < \pi \leq \frac{1}{3} \,, \end{cases} \tag{3.13}$$

where α_M^{N} *and* α_R^{N} *are defined in Proposition 3.1 and*

$$\alpha_M^{\mathrm{S}} = \frac{8c_r \pi \mu + (1+\pi)^2 \gamma^2}{8c_r(\rho_m + \delta)} \,,$$

$$\alpha_R^{\mathrm{S}} = \frac{4c_r(1-\pi)\mu + (1-\pi^2)\gamma^2}{4c_r(\rho_r + \delta)} \,.$$

Proof. See the Appendix. □

The manufacturer offers to support part of the retailer's advertising cost only in the case where her revenue sharing rate is sufficiently high. Otherwise, it is not profitable to pay an additional cost and the best she could do is to compete with the retailer, thus resulting the same outcome as that of in scenario Nash. This finding coincides with Jørgensen et al. (2000) and Jørgensen et al. (2003).

Now let us focus on the case of $1/3 < \pi < 1$, when a cost sharing program is conducted.

We can see some common properties between (3.5) and (3.9). Specifically, in both scenarios N and S, the manufacturer responses in a similar way to the efficiency ratio k_m/c_m, goodwill depreciation rate δ, and time preference ρ_m. However, unlike Scenario N, where a lower global advertising budget might be associated with a higher revenue sharing rate because of (relative) inefficiency and limited local advertising, here the larger proportion of revenue the manufacturer takes, the more resources she would spend in global advertising ($\partial A_M^{\mathrm{S}}/\partial\pi > 0$). With an active cost sharing program, the local advertising is greater than that in competition setting (we present a more detailed comparison in Section 3.4.1). This allows the manufacturer to invest more in advertisement, independently of how efficient the follower's marketing tool is.

With respect to the retailer's advertising, as in the competition scenario N, it is proportional to γ/c_r and increases in the goodwill with a decreasing marginal effect. What may be surprising at first sight is that A_R^{S} (equation (3.11)) is increasing in π, the manufacturer's revenue sharing rate. However, note that the higher fraction the manufacturer gets from the revenue, the higher percentage she would pay to the retailer ($\partial\Phi^{\mathrm{S}}/\partial\pi > 0$). Indeed, the actual cost the retailer is paying is decreasing in the manufacturer's revenue sharing rate $\left(\partial[A_R^{\mathrm{S}}(1-\Phi^{\mathrm{S}})]/\partial\pi < 0\right)$.

3.3.3 Determination of the Time-Consistent Cooperative Solution under Vertical Integration

In the vertical integration scenario, the manufacturer and the retailer form a coalition to maximize the sum of their individual payoffs defined in (3.3) and (3.4). Using the superscript J to represent "Joint maximization", we are facing a problem with the objective functional:

$$\begin{aligned} J^{\mathrm{J}} =& J_M + J_R \\ =& \int_0^{\infty} e^{-\rho_m t}\left\{\pi S + e^{-(\rho_r-\rho_m)t}(1-\pi)S - \frac{c_m}{2}(A_M)^2 \right. \\ & \left. -\frac{c_r}{2}\left[\Phi(A_R)^2 + e^{-(\rho_r-\rho_m)t}(1-\Phi)(A_R)^2\right]\right\} dt. \end{aligned} \tag{3.14}$$

Notice that when $\rho_m \neq \rho_r$, terms depending on π and Φ don't vanish, unlike the standard case. When agents are cooperating, π and Φ are side payments which can be negotiated in order to sustain the cooperation. However, when they discount them heterogeneously, as we will show later, these side payments do have impacts on their behaviors and the outcome.

As illustrated in de-Paz et al. (2013), the aggregated time preferences given by (3.14) are time-inconsistent. Similar to hyperbolic discounting, time-consistent solutions can be defined for this kind of problems. Applying the approach proposed in the chapter mentioned above, the time-consistent cooperative solution is characterized by the following proposition.

Proposition 3.3. *The time-consistent cooperative advertising strategies are determined by*

$$A_M^{\mathrm{J}} = \begin{cases} \frac{k_m}{c_m}(\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}}) & if \quad \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0\,, \\ 0 & if \quad \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} < 0\,, \end{cases} \tag{3.15}$$

$$A_R^{\mathrm{J}} = \frac{\gamma}{c_r}\sqrt{G}\,, \tag{3.16}$$

and the corresponding value functions are given by

$$V_M^{\rm J} = \alpha_M^{\rm J} G + \begin{cases} \frac{k_m^2}{2c_m\rho_m}((\alpha_M^{\rm J})^2 - (\alpha_R^{\rm J})^2) + \frac{\pi\theta}{\rho_m} & \text{if} \quad \alpha_M^{\rm J} + \alpha_R^{\rm J} \geq 0\,, \\ \frac{\pi\theta}{\rho_m} & \text{if} \quad \alpha_M^{\rm J} + \alpha_R^{\rm J} < 0\,, \end{cases} \tag{3.17}$$

$$V_R^{\rm J} = \alpha_R^{\rm J} G + \begin{cases} \frac{k_m^2}{c_m\rho_r}(\alpha_M^{\rm J}\alpha_R^{\rm J} + (\alpha_R^{\rm J})^2) + \frac{(1-\pi)\theta}{\rho_r} & \text{if} \quad \alpha_M^{\rm J} + \alpha_R^{\rm J} \geq 0\,, \\ \frac{(1-\pi)\theta}{\rho_r} & \text{if} \quad \alpha_M^{\rm J} + \alpha_R^{\rm J} < 0\,, \end{cases} \tag{3.18}$$

where

$$\alpha_M^{\rm J} = \frac{2c_r\pi\mu + (2\pi - \Phi^{\rm J})\gamma^2}{2c_r(\rho_m + \delta)}\,,$$

$$\alpha_R^{\rm J} = \frac{2c_r(1-\pi)\mu + (1 + \Phi^{\rm J} - 2\pi)\gamma^2}{2c_r(\rho_r + \delta)}\,.$$

Proof. See the Appendix. □

We first look at the standard case, where both players share the same discount rate. In this case, the condition $\alpha_M^{\rm J} + \alpha_R^{\rm J} \geq 0$ is verified. We rewrite the non-zero manufacturer's strategy as

$$A_M^{\rm J} = \frac{k_m\left\{(\rho_m - \rho_r)\left[-2(\gamma^2 + c_r\mu)\pi + \gamma^2\Phi^{\rm J}\right] + (\rho_m + \delta)(\gamma^2 + 2c_r\mu)\right\}}{2c_m c_r(\rho_m + \delta)(\rho_r + \delta)}\,. \tag{3.19}$$

When $\rho_m = \rho_r = \rho$,

$$A_M^{\rm J} = \frac{k_m(\gamma^2 + 2c_r\mu)}{2c_m c_r(\rho + \delta)}\,,$$

and the payoffs of the grand coalition are

$$V_M^{\rm J} + V_R^{\rm J} = \frac{\gamma^2 + 2c_r\mu}{2c_r(\rho + \delta)}G + \frac{k_m^2}{2c_m\rho}\left[\frac{\gamma^2 + 2c_r\mu}{2c_r(\rho + \delta)}\right]^2 + \frac{\theta}{\rho}\,.$$

Note that none of them is affected by the revenue sharing rate nor the cost sharing rate. As we mentioned before, in the standard case, side payments given from π and Φ cancel in the coalition's objective functional. As a consequence, even though agents' individual payoffs vary with different values of side payments, their strategies and the joint payoffs are not affected.

However, when the two players discount future payoffs differently, the situation changes. In this setting with full cooperation among agents, giving rise to a vertical integration, we assume values of π and $\Phi^{\rm J}$ as given, as the result, e.g., of a previous agreement (contract) among the manufacturer and the retailer. Note that in some extreme case (it is easy to check that $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$ is fulfilled unless ρ_m is very different to ρ_r), the manufacturer would choose not to invest in advertising at all. When the parameters setting is in such a way that $A_M^{\mathrm{J}} > 0$, the manufacturer's advertising (3.19) is monotone in π and Φ^{J}, depending on the relation between ρ_m and ρ_r. From the shadow prices α_M^{J} and α_R^{J}, we can see that a higher revenue sharing rate and a lower cost sharing rate are beneficial to the manufacturer, but damaging to the retailer (and these effects decrease in their corresponding discount rate). Concretely, when $\rho_m > \rho_r$, the overall effect of higher π turns out to be unfavorable to the coalition, thus implying lower A_M^{J}. On the contrary, Φ^{J} and A_M^{J} move in the same direction due to the same reason. If $\rho_r > \rho_m$, the previous results are reversed. A more detailed analysis of the effects of π and Φ is provided in Section 3.4.3.

The property of being proportional to k_m/c_m also appears in this scenario, as in the other two settings. However, unlike (3.5) and (3.9), the effects of δ and ρ_m are not so straightforward. A_M^{J} (in the non-zero case) is increasing in ρ_m if $\Phi^{\mathrm{J}}/\pi > 2 + c_r\mu/\gamma^2$. Since the manufacturer has larger influence on the market, the above condition (implying a cost sharing rate more than twice the revenue sharing rate) seems unrealistic. Hence, it is more likely that A_M^{J} decreases in ρ_m. Another relevant difference is that the time preference of the retailer ρ_r also plays a role in determining the manufacturer's decision. We can observe that ρ_r and ρ_m have opposite effects on A_M^{J}.

The retailer's advertisement expenditure does not depend on the time preferences. Compared to the policies in the other two scenarios N and S, some similar properties remain: it is state dependent and proportional to γ/c_r. Nevertheless, it is not subject to the revenue sharing rate as in (3.6) and (3.11). Subsequently, if π and the goodwill level are sufficiently high, it can happen that the retailer is forced to invest too much but gets too little. In this situation, high goodwill level is harmful to the retailer, and this explains why the shadow price α_R^{J} can be even negative for large π, which is not so frequent in the literature.

It is hard to conclude the side-payments' impact on the joint payoffs but, as shown in (3.17) and (3.18), they do have their influence.

3.4. Analysis of the Results

3.4.1 Comparison of the Equilibrium Strategies and Payoffs

While seeking the time-consistent solutions in the joint maximization setting (Scenario J), we have kept Φ^{J} as an arbitrary constant to see how it affects the coalition's behaviors and payoffs. Here we choose, for Scenario J, a critical value: $\Phi^{\mathrm{J}} = 0$, as in the (feedback Nash) non-cooperative case. This natural choice implies that side payments, in the case of vertical integration, solely come via the revenue sharing rate π. A pairwise comparison between Scenario S and J with $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$ will be briefly represented in Remark 3.1.

We first compare the manufacturer's advertising strategy among three scenarios.

Proposition 3.4. *The manufacturer's strategies are related as follows:*

1. *If* $\rho_m < \rho_r$, $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}} < A_M^{\mathrm{J}}$ *for all* $\pi \in (0, 1)$ *(with the first inequality strict for* $\pi \in (\frac{1}{3}, 1)$*);*
2. *If* $\rho_r < \rho_m < 2\rho_r + \delta$*,*
   - $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}} < A_M^{\mathrm{J}}$ *for all* $\pi \in (0, \pi^*)$ *(with the first inequality strict for* $\pi \in (\frac{1}{3}, 1)$*),*
   - $A_M^{\mathrm{N}} < A_M^{\mathrm{J}} < A_M^{\mathrm{S}}$ *for all* $\pi \in (\pi^*, 1)$*, where* $\pi^* \in (\frac{1}{3}, 1)$ *solves*

   $$-\gamma^2(\rho_r + \delta)\pi^2 + \left[6\gamma^2(\rho_r + \delta) - 8(\gamma^2 + c_r\mu)(\rho_m + \delta)\right]\pi + 4(\gamma^2 + 2c_r\mu)(\rho_m + \delta) - \gamma^2(\rho_r + \delta) = 0\ ;$$

3. *If* $2\rho_r + \delta < \rho_m$*,*
   - $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}} < A_M^{\mathrm{J}}$ *for all* $\pi \in (0, \pi^*)$ *(with the first inequality strict for* $\pi \in (\frac{1}{3}, \pi^*)$*),*
   - $A_M^{\mathrm{N}} < A_M^{\mathrm{J}} < A_M^{\mathrm{S}}$ *for all* $\pi \in (\pi^*, \pi^{**})$*,*
   - $A_M^{\mathrm{J}} < A_M^{\mathrm{N}} < A_M^{\mathrm{S}}$ *for all* $\pi \in (\pi^{**}, 1)$*, where* $\pi^{**} \in (\frac{1}{3}, 1)$ *is the solution to*

   $$-2\gamma^2(\rho_r + \delta)\pi^2 + 2(\gamma^2 + c_r\mu)(\rho_m + \delta)\pi - (\gamma^2 + 2c_r\mu)(\rho_m + \mu) = 0$$

   and $\pi^* < \pi^{**}$*.*

Proof. See the Appendix. □

The comparison between Nash and Stackelberg equilibria is clear. For $\pi > \frac{1}{3}$, the manufacturer's advertising is lower in Scenario N than in Scenario S. However, the nature of global advertising in the vertical integration (joint maximization) setting is more complex and highly depend on the agents' time preferences.

As a side note, large firms usually tend to have low discount rates, because they are very often associated with less financial constraints, lower potential crisis likelihood and higher survival probability. On the contrary, small firms are generally more eager for current payments due to the necessity of development[1]. If the manufacturer is more powerful and farsighted than the retailer, the global advertising would be the highest in vertical integration than in any other two cases. A similar result can arise if the manufacturer is slightly more myopic and the revenue sharing rate is small-intermediate. However, as discussed in the previous section with respect to (3.19), when $\rho_m > \rho_r$ holds, A_M^{J} is decreasing in π due to the joint shadow price. As a consequence, the manufacturer's advertisement expenditure in Scenario J would be between that in Scenario N and S for π considerably high ($\pi > \pi^*$). In the case where the manufacturer is much more shortsighted than the retailer ($\rho_m > 2\rho_r + \delta$), if π is sufficiently near to 1 ($\pi > \pi^{**}$), the global advertising in centralized coordination might be insufficient and become the minimum among all three scenarios.

We next compare the retailer's policies in all the three scenarios.

Proposition 3.5. *The retailer's strategies have the following properties (with the first inequality strict for $\pi \in (\frac{1}{3}, 1)$):*

$A_R^{\mathrm{N}} \leq A_R^{\mathrm{S}} < A_R^{\mathrm{J}}$.

Proof. It follows from (3.6), (3.11) and (3.16). □

For a given goodwill level, the retailer's advertising is the highest under vertical integration, and the lowest in competition setting.

Finally, we compare individual payoffs.

Proposition 3.6. *Equilibrium payoffs are related as follows (with strict inequality for $\pi \in (\frac{1}{3}, 1)$):*

[1]Obviously this is not always true, time preferences also depend on the financial health of the company.

1. *$V_M^{\mathrm{N}}(G) \leq V_M^{\mathrm{S}}(G)$, $V_R^{\mathrm{N}}(G) \leq V_R^{\mathrm{S}}(G)$, for all $G \geq 0$.*

2. *Shadow prices are ranked as $\alpha_M^{\mathrm{N}} \leq \alpha_M^{\mathrm{S}} < \alpha_M^{\mathrm{J}}$ and $\alpha_R^{\mathrm{J}} < \alpha_R^{\mathrm{N}} \leq \alpha_R^{\mathrm{S}}$.*

Proof. See the Appendix. □

Clearly, both manufacturer and retailer get better off in the cost-sharing setting than in the fully non-cooperative game. From Propositions 3.4 and 3.5, both firms invest more in the Stackelberg Scenario than in the Nash Scenario. Acting in this way together, they reach a larger market size. As for the vertical integration, we can conclude that the shadow price of goodwill (α_i^{J} with $i = M, R$ and j =N, S, J) in Scenario J is the greatest for the manufacturer and the smallest for the retailer than in the other two settings. Since the shadow price represents the increase of the payoff when increasing the initial goodwill by one unit, when the initial goodwill level is sufficiently high, vertical integration would be more preferred by the manufacturer and less preferred by the retailer compared to any other program.

Remark 3.1. *Although $\Phi^{\mathrm{J}} = 0$ seems to be the natural choice in the cooperative setting (agents share profits, not costs), there are situations in which it could make sense to consider $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$. Take, for instance, the case of a supply chain with an active cost sharing program ($\Phi^{\mathrm{S}} > 0$) that is looking at the feasibility of carrying out a vertical merger. It can be checked (see the Appendix) that, for $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$ and $\pi \in (\frac{1}{3}, 1)$, the manufacturer's strategies, the retailer's strategies and the shadow prices are ranked as:*

1. *$A_M^{\mathrm{S}} < A_M^{\mathrm{J}}$.*

2. *$A_R^{\mathrm{S}} < A_R^{\mathrm{J}}$.*

3. *$\alpha_M^{\mathrm{S}} < \alpha_M^{\mathrm{J}}$ and $\alpha_R^{\mathrm{J}} < \alpha_R^{\mathrm{S}}$.*

Proof. See the Appendix. □

3.4.2 Existence of Group Inefficiency

In this section we study the possible existence of group inefficiency in the vertical integration setting. By group inefficiency we mean a situation in which the joint payment if players cooperate is smaller than the sum of the individual payoffs of all members in the coalition when no cooperation happens. By construction, in the case of equal discount rates, group inefficiency cannot appear. However, if discount rates are heterogeneous, the restriction to the search of time-consistent solutions in a cooperative framework can have as a price the loss of group efficiency. This property was illustrated with a simple example in Marín-Solano (2015). Note that, although utilities are transferable, in the case of group inefficiency, no side payments exist such that all players can get, at least, what they obtain in the non-cooperative framework.

Since, from Proposition 3.6, payments for both players in Scenario S (where we are considering a partial cooperation via Φ^{S}) are higher than those in Scenario N, we center our analysis on the more demanding comparison between Scenario N and Scenario J. *A priori*, it is hard to conclude the overall effect of vertical integration: the manufacturer benefits more from the high initial goodwill level in Scenario J than in Scenario N, whereas the contrary happens to the retailer.

Hence, we proceed to check if the following relation holds:

$$\begin{aligned} &\left(V_M^{\mathrm{J}}(G) + V_R^{\mathrm{J}}(G)\right) - \left(V_M^{\mathrm{N}}(G) + V_R^{\mathrm{N}}(G)\right) \\ &= -\frac{\gamma^2\pi^2(\rho_m - 2\rho_r - \delta)}{2c_r(\rho_m+\delta)(\rho_r+\delta)}G + \begin{cases} \Delta(\pi) & \text{if} \quad \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0 \\ -\frac{k_m^2}{c_m}\left[\frac{(\alpha_M^{\mathrm{N}})^2}{2\rho_m} + \frac{\alpha_M^{\mathrm{N}}\alpha_R^{\mathrm{N}}}{\rho_r}\right] & \text{if} \quad \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} < 0 \end{cases} \\ &< 0\,, \end{aligned} \tag{3.20}$$

where α_M^{N}, α_R^{N}, α_M^{J} and α_R^{J} are defined in Proposition 3.1 and 3.3, and

$$\begin{aligned} \Delta(\pi) &= \frac{k_m^2}{8c_m c_r^2\rho_m\rho_r(\rho_m+\delta)^2(\rho_r+\delta)^2}(a\pi^4 + b\pi^3 + c\pi^2 + d\pi + e)\,, \text{with} \\ a &= 4\gamma^4(\rho_m - \rho_r)(\rho_r+\delta)(\rho_m+\rho_r+\delta)\,, \\ b &= -4\gamma^2(\gamma^2 + c_r\mu)(\rho_r+\delta)\left[3\rho_m(\rho_m+\delta) - 2\rho_r(\rho_r+\delta)\right]\,, \\ c &= 4\left[c_r^2\mu^2(\rho_m+\delta)(2\rho_m-\rho_r) + \gamma^2(\gamma^2+2c_r\mu)(2\rho_m^2 + 3\rho_m\delta - \rho_r\delta)\right] \\ &\quad \times(\rho_m+\delta)\,, \\ d &= -4(\gamma^2+c_r\mu)(\gamma^2+2c_r\mu)(2\rho_m-\rho_r)(\rho_m+\delta)^2\,, \\ e &= (\gamma^2+2c_r\mu)^2(\rho_m+\delta)^2(2\rho_m-\rho_r)\,. \end{aligned} \tag{3.21}$$

Table 3.1: Benchmark Parameter Setting

ρ_m	ρ_r	δ	c_m	c_r	k_m	μ	γ	θ
0.15	0.03	0.03	2	2	1	1	1	1

In the case of identical time preferences, $\rho_m = \rho_r = \rho$, given that the set of non-cooperative strategies is included in the set of time-consistent strategies, the total outcome of joint maximization is always equal or larger than that of Nash competition case. Nevertheless, when agents exhibit divergent discount rates, the aggregated time preferences become time-inconsistent and it could happen that (3.20) holds. Specifically, it becomes clear that, if $\rho_m > 2\rho_r + \delta$, for any initial goodwill level, the coalition is inefficient when parameters are such that $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} < 0$, which implies zero manufacturer's advertising; a sufficiently high initial goodwill level will also give rise to group inefficiency when $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$, independently on the sign of $\Delta(\pi)$. Moreover, since $\Delta(0) < 0$ if $\rho_r > 2\rho_m$, group inefficiency can also happen when the initial goodwill level and the revenue sharing rate are sufficiently small.

We provide some numerical illustrations to throw light on the existence of group inefficiency. We confine our interest to the case when $\rho_m > 2\rho_r + \delta$ and $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$. Accordingly, if $\Delta(\pi)$ in (3.20) takes a negative value, the cooperation is group inefficient no matter how the initial goodwill level is. Under this parameter setting, we have $\Delta(0) > 0$ and $\Delta(1) < 0$[2], which assures that there exists solution(s) of $\Delta(\pi) = 0$ for $\pi \in (0, 1)$. Due to the complexity of the quartic function $\Delta(\pi)$, it is difficult to prove formally the uniqueness of the solution in the interval $(0, 1)$. But after running many simulations, we found that there exists $\hat{\pi} \in (0, 1)$ such that $\Delta(\pi) > 0$ for all $\pi \in (0, \hat{\pi})$ and $\Delta(\pi) < 0$ for all $\pi \in (\hat{\pi}, 1)$.

In Table 3.1 we summarize the parameter values used as the benchmark case (corresponding to the solid line in all the figures). For simplicity the effectiveness parameters k_m, μ and γ are normalized to 1, and the cost parameters c_m, c_r are set to be 2. This benchmark sample is somehow symmetric, except for the agents' time preferences. Furthermore, by altering one single parameter value from the benchmark case, we have conducted some sensitivity analysis of how each parameter can affect the interval $(\hat{\pi}, 1)$

[2]Proof: see the Appendix.

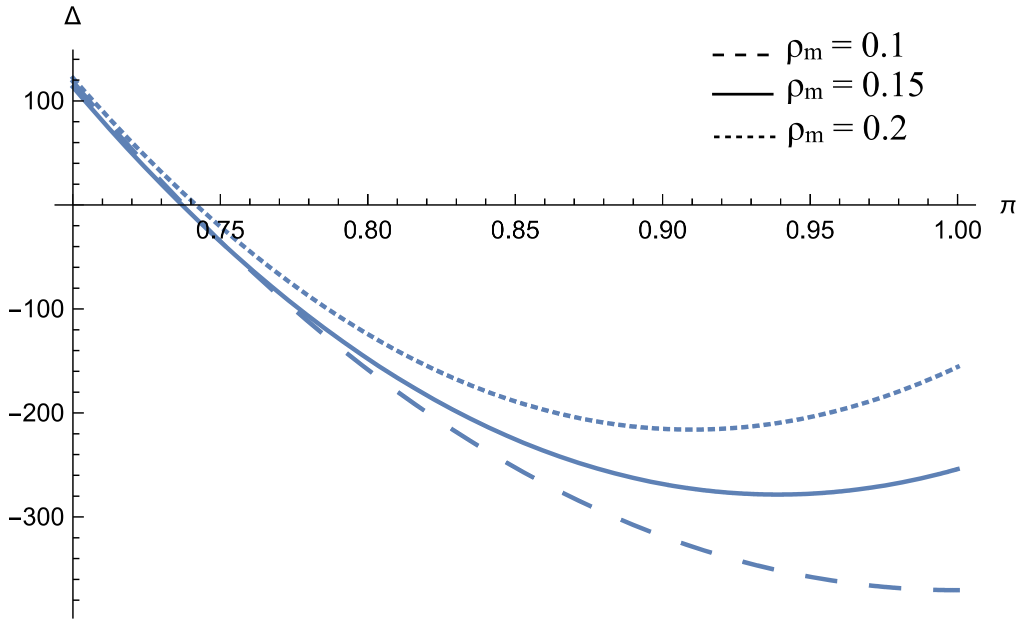

Figure 3.1: Group Inefficiency (ρ_m)

such that $\Delta(\pi) < 0$, which to some extent measures how likely an inefficient cooperation is to happen, as well as the group inefficiency level.

Some conclusions can be drawn directly from (3.20) and (3.21). Although the baseline sales θ, the manufacturer's advertising effectiveness k_m and cost parameter c_m determine the players' strategies and payoffs in both scenarios N and J, they do not affect the group inefficiency likelihood. However, for any given initial goodwill level, more effective (larger k_m) and/or less costly (smaller c_m) global advertising will imply a higher group inefficiency level.

Figures 3.1 to 3.6 represent the sensitivity analysis of ρ_m, ρ_r, δ, c_r, μ and γ, respectively[3]. As shown in all the figures, there exists group inefficiency for high levels of π. As explained previously, in the vertical integration setting, the retailer's advertising level is independent of the revenue sharing rate. A small participation on revenues may induce the retailer to earn less than what she spends and, as a result, she suffers a great loss of profit compared to the Nash setting. When the retailer loses so much that the improvement of the

[3]Please notice that in order to show with more details the group inefficiency, we do not present the whole range of $\pi \in (0, 1)$. However, $\Delta(\pi)$ is strictly decreasing in the omitted interval.

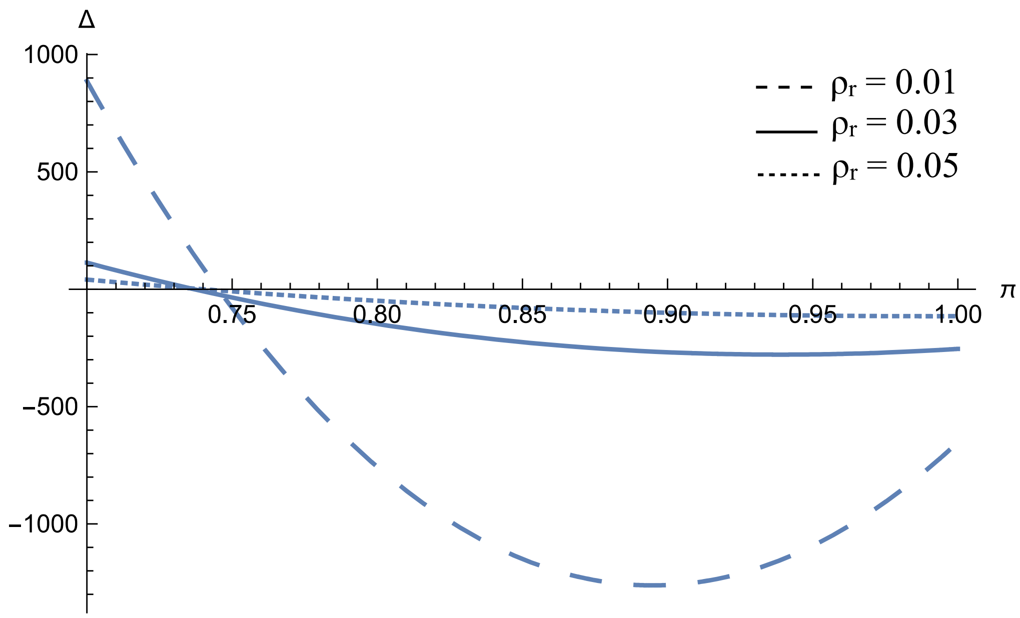

Figure 3.2: Group Inefficiency (ρ_r)

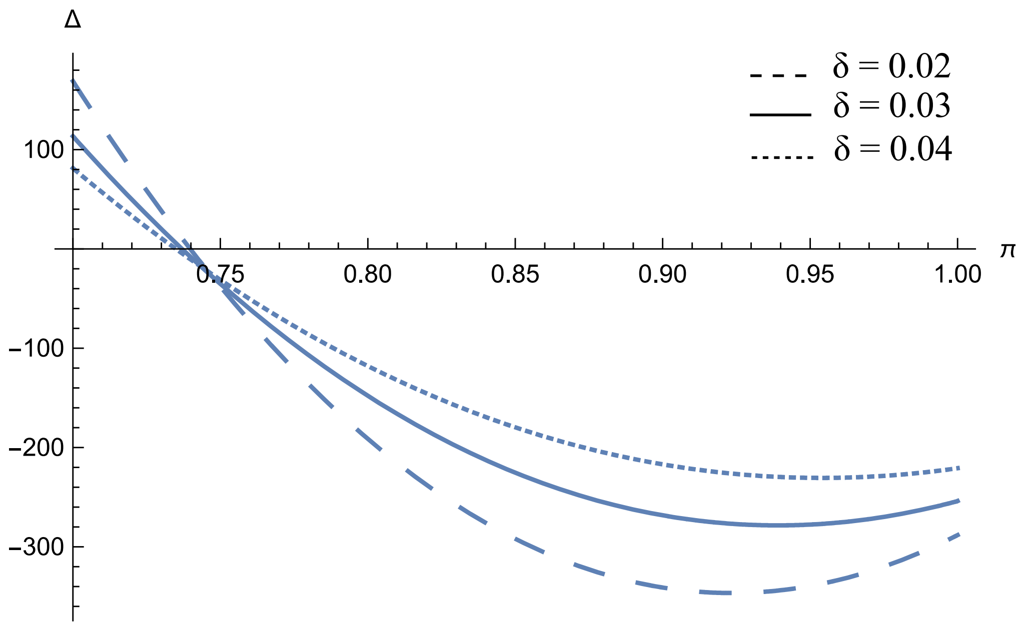

Figure 3.3: Group Inefficiency (δ)

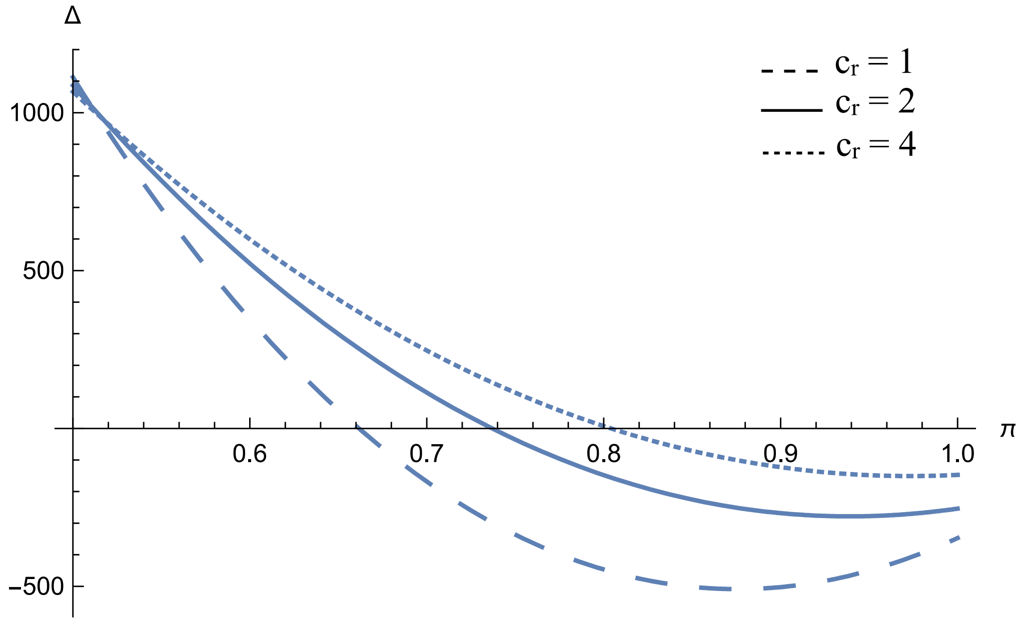

Figure 3.4: Group Inefficiency (c_r)

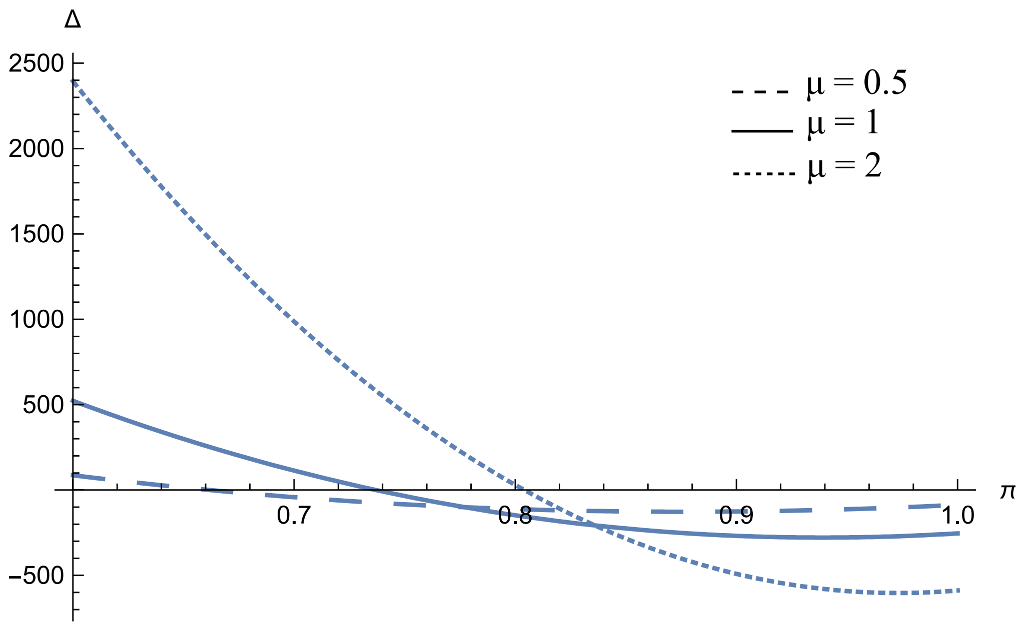

Figure 3.5: Group Inefficiency (μ)

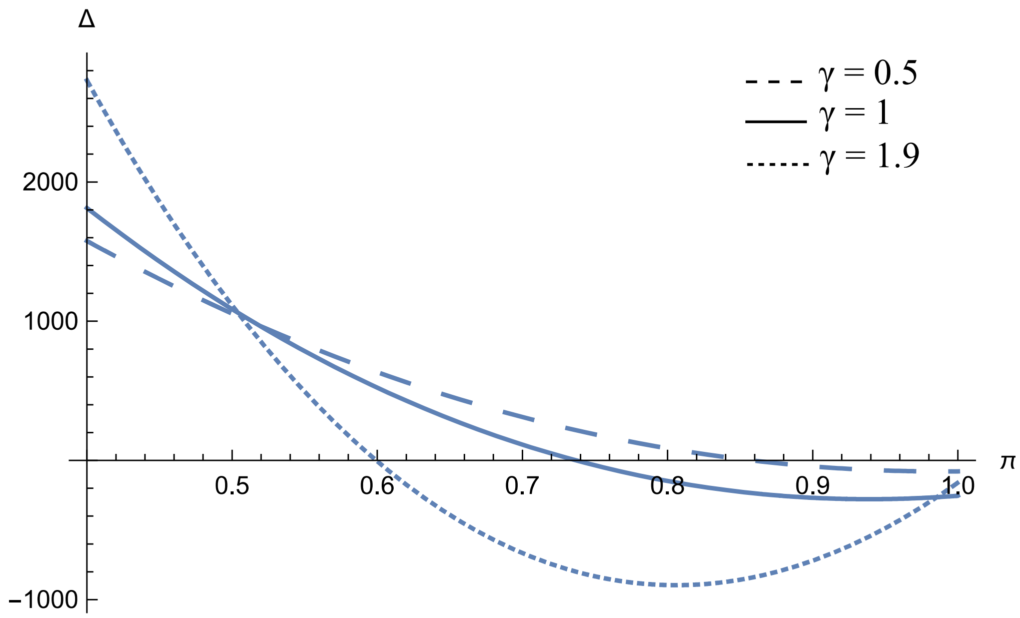

Figure 3.6: Group Inefficiency (γ)

manufacturer's utility can not compensate, the group inefficiency arises. One may notice that joint payoffs are much higher when π is extremely small. This phenomenon derives from the fact that the manufacturer's advertising is decreasing in π for $\rho_m > \rho_r$. A smaller value of π would imply a higher national advertising level and this is beneficial to the whole supply chain. However, in our model setting, as the manufacturer has larger influence on the market, extremely small π would be less realistic.

Table 3.2 gives the parameter values we used for sensitivity analysis, and the corresponding $\hat{\pi}$ and π^{**}[4]. It is clear that in all of the cases, $\hat{\pi} < \pi^{**}$. For a revenue sharing rule $\hat{\pi} < \pi < \pi^{**}$, both manufacturer and retailer exert higher advertising effort in vertical integration than in Nash, however, this does not yield a higher joint payoffs, as one may expect. For $\pi > \pi^{**}$, we have $A^{\mathrm{J}}_M < A^{\mathrm{N}}_M$, implying a slower goodwill accumulation, and the group inefficiency follows. Moreover, smaller c_r and μ, greater δ and γ would induce a larger interval $(\hat{\pi}, 1)$[5]. As a result, group inefficiency is more likely

[4] If $\rho_m > 2\rho_r + \delta$, $A^{\mathrm{J}}_M > A^{\mathrm{N}}_M$ for all $\pi \in (0, \pi^{**})$, and $A^{\mathrm{J}}_M < A^{\mathrm{N}}_M$ for all $\pi \in (\pi^{**}, 1)$. For the definition of π^{**} and more details, please check Proposition 3.4.

[5] We have run more simulations to confirm this interrelation.

Table 3.2: Values of $\hat{\pi}$ and π^{**} under Different Parameters Setting

	Figure 1			Figure 2		
	$\rho_m = 0.1$	$\rho_m = 0.15$	$\rho_m = 0.2$	$\rho_r = 0.01$	$\rho_r = 0.03$	$\rho_r = 0.05$
$\hat{\pi}$	0.737676	0.736966	0.741752	0.74525	0.736966	0.740058
π^{**}	0.981557	0.929286	0.904469	0.892313	0.929286	0.973828
	Figure 3			Figure 4		
	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.04$	$c_r = 1$	$c_r = 2$	$c_r = 4$
$\hat{\pi}$	0.739746	0.736966	0.734898	0.661669	0.736966	0.80407
π^{**}	0.915505	0.929286	0.942401	0.87868	0.929286	0.961652
	Figure 5			Figure 6		
	$\mu = 0.5$	$\mu = 1$	$\mu = 2$	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 1.9$
$\hat{\pi}$	0.661669	0.736966	0.80407	0.857932	0.736966	0.599279
π^{**}	0.87868	0.929286	0.961652	0.980016	0.929286	0.823833

to happen when the retailer's advertising is less costly, the synergistic effect of $A_R\sqrt{G}$ is stronger, the goodwill depreciates faster, or the goodwill's single contribution to revenue is trivial. The effects of ρ_m and ρ_r on $\hat{\pi}$ are less clear and we do not observe a straightforward relationship. They mainly act as the sufficient condition for the existence of group inefficiency.

Regarding the group inefficiency level, as we can see, the curves' intersection can be found in $(0, \hat{\pi})$ (as in Figure 3.6) and in $(\hat{\pi}, 1)$ (Figure 3.5). Besides, $\Delta(\pi)$ is not always monotonic in π and the local minimum can be in the interval $(0, 1)$ (see, for example, Figure 3.2). Moreover, the parameters' effects are subject to the initial goodwill level. All these properties substantially increase the complexity of the sensitivity analysis, and there is no clear conclusion about under what circumstances a higher group inefficiency level would be generated.

3.4.3 Discussion on the Effects of the Cost and Revenue Sharing Rates

In our model, we have assumed that the revenue sharing rate is exogenously given. This is in agreement with previous literature on the topic and, also, with most of market conditions: each agent obtains a given and previously known percentage of benefits from sales. As for the cost sharing rate, in the vertical integration setting, we have computed the optimal solution for every possible value of Φ^{J}, which is assumed to be constant and exogenous. The aim of this section is to analyze what are the effects of the cost and revenue sharing rates if we relax these assumptions.

Cost Sharing Rate

In Scenario N (which coincides with Scenario S if cost sharing is not allowed), it is natural so assume that $\Phi^{\mathrm{N}} = 0$. In Scenario S (cost sharing), Φ^{S} is a decision variable of the manufacturer. As illustrated in Proposition 3.2, if π is not too small, it is profitable for the manufacturer to support part of the retailer's advertising cost. When we move to the vertical integration scenario, if discount rates are equal, the joint effect of Φ^{J} is null, as expected. However, this is not the case if discount rates are different, as illustrated in Equation (3.14). Different cost sharing rates will give rise to different values of $V_M^{\mathrm{J}} + V_R^{\mathrm{J}}$ (see Proposition 3.3). Hence, a natural question emerges in this context: if, in Scenario J, Φ^J is treated as a decision variable of the coalition, when computing the time-consistent cooperative solution, is there an optimal value of Φ^J? An inspection of Equation (3.34) shows that the answer is negative: Φ^J does not appear in the right hand term of that expression. Therefore, the solution to the problem (3.14) by including Φ^J as a decision variable is again given just by Proposition 3.3. There is an infinite number (a continuum of them) of cooperative equilibria in Scenario J, each of them corresponds to a different $\Phi^J \in [0, 1]$, giving rise to a different steady state. Such multiplicity of equilibria is a property well-documented in problems with time inconsistent preferences in an infinite horizon setting. In any case, it is interesting to study if one of these equilibria provide higher payments to the joint coalition or not. In the following, we analyze the results in Proposition 3.3 with respect to the changes in the cost sharing rate Φ^J.

First, notice that α_M^J is decreasing in Φ^J, whereas α_R^J increases in Φ^J. If we express the advertising effort of the manufacturer as a function of the cost sharing rate, $A_M^J(\Phi^J)$, it is straightforward to check that, if $\alpha_M^J + \alpha_R^J > 0$ (which is the more interesting case), then $A_M^J(\Phi^J)$ is increasing if $\rho_m > \rho_r$ and decreasing for $\rho_m < \rho_r$. On the contrary, A_R^J (and A_M^J for $\alpha_M^J + \alpha_R^J < 0$) are independent of Φ^J.

Next, we can analyze if there is a value of Φ^J providing a solution that is Pareto superior to the others. It is easy to check that the answer is, in general, negative.

Finally, let us study the effects of Φ^J on the joint payoffs. We can distinguish three different situations:

1. In the less interesting case when $\alpha_M^J + \alpha_R^J < 0$, since
$$\frac{\partial}{\partial \Phi^J}\left(\alpha_M^J + \alpha_R^J\right) = \frac{\gamma^2(\rho_m - \rho_r)}{2c_r(\rho_m + \delta)(\rho_r + \delta)},$$
then, from Equations (3.17) and (3.18), if $\rho_m > \rho_r$, joint payments are higher if the manufacturer finances the whole cost of retailer advertising. On the contrary, if $\rho_m < \rho_r$, it will be profitable for the coalition that the retailer covers the totality of her advertising cost.

2. If $\alpha_M^J > 0$ and $\alpha_R^J > 0$, let $V_M^J + V_R^J = \alpha^J G + \beta^J$, where $\alpha^J = \alpha_M^J + \alpha_R^J$ and β^J contains all the terms in Equations (3.17) and (3.18) not depending on G, after several calculations, it can be proved that

- If $\rho_m > \rho_r$, then $\dfrac{\partial \alpha^{\mathrm{J}}}{\partial \Phi^{\mathrm{J}}} > 0$ and $\dfrac{\partial \beta^{\mathrm{J}}}{\partial \Phi^{\mathrm{J}}} > 0$.
- If $\rho_m < \rho_r$, then $\dfrac{\partial \alpha^{\mathrm{J}}}{\partial \Phi^{\mathrm{J}}} < 0$ and $\dfrac{\partial \beta^{\mathrm{J}}}{\partial \Phi^{\mathrm{J}}} < 0$.

As a result, independently of the initial goodwill level G, joint payments are higher for $\Phi^{\mathrm{J}} = 1$ if $\rho_m > \rho_r$ and, on the contrary, if $\rho_m < \rho_r$, it is better for the coalition to take $\Phi^{\mathrm{J}} = 0$.

3. The situation becomes much more complicated if $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} > 0$ but α_M^{J} or α_R^{J} is strictly negative. In such a case, the value of Φ^{J} maximizing $V_M^{\mathrm{J}} + V_R^{\mathrm{J}}$ can be at any point in the interval $[0,1]$. If the optimal solution is interior, then the "optimal" value of Φ^{J} will be, in general, a linear function of G. But the goodwill level evolves along time. Therefore, there is not a (constant) value of Φ^{J} maximizing the joint payments. In addition, by taking Φ^{J} as a linear function of G, say $\Phi^{\mathrm{J}} = aG + b \in [0,1]$, with the idea of looking for later on the values of the parameters a and b maximizing the joint payments, we lose linearity of the decision rule for the manufacturer and the quadratic structure of the value functions. It is unclear how this problem could be solved.

Summarizing, in the search of time-consistent equilibria in the vertical integration setting, if the cost sharing rate is treated as a decision variable, there is a continuum of solutions obtained for different values of Φ^{J}. This property, that seems to be surprising, is one of the effects of introducing time inconsistent preferences. It is not possible, in general, to select a particular value of Φ^{J} giving rise to an equilibrium Pareto dominating the others. As for the joint payments, in some cases (the second situation we discussed above), it is possible to identify a value $\Phi^{\mathrm{J}} \in \{0,1\}$ maximizing the joint payments. This appropriate selection of an "optimal" (when there exists) value of the cost sharing rate can mitigate (but not completely eliminate, depending on the values of the parameters) the group inefficiency effect. This result can be checked from an inspection of the difference $(V_M^{\mathrm{J}}(G) + V_R^{\mathrm{J}}(G)) - (V_M^{\mathrm{N}}(G) + V_R^{\mathrm{N}}(G))$ for all values of Φ^{J}. In particular, the linear term in G in Equation

(3.20) becomes

$$\frac{\gamma^2}{2c_r(\rho_m+\delta)(\rho_r+\delta)}\left[(\rho_m-\rho_r)\Phi^{\mathrm{J}}-(\rho_m-2\rho_r-\delta)\pi^2\right] \ .$$

For every Φ^{J} there are values of the parameters ρ_m, ρ_r, π and δ guaranteeing that the expression above is negative, so if the goodwill level G is high enough, there will be group inefficiency for every $\Phi^{\mathrm{J}} \in [0,1]$.

Revenue Sharing Rate

In Scenarios N and S, if the retailer can decide the revenue sharing rate, she will take it at its lowest possible value (Equations (3.26) and (3.30)). On the contrary, if this variable is decided by the manufacturer, in Scenario N, she will take it at its highest value (Equation (3.25)). The situation is less clear in the cost sharing scenario, if π can be decided by the manufacturer. An inspection to Equation (3.31) shows that, in that case, the maximum of the right hand term in π is achieved when

$$\pi^* = \frac{1}{2-\Phi^{\mathrm{S}}}\left[1+\frac{\mu c_r\left(1-\Phi^{\mathrm{S}}\right)^2}{\gamma^2}+\frac{c_r\left(1-\Phi^{\mathrm{S}}\right)^2\theta}{\gamma^2 G}\right] \ .$$

If $\pi^* < 1$, then it will be profitable for the manufacturer to choose this revenue sharing rate. However, it can be shown that this can not happen for the values of $\Phi^{\mathrm{S}*}$. Hence, as in Scenario N, the manufacturer will choose the maximum possible revenue sharing rate.

With respect to the vertical integration scenario, we obtain similar results to the previous ones on the cost sharing rate: if the revenue sharing rate is treated as a decision variable, there is a continuum of time-consistent cooperative equilibria obtained for all the different values of π. It is also not possible, in general, to select a particular value of π providing an equilibrium Pareto dominating the others. As for the joint payments, the discussion follows the similar patterns as in that of the cost sharing rate.

3.5. Concluding Remarks

In this chapter, we have considered a differential game of advertising in a two-echelon supply chain. We have contributed to the literature by introducing two extensions. First of all, we allow for the possibility that each chain member could exhibit distinct time discount rates, which could result in a trade-off between efficiency and time-consistency. Besides, we have extended the sales model by combining the separate effect of goodwill, the synergistic effect of goodwill and retailer's advertising, and the decreasing marginal returns to goodwill. We have characterized the feedback Nash equilibrium, the stage-wise Stackelberg equilibrium and the time-consistent cooperative solution for scenarios of non-cooperation, cost-sharing program and vertical integration, respectively. We have made a detailed comparison of the advertising strategies and outcomes among different cooperative and non-cooperative settings.

Our results reveal that when the manufacturer is the leader and the retailer has limited influence on the sales, the cost-sharing program will be implemented if the revenue sharing rate of the retailer is not much larger than that of the manufacturer. Both members of the supply chain increase their advertising budget when the cost-sharing program is applied, which generates a bigger market size, thus implying a Pareto superior outcome to the one under non-cooperation. Similar results can be found in the literature (for instance, Jørgensen et al., 2000, 2001, 2003; Buratto et al., 2007). Recall that, by modifying the sales function, we have enlarged the difference of the marketing influence of each member in the supply chain. As in the other leader-follower games where all agents' marketing activities affect goodwill and/or sales, we also find that the necessary condition for giving positive advertising support is related to the margins of each member, and the Stackelberg payments are Pareto superior to Nash payments.

Under vertical integration, when the two agents differ in their time preference rates but act in a time-consistent way, depending on the parameters of the model, the manufacturer's advertising rate may be zero. If we focus on the parameter set such that the manufacturer's investment is positive, it depends on the revenue and cost sharing rates. This result differs from that of the standard case, where agents have the same discount rate. Moreover, even the payoffs of the grand coalition is affected by these two side payments.

The most novel result of our study is related to the efficiency analysis. There is a consensus that a centralized channel is more efficient, according to all the studies up to date. Nonetheless, under the hypothesis of asymmetric time-discounting, the vertical integration scenario does not necessarily yield a better outcome for the coalition. Particularly, we find that if the retailer is much more impatient than the manufacturer, group inefficiency emerges when both initial goodwill level and revenue sharing rate are small. On the contrary, if the manufacturer's discount rate is much greater than that of the retailer, there exists group inefficiency for all levels of initial goodwill when the revenue sharing rate does not prioritize the retailer. We also observe that, for the latter case, the group inefficiency likelihood is increased by the retailer's advertising's higher cost-effectiveness and larger contribution to revenue, as well as the goodwill's higher depreciation rate and smaller influence on revenue; whereas the group inefficiency level is elevated by the manufacturer's better advertising performance (more effective and less costly).

We believe that our research could be a useful aid for managers to decide whether to cooperate and which coordination mechanism to choose. From our model, a co-op advertising program is promising and offers mutual prosperity. Consequently, we would advise the manufacturer to subsidize the retailer's advertising campaigns, as long as she has advantage in revenue sharing. Furthermore, our findings suggest that the decision makers should take into account the possible divergence in the discount rates (for instance, due to differences in firm size, financial health, legislative restrictions, crisis intensity rate, and so on) between the two entities when considering a vertical integration in the form of a coalition, merger, acquisition, etc. A channel centralization is not recommended if the retailer is shortsighted and takes large part of the revenue, unless the brand is well known. Similarly, this coordination mechanism is not advantageous for a supply chain consisting of one relatively myopic manufacturer, i.e., with a high discount rate, and one farsighted retailer when the revenue sharing rule does not favor the retailer.

Appendix

Proof of Lemma 3.1. First, note that, if $A_M(s) = A_m G(s) + B_m$, for all $s \in [t, \infty)$, $t \geq 0$, then the solution to $\dot{G}(s) = k_m A_M(s) - \delta G(s)$, with initial condition $G(t) = G$, is

$$G(s) = \left(G - \frac{k_m A_m}{\delta - A_m k_m}\right) e^{-(\delta - A_m k_m)(s-t)} + \frac{k_m B_m}{\delta - A_m k_m} \ . \tag{3.22}$$

Next, if $A_M(s) = A_m G(s) + B_m$ and $A_R(s) = A_r\sqrt{G(s)}$, then

$$V_M(G) = \int_t^\infty e^{-\rho_m(s-t)} \Big\{\pi\left[\theta + (\mu + \gamma A_r)G(s)\right] - \frac{c_m}{2}(A_m G(s) + B_m)^2 \\ -\frac{c_r}{2}\Phi(s)(A_r)^2 G(s)\Big\}\ ds \tag{3.23}$$

and

$$V_R(G) = \int_t^\infty e^{-\rho_r(s-t)} \{(1-\pi)\left[\theta + (\mu + \gamma A_r)G(s)\right] \\ -\frac{c_r}{2}(1-\Phi(s))(A_r)^2 G(s)\Big\}\ ds\ . \tag{3.24}$$

The result follows by substituting equation (3.22) in (3.23) and (3.24). □

Proof of Proposition 3.1. Denoting $V_M^{\mathrm{N}}(G)$, $V_R^{\mathrm{N}}(G)$ the value functions of the manufacturer and the retailer respectively, the Hamilton-Jacobi-Bellman (HJB) equations are

$$\rho_m V_M^{\mathrm{N}} = \max_{\{A_M^{\mathrm{N}}\geq 0\}} \Big\{\pi\left(\theta + \mu G + \gamma A_R^{\mathrm{N}}\sqrt{G}\right) - \frac{c_m}{2}(A_M^{\mathrm{N}})^2 \\ +(V_M^{\mathrm{N}})'(k_m A_M^{\mathrm{N}} - \delta G)\Big\}\ , \tag{3.25}$$

$$\rho_r V_R^{\mathrm{N}} = \max_{\{A_R^{\mathrm{N}}\geq 0\}} \Big\{(1-\pi)\left(\theta + \mu G + \gamma A_R^{\mathrm{N}}\sqrt{G}\right) - \frac{c_r}{2}(A_R^{\mathrm{N}})^2 \\ +(V_R^{\mathrm{N}})'(k_m A_M^{\mathrm{N}} - \delta G)\Big\}\ . \tag{3.26}$$

Assuming an interior solution, maximizing the right-hand sides of these two equations yields $A_M^{\mathrm{N}*} = k_m(V_M^{\mathrm{N}})'/c_m$ and $A_R^{\mathrm{N}*} = (1-\pi)\gamma\sqrt{G}/c_r$. Note that the strategy of the retailer is already fixed. As a result, when substituting it in the objective functional of the manufacturer (3.3), we obtain a standard linear-quadratic optimal control problem, whose unique solution is known to be linear. For these strategies, from Lemma 3.1 we know that, in such a case, value functions must be of the form $V_M^{\mathrm{N}}(G) = (\lambda_M^{\mathrm{N}}/2)G^2 + \alpha_M^{\mathrm{N}}G + \beta_M^{\mathrm{N}}$, $V_R^{\mathrm{N}}(G) = \alpha_R^{\mathrm{N}}G + \beta_R^{\mathrm{N}}$. Substituting $A_M^{\mathrm{N}*}$, $A_R^{\mathrm{N}*}$, together with $V_M^{\mathrm{N}}(G)$, into (3.25), we obtain

$$\rho_m \left(\frac{\lambda_M^{\mathrm{N}}}{2} G^2 + \alpha_M^{\mathrm{N}} G + \beta_M^{\mathrm{N}} \right) = \pi \left(\theta + \mu G + \frac{\gamma^2(1-\pi)}{c_r} G \right) - \frac{k_m^2}{2c_m} \left((\lambda_M^{\mathrm{N}})^2 G^2 + 2\lambda_M^{\mathrm{N}} \alpha_M^{\mathrm{N}} G + (\alpha_M^{\mathrm{N}})^2 \right) + (\lambda_M^{\mathrm{N}} G + \alpha_M^{\mathrm{N}}) \left(\frac{k_m^2(\lambda_M^{\mathrm{N}} G + \alpha_M^{\mathrm{N}})}{c_m} - \delta G \right) . \tag{3.27}$$

By identifying the terms in G^2 in the equation above we obtain

$$\frac{\rho_m}{2} \lambda_M^{\mathrm{N}} = \frac{(\lambda_M^{\mathrm{N}})^2 k_m^2}{2c_m} - \lambda_M^{\mathrm{N}} \delta ,$$

which has two solutions: $\lambda_M^{\mathrm{N}} = 0$ and $\lambda_M^{\mathrm{N}} = (\rho_m + 2\delta)c_m/(k_m)^2$. We analyze first the existence of a feedback Nash equilibrium in constant strategies for the manufacturer. For $\lambda_M^{\mathrm{N}} = 0$, after rearranging terms, equations (3.25) and (3.26) become

$$\left\{ \rho_m \alpha_M^{\mathrm{N}} - \frac{\pi \left[c_r \mu + (1-\pi)\gamma^2 \right]}{c_r} + \delta \alpha_M^{\mathrm{N}} \right\} G = -\rho_m \beta_M^{\mathrm{N}} + \pi\theta + \frac{k_m^2}{2c_m} (\alpha_M^{\mathrm{N}})^2 , \tag{3.28}$$

$$\left\{ \rho_r \alpha_R^{\mathrm{N}} - \frac{(1-\pi) \left[2c_r \mu + (1-\pi)\gamma^2 \right]}{2c_r} + \delta \alpha_R^{\mathrm{N}} \right\} G = -\rho_r \beta_R^{\mathrm{N}} + (1-\pi)\theta + \frac{k_m^2}{c_m} \alpha_M^{\mathrm{N}} \alpha_R^{\mathrm{N}} . \tag{3.29}$$

It is straightforward to check that

$$\alpha_M^{\mathrm{N}} = \frac{c_r \pi \mu + \pi(1-\pi)\gamma^2}{c_r(\rho_m + \delta)}, \quad \alpha_R^{\mathrm{N}} = \frac{2c_r(1-\pi)\mu + (1-\pi)^2\gamma^2}{2c_r(\rho_r + \delta)},$$
$$\beta_M^{\mathrm{N}} = \frac{k_m^2}{2c_m \rho_m} (\alpha_M^{\mathrm{N}})^2 + \frac{\pi\theta}{\rho_m}, \quad \beta_R^{\mathrm{N}} = \frac{k_m^2}{c_m \rho_r} \alpha_M^{\mathrm{N}} \alpha_R^{\mathrm{N}} + \frac{(1-\pi)\theta}{\rho_r}$$

satisfy (3.28) and (3.29). It is straightforward to check that the sufficient transversality conditions $\lim_{t \to \infty} e^{-\rho_m t} V_i(G(t)) = 0, i = M, R$ is met (the solution converges to a steady state). Note also that, since $\alpha_M^{\mathrm{N}} > 0$, then $A_M^{\mathrm{N}*} > 0$, in agreement with our hypothesis concerning the existence of an interior solution. Finally, it can be checked that, for the other candidate $\lambda_M^{\mathrm{N}} = (\rho_m + 2\delta)c_m/(k_m)^2$, $\lim\limits_{t\to\infty} e^{-\rho_m t} V_M(G(t)) = \infty$. □

Proof of Proposition 3.2. We solve the problem by backward induction. Denoting $V_M^{\mathrm{S}}(G)$, $V_R^{\mathrm{S}}(G)$ the value functions of the manufacturer and the retailer respectively, we start from determining the retailer's advertising strategies. The retailer's HJB equation is

$$\begin{aligned}\rho_r V_R^{\mathrm{S}} = \max_{\{A_R^{\mathrm{S}}\geq 0\}} \Big\{ &(1-\pi)\left(\theta + \mu G + \gamma A_R^{\mathrm{S}}\sqrt{G}\right) - \frac{c_r}{2}(1-\Phi^{\mathrm{S}})(A_R^{\mathrm{S}})^2 \\ &+(V_R^{\mathrm{S}})'(k_m A_M^{\mathrm{S}} - \delta G)\Big\} \; .\end{aligned} \tag{3.30}$$

If $\Phi^{\mathrm{S}} \neq 1$, maximizing the right-hand side yields $A_R^{\mathrm{S}*} = \dfrac{(1-\pi)\gamma\sqrt{G}}{c_r(1-\Phi^{\mathrm{S}})}$. Substituting $A_R^{\mathrm{S}*}$ into the manufacturer's HJB equation we obtain

$$\begin{aligned}\rho_m V_M^{\mathrm{S}} = \max_{\{A_M^{\mathrm{S}}\geq 0,\, 0\leq\Phi^{\mathrm{S}}\leq 1\}} \Bigg\{ &\pi\left[\theta + \mu G + \frac{(1-\pi)\gamma^2}{c_r(1-\Phi^{\mathrm{S}})}G\right] - \frac{c_m}{2}(A_M^{\mathrm{S}})^2 \\ &-\frac{(1-\pi)^2\gamma^2\Phi^{\mathrm{S}}}{2c_r(1-\Phi^{\mathrm{S}})^2}G + (V_M^{\mathrm{S}})'(k_m A_M^{\mathrm{S}} - \delta G)\Bigg\} \; .\end{aligned} \tag{3.31}$$

The manufacturer's strategies are derived by maximizing the right-hand side of (3.31), whose result is, in the case of interior solutions,

$$A_M^{\mathrm{S}*} = \frac{k_m}{c_m}(V_M^{\mathrm{S}})', \quad \Phi^{\mathrm{S}*} = \begin{cases} \frac{3\pi-1}{\pi+1} & \text{if } \; \frac{1}{3} < \pi < 1, \\ 0 & \text{if } \; 0 < \pi \leq \frac{1}{3}. \end{cases}$$

In accordance with our hypothesis, $\Phi^{\mathrm{S}*} \neq 1$, since $\Phi^{\mathrm{S}*} = 1$ only happens when $\pi = 1$. Note also that when $0 < \pi \leq \frac{1}{3}$, the outcome is consistent with that of scenario N. In the case of $\frac{1}{3} < \pi < 1$, since the strategy of the retailer is already fixed, from (3.3) and (3.1) we have to look for the linear solution of the corresponding linear-quadratic optimal control problem for the manufacturer. From Lemma 3.1, value functions are of the form $V_M^{\mathrm{S}}(G) = (\lambda_M^{\mathrm{S}})G^2 + \alpha_M^{\mathrm{S}} G + \beta_M^{\mathrm{S}}$, $V_R^{\mathrm{S}}(G) = \alpha_R^{\mathrm{S}} G + \beta_R^{\mathrm{S}}$.

Substituting A_M^{S*}, A_R^{N*}, together with $V_M^N(G)$, into (3.31), we obtain

$$\rho_m \left(\frac{\lambda_M^S}{2} G^2 + \alpha_M^S G + \beta_M^S \right)$$
$$= \pi \left(\theta + \mu G + \frac{\gamma^2(1-\pi)}{c_r(1-\phi^{S*})} G \right) - \frac{k_m^2}{2c_m} \left[(\lambda_M^S)^2 G^2 + 2\lambda_M^S \alpha_M^S G + (\alpha_M^S)^2 \right]$$
$$- \frac{(1-\pi)^2\gamma^2\Phi^{S*}}{2c_r(1-\Phi^{S*})^2} G + (\lambda_M^S G + \alpha_M^S) \left(\frac{k_m^2(\lambda_M^S G + \alpha_M^S)}{c_m} - \delta G \right) .$$

By identifying the terms in G^2 in the equation above we obtain

$$\frac{\rho_m}{2} \lambda_M^S = \frac{(\lambda_M^S)^2 k_m^2}{2c_m} - \lambda_M^S \delta ,$$

which has two solutions: $\lambda_M^S = 0$ and $\lambda_M^S = (\rho_m + 2\delta)c_m / k_m^2$. So, in particular, there exists a feedback Stackelberg equilibrium in constant strategies for the manufacturer. In order to compute this equilibrium, we substitute, for $\lambda_M^S = 0$, A_R^{S*}, A_M^{S*} and Φ^{S*} into (3.30) and (3.31) to yield

$$\left[\rho_r \alpha_R^S - (1-\pi)\mu - \frac{(1-\pi^2)\gamma^2}{4c_r} + \delta \alpha_R^S \right] G = -\rho_r \beta_R^S + (1-\pi)\theta + \frac{k_m^2}{c_m} \alpha_M^S \alpha_R^S , \tag{3.32}$$

$$\left[\rho_m \alpha_M^S - \pi\mu - \frac{(1+\pi)^2\gamma^2}{8c_r} + \delta \alpha_M^S \right] G = -\rho_m \beta_M^S + \pi\theta + \frac{k_m^2}{2c_m} (\alpha_M^S)^2 . \tag{3.33}$$

It is easy to check that

$$\alpha_M^S = \frac{8c_r\pi\mu + (1+\pi)^2\gamma^2}{8c_r(\rho_m + \delta)}, \quad \alpha_R^S = \frac{4c_r(1-\pi)\mu + (1-\pi^2)\gamma^2}{4c_r(\rho_r + \delta)},$$
$$\beta_M^S = \frac{k_m^2}{2c_m\rho_m}(\alpha_M^S)^2 + \frac{\pi\theta}{\rho_m}, \quad \beta_R^S = \frac{k_m^2}{c_m\rho_r}\alpha_M^S\alpha_R^S + \frac{(1-\pi)\theta}{\rho_r}$$

satisfy (3.32) and (3.33). For this solution, it is straightforward to check that $\lim_{t\to\infty} e^{-\rho_m t} V_i(G(t)) = 0$, $i = M, R$ is met (the solution converges to a steady state). These conditions are not satisfied by the other (nonlinear) decision rule for the manufacturer. Finally, notice that $\alpha_M^S > 0$ and then $A_M^{S*} > 0$, i.e., the solution is interior, as we had assumed. □

Proof of Proposition 3.3. We follow the approach of de Paz et al. (2013) to obtain the time-consistent equilibria. Denoting $V_M^J(G)$, $V_R^J(G)$ the value functions of the manufacturer and the retailer respectively, the Dynamic Programming Equation (DPE) of the coalition is

$$\rho_m V_M^{\mathrm{J}} + \rho_r V_R^{\mathrm{J}} = \max_{\{A_M^{\mathrm{J}}\geq 0,\, A_R^{\mathrm{J}}\geq 0\}} \Big\{\theta + \mu G + \gamma A_R^{\mathrm{J}}\sqrt{G} - \frac{c_m}{2}(A_M^{\mathrm{J}})^2 - \frac{c_r}{2}(A_R^{\mathrm{J}})^2 \\ +((V_M^{\mathrm{J}})' + (V_R^{\mathrm{J}})')\left(k_m A_M^{\mathrm{J}} - \delta G\right)\Big\} \ . \tag{3.34}$$

Maximization gives, in case of interior solution, $A_M^{\mathrm{J}*} = k_m((V_M^{\mathrm{J}})' + (V_R^{\mathrm{J}})')/c_m$, $A_R^{\mathrm{J}*} = \gamma\sqrt{G}/c_r$. As in the previous scenarios, we focus our attention in the existence of linear strategies for the manufacturer. Since $V_M^{\mathrm{J}}(G) = (\lambda_M^{\mathrm{J}})G^2 + \alpha_M^{\mathrm{J}} G + \beta_M^{\mathrm{J}}$, $V_R^{\mathrm{J}}(G) = \alpha_R^{\mathrm{J}} G + \beta_R^{\mathrm{J}}$ (Lemma 3.1), the dynamic programming equation for the manufacturer becomes

$$\rho_m\left(\frac{\lambda_M^{\mathrm{J}}}{2}G^2 + \alpha_M^{\mathrm{J}} G + \beta_M^{\mathrm{J}}\right)$$
$$= \pi(\theta + \mu G + \gamma A_R^{\mathrm{J}*}\sqrt{G}) - \frac{c_m}{2}(A_M^{\mathrm{J}*})^2 - \frac{c_r}{2}\Phi(A_R^{\mathrm{J}*})^2 + \alpha_M^{\mathrm{J}}\left[k_m^2 A_M^{\mathrm{J}*} - \delta G)\right]$$
$$= \pi\left(\theta + \mu G + \frac{\gamma^2}{c_r}G\right) - \frac{k_m^2}{2c_m}(\lambda_M^{\mathrm{J}} G + \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}})^2 - \frac{\gamma^2\Phi}{2c_r}G$$
$$+ (\lambda_M^{\mathrm{J}} G + \alpha_M^{\mathrm{J}})\left[\frac{k_m^2}{c_m}(\lambda_M^{\mathrm{J}} G + \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}}) - \delta G)\right] \ .$$

By identifying the terms in G^2 in the equation above we obtain

$$\frac{\rho_m}{2}\lambda_M^{\mathrm{J}} = \frac{(\lambda_M^{\mathrm{J}})^2 k_m^2}{2c_m} - \lambda_M^{\mathrm{J}}\delta \ ,$$

which has two solutions: $\lambda_M^{\mathrm{J}} = 0$ and $\lambda_M^{\mathrm{J}} = (\rho_m + 2\delta)c_m/k_m^2$. Let us compute the time-consistent cooperative equilibrium in constant strategies for the manufacturer ($\lambda_M^{\mathrm{J}} = 0$). By substituting $A_M^{\mathrm{J}*}$ and $A_R^{\mathrm{J}*}$ into the individual DPEs, we obtain

$$\rho_m(\alpha_M^{\mathrm{J}} G + \beta_M^{\mathrm{J}}) = \pi\left(\theta + \mu G + \frac{\gamma^2}{c_r}G\right) - \frac{k_m^2}{2c_m}(\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}})^2 - \frac{\gamma^2\Phi}{2c_r}G \\ + \alpha_M^{\mathrm{J}}\left[\frac{k_m^2}{c_m}(\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}}) - \delta G)\right] , \tag{3.35}$$

$$\rho_r(\alpha_R^{\mathrm{J}} G + \beta_R^{\mathrm{J}}) = (1-\pi)\left(\theta + \mu G + \frac{\gamma^2}{c_r}G\right) - \frac{\gamma^2(1-\Phi)}{2c_r}G \\ + \alpha_R^{\mathrm{J}}\left[\frac{k_m^2}{c_m}(\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}}) - \delta G)\right] \ . \tag{3.36}$$

By identifying terms, the coefficients are given by

$$
\begin{aligned}
\alpha_M^{\mathrm{J}} &= \frac{2c_r\pi\mu + (2\pi - \Phi)\gamma^2}{2c_r(\rho_m + \delta)}, \\
\alpha_R^{\mathrm{J}} &= \frac{2c_r(1-\pi)\mu + (1 + \Phi - 2\pi)\gamma^2}{2c_r(\rho_r + \delta)}, \\
\beta_M^{\mathrm{J}} &= \frac{k_m^2}{2c_m\rho_m}((\alpha_M^{\mathrm{J}})^2 - (\alpha_R^{\mathrm{J}})^2) + \frac{\pi\theta}{\rho_m}, \\
\beta_R^{\mathrm{J}} &= \frac{k_m^2}{c_m\rho_r}(\alpha_M^{\mathrm{J}}\alpha_R^{\mathrm{J}} + (\alpha_R^{\mathrm{J}})^2) + \frac{(1-\pi)\theta}{\rho_r}.
\end{aligned}
$$

This is the solution when $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$. Note that in this case (with $\lambda_M^{\mathrm{J}} = 0$) the solution converges to a steady state, as needed. On the contrary, it can be checked that the solution obtained for $\lambda_M^{\mathrm{J}} = (\rho_m + 2\delta)c_m/k_m^2$ does not converge to a steady state.

It remains to compute the corner solution. By reproducing the same calculations for $A_M^{\mathrm{J}*} = 0$, $A_R^{\mathrm{J}*} = \dfrac{\gamma\sqrt{G}}{c_r}$, from the individual DPEs we derive the same values of α_M^{J}, α_R^{J}, and $\beta_M^{\mathrm{J}} = \dfrac{\pi\theta}{\rho_m}$, $\beta_R^{\mathrm{J}} = \dfrac{(1-\pi)\theta}{\rho_r}$. □

Proof of Proposition 3.4.

1. We first compare A_M^{N} and A_M^{S}. From (3.5) and (3.9), it is straightforward to get $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}}$ (when $\pi \in (\frac{1}{3}, 1)$ a strict inequality holds).

2. Then we compare A_M^{S} and A_M^{J}. From (3.9) and (3.15), we obtain

$$
A_M^{\mathrm{J}} - A_M^{\mathrm{S}} = \frac{k_m}{8c_mc_r(\rho_m + \delta)(\rho_r + \delta)} f_1(\pi),
$$

where

$$
\begin{aligned}
f_1(\pi) = &- \gamma^2(\rho_r + \delta)\pi^2 + \left[6\gamma^2(\rho_r + \delta) - 8(\gamma^2 + c_r\mu)(\rho_m + \delta)\right]\pi \\
&+ 4(\gamma^2 + 2c_r\mu)(\rho_m + \delta) - \gamma^2(\rho_r + \delta)\ .
\end{aligned}
$$

Note that $f_1\left(\dfrac{1}{3}\right) = \dfrac{8}{9}\gamma^2(\rho_r + \delta) + \left(\dfrac{4}{3}\gamma^2 + \dfrac{16}{3}c_r\mu\right)(\rho_m + \delta) > 0$, and $f_1(1) = 4\gamma^2(\rho_r - \rho_m)$. Then, if $\rho_r > \rho_m$, $f_1(\pi) > 0$ for all $\pi \in (\frac{1}{3}, 1)$. If $\rho_r < \rho_m$, there exists a (unique) root $\pi^* \in (\frac{1}{3}, 1)$ of $f_1(\pi)$ such that $f_1(\pi) > 0$ for $\pi \in (\frac{1}{3}, \pi^*)$ and $f_1(\pi) < 0$ for $\pi \in (\pi^*, 1)$. Since $f_1(0) > 0$ for $\rho_r < \rho_m$, then $f_1(\pi) > 0$ for $\pi \in (0, \pi^*)$ in this case.

3. Then we compare A_M^{N} and A_M^{J}. Use (3.5) and (3.15) to compute

$$A_M^{\mathrm{N}} - A_M^{\mathrm{J}} = \frac{k_m}{2c_m c_r(\rho_m + \delta)(\rho_r + \delta)} f_2(\pi), \tag{3.37}$$

where

$$f_2(\pi) = -2\gamma^2(\rho_r + \delta)\pi^2 + 2(\gamma^2 + c_r\mu)(\rho_m + \delta)\pi - (\gamma^2 + 2c_r\mu)(\rho_m + \delta). \tag{3.38}$$

Note that $f_2(0) = -(\gamma^2 + 2c_r\mu)(\rho_m + \delta) < 0$ and $f_2(1) = \gamma^2(\rho_m - 2\rho_r - \delta)$. If $\rho_m > 2\rho_r + \delta$, then $f_2(1) > 0$ and there must exist $\pi^{**} \in (0, 1)$ solving $f_2(\pi) = 0$. Since $f_2(\pi)$ is a second degree polynomial, π^{**} is unique. This implies that (3.37) is negative for $\pi \in (0, \pi^{**})$ and positive for $\pi \in (\pi^{**}, 1)$.

It remains to analyze the case when $f_2(1) < 0$. If $\rho_m < 2\rho_r + \delta$, i.e. $\rho_m + \delta < 2(\rho_r + \delta)$, let us compute the maximum of the second-degree polynomial $f_2(\pi)$. The solution to $f_2'(\pi) = 0$ is

$$\bar{\pi} = \frac{(\gamma^2 + c_r\mu)(\rho_m + \delta)}{2\gamma^2(\rho_r + \delta)}\,. \tag{3.39}$$

A necessary condition for the existence of $\pi^{**} \in (0, 1)$ such that $f_2(\pi^{**}) = 0$ is that $\bar{\pi} < 1$, so

$$\rho_r + \delta > \frac{(\gamma^2 + c_r\mu)(\rho_m + \delta)}{2\gamma^2}\,. \tag{3.40}$$

Therefore, if $(\rho_m + \delta)/2 < \rho_r + \delta < (\gamma^2 + c_r\mu)(\rho_m + \delta)/(2\gamma^2)$, then $\bar{\pi} > 1$ and $f_2(\pi)$ is negative for all $\pi \in (0, 1)$. It remains to consider the case when condition (3.40) is verified. In that case, $\bar{\pi} \in (0, 1)$ and a necessary and sufficient condition for the existence of a root of $f_2(\pi)$ in the interval $(0, 1)$ is that $f_2(\bar{\pi}) > 0$. By substituting (3.39) in (3.38) we obtain that $f_2(\bar{\pi}) > 0$ if, and only if,

$$\rho_r + \delta < \frac{(\gamma^2 + c_r\mu)^2(\rho_m + \delta)}{2\gamma^2(\gamma^2 + 2c_r\mu)}\,,$$

but this is in contradiction with condition (3.40). Therefore, for $\rho_m < 2\rho_r + \delta$, $f_2(\pi)$ is negative for all $\pi \in (0, 1)$.

From the previous proof in points 2 and 3, it is straightforward that when $\rho_r < \rho_m < 2\rho_r + \delta$, $\frac{1}{3} < \pi^* < 1 < \pi^{**}$.

When $\rho_m > 2\rho_r + \delta$, $\pi^* \in (\frac{1}{3}, 1)$ and $\pi^{**} \in (0, 1)$. Assume $\pi^{**} < \pi^*$, then from the pairwise comparison between A_M^{J} and A_M^{S} (in point 2), we have $A_M^{\mathrm{J}} > A_M^{\mathrm{S}}$ if $\pi \in (0, \pi^*)$. From the pairwise comparison between A_M^{J} and A_M^{N} (in point 3), we have $A_M^{\mathrm{N}} > A_M^{\mathrm{J}}$ if $\pi \in (\pi^{**}, 1)$. Summarizing, if $\pi \in (\pi^{**}, \pi^*)$, $A_M^{\mathrm{N}} > A_M^{\mathrm{J}} > A_M^{\mathrm{S}}$. However, it is contradictory to the result in point 1 where we obtain $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}} \forall \pi \in (0, 1)$. Therefore, $\pi^* < \pi^{**}$.

Summarizing all the pairwise comparison we made previously, the results follow. □

Proof of Proposition 3.6.

1. For $\pi \in (0, \frac{1}{3}]$, $V_M^{\mathrm{N}} = V_M^{\mathrm{S}}$ and $V_R^{\mathrm{N}} = V_R^{\mathrm{S}}$. For $\pi \in (\frac{1}{3}, 1)$ use (3.7) and (3.12) to compute

$$V_M^{\mathrm{N}} - V_M^{\mathrm{S}} = -(3\pi - 1)^2 \gamma^2 \times \left\{ \frac{1}{8c_r(\rho_m + \delta)} G + \frac{k_m^2 \left[(-7\pi^2 + 10\pi + 1)\gamma^2 + 16c_r \pi \mu \right]}{128 c_m c_r^2 \rho_m (\rho_m + \delta)^2} \right\}$$

where $-7\pi^2 + 10\pi + 1 > 0 \quad \forall \pi \in (\frac{1}{3}, 1)$, implying $V_M^{\mathrm{N}} - V_M^{\mathrm{S}} < 0$. Next use (3.8) and (3.13) to compute

$$V_R^{\mathrm{N}} - V_R^{\mathrm{S}} = -(1 - \pi)(3\pi - 1)\gamma^2 \times \left\{ \frac{1}{4c_r(\rho_r + \delta)} G + \frac{k_m^2 \left[(-5\pi^2 + 10\pi - 1)\gamma^2 + 4c_r \mu (5\pi - 1) \right]}{32 c_m c_r^2 \rho_r (\rho_m + \delta)(\rho_r + \delta)} \right\}$$

where $-5\pi^2 + 10\pi - 1 > 0 \quad \forall \pi \in (\frac{1}{3}, 1)$, implying $V_R^{\mathrm{N}} - V_R^{\mathrm{S}} < 0$.

2. From the previous proof, we have $\alpha_M^{\mathrm{S}} \geq \alpha_M^{\mathrm{N}}$ and $\alpha_R^{\mathrm{S}} \geq \alpha_R^{\mathrm{N}}$ (when $\pi \in (\frac{1}{3}, 1)$ the strict inequality holds).

   Next, for $\pi \in (\frac{1}{3}, 1)$, use α_M^{S} and α_M^{J} as defined in Propositions 3.2 and 3.3 and take $\Phi^{\mathrm{J}} = 0$ to compute

$$\alpha_M^{\mathrm{S}} - \alpha_M^{\mathrm{J}} = \frac{\gamma^2}{8c_r(\rho_m + \delta)} \left[\pi^2 - 6\pi + 1 \right] < 0 \, .$$

   In a similar way, by using α_R^{N} and α_R^{J} defined in Propositions 3.1 and 3.3,

$$\alpha_R^{\mathrm{N}} - \alpha_R^{\mathrm{J}} = \frac{\pi^2 \gamma^2}{2c_r(\rho_r + \delta)} > 0 \, .$$

$\square$

Proof of Remark 3.1.

1. If $\pi \in (\frac{1}{3}, 1)$, use (3.9) and (3.15) and take $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$ in (3.10) to compute

$$A_M^{\mathrm{J}} - A_M^{\mathrm{S}} = \frac{k_m(1-\pi)}{8c_m c_r(1+\pi)(\rho_m+\delta)(\rho_r+\delta)} f_3(\pi), \tag{3.41}$$

where

$$\begin{aligned} f_3(\pi) =& \gamma^2(\rho_r+\delta)\pi^2 + \left[8c_r\mu(\rho_m+\delta) + 4\gamma^2(2\rho_m - \rho_r + \delta)\right]\pi \\ &+ 8c_r\mu(\rho_m+\delta) + 3\gamma^2(\rho_r+\delta). \end{aligned}$$

It is straightforward to check that $f_3(1)$ and $f_3(\frac{1}{3})$ are positive. It suffices to check that there is no $\widetilde{\pi}$ verifying $f_3'(\widetilde{\pi}) = 0$ with $f_3(\widetilde{\pi}) < 0$ in the interval $\widetilde{\pi} \in (\frac{1}{3}, 1)$. First, the stationary point of function $f_3(\pi)$ is

$$\widetilde{\pi} = 2 - 4\left(\frac{\rho_m+\delta}{\rho_r+\delta}\right)\left(\frac{c_r\mu}{\gamma^2}+1\right).$$

Condition $\frac{1}{3} < \widetilde{\pi} < 1$ becomes

$$\frac{12}{5}\left(\frac{c_r\mu}{\gamma^2}+1\right) < \frac{\rho_r+\delta}{\rho_m+\delta} < 4\left(\frac{c_r\mu}{\gamma^2}+1\right). \tag{3.42}$$

Define $g(\pi) = f_3(\pi) - \gamma^2(\rho_r+\delta)\pi^2$. It is clear that a necessary condition for the existence of negative values of $f_3(\pi)$ (and also for the existence of a positive $\widetilde{\pi}$) is that the coefficient in the linear term must be negative, hence $g(\pi)$ is decreasing. As a result,

$$f_3(\widetilde{\pi}) = \gamma^2(\rho_r+\delta)\widetilde{\pi}^2 + g(\widetilde{\pi}) > g(\widetilde{\pi}) > g(1).$$

Condition $f_3(\widetilde{\pi}) < 0$ implies $g(1) < 0$, i.e.

$$\frac{\rho_r+\delta}{\rho_m+\delta} > 8 + 16\left(\frac{c_r\mu}{\gamma^2}\right),$$

in contradiction with (3.42), so there is no solution for $f_3(\pi) = 0$ in the interval $(\frac{1}{3}, 1)$ and (3.41) is positive for $\pi \in (\frac{1}{3}, 1)$.

2. It follows from (3.11) and (3.16).

3. If $\pi \in (\frac{1}{3}, 1)$, use α_M^{S}, α_M^{J}, α_R^{S} and α_R^{J} as defined in Propositions 3.2 and 3.3, and take $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$ in (3.10) to compute

$$\alpha_M^{\mathrm{S}} - \alpha_M^{\mathrm{J}} = -\frac{(3-\pi)(1-\pi)^2\gamma^2}{8c_r(1+\pi)(\rho_m+\delta)} < 0\,,$$

and

$$\alpha_R^{\mathrm{S}} - \alpha_R^{\mathrm{J}} = \frac{\gamma^2(1-\pi)^3}{4c_r(\rho_r+\delta)(\pi+1)} > 0\,.$$

□

Proof of $\Delta(0) > 0$ *and* $\Delta(1) < 0$ *in Section* 3.4.2.

1.

$$\Delta(0) = \frac{k_m^2}{8c_m c_r^2 \rho_m \rho_r (\rho_m+\delta)^2(\rho_r+\delta)^2}\left[(\gamma^2+2c_r\mu)^2(\rho_m+\delta)^2(2\rho_m-\rho_r)\right]\,,$$

$\rho_m > 2\rho_r + \delta$ implies $\Delta(0) > 0$.

2.

$$\begin{aligned}\Delta(1) = &-\frac{\gamma^2 k_m^2}{8c_m c_r^2 \rho_m \rho_r (\rho_m+\delta)^2(\rho_r+\delta)^2}\\ &\{2c_r\mu(\rho_r+\delta)\left[2\rho_m(\rho_m+\delta)-4\rho_r(\rho_r+\delta)\right]\\ &-\gamma^2(\rho_m-2\rho_r-\delta)\left[(2\rho_m-\rho_r)(\rho_m+\delta)-2\rho_r(\rho_r+\delta)\right]\}\,.\end{aligned} \tag{3.43}$$

$$\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} = \frac{\left[-2(\rho_m-\rho_r)(\gamma^2+c_r\mu)\pi + (\rho_m+\delta)(\gamma^2+2c_r\mu)\right]}{2c_r(\rho_m+\delta)(\rho_r+\delta)}$$

is decreasing in π when $\rho_m > \rho_r$. The assumption $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$ for all $\pi \in (0,1)$ is equivalent to

$$\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}}|_{\pi=1} = \frac{2(c_r\mu+\gamma^2)(\rho_r+\delta) - \gamma^2(\rho_m+\delta)}{2c_r(\rho_m+\delta)(\rho_r+\delta)} \geq 0\,,$$

implying

$$2c_r\mu(\rho_r+\delta) \geq \gamma^2(\rho_m - 2\rho_r - \delta)\,. \tag{3.44}$$

Using (3.44) and $\rho_m > 2\rho_r + \delta$, we have that (3.43) is negative.

□

CHAPTER 4

Advertising and Quality Improving Strategies in a Marketing Channel When Facing Potential Crises

4.1. Introduction

It is well accepted that quality is a concept of multi-dimensional structure. The most influential definition can date back to Garvin (1984), who proposed an eight-dimension framework to understand the fundamental elements of quality: performance, features, reliability, conformance, durability, serviceability, aesthetics, and perceived quality. Scholars in dynamic game theory have extensively studied this subject focusing on one or multiple specific perspectives and its interaction with other managerial tools such as pricing, advertising, cost savings and so on.

In a monopolistic setting, Chand et al. (1996) firstly introduced quality by describing the fraction of non-defective products produced by the company, which be improved by process enhancement activities. This concept, years later, is defined as conformance quality by El Ouardighi & Pasin (2006). They adapt the Lanchester model (Kimball, 1957) to study an advertising battle, where only customers who have experienced defective products can be attracted by the other company. Following the same idea, El Ouardighi et al. (2008) place this issue in a supply chain environment and compare the cases with and without cooperation. Using conformance quality to differentiate between positive and negative word-of-mouth effects, El Ouardighi et al. (2016) analyse how they influence the advertising attracting effectiveness. In all the studies mentioned above, firms try to raise the conformance quality by making effort only on defective products, however, De Giovanni (2019) emphasizes the necessity of appraisal and prevention effort put into the non-defective fragment, which forms part of total quality management.

Some other dimensions of quality have also been tackled in economic literature. For instance, design quality, an aggregation of performance, features and aesthetics. Assuming that defective items cause full refund, El Ouardighi & Kogan (2013) study the interaction of design quality and conformance quality in supply chain management. Liu et al. (2015) consider different mechanisms for the manager to coordinate the operations department, which is in charge of design quality, and the marketing department, which controls price and advertising. Furthermore, Fruchter (2009) discusses the situation where the price and advertisement are used as a signal to influence the consumers' perceived quality, whereas in Xue et al. (2017), the demand is determined by the difference between perceived quality and the real product quality. Vörös (2006) suggests another two-dimension framework to comprehend the quality build-up, where one requires a development path (such as the knowledge of workers) and the other does not (like the choice of raw materials).

However, most of the researchers consider quality from a more general perspective, rather than engaging in one or several dimensions. Quality investment is believed to be able to contribute directly to the goodwill accumulation (Nair & Narasimhan, 2006; De Giovanni, 2011), to slow down the customer attrition (Ringbeck, 1985), to prevent potential demand reduction caused by unsatisfied experience related to quality deficits (Caulkins et al., 2017), and so on. Moreover, Reddy et al. (2016) investigate the application of impulse control, assuming that quality can mitigate the decay effect of goodwill and sales, and the linkage between advertising and quality level is revised by Chenavaz & Jasimuddin (2017).

With respect to the crisis management, it has been substantially analyzed in business literature. Covered issues include how and to what extent crisis can hurt a firm (MacKenzie & Lutz, 1989; Ahluwalia, 2000; Van Heerde et al., 2007), factors moderating the crisis impact (Dawar & Pillutla, 2000; Cleeren et al., 2008), possible strategic moves that agents take when facing a crisis (Souiden & Pons, 2009; Chen et al., 2009; Gao et al., 2015), and so on. While in fields like finance or environmental economics the use of stochastic optimization techniques or stochastic dynamic games has been extensively used to model problems with inherent uncertainty (for example, see Josa-Fombellida & Rincón-Zapatero, 2012; Ngwira & Gerrard, 2007; Polasky et al., 2011; van der Ploeg, 2014), this has not been the case in the study of brand crisis management. To the best of our knowledge, Rubel et al. (2011) is the only exception. They extend the model of Sethi (1983), where a monopoly influences the sales by advertising, to a stochastic setting, and analyze both theoretically and empirically the effects of a crisis. They also discuss, in brief, the case where firms can choose between a low and a high type of quality investment, which correspond to high and low instantaneous crisis rate respectively. Their results reveal that a higher crisis hazard rate would decrease the pre-crisis advertising, but increase the post-crisis budget.

The objective of this chapter is threefold. First, to introduce quality management into supply chain management in an intertemporal setting. Quality improvement has been playing an important role in both practice and academy. As the interaction among members in the supply chain differs from that in another market structure, the quality strategies may also differ. Special attention is required, yet studies placing this strategy in a supply chain environment are scarce (we refer the readers to Leng & Parlar, 2005, for a survey of game theoretic models in supply chain management). Most of the previous work has studied this issue using a static setting (see, for example, Moorthy, 1988; Reyniers & Tapiero, 1995; Wang et al., 2017), whereas El Ouardighi et al. (2008), El Ouardighi & Kogan (2013) and De Giovanni (2011) are the only exceptions that apply a dynamic approach. Second, to study crisis management policies as a piecewise deterministic dynamic game, where strategic moves, dynamic evolution and uncertainty could be captured together. Using this framework, we try to shed light on how the manufacturer and retailer adjust their strategies when anticipating a potential crisis, as well as the crisis impact on the supply chain by studying the pre-crisis and post-crisis optimal strategies. Third, to analyze the interaction among operations management (quality), marketing (advertising), and crisis management, since all of them are crucial challenges for managers nowadays.

In our model, we characterize the rules to decide in which regime (pre-crisis or post-crisis) to allocate more quality improving resources, and to adjust the global and local advertising when the crisis happens, which involve the consideration of short-term, long-term damages and hazard rate of crisis.

As a result, we generalize the results of Rubel et al. (2011) and we obtain a broader casuistic of the effects of crisis on both strategies and payoffs. Moreover, with the combination of quality and advertising, we identify some circumstances under which the enterprises can mitigate the crisis damage by proactively anticipating the crisis, thus offering some theoretical support to the benefits of voluntary recalls.

The rest of the chapter proceeds as follows. First we describe a piecewise deterministic game in which the supply chain members face a potential brand crisis in Section 4.2. The feedback equilibria are characterized in the following section. We then make a detailed analysis of the strategies and payoffs obtained for different regimes, followed by some numerical simulations to cast light on how the crisis influences the agents' behaviors and payoffs. We also offer a discussion on if a cooperation scheme makes the supply chain more resistant to crisis in Section 4.5. Finally, some concluding remarks are presented in Section 4.6.

4.2. Model Formulation

4.2.1 Advertising and Quality Management in a Supply Chain

Game theoretic attempts to incorporate quality control into management activities include different manners. One common way to introduce quality management into management activities is to consider the quality investment as a control variable contributing to goodwill build-up, customer retention, potential market size and so on (see, e.g., Ringbeck, 1985; Nair & Narasimhan, 2006; De Giovanni, 2011; Caulkins et al., 2017).

Other research line believes that the quality improvement activities can create an intangible stock, which evolves over time and whose dynamics are subject to the investment and depreciation. In line with this idea, some researchers mainly address the conformance quality. For instance, El Ouardighi & Pasin (2006) proposed the following conformance quality evolution assuming that firms work only on improving defective products to increase the perfection rate,

$$\dot{Q}(t) = q(t)[1 - Q(t)] ,$$

and that only the customers who have experienced defective products can be attracted by the rival company. Some related variations include an extension by setting the same idea in the supply chain management context (El Ouardighi et al., 2008), by introducing the word-of-mouth effect (El Ouardighi et al., 2016), and by incorporating the appraisal and prevention on the non-defective products (De Giovanni, 2019).

Another widely spread research stream, which we are following in this chapter, looks at the quality stock from a more integrated and knowledge-alike aspect, and takes a more general accumulation structure (for example, Vörös, 2006; Roselli & De Giovanni, 2012; Reddy et al., 2016; Xue et al., 2017),

$$\dot{Q}(t) = k_q q(t) - \epsilon Q(t), \quad Q(t) \geq 0 \ \forall \ t, \quad Q(0) = Q_0 \,, \tag{4.1}$$

where k_q and ϵ are positive constants representing the effectiveness of quality investment and the depreciation rate, respectively. Here, the state variable $Q(t)$ can be considered as total quality, and its evolution is attributed to all kinds of quality improvement effort $q(t)$ on product quality, product process quality, service quality, service quality process, and business planning (Juran & Godfrey, 1951). As any other intangible asset, it also suffers a depreciation proportional to the current state.

The market-based and cost-based linkages between quality and higher profitability are well supported by business literature. On the one hand, higher quality level, together with increased advertising, can improve the firm's reputation, and thereby lead to a higher market share and/or prices and higher profitability results. On the other hand, a better quality is connected with an increased productivity, lower rework and scrap costs, and lower warranty and product liability costs, thus greater profitability is achieved through cost reduction (Garvin, 1984).

In order to incorporate the market gains caused by quality, we adopt the Nerlove & Arrow (1962) goodwill model. Different from what Liu et al. (2015) do in their study, where the quality contributes to goodwill exactly in the same linear way as advertising, we propose the following dynamics,

$$\dot{G}(t) = k_m A_M(t) \sqrt{Q(t)} - \delta G(t) \,, \tag{4.2}$$

where $k_m > 0$ denotes the advertising effectiveness, $\delta \in (0, 1)$ measures the consumers' forgetting effect. In (4.2) we suggest that the quality level determines how effective advertising can be. The main idea here derives from Nelson (1974), where he argues that firms generally delivers quality information via advertisement, whereas consumers receive such information and validate it by searching or experiencing. Consider a firm providing products of absolutely poor quality ($Q(t) = 0$), the goodwill build-up would be extremely unlikely. Although advertising providing wrong quality information can induce trial purchases, it does not contribute to, and may even damage to the seller's reputation. On the contrary, repeat purchases usually happen to a firm offering products of high quality, and therefore her advertising is more effective in the long run.

We extend the revenue function in Lu et al. (2019) and define the revenue as follows:

$$R(t) = \theta + \mu G(t) + \gamma A_R(t)\sqrt{G(t)} + \eta Q(t) \,, \tag{4.3}$$

where the extra term $\eta Q(t)$ stands for the cost savings caused by improved quality with effectiveness parameter η, θ represents the baseline sales, and μ and γ refer to the influential factors of goodwill and the synthetic product of goodwill and retailer's advertising. Note that the quality's market-based contribution is reflected in the positive correlation between goodwill and revenue.

The advertising and quality cost are assumed to be of quadratic form,

$$C(q) = \frac{c_q}{2} q^2 \,, \quad C(A_M) = \frac{c_m}{2} A_M(t)^2 \,, \quad C(A_R) = \frac{c_r}{2} A_R(t)^2 \,,$$

where c_q, c_m, and c_r are positive constant cost parameters.

4.2.2 A Two-regime Game with Crisis

Now we proceed to incorporate the crisis management.

In this chapter, for the sake of simplicity, we assume that the crisis just happens once, at a random time instant τ. This is a common setting in the literature related to regime shifts (for example, Josa-Fombellida & Rincón-Zapatero, 2012; Polasky et al., 2011; van der Ploeg, 2014; Rubel et al., 2011). The common way to model this continuous random variable τ is through hazard rate $\lambda(t)$, defined as

$$\lambda(t) = \lim_{\delta t \to 0} \frac{Pr\{t \leq \tau < t + \delta t \mid \tau \geq t\}}{\delta t} \,,$$

which is the conditional probability that the crisis will take place in the interval $[t, t + \delta t)$, given that it has not occurred before. We confine our interest to the case of constant hazard rate $\lambda(t) = \lambda$, then the corresponding probability density function is exponential $f(t) = \lambda e^{-\lambda t}$. Besides, $1/\lambda$ (the mean of the exponential probability distribution) is the expected time when the crisis takes place ($E(\tau) = 1/\lambda$).

The crisis results in an instantaneous goodwill downturn (a shock):

$$G(\tau+) = (1 - \Phi)G(\tau-)\,, \tag{4.4}$$

which in turn implies a loss of revenue. In addition, this short-term damage also incorporates the lump-sum cost induced by the crisis. Take a product-harm crisis for example, the firms may need to pay the recall expenses, consumer compensation, lawsuit etc. Since the goodwill level somehow captures the market size, it is appropriate to assume this cost to be linear in goodwill. Similar idea appears in Rubel et al. (2011), where, instead of goodwill, they focus on the sales dynamics and the sales drop-down caused by crisis is also proportional to the state at the time τ.

The crisis might also make the subsequent goodwill accumulation less efficient. First of all, Consumers could be more skeptical when receiving the information of the advertising as the firm loses her credibility (MacKenzie & Lutz, 1989). Similarly, with the product harm crisis (negative information) in mind, they are more resistant to persuasion (Ahluwalia, 2000). Besides, dissatisfied customers can share their bad experience to potential customers, in other words, the negative word-of-mouth effect increases the difficulty in attracting new clients (El Ouardighi et al., 2016). Moreover, the crisis could induce a brand's equity damage (Dawar & Pillutla, 2000) and brand advertising is more effective for strong brand (Cleeren et al., 2008). The empirical test run by Van Heerde et al. (2007) shows that the advertising effectiveness is significantly positive before the crisis, and non-significant and of smaller magnitude after the crisis. Thus, the game is divided into pre-crisis and post-crisis regimes, with the following goodwill dynamics:

$$\dot{G}(t) = \begin{cases} k_{m1}A_M(t)\sqrt{Q(t)} - \delta_1 G(t) & \text{for} \quad 0 \le t \le \tau, \\ k_{m2}A_M(t)\sqrt{Q(t)} - \delta_2 G(t) & \text{for} \quad t \ge \tau, \end{cases} \tag{4.5}$$
$$G(t) \ge 0 \;\forall\, t, \quad G(0) = G_0\,,$$

where $k_{m1} \geq k_{m2}$ and $\delta_1 \leq \delta_2$, denoting a higher advertising effectiveness and a lower depreciation rate in the pre-crisis regime.

Finally, the manufacturer and retailer aim to maximize their expected profits over time, given by

$$J_M = E\left[\int_0^\tau e^{-\rho t}\left(\pi R(t) - \frac{c_m}{2}A_M(t)^2 - \frac{c_q}{2}q(t)^2\right)dt + e^{-\rho\tau}V_M(2,G,Q)\right], \tag{4.6}$$

and

$$J_R = E\left[\int_0^\tau e^{-\rho t}\left((1-\pi)R(t) - \frac{c_r}{2}A_R(t)^2\right)dt + e^{-\rho\tau}V_R(2,G,Q)\right], \tag{4.7}$$

where $R(t)$ is defined in (4.3), and $V_M(2,G,Q)$ and $V_R(2,G,Q)$ stand for the manufacturer's and retailer's post-event value functions, respectively.

Equations (4.1), (4.4), (4.5), (4.6) and (4.7) define a two-player piecewise differential game with two state variables $Q(t) \geq 0$ and $G(t) \geq 0$, where the manufacturer controls $q(t) \geq 0$ and $A_M(t) \geq 0$, and the retailer controls $A_R(t) \geq 0$.

4.3. Determination of Feedback Nash Equilibria

Following the approach in Seierstad (2013), the Hamilton-Jacobi-Bellman (HJB) equations for the two players are:

$$\begin{aligned}
&\rho V_M(1,G,Q)\\
&= \max_{\substack{A_M(1)\geq 0\\ q(1)\geq 0}}\left\{\pi\left[\theta + \mu G + \gamma A_R(1)\sqrt{G} + \eta Q\right] - \frac{c_m}{2}A_M(1)^2 - \frac{c_q}{2}q(1)^2\right.\\
&+\frac{\partial V_M(1,G,Q)}{\partial G}\left[k_{m1}A_M(1)\sqrt{Q} - \delta_1 G\right] + \frac{\partial V_M(1,G,Q)}{\partial Q}\left[k_q q(1) - \epsilon Q\right]\\
&\left.+\lambda\left[V_M(2,(1-\Phi)G,Q) - V_M(1,G,Q)\right]\right\},
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\rho V_R(1,G,Q) = &\max_{\{A_R(1)\geq 0\}}\left\{(1-\pi)\left[\theta + \mu G + \gamma A_R(1)\sqrt{G} + \eta Q\right] - \frac{c_r}{2}A_R(1)^2\right.\\
&+\frac{\partial V_R(1,G,Q)}{\partial G}\left[k_{m1}A_M(1)\sqrt{Q} - \delta_1 G\right] + \frac{\partial V_R(1,G,Q)}{\partial Q}\left[k_q q(1) - \epsilon Q\right]\\
&\left.+\lambda\left[V_R(2,(1-\Phi)G,Q) - V_R(1,G,Q)\right]\right\},
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&\rho V_M(2,G,Q)\\
&= \max_{\substack{A_M(2)\geq 0\\ q(2)\geq 0}} \left\{\pi \left[\theta + \mu G + \gamma A_R(2)\sqrt{G} + \eta Q\right] - \frac{c_m}{2} A_M(2)^2 - \frac{c_q}{2} q(2)^2 \right.\\
&\left. + \frac{\partial V_M(2,G,Q)}{\partial G}\left[k_{m2}A_M(2)\sqrt{Q} - \delta_2 G\right] + \frac{\partial V_M(2,G,Q)}{\partial Q}\left[k_q q(2) - \epsilon Q\right]\right\},
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
\rho V_R(2,G,Q) = \max_{\{A_R(2)\geq 0\}} &\left\{(1-\pi)\left[\theta + \mu G + \gamma A_R(2)\sqrt{G} + \eta Q\right] - \frac{c_r}{2} A_R(2)^2\right.\\
&\left. + \frac{\partial V_R(2,G,Q)}{\partial G}\left[k_{m2}A_M(2)\sqrt{Q} - \delta_2 G\right] + \frac{\partial V_R(2,G,Q)}{\partial Q}\left[k_q q(2) - \epsilon Q\right]\right\},
\end{aligned}
\tag{4.11}
$$

where 1 and 2 in parenthesis following strategies/value functions denote the pre-crisis and post-crisis regimes, respectively. Notice that the post-crisis regime game is equivalent to a deterministic game, so $V_M(2,G,Q)$ and $V_R(2,G,Q)$ can be computed using the corresponding method. Different from the usual HJB, equations (4.8) and (4.9) have an additional term $\lambda\left[V_i(2,(1-\Phi)G,Q) - V_i(1,G,Q)\right]$ $(i = M,R)$, indicating the expected revenue change by jumping from pre-crisis to post-crisis regime. Maximizing the right-hand-side of equations (4.8) - (4.11) yields $q(j)^* = \frac{k_q}{c_q}\frac{\partial V_M}{\partial Q}(j,G,Q)$, $A_M(j,Q)^* = \frac{k_{mj}}{c_m}\frac{\partial V_M}{\partial G}(j,G,Q)\sqrt{Q}$ and $A_R(j,G)^* = \frac{(1-\pi)\gamma}{c_r}\sqrt{G}$ $(j = 1,2)$. We guess that value functions in both regimes are linear in G and Q. After substituting them into (4.8) - (4.11), by identifying the parameters of G, Q and constant parts, the feedback Nash equilibrium strategies in both regimes are given in the following two propositions.

Proposition 4.1. *A Feedback Nash equilibrium in the post-crisis regime is given by the strategies*

$$q(2) = \frac{k_q}{c_q}\beta_{M2}\,, \tag{4.12}$$

$$A_M(2,Q) = \frac{k_{m2}}{c_m}\alpha_{M2}\sqrt{Q}\,, \tag{4.13}$$

$$A_R(2,G) = \frac{(1-\pi)\gamma}{c_r}\sqrt{G}\,, \tag{4.14}$$

and the corresponding value functions are given by

$$V_M(2,G,Q) = \alpha_{M2}G + \beta_{M2}Q + \tau_{M2}\,, \tag{4.15}$$

$$V_R(2,G,Q) = \alpha_{R2}G + \beta_{R2}Q + \tau_{R2}\,, \tag{4.16}$$

where $\alpha_{M2} = \dfrac{c_r\pi\mu + \pi(1-\pi)\gamma^2}{c_r(\rho+\delta_2)}$, $\beta_{M2} = \dfrac{(k_{m2})^2}{2c_m(\rho+\epsilon)}(\alpha_{M2})^2 + \dfrac{\pi\eta}{\rho+\epsilon}$,

$$\tau_{M2} = \frac{(k_q)^2}{2c_q\rho}(\beta_{M2})^2 + \frac{\pi\theta}{\rho}, \quad \alpha_{R2} = \frac{2c_r(1-\pi)\mu + (1-\pi)^2\gamma^2}{2c_r(\rho+\delta_2)},$$

$$\beta_{R2} = \frac{(k_{m2})^2}{c_m(\rho+\epsilon)}\alpha_{M2}\alpha_{R2} + \frac{(1-\pi)\eta}{\rho+\epsilon}, \quad \tau_{R2} = \frac{(k_q)^2}{c_q\rho}\beta_{M2}\beta_{R2} + \frac{(1-\pi)\theta}{\rho}.$$

All three strategies given by (4.12) - (4.14) are proportional to the ratio of their corresponding effectiveness parameter to cost parameter k_q/c_q, k_{m2}/c_m and γ/c_r. The investment in quality improvement $q(2)$ is increasing in μ, γ and η, i.e., when the i) goodwill, ii) the synergistic effect of goodwill and local advertising, and/or iii) the quality contribute to the sales to a larger extent, the manufacturer will invest more. It also increases in k_{m2}, which measures how large the quality level's influence is on the dynamics of goodwill. On the contrary, higher cost parameters of marketing (of both agents), larger depreciation rates of quality and goodwill, and greater discount rates will result in a decrease of quality improvement expenditure. Regarding the manufacturer's advertising strategy $A_M(2,Q)$, it is increasing in the quality level with an elasticity of 0.5: 1% increase of Q will lead to an increase of 0.5% in A_M.

Concerning the retailer's advertising $A_R(2,G)$, it is goodwill-state dependent in a similar way as A_M with respect to Q. Besides, when she takes a higher part of revenue, she spends more in local advertising.

If the manufacturer only had a marketing tool A_M, as in Lu et al. (2019), she would invest in a constant way, whereas the retailer would decide the local advertising depending on the goodwill level. These properties might be a result of the influential mechanism: the revenue is highly dependent on the goodwill level, and the global advertising A_M is the unique way to enhance it. The manufacturer's strategies change qualitatively when she has the option to improve quality. On the one hand, since the quality level determines the advertising's effectiveness, it is beneficial to make a positive effort on it. On the other hand, she gains sort of responsiveness by being able to adapt her advertising budget depending on the quality level achieved. The retailer reacts in exactly the same way (for a given goodwill level) in these two models because she has limited influential power in both settings, in the sense that A_R works solely together with goodwill.

Proposition 4.2. *A Feedback Nash equilibrium in the pre-crisis regime is given by the pair of strategies*

$$q(1) = \frac{k_q}{c_q}\beta_{M1} , \tag{4.17}$$

$$A_M(1,Q) = \frac{k_{m1}}{c_m}\alpha_{M1}\sqrt{Q} , \tag{4.18}$$

$$A_R(1,G) = \frac{(1-\pi)\gamma}{c_r}\sqrt{G} , \tag{4.19}$$

and the corresponding value functions are given by

$$V_M(1,G,Q) = \alpha_{M1}G + \beta_{M1}Q + \tau_{M1} , \tag{4.20}$$

$$V_R(1,G,Q) = \alpha_{R1}G + \beta_{R1}Q + \tau_{R1} , \tag{4.21}$$

where $\alpha_{M1} = \dfrac{\rho+\delta_2+\lambda(1-\Phi)}{\rho+\delta_1+\lambda}\alpha_{M2}$, $\alpha_{R1} = \dfrac{\rho+\delta_2+\lambda(1-\Phi)}{\rho+\delta_1+\lambda}\alpha_{R2}$,

$\beta_{M1} = \dfrac{1}{2c_m(\rho+\epsilon+\lambda)}\left[(k_{m1})^2(\alpha_{M1})^2 + \dfrac{\lambda}{(\rho+\epsilon)}(k_{m2})^2(\alpha_{M2})^2\right] + \dfrac{\pi\eta}{\rho+\epsilon}$,

$\beta_{R1} = \dfrac{1}{c_m(\rho+\epsilon+\lambda)}\left[(k_{m1})^2\alpha_{M1}\alpha_{R1} + \dfrac{\lambda}{(\rho+\epsilon)}(k_{m2})^2\alpha_{M2}\alpha_{R2}\right] + \dfrac{(1-\pi)\eta}{\rho+\epsilon}$,

$\tau_{M1} = \dfrac{(k_q)^2}{2c_q(\rho+\lambda)}\left[(\beta_{M1})^2 + \dfrac{\lambda}{\rho}(\beta_{M2})^2\right] + \dfrac{\pi\theta}{\rho}$,

$\tau_{R1} = \dfrac{(k_q)^2}{c_q(\rho+\lambda)}\left(\beta_{M1}\beta_{R1} + \dfrac{\lambda}{\rho}\beta_{M2}\beta_{R2}\right) + \dfrac{(1-\pi)\theta}{\rho}$,

and α_{M2}, β_{M2}, α_{R2} *and* β_{R2} *are defined in Proposition 4.1.*

Strategies taken in the pre-event regime show similar structures as those in the post-event regime. Some properties such as being proportional to the efficiency ratio and being positive state-dependent with decreasing marginal effect also apply here. It is worth mentioning that parameters in the second regime are also involved in the decision making. For example, $A_M(1,Q)$ is decreasing in both δ_1 and δ_2 at different rates (but only increasing in k_{m1}). Besides, high advertising effectiveness in both regimes (k_{m1} and k_{m2}) would induce a higher investment in quality $q(1)$. Note that when the crisis happens, it is the goodwill $G(t)$ which suffers a sharp decrease, whereas the quality stock remains and will continue serving as a booster to the goodwill accumulation. Thus the manufacturer would be also motivated to invest more in quality by high advertising effectiveness in the second regime. Besides, in the pre-crisis regime the manufacturer has to take into account the potential crisis, namely, the hazard rate λ, and the shock in goodwill Φ while deciding the quality improvement and the global advertising.

The response in global advertising $A_M(1,Q)$ to λ is straightforward. As

$$\frac{\partial A_M(1,Q)}{\partial \lambda} = -\frac{k_{m1}[\Phi\rho + (\delta_2 - \delta_1) + \Phi\delta_1]}{c_m(\rho + \delta_1 + \lambda)^2}\alpha_{M2}\sqrt{Q} < 0\,,$$

the manufacturer invests less in marketing when anticipating a greater chance of the crisis happening. Since the shock is proportional to the goodwill state and $A_M(1,Q)$ directly acts on the goodwill accumulation, it is reasonable to slower the build-up process before crisis in order to minimize the loss.

However, the case of quality investment $q(1)$ is more complicated. By rewriting

$$q(1) = \frac{k_q}{c_q}(\alpha_{M2})^2\left[f_1(\lambda)(k_{m1})^2 + f_2(\lambda)(k_{m2})^2\right] + \frac{k_q\pi\eta}{c_q(\rho+\epsilon)}\,,\ \text{with}$$
$$f_1(\lambda) = \frac{[\rho + \delta_2 + \lambda(1-\Phi)]^2}{2c_m(\rho+\epsilon+\lambda)(\rho+\delta_1+\lambda)^2}\,,\ f_2(\lambda) = \frac{\lambda}{2c_m(\rho+\epsilon+\lambda)(\rho+\epsilon)}\,, \tag{4.22}$$

we can observe a conflicting influence of λ on $q(1)$ in (4.22), since $\frac{\partial f_1(\lambda)}{\partial\lambda} < 0$ whereas $\frac{\partial f_2(\lambda)}{\partial\lambda} > 0$. Let us consider the dual contribution of $q(1)$. On the one hand, it accelerates the goodwill build-up before the crisis, and a larger hazard rate results intuitively harmful and lowers the quality investment. On the other hand, it has a carryover effect on the recovery after the crisis, and the player has incentives to increase the budget. The manufacturer needs to balance these two impacts while deciding the quality improvement effort. Particularly, a larger hazard rate corresponds to an earlier expected crisis occurrence time, therefore a shorter pre-crisis regime and a longer post-crisis period, which prioritize the carryover effect, and *vice versa*. Consequently, the overall effect turns out negative when the hazard rate is small, and positive for larger hazard rates. The only exception is when the advertising effectiveness decreases so much after the crisis ($k_{m1} \gg k_{m2}$) that the carryover effect is trivial, the manufacturer would always invest less in quality improvement in the first regime when facing a larger hazard rate. We will offer some numerical illustration of $q(1)$ in Section 4.4.2.

As to the crisis magnitude Φ, it manifests a negative influence in both quality improvement $q(1)$ and global advertising $A_M(1, Q)$, due to the same idea of slowing down the goodwill accumulation so that the crisis would be less catastrophic.

Notwithstanding, unlike the reactive manufacturer, the retailer's policies are similar before and after crisis (though the global expenditure varies as the goodwill level is evolving), because her highly limited influencing power makes her not able to respond to the crisis, though we can observe that at the moment of the crisis happening, she adjusts her local advertising through a reduction of $(1 - \pi)\gamma \left(1 - \sqrt{1 - \Phi}\right) \sqrt{G}/c_r$.

4.4. Analysis of the Results

We start our analysis by presenting two benchmark cases in order to have a better understanding about the changes of introducing quality management and crisis.

We first compare how the supply chain members adjust their strategies when the manufacturer gets an additional operational tool, the quality improvement, and if they benefit from it. To do so, we extend the model of Lu et al. (2019) to a crisis setting. The extended model and its equilibria are briefly given in the Appendix, and the comparison is summarized in Remark 4.1.

Remark 4.1. *Let $A_M^{NQ}(i)$ and $A_R^{NQ}(i)$ ($i = 1, 2$) denote the manufacturer's advertising, the retailer's advertising in pre-crisis ($i = 1$) and post-crisis ($i = 2$) regimes with the absence of quality. The agents' behaviors and payoffs in feedback Nash equilibria can be related as follows:*

1. $A_M^{NQ}(i) \leq A_M(i, Q)$ *if* $Q \geq 1$ ($i = 1, 2$).
2. $A_R^{NQ}(i, G) = A_R(i, G)$.
3. $V_j^{NQ}(1, G) \leq V_j(1, G, Q)$ *if* $Q \geq \frac{\rho}{\rho + \epsilon} - \frac{\tau_{j1}}{\beta_{j1}}$ ($j = M, R$).

Proof. See Appendix. □

In general, if the manufacturer's product has a superior quality compared to the industry standard, she would invest more in global advertising. The retailer uses the same responsive strategy, thought the goodwill accumulation would be of different paths. In most of the cases, both members are better off even if the game starts with a zero quality level[1].

Next, we analyze how the existence of potential crisis could affect the agents' behaviors and the payoffs. We start by studying the deterministic model where the supply chain does not face a potential crisis. It can be represented by the special case of $\lambda = 0$. The equilibrium corresponding to such deterministic game is characterized in Proposition 4.2 with $\lambda = 0$.

Remark 4.2. *Let* $\alpha_i = \alpha_{i1}|_{\lambda=0}$, $\beta_i = \beta_{i1}|_{\lambda=0}$, $\tau_i = \tau_{i1}|_{\lambda=0}$ *(*$i = M, R$*), and* q, A_M *and* A_R *denote the quality improvement effort, the manufacturer's advertising, the retailer's advertising when facing no potential crisis. The agents' behaviors and payoffs can be related as follows:*

1. $q > q(1)$ *for all* $\lambda > 0$.
2. $A_M(Q) \geq A_M(1, Q)$ *for all* $\lambda > 0$ *(the equality holds if and only if* $\Phi = 0$ *and* $\delta_1 = \delta_2$*).*
3. $A_R(G) = A_R(1, G)$ *for all* $\lambda > 0$.
4. $V_M(G, Q) > V_M(1, G, Q)$ *and* $V_R(G, Q) > V_R(1, G, Q)$ *for all* $\lambda > 0$.

Proof. See Appendix. □

If there is no potential crisis, the manufacturer would invest more in quality improvement, and both types of advertising budgets are higher for a given quality/goodwill level. Furthermore, both the manufacturer and the retailer are better off, which is intuitive.

4.4.1 Pro-Efficiency vs. Pro-Recovery

We now compare the players' strategies before and after crisis. As the quality and goodwill levels are dynamic, so are both agents' advertising budgets (which are state dependent). Instead of comparing the advertising strategies

[1] It can be easily checked that $Q < \frac{\rho}{\rho+\epsilon} - \frac{\tau_{j1}}{\beta_{j1}}$ ($j = M, R$) can hold only under extremely unreasonable parameters setting.

in both regimes from a global aspect, as how we dealt with $q(1)$ and $q(2)$, we focus on the moment τ and we are able to show the players' immediate reaction in their marketing strategies when come up against a crisis.

Proposition 4.3. *The agents' strategies in the two regimes are related as follows:*

1. $q(1) = q(2) + \dfrac{k_q}{2c_m c_q(\rho + \epsilon + \lambda)} \left[(k_{m1}\alpha_{M1})^2 - (k_{m2}\alpha_{M2})^2\right]$, *and*
$q(1) \geq q(2)$ *if* $\dfrac{k_{m2}}{k_{m1}} \leq \Omega = \dfrac{\rho + \delta_2 + \lambda(1-\Phi)}{\rho + \delta_1 + \lambda}$,
$q(1) < q(2)$ *otherwise.*

2. *When crisis occurs,* $A_M(1, Q(\tau)) = \left(\dfrac{k_{m1}\alpha_{M1}}{k_{m2}\alpha_{M2}}\right) A_M(2, Q(\tau))$, *and*
$A_M(1, Q(\tau)) \geq A_M(2, Q(\tau))$ *if* $\dfrac{k_{m2}}{k_{m1}} \leq \Omega = \dfrac{\rho + \delta_2 + \lambda(1-\Phi)}{\rho + \delta_1 + \lambda}$,
$A_M(1, Q(\tau)) < A_M(2, Q(\tau))$ *otherwise.*

3. *When crisis occurs,* $A_R(1, G(\tau-)) \geq A_R(2, G(\tau+))$ *(with strict inequality for* $\Phi > 0$*).*

Proof. It follows from (4.12), (4.13), (4.14), (4.17), (4.18) and (4.19). □

As shown in Proposition 4.3, the difference between $q(1)$ and $q(2)$ is a constant, whereas at the moment when crisis happens, the *ex-post* global advertising $A_M(2, Q(\tau)$ is proportional to the *ex-ante* $A_M(1, Q(\tau)$. Although we compare the manufacturer's strategies in the pre- and post-crisis regimes in different manners, the manufacturer has a quite clear regime-based-priority, in the sense that in one of the pre- and post-event regimes/instants, she invests more in both quality and global advertising. This consistency stems from the two-sided effects of pre-crisis quality improvement. As explained previously, $q(1)$ contributes to the goodwill accumulation in both pre- and post-event regimes, since the quality state is not affected by the crisis. Therefore, while deciding in which regime to allocate more quality investment, the manufacturer has to take into account the advertising effectiveness. Specifically, when k_{m2}/k_{m1}, the fraction of the post-crisis global advertising effectiveness per unit of its pre-crisis value, is smaller than the threshold Ω, which is decreasing in λ, Φ and $\delta_2 - \delta_1$, the priority will be investing before the crisis occurs, and *vice versa*.

We call it *pro-efficiency* if she decreases both quality and global advertising budget when crisis occurs, and *pro-recovery* for the contrary case.

The retailer always lowers the local advertising at the moment when crisis happens, as long as it harms the product's reputation. Consider that the local advertising mainly consists of promotion, fliers, point-of-sale display etc., and normally does not convey quality-related information, it would not be a good idea increasing the local advertising intensity when the crisis is brewing, as it could make consumers even more impressed by the crisis. However, this does not apply to the manufacturer, because she can use the global advertising to deliver information like what they would do to compensate the consumers, the future plan to avoid same thing happening again, the product quality, and so on. She has to take more factors into account, as we will present in the following.

Since

$$\frac{\partial \Omega}{\partial \lambda} = \frac{\delta_1(1-\Phi) - \delta_2 - \rho(1+\Phi)}{(\rho + \delta_1 + \lambda)^2} \leq 0\,,$$

we have

$$1 - \Phi \leq \Omega \leq \frac{\rho + \delta_1}{\rho + \delta_2}\,,$$

and we can classify some special cases as described in Remark 4.3.

Remark 4.3. *Depending on the short-term and long-term damages caused by the crisis, we have some special cases:*

(I) If $\frac{k_{m2}}{k_{m1}} = 1$ *and* $\delta_1 = \delta_2$*, then* $q(1) \leq q(2)$ *and* $A_M(1, Q(\tau)) \leq A_M(2, Q(\tau))$ *for all* $\lambda > 0$ *(with strict inequality for* $\Phi > 0$*).*

(II) If $\Phi \neq 0$ *and* $1 - \Phi < \frac{k_{m2}}{k_{m1}} < 1$*, then* $q(1) > q(2)$ *and* $A_M(1, Q(\tau)) > A_M(2, Q(\tau))$ *for* $0 < \lambda < \hat{\lambda}$*, and* $q(1) < q(2)$ *and* $A_M(1, Q(\tau)) < A_M(2, Q(\tau))$ *for* $\lambda > \hat{\lambda}$*, where* $\hat{\lambda} = \frac{k_{m2}(\rho + \delta_1) - k_{m1}(\rho + \delta_2)}{k_{m1}(1-\Phi) - k_{m2}}$.

(III) If $\frac{k_{m2}}{k_{m1}} \leq 1 - \Phi$*, then* $q(1) \geq q(2)$ *and* $A_M(1, Q(\tau)) \geq A_M(2, Q(\tau))$ *for all* $\lambda > 0$ *(with strict inequality for* $\Phi > 0$*).*

Case (I) is a special case where after the crisis the goodwill stock evolves exactly in the same way as how it is before ($k_{m1} = k_{m2}$ and $\delta_1 = \delta_2$), namely, the crisis does instantaneous harm to the companies (if $\Phi \neq 0$) without causing any other long-term effect. Under these circumstances, it is worthier to make relatively more effort after the crisis in order to recover from the shock as soon as possible. As a consequence, independently of the crisis hazard rate, the manufacturer always invests more in quality in the post-crisis regime, and increase the global advertising when the crisis happens. In the empirical study by Cleeren et al. (2013), they recommend an advertising increase in a high-publicity product-harm case where the fault is not attributed to the firm (and thus there should not be long-term damages).

On the contrary, case (III) describes a situation where the long-term damage dominates the short-term loss. Accordingly, the emphasis will be always placed in the first regime regardless of the crisis intensity rate, in this way the manufacturer can profit from the high efficiency before crisis. It is interesting to note that the influence of the change in advertising effectiveness k_{m2}/k_{m1} is larger than that in goodwill depreciation $\delta_2 - \delta_1$, for the reason that k_{m1} and k_{m2} modify directly the effect of the strategies. This situation coincides with the empirical results of Cleeren et al. (2013), which suggest an advertising decrease when the product-harm crisis is of low publicity but the firm needs to acknowledge the fault.

As an intermediate case, in (II) the crisis causes an instantaneous loss, and also reshapes the goodwill accumulation path in the way that it becomes more difficult to strengthen the goodwill by advertisement and/or the goodwill suffers a faster depreciation. However, it is hard to conclude which impact is more destructive. The manufacturer exhibits higher interests in the first regime for a sufficiently small instantaneous crisis rate $\lambda < \hat{\lambda}$, which indicates a later expected occurrence time and thus a longer pre-crisis period. Whereas she would switch to *pro-recovery* strategies if the crisis is estimated to happen in the early stage ($\lambda > \hat{\lambda}$).

To sum up, the instantaneous injury generated by the crisis makes the manufacturer incline towards a set of *pro-recovery* strategies, while a strong long-term damage may lead to *pro-efficiency* policies. When making decisions, the manufacturer has to face a trade-off between higher efficiency in the pre-event regime and faster recovery in the post-event regime, apart from

Table 4.1: Parameter Setting

ρ	θ	μ	γ	η	c_m	c_q	c_r
0.1	1	1	1	1	2	2	2
k_q	k_r	π	ϵ	δ_1	δ_2	Φ	k_{m1}
1	1	0.75	0.1	0.05	0.05	0.3	0.5

Figure	**4.1**	**4.2**	**4.3**	**4.4**	**4.5**	**4.6**
k_{m2}	0.5	0.43	0.3	0.45	0.45	0.45
G	–	–	–	5	20	70
Q	–	–	–	3	3	3

considering the crisis instantaneous rate.

4.4.2 The Three "Impact Factors" of Crisis: Numerical Illustration

In this section we present some numerical illustrations to throw light on how the crisis influences the quality and advertising strategies. There are three underlying "impact factors" capturing the nature of the crisis: the hazard rate λ, which is inversely proportional to the average time when the crisis occurs; the shock Φ, which exhibits the immediate loss; and the changes in effectiveness and in depreciation, which picture the permanent damage. Specifically, based on the previous analysis in Remark 4.3, the change in advertising effectiveness k_{m2}/k_{m1} is more representative of the long-term injury. The parameters used are summarized in Table 4.1.

Figures 4.1-4.3 show how the quality improvement expenditure changes with the hazard rate. Quality investment policies in a game without crisis (q) are also graphed to serve as a benchmark. To interpret better these figures, we firstly discuss two extreme points: $\lambda = 0$ and $\lambda \to \infty$. For zero hazard rate, $q(1)$ coincides with q. As to the other extreme point, from Proposition

4.3,

$$\lim_{\lambda\to\infty}[q(1)-q(2)] = \lim_{\lambda\to\infty}\left\{\frac{k_q}{2c_m c_q(\rho+\epsilon+\lambda)}\left[(k_{m1}\alpha_{M1})^2-(k_{m2}\alpha_{M2})^2\right]\right\} = 0, \tag{4.23}$$

so $q(1)$ converges to $q(2)$ as λ approaches ∞. Note that the case in which the crisis is expected to happen in the beginning of the time horizon can be considered as another deterministic game of the same set of parameters in the post-crisis regime, which explains why (4.23) holds. With the two extreme points fixed, we can observe how the manufacturer adapts $q(1)$ taking into account the hazard rate. These three figures correspond to the three cases described in Remark 4.3. Recall that $q(1)$ has double effects in both regimes, as discussed in Section 4.3, and in some cases, it exhibits non-monotonic tendency under the conflicting influences from hazard rate, as shown in Figure 4.1 and 4.2. However, when the long-term damage absolutely wins over the instantaneous loss (case (III) in Remark 4.3), the *pro-efficiency* strategies are applied independently of the hazard rate. It is also in this case when the carryover effect of $q(1)$ is irrelevant and the quality investment in pre-crisis regime is monotonically decreasing.

Next we discuss the effect of λ on value functions. Following the same idea of analyzing the quality investment, we also focus on two extreme cases, $\lambda = 0$ and $\lambda \to \infty$, representing situations where there is no potential crisis, and the crisis is estimated to occur at the beginning of planning horizon, respectively. On the one hand, it is clear that, for both agents in the supply chain, the payoffs of a game with crisis are always inferior to that of the situation without crisis, as explained in Remark 4.2. Moreover,

$$\lim_{\lambda\to\infty} V_i(1,G,Q) = (1-\Phi)\alpha_{i2}G+\beta_{i2}Q+\tau_{i2} = V_i(2,(1-\Phi)G,Q)\,, \quad i = M,R.$$

If the crisis occurs immediately, the players will get as much as that in a deterministic game with initial goodwill state $(1-\Phi)G$ and under the parameters setting in the post-crisis regime.

Since $V_i(2,(1-\Phi)G,Q) < V_i(G,Q)$ holds for all $\lambda > 0$[2], the behaviors of $V_i(1,G,Q)$ will be determined by its value when λ tends to

[2] Except for $\Phi = 0$, $k_{m1} = k_{m2}$ and $\delta_1 = \delta_2$. However, the crisis would have no effect in this case, so we do not include it into our discussion. That is why we will not consider the Case (Ia) presented in Table 4.2 in the following analysis.

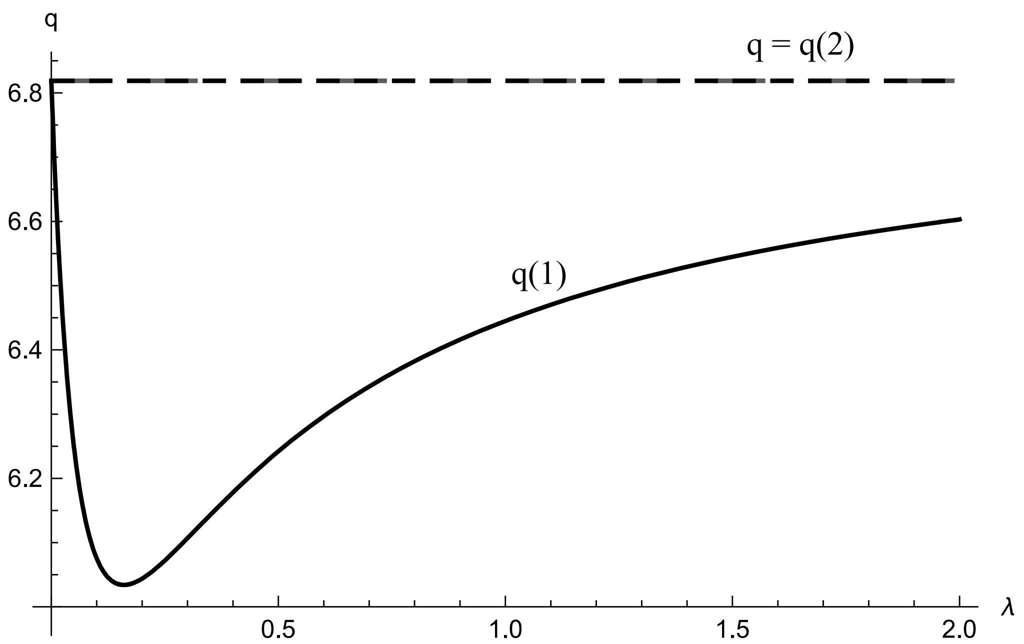

Figure 4.1: Quality Investment. Case (I)

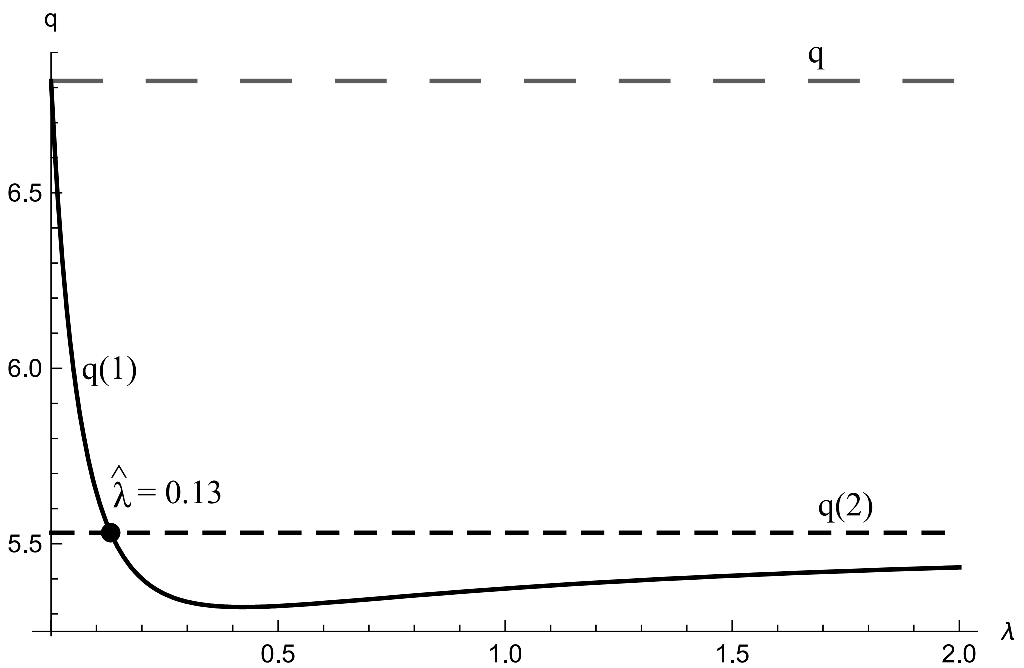

Figure 4.2: Quality Investment. Case (II)

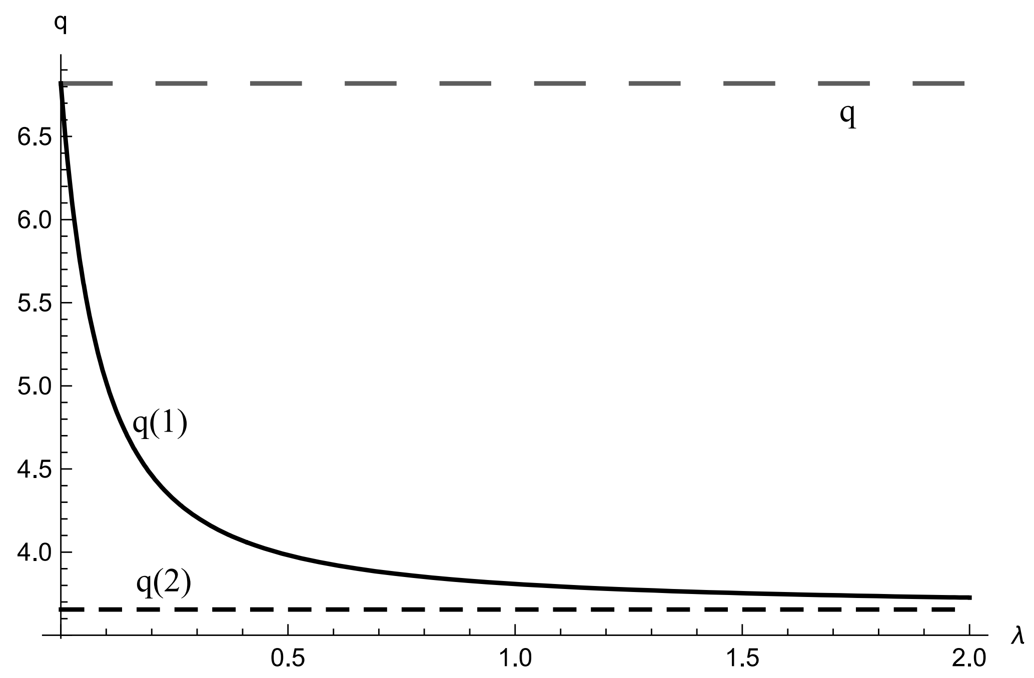

Figure 4.3: Quality Investment. Case (III)

infinity. Specifically, if $V_i(1, G, Q) = V_i(2, (1-\Phi)G, Q)$ has no solution, $V_i(1, G, Q)$ will be monotone and decreasing in λ, whereas for the case where $V_i(1, G, Q) = V_i(2, (1-\Phi)G, Q)$ has a unique solution, $V_i(1, G, Q)$ will be decreasing firstly, then increasing.

Take the manufacturer as an example (the retailer's value function has a similar behavior so it suffices to concentrate our analysis on the manufacturer), and compute

$$\begin{aligned}
&V_M(1, G, Q) - V_M(2, (1-\Phi)G, Q)\\
&= [\alpha_{M1} - (1-\Phi)\alpha_{M2}]G + (\beta_{M1} - \beta_{M2})Q + (\tau_{M1} - \tau_{M2})\\
&= \frac{\rho\Phi + \delta_2 - (1-\Phi)\delta_1}{\rho + \delta_1 + \lambda} G + (\beta_{M1} - \beta_{M2})Q + \frac{(k_q)^2}{2c_q(\rho + \lambda)} \left[(\beta_{M1})^2 - (\beta_{M2})^2\right],
\end{aligned} \tag{4.24}$$

and let $\tilde{\lambda}_M$ be the solution to $V_M(1, G, Q) = V_M(2, (1-\Phi)G, Q)$, when it exists. From (4.24), it is clear that the existence of $\tilde{\lambda}_M$ mainly depends on the initial goodwill level and the relationship between β_{M1} and β_{M2}, which determine $q(1)$ and $q(2)$ respectively. In particular, we can characterize three

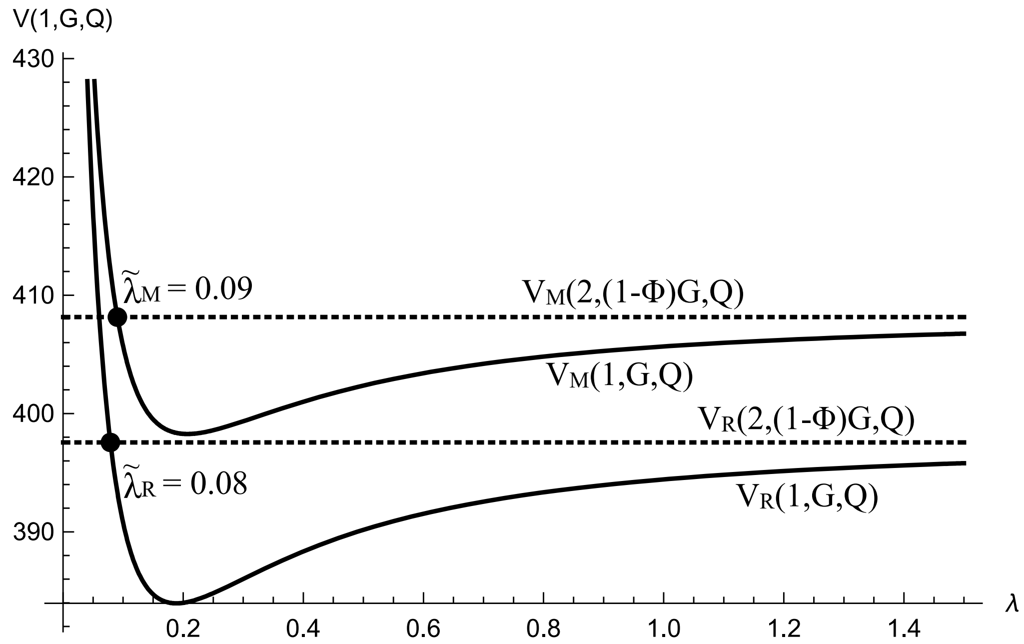

Figure 4.4: Value Functions. Case (IIb)

scenarios of zero, positive and very high initial goodwill levels, which are summarized, together with the three cases in Remark 4.3, in Table 4.2.

It is straightforward that for zero initial goodwill[3], which could be the case of a start-up manufacturer, the sign of (4.24) depends only on the relationship between β_{M1} and β_{M2}. Accordingly, there are three special cases, which are consistent with those described in Remark 4.3. If the initial goodwill level is positive, then even in the case (I) where the crisis has no long-term damage, there is also a single $\tilde{\lambda}_M$ solving $V_M(1, G, Q) = V_M(2, (1 - \Phi)G, Q)$. As to the case (II) where none of the short-term and long-term damages is strictly dominant for all possible hazard rates, we can find a unique solution $\tilde{\lambda}_M$, which is greater than $\hat{\lambda}$, the solution to $q(1) = q(2)$. Moreover, if the initial goodwill level is much higher than the initial quality level, it can happen that $V_M(1, G, Q) > V_M(2, (1 - \Phi)G, Q)$ holds for all $\lambda > 0$, no matter which is the dominance between short-term and long-term damages.

Figure 4.4 represents the case (IIb) and Figures 4.5 and 4.6 demonstrate

[3]Note that in this case, $V_M(2, (1 - \Phi)G, Q)$ is "immune" from the instantaneous loss of the crisis.

Table 4.2: Existence of $\tilde{\lambda}_M$

	Scenario a: $G = 0$ or $\Phi = 0$, $\delta_2 = \delta_1$	Scenario b: $G > 0$, $\Phi \geq 0$, $\delta_2 \geq \delta_1$ (at least one $\geq$ is strict)	Scenario c: $G \gg Q$, $\Phi \geq 0$, $\delta_2 \geq \delta_1$ (at least one $\geq$ is strict)
Case (I): $\frac{k_{m2}}{k_{m1}} = 1$, $\delta_1 = \delta_2$	(4.24) < 0, no solution	(4.24) > 0 for $0 < \lambda < \tilde{\lambda}_M$ (4.24) < 0 for $\lambda > \tilde{\lambda}_M$	(4.24) > 0, no solution
Case (II): $1 - \Phi < \frac{k_{m2}}{k_{m1}} < 1$	(4.24) > 0 for $0 < \lambda < \tilde{\lambda}_M$ (4.24) < 0 for $\lambda > \tilde{\lambda}_M$ ($\tilde{\lambda}_M = \hat{\lambda}$)	(4.24) > 0 for $0 < \lambda < \tilde{\lambda}_M$ (4.24) < 0 for $\lambda > \tilde{\lambda}_M$ ($\tilde{\lambda}_M > \hat{\lambda}$)	(4.24) > 0, no solution
Case (III): $\frac{k_{m2}}{k_{m1}} \leq 1 - \Phi$	(4.24) > 0, no solution	(4.24) > 0, no solution	(4.24) > 0, no solution

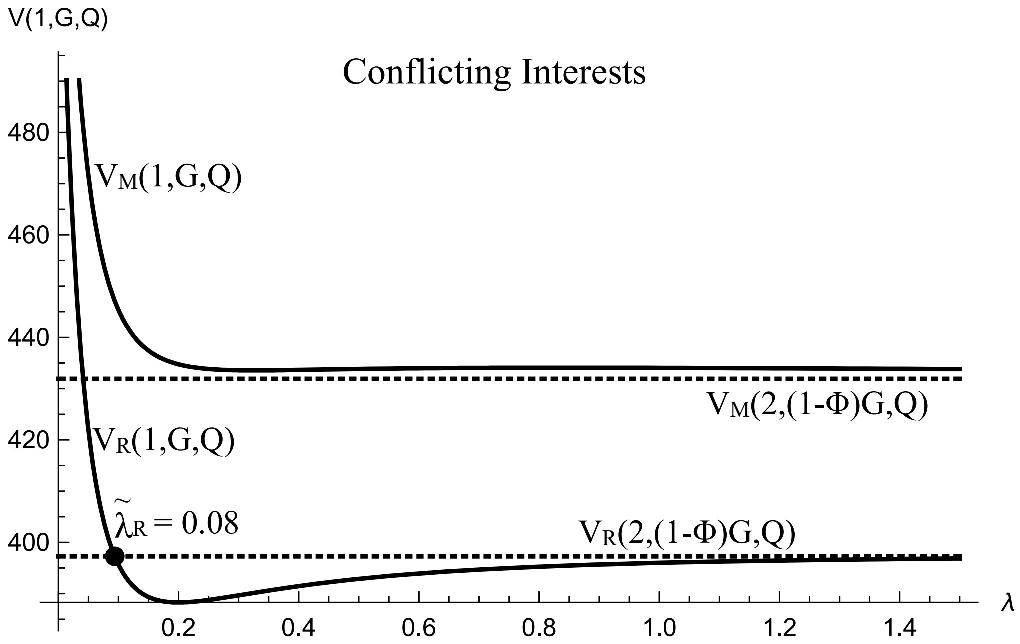

Figure 4.5: Value Functions. Case (IIc)

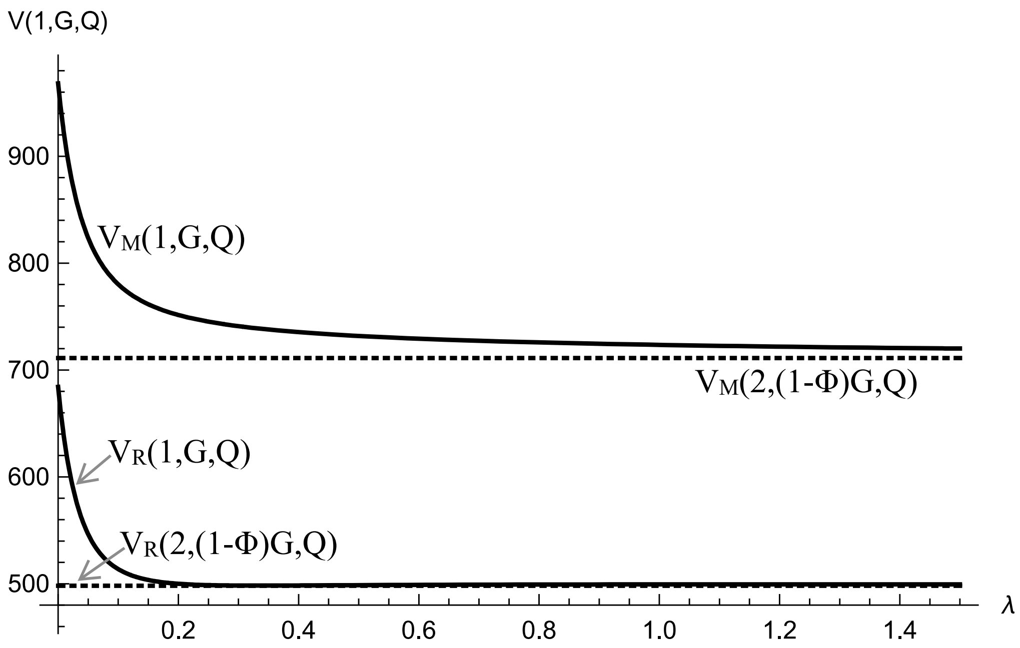

Figure 4.6: Value Functions. Case (IIc)

the case (IIc)[4]. When (4.24) has a solution, as shown in Figure 4.4, there exists a λ minimizing the manufacturer's value function. This non-monotonicity derives from the binary effect of $q(1)$. It is worthwhile noting that, under these circumstances, a larger λ may be beneficial to the players. Compared with any instantaneous crisis rate such that $\lambda > \tilde{\lambda}_M$, the manufacturer would prefer $\lambda \to \infty$.

A managerial implication derived from the discussion above is that, in this situation (case (IIb) or other similar cases), if the manufacturer is facing a hazard rate greater than $\tilde{\lambda}_M$, she would be interested in anticipating the crisis. The break-out of a crisis can be interpreted as the moment when the negative information is disclosed to the public and it has broad impact and causes unpleasant reaction, which is usually later than when the problem occurs. In practice, firms and costumers very often have asymmetric access to certain information. In this sense, the company can anticipate the crisis by sending out private (negative) information, one example is announcing a recall. Consider the case that a company estimates that a crisis related to quality would happen in the near future (e.g. less than 10 years - according to our simulation) due to some private information such as quality test reports, consumer complaints and so on, which are not transparent to the public, it would be a better strategy announcing a recall policy. Although a recall is generally considered as a crisis and it harms the cooperate's reputation and sales, it is still better than waiting for its disclosure. The retailer also has incentives to anticipate the crisis, although she has another benchmark $\tilde{\lambda}_R$ different from $\tilde{\lambda}_M$.

To explain better this surprising result, we need to discuss some underlying properties. In this model we assume that the crisis happens only once, which means that if for some reason, the players anticipate the crisis, they can get rid of it forever. Besides, this kind of strategic anticipation does not work when the long-term damage plays a determinant role (Case III). It also seems more feasible to company that is not well known, as it does not work either when the firm has a strong initial goodwill (Scenario c). Lastly, $V_M()$ and $V_R()$ increase in λ does not happen when λ is very small, which is, by

[4]Note that the behaviors of $V_M(1, G, Q)$ in Figure 4.4 coincides with that of the case (Ib) and (IIa), whereas those in Figure 4.5 and 4.6 are consistent with that in the case (Ic), (IIIa), (IIIb) and (IIIc).

intuition, more preferred by the players. This phenomenon only appears for λ being moderately small or large, that is to say, the crisis is expected to happen in the short run with a considerable chance. Under all these assumptions, it is true that the players would get better off when λ takes a greater value.

The benefits of voluntary recall are supported in the literature. Kong et al. (2019) show that voluntary recalls result in less loss and quick recovery of normal stock returns compared with mandatory recalls. Souiden & Pons (2009) argue that voluntary recall can have positive impacts on the firms' image, and consumers' loyalty and purchase intentions.

In the other case represented in Figures 4.5 and 4.6, the manufacturer would prefer the hazard rate to be as little as possible, which is coherent to our intuition. However, as we can see in Figure 4.5, the manufacturer and the retailer could have conflict of interests, as the retailer would prefer to anticipate the crisis.

Finally, we summarize the indirect effects of the crisis on some strategies in Figure 4.7. From Section 4.3 we can see that the retailer's advertising in both regimes $A_R(1,G)$ and $A_R(2,G)$, as well as the post-crisis global advertising $A_M(2,Q)$ are not determined by crisis. However, as the crisis changes the goodwill and quality trajectories in the pre-crisis regime, these state-dependent strategies will also change accordingly. Summarizing, higher instantaneous damage rate will imply lower local advertising in pre- and post-crisis regime, and lower *ex-post* global advertising, which coincides what is found in Rubel et al. (2011). However, different from the study mentioned above, the indirect effect of hazard rate λ on these policies can be positive when the carryover effect of pre-event quality investment $q(1)$ is strong enough and the crisis intensity rate is large.

4.5. Discussion: Does Vertical Integration Make the Supply Chain More Resistant to Crisis?

In this section we analyze the situation where the supply chain members decide to carry out a vertical integration/centralized coordination, i.e., they form a coalition and act cooperatively aiming to maximize the collective profit. Using "C" to refer to "Coordination", $V_C(2,G,Q)$ to denote the post-crisis value function of the coordinated channel, then it becomes an optimal

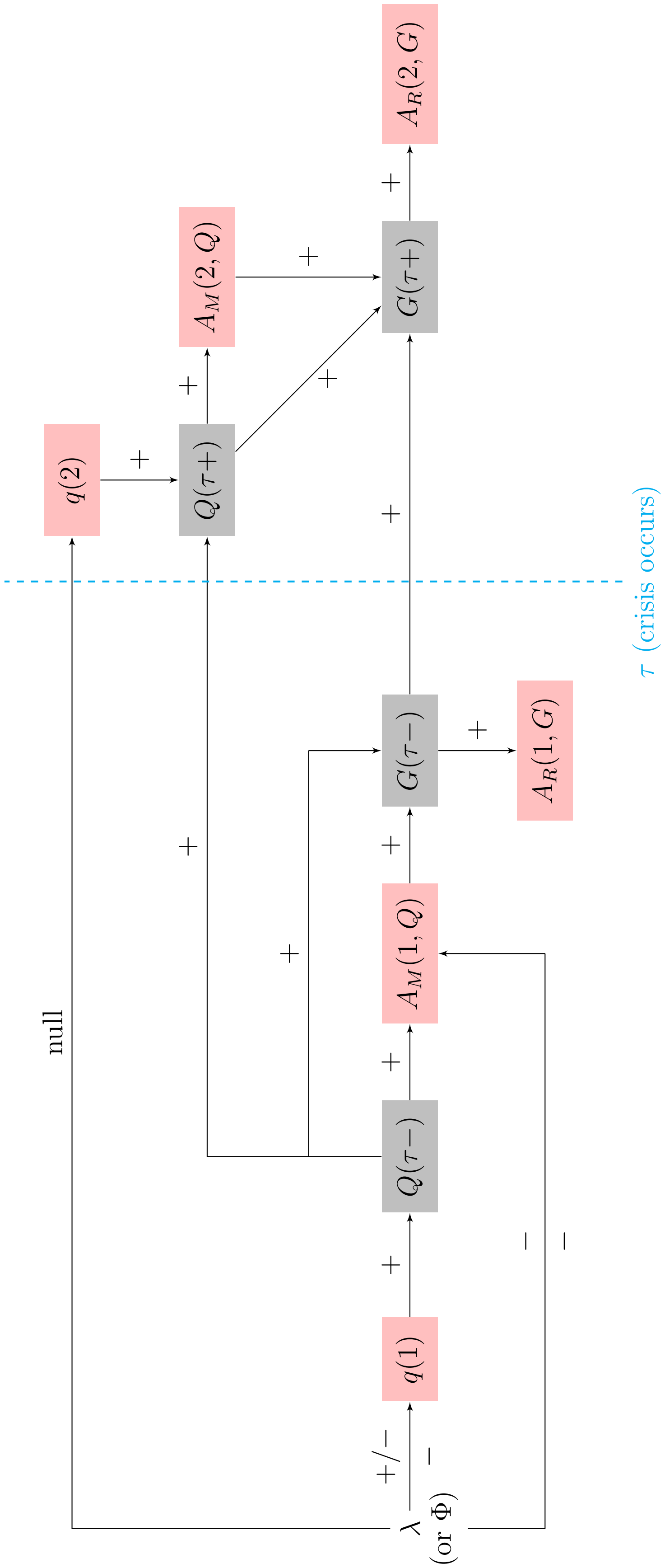

Figure 4.7: Indirect Effects of the Crisis

control problem, with the objective functional:

$$
\begin{aligned}
J_C &= J_M + J_R \\
&= E\left[\int_0^{\tau} e^{-\rho t}\left(R(t) - \frac{c_m}{2}A_{MC}(t)^2 - \frac{c_q}{2}q_C(t)^2 - \frac{c_r}{2}A_{RC}(t)^2\right)dt\right. \\
&\quad \left. + e^{-\rho\tau}V_C(2, G, Q)\right],
\end{aligned}
$$

where $R(t)$ is given in (4.3).

Applying the same approach used for the non-cooperative case (Seierstad, 2013), the HJB equations for post- and pre-crisis regimes are:

$$
\begin{aligned}
\rho V_C(2, G, Q) = \max_{\{A_{MC}(2)\geq 0,\ q_C(2)\geq 0,\ A_{RC}(2)\geq 0\}} &\left\{\theta + \mu G + \gamma A_{RC}(2)\sqrt{G} + \eta Q \right. \\
&- \frac{c_m}{2}A_{MC}(2)^2 - \frac{c_q}{2}q_C(2)^2 - \frac{c_r}{2}A_{RC}(2)^2 + \frac{\partial V_C(2, G, Q)}{\partial Q}\left[k_q q_C(2) - \epsilon Q\right] \\
&\left. + \frac{\partial V_C(2, G, Q)}{\partial G}\left[k_{m2}A_{MC}(2)\sqrt{Q} - \delta_2 G\right]\right\},
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
\rho V_C(1, G, Q) = \max_{\{A_{MC}(1)\geq 0,\ q_C(1)\geq 0,\ A_{RC}(1)\geq 0\}} &\left\{\theta + \mu G + \gamma A_{RC}(1)\sqrt{G} + \eta Q \right. \\
&- \frac{c_m}{2}A_{MC}(1)^2 - \frac{c_q}{2}q_C(1)^2 - \frac{c_r}{2}A_{RC}(1)^2 + \frac{\partial V_C(1, G, Q)}{\partial Q}\left[k_q q_C(1) - \epsilon Q\right] \\
&\left. + \frac{\partial V_C(1, G, Q)}{\partial G}\left[k_{m1}A_{MC}(1)\sqrt{Q} - \delta_1 G\right] + \lambda\left[V_C(2, (1-\Phi)G, Q) - V_C(1, G, Q)\right]\right\}.
\end{aligned}
\tag{4.26}
$$

Maximizing the right-hand-side, we obtain the expressions of the optimal strategies. We conjecture linear structure of the coordinated channel's value functions $V_C(i, G, Q) = \alpha_{Ci}G + \beta_{Ci}Q + \tau_{Ci}$ $(i = 1, 2)$. Substituting the strategies and value functions into (4.25) and (4.26), by identifying the parameters values such that the HJBs are satisfied, we characterize the cooperative feedback solutions in the following proposition.

Proposition 4.4. *The cooperative feedback strategies in the post-crisis regime are given by*

$$
q_C(2) = \frac{k_q}{c_q}\beta_{C2}, \tag{4.27}
$$

$$
A_{MC}(2, Q) = \frac{k_{m2}}{c_m}\alpha_{C2}\sqrt{Q}, \tag{4.28}
$$

$$A_{RC}(2,G) = \frac{\gamma}{c_r}\sqrt{G}\,, \tag{4.29}$$

and the corresponding value functions are given by

$$V_C(2,G,Q) = \alpha_{C2}G + \beta_{C2}Q + \tau_{C2}\,, \tag{4.30}$$

where
$\alpha_{C2} = \dfrac{2c_r\mu+\gamma^2}{2c_r(\rho+\delta_2)}$, $\beta_{C2} = \dfrac{(k_{m2})^2}{2c_m(\rho+\epsilon)}(\alpha_{C2})^2 + \dfrac{\eta}{\rho+\epsilon}$, $\tau_{C2} = \dfrac{(k_q)^2}{2c_q\rho}(\beta_{C2})^2 + \dfrac{\theta}{\rho}$.

The cooperative feedback solutions in the pre-crisis regime are characterized by

$$q_C(1) = \frac{k_q}{c_q}\beta_{C1}\,, \tag{4.31}$$

$$A_{MC}(1,Q) = \frac{k_{m1}}{c_m}\alpha_{C1}\sqrt{Q}\,, \tag{4.32}$$

$$A_{RC}(1,G) = \frac{\gamma}{c_r}\sqrt{G}\,, \tag{4.33}$$

and the corresponding value functions are given by

$$V_C(1,G,Q) = \alpha_{C1}G + \beta_{C1}Q + \tau_{C1}\,, \tag{4.34}$$

where
$\alpha_{C1} = \dfrac{\rho+\delta_2+\lambda(1-\Phi)}{\rho+\delta_1+\lambda}\alpha_{C2}$, $\tau_{C1} = \dfrac{(k_q)^2}{2c_q(\rho+\lambda)}\left[(\beta_{C1})^2 + \dfrac{\lambda}{\rho}(\beta_{C2})^2\right] + \dfrac{\theta}{\rho}$,
and $\beta_{C1} = \dfrac{1}{2c_m(\rho+\epsilon+\lambda)}\left[(k_{m1})^2(\alpha_{C1})^2 + \dfrac{\lambda}{(\rho+\epsilon)}(k_{m2})^2(\alpha_{C2})^2\right] + \dfrac{\eta}{\rho+\epsilon}$.

The coalition also has a clear emphasis in one of the regimes, exactly as in the non-cooperative case, which is summarized in Proposition 4.3 and Remark 4.3.

The comparison of the equilibria outcomes in cooperative and non-cooperative contexts is presented in the following proposition.

Proposition 4.5. *The agents' strategies and payoffs, stationary states in non-cooperative setting $\bar{Q}$ and $\bar{G}$, and those in cooperative context $\bar{Q}_C$ and $\bar{G}_C$ are related as follows:*

1. $q_C(i) > q(i)$, $A_{MC}(i,Q) > A_M(i,Q)$, $A_{RC}(i,Q) > A_R(i,Q)$ *($i = 1,2$).*
2. $V_C(1,G,Q) > V_M(1,G,Q) + V_R(1,G,Q)$.

3. $\bar{Q}_C > \bar{Q}$, $\bar{G}_C > \bar{G}$.

Proof. It follows from Propositions 4.1, 4.2 and 4.4. □

In both pre- and post-crisis regimes, the supply chain increase the budget of all types of investment under vertical integration, thus the stationary levels of quality and goodwill reach higher level and larger total profits are generated. The outcome of cooperation is Pareto superior to that of non-cooperation.

4.6. Concluding Remarks

In this chapter we have developed a piecewise deterministic differential game in a two-level supply chain, where the manufacturer can decide quality improvement and global advertising levels, and the retailer determines local advertising effort. We have enriched the discussion about the quality management in supply chain operations, and have contributed to the research line of crisis management by using a differential game framework. Moreover, we have analyzed the interaction among quality control, advertising and crisis management, which, as far as we know, is a novelty in the literature. The feedback Nash equilibria for both pre- and post-crisis regimes are determined. We then have analyzed, in detail, the agents' behaviors and illustrated graphically the impact of crisis. A brief discussion of what a vertical integration program would entail is also presented.

Our results reveal that when the supply chain faces a potential crisis, their strategies change accordingly under the overall interactive effect of crisis intensity rate, short-term damage and long-term damage. Besides, due to the fact that pre-crisis quality investment also helps the recovery in post-crisis regime, the manufacturer will invest more in both quality and advertising in one of the regimes. Specifically, if the advertising effectiveness decreases sufficiently after the crisis, the manufacturer needs to apply *pro-efficiency* strategies, otherwise *pro-recovery* strategies are more preferred.

The carryover effect of pre-event quality investments also gives rise to a non-monotonicity of quality improvement effort and value functions with respect to the instantaneous crisis rate. These properties allow both agents in the supply chain to strategically choose the crisis occurrence time under certain circumstances. Particularly, if the initial goodwill level is not much higher than the initial quality level, and the crisis long-term damage is not dominant to the short-term damage, the players could have the chance to reduce the loss by anticipating the crisis. This managerial implication is supported by some studies related to voluntary recalls (for instance, Kong et al., 2019; Souiden & Pons, 2009). It is also worth mentioning that, in some cases, the supply chain members may have conflicting interests as the retailer would like to anticipate the crisis whereas the manufacturer prefers not to.

The intervention effect of the hazard rate on post-crisis national advertising, pre- and post-crisis local advertising is consistent with its direct effect on pre-event quality investment $q(1)$, which can be positive or negative, depending on the three crisis impact factors. These findings generalize those of Rubel et al. (2011). As to the instantaneous damage rate, both its direct impact (on manufacturer's pre-crisis strategies) and intervention effect (on retailer's strategies in two regimes and post-crisis national advertising) are negative: a higher damage rate induces lower investment.

Finally, we have shown that vertical integration results in a Pareto superior outcome, making a centralized channel more resistant to the crisis.

Appendix

The current model without quality.

Let $V_M(2, G)$ and $V_R(2, G)$ denote the manufacturer's and retailer's post-crisis value functions, the game is defined by

$$\max_{A_M(t)\geq 0} E\left[\int_0^{\tau} e^{-\rho t}\left[\pi R(t) - \frac{c_m}{2}A_M(t)^2\right]dt + e^{-\rho\tau}V_M(2, G)\right],$$

$$\max_{A_R(t)\geq 0} E\left[\int_0^{\tau} e^{-\rho t}\left[(1-\pi)R(t) - \frac{c_r}{2}A_R(t)^2\right]dt + e^{-\rho\tau}V_R(2, G)\right],$$

subject to

$$R(t) = \theta + \mu G(t) + \gamma A_R(t)\sqrt{G(t)}\,,$$

$$\dot{G}(t) = k_m A_M(t) - \delta G(t)\,,\ G(0) = G_0\,.$$

The strategies in regime 1 (pre-crisis) and in regime 2 (post-crisis) are given by

$$A_M^{NQ}(1) = \frac{k_{m1}}{c_m}\alpha_{M1}\,, \quad A_M^{NQ}(2) = \frac{k_{m2}}{c_m}\alpha_{M2}\,, \tag{4.35}$$

$$A_R^{NQ}(1,G) = A_R^{NQ}(2,G) = \frac{(1-\pi)\gamma}{c_r}\sqrt{G}\,, \tag{4.36}$$

and the corresponding value functions are determined by

$$V_M^{NQ}(1,G,Q) = \alpha_{M1}G + \frac{\rho}{\rho+\epsilon}\beta_{M1}\,, \tag{4.37}$$

$$V_R^{NQ}(1,G,Q) = \alpha_{R1}G + \frac{\rho}{\rho+\epsilon}\beta_{R1}\,, \tag{4.38}$$

where α_{M2} and α_{R2} are defined in Proposition 4.1, α_{M1}, α_{R1}, β_{M1} and β_{R1} are defined in Proposition 4.2.

Proof of Remark 4.1. 1. It follows from (4.13), (4.18) and (4.35).

2. It follows from (4.14), (4.19) and (4.36).

3. It follows from (4.20), (4.21), (4.37) and (4.38).

□

Proof of Remark 4.2.

$$\begin{aligned}
\alpha_M - \alpha_{M1} &= \frac{\lambda\left[\rho\Phi + \delta_2 - (1-\Phi)\delta_1\right]\left[c_r\pi\mu + \pi(1-\pi)\gamma^2\right]}{c_r(\rho+\delta_1)(\rho+\delta_2)(\rho+\delta_1+\lambda)} \geq 0\,,\\
\alpha_M - \alpha_{M2} &= \frac{(\delta_2-\delta_1)\left[c_r\pi\mu + \pi(1-\pi)\gamma^2\right]}{(\rho+\delta_1)(\rho+\delta_2)} \geq 0\,,\\
\beta_M - \beta_{M1} &= \frac{(k_{m1})^2(\rho+\epsilon+\lambda)(\alpha_M)^2 - (k_{m1})^2(\rho+\epsilon)(\alpha_{M1})^2 - (k_{m2})^2\lambda(\alpha_{M2})^2}{2c_m(\rho+\epsilon)(\rho+\epsilon+\lambda)}\\
&= \frac{(k_{m1})^2(\rho+\epsilon)\left[(\alpha_M)^2 - (\alpha_{M1})^2\right] + \lambda\left[(k_{m1})^2(\alpha_M)^2 - (k_{m2})^2(\alpha_{M2})^2\right]}{2c_m(\rho+\epsilon)(\rho+\epsilon+\lambda)}\\
&\geq 0\,,\\
\beta_M - \beta_{M2} &= \frac{(k_{m1})^2(\alpha_M)^2 - (k_{m2})^2(\alpha_{M2})^2}{2c_m(\rho+\epsilon)} \geq 0\,,\\
\tau_M - \tau_{M1} &= \frac{(k_q)^2\left[(\rho+\lambda)(\beta_M)^2 - \rho(\beta_{M1})^2 - \lambda(\beta_{M2})^2)\right]}{2c_q\rho(\rho+\lambda)}\\
&= \frac{(k_q)^2\left\{\rho\left[(\beta_M)^2 - (\beta_{M1})^2\right] + \lambda\left[(\beta_M)^2 - (\beta_{M2})^2\right]\right\}}{2c_q\rho(\rho+\lambda)} \geq 0\,.
\end{aligned}$$

Similarly, we have $\alpha_R - \alpha_{R1} \geq 0$, $\beta_R - \beta_{R1} \geq 0$ and $\tau_R - \tau_{R1} \geq 0$. Moreover, $\alpha_j = \alpha_{j1}$ holds if and only if $\Phi = 0$ and $\delta_1 = \delta_2$. $\beta_j = \beta_{j1}$ or $\tau_j = \tau_{j1}$ holds if and only if $\Phi = 0$, $\delta_1 = \delta_2$ and $k_{m1} = k_{m2}$ (and the crisis has no effects) ($j = M, R$). Thus the results follow. □

CHAPTER 5

Conclusions

Advertising is recognized as being one of the most crucial activities for a business, and it is among the most widespread topics in the academic world. The aim of this thesis has been to broaden current knowledge of decision making in advertising using a differential game approach. To this end, we have addressed the issue from two perspectives. One is the temporal bias exhibited in reality and evidenced in literature, which is novel in marketing research. Another one is the interface among different functional areas, which has attracted certain attention, yet deserves more consideration.

Departing from a simple horizontal advertising competition between two firms, in the first study (Chapter 2) we have introduced two alternatives to the standard exponential discounting in order to capture some additional descriptive realism. The heterogeneous discounting describes the scene where a firm can have an increasing/decreasing valuation of the state (market share, in our case) at the end of planning horizon with the passage of time, whereas the hyperbolic discounting depicts the tendency to value more the payoffs that are closer to the present. We have derived three different types of feedback Nash strategies, depending on how agents deal with their time-varying preferences. The pre-commitment solutions are employed by firms that are not aware of future changes or have a strong commitment power. Another option is to make decisions at every instant of time based on the corresponding instantaneous preferences, and only apply them at the very same moment (naive). The third action is to anticipate such variation and to include it into the decision making (sophisticated/time-consistent). Clear discrepancy is found in the advertising paths corresponding to different strategies.

The game starts with a battle: the company which is at a disadvantage in the beginning advertises intensively since the target market (the rival's portion) is large, then eventually reduces the budget. The firm initially of larger size would do the other way around. This battle stage is present under all kinds of discounting and solution types, though the exact values might differ. The battle ends up near the steady state, if the planning period is long enough. Another property in common is the similarity between naive and sophisticated solutions.

Heterogeneous discounting would lead to some last-minute changes. We can observe, in the last years, the adapting behaviors in accordance with their increasing/decreasing valuations of the final state. Under some circumstances, the change can be so radical that the pre-commitment solution takes the contrary path of time-consistent strategies.

Concerning the competition under hyperbolic discounting, the time dependence of advertising efforts show a quite different nature. The rate of time preference decreases rapidly in early periods, then slowly in the long term and converges to a constant rate. For this reason, different strategies exhibit disparity in the beginning, and encounter in the neighborhood in the end, which is contrary to the heterogeneous discounting. It is worth mentioning that, the lack of information about declining discount rates in the future or strong commitment power would lead to over investment.

After getting some insights from the first study into the mechanism of general time preference at individual level and in a competitive environment, we place this issue into a supply chain at collective level. Specifically, when both members in the supply chain have constant but different discount rates, the centralized channel will behave like an agent with heterogeneous discounting and face the trade-off between time-consistency and efficiency. Then arises the research question: on the premise that time-consistency is guaranteed, can cooperation be inefficient? In order to answer this question, in Chapter 3, we have analyzed three different scenarios where coordination is absent, where a cost-sharing program is applied, and where the channel is vertically integrated. We have compared, in detail, the advertising strategies and outcomes among these cooperative and non-cooperative settings.

Our results show that the manufacturer would be willing to pay a subsidy to the retailer if her revenue sharing rate is sufficiently high. If the program is active, both members of the marketing channel employ higher advertising policies, thus a bigger market size is achieved and both of them are better off compared with non-cooperation. If the manufacturer and the retailer form alliance to maximize the joint profit, the revenue and cost sharing rates will determine the manufacturer's global advertising rate. Furthermore, they even alter the total profit of the channel, even though they are generally considered as side payments.

The most striking result to emerge from our analysis is that the centralized coordination is not necessarily beneficial. All the existing studies have agreed that a vertical integrated supply chain is more efficient, as a matter of fact, it usually serves as a benchmark to assess the performance of other coordination mechanisms. Nonetheless, such consensus can be revoked by simply allowing both participants to be asymmetric in time discounting.

We identified the circumstances under which group inefficiency emerges. One case that leads to inefficient coordination is when the retailer is much more impatient than the manufacturer, coupled with low initial goodwill level and revenue sharing rate. The contrary case where the manufacturer's rate of time preference is much higher than the retailer's might also cause similar result, if the revenue sharing rule does not favor the retailer. However, unlike the other case, even a large initial goodwill cannot avoid the inefficiency. Besides, for the latter case, we find that if the retailer is more effective, in the sense that she has higher advertising effectiveness, and/or lower cost parameter, the group inefficiency likelihood would increase. A larger contribution of the synergy to revenue, and the goodwill's faster depreciation also yield a larger likelihood. Besides, the inefficiency level is severer if the manufacturer's advertising is more effective and/or less costly.

So far we have only considered a single marketing tool, the advertising. However, coordinating activities in different functional areas is indispensable to the success. Therefore, once reached a better understanding of how advertising alone contributes to sales/goodwill in different environment under different time discounting, we tend to investigate the interface among marketing, operations research, and public relations under uncertainty.

In the third study (Chapter 4), we have analyzed a two-level supply chain that faces a potential crisis, with one manufacturer deciding quality improvement and global advertising levels, and one retailer determining local advertising effort. The feedback Nash equilibria for both pre- and post-crisis regimes are characterized. Both marketing channel members' strategies and the impacts of the crisis are fully analyzed.

Our results reveal that the pre-crisis quality improvement accelerates the goodwill build-up before the crisis, and also helps the recovery in post-crisis regime. Its twofold function suggests that one of the pre- and post-crisis regimes/instants ought to be matched with more intense investment in both quality and global advertising, i.e., the manufacturer shall choose between *pro-efficiency* and *pro-recovery* strategies. By analyzing the overall effect of instantaneous crisis rate, short-term damage and long-term damage, we have provided some instructions to make the choice.

This carryover effect also brings a non-monotonicity of quality improvement effort and value functions with respect to the instantaneous crisis rate. These properties leave the chance to mitigate the loss by anticipating crisis for both members under certain circumstances. Particularly, such strategy is applicable to the conditions where the initial goodwill level does not far exceed the initial quality level, and the crisis long-term effect is not much severer than the instantaneous one. However, they may not always agree on the anticipation, since in some cases it is only beneficial to one of them.

This thesis has gone some way towards enhancing our understanding of advertising from the perspectives of time and functional interaction. We believe that the implications derived from our research could possibly support decision makers.

We then propose some future tasks that might be of interest. Following the horizontal competition line, one future task is to consider the general time preferences in an oligopolistic market, and/or under uncertainty. We could also combine heterogeneous discounting with market size properties. For instance, an increasing valuation of final state together with an expanding market, or *vice versa*. Whereas in the supply chain environment, it would be worth introducing competition among manufacturers applying the Lanchester dynamics, since this battle specification has been seldom applied in vertical channel (only Rubel & Zaccour, 2007, has done so), and the prevalent competition introduced mainly happens among retailers (with the exceptions of Kim & Staelin, 1999 and Karray & Zaccour, 2007). Besides, as an extension of our third study, instead of a general crisis, we can focus on the product-harm crisis, whose intensity rate depends on the quality of products.

Moreover, the speedily expanding retail chains call for attention paid on another kind of market power distribution, where retailer is the dominant. We also expect future investigation would focus on the interaction among different functional areas, for instance, by including more marketing/operations management tools, or by introducing horizontal competition. It is our sincere hope that this thesis could serve as inspiration for new research.

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