An agreeable collusive equilibrium in differential games with asymmetric players

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Abstract

We study a class of collusive equilibria in differential games with asymmetric players discounting the future at different rates. For such equilibria, at each moment, weights of players can depend on the state of the system. To fix them, we propose using a bargaining procedure according to which players can bargain again at every future moment. By choosing as threat point the feedback noncooperative outcome, the corresponding solution, if it exists, is agreeable. An exhaustible resource game illustrates the results.

Keywords: Differential games, collusive equilibrium, agreeability, asymmetric players, heterogeneous discounting, dynamic bargaining

1. Introduction

Consider a differential game where players share the property of a resource. Players are typically asymmetric, exhibiting different utility functions and discount rates. In the search of Pareto optimal solutions (with constant weights), two problems arise. First, if utilities are non transferable, not all weights guarantee the dynamic consistency of the solution. At future moments, payments for some players can be higher by playing in a noncooperative way. Second, if discount rates do not coincide, what is optimal for the coalition at time t will be no longer optimal at time s, for $s > t$.

As a way to overcome these problems, [7] proposed to introduce nonconstant weights (we refer also to [11]). We think that this is a natural choice in a context with asymmetric players. Note that, if there are N players, for $i \in \{1, \ldots, N\}$, if $t < s < \tau$, players at time t discount payments at time τ via the discount functions at time t $e^{-\rho_i(\tau-t)}$, whereas the same players at time s discount payments at time τ by using the discount functions $e^{-\rho_i(\tau-s)}$. But, if $\rho_i \neq \rho_j$, $e^{-\rho_i(\tau-t)}/e^{-\rho_j(\tau-t)} \neq e^{-\rho_i(\tau-s)}/e^{-\rho_j(\tau-s)}$. In particular, for $s = \tau$, $e^{-\rho_i(s-t)}/e^{-\rho_j(s-t)} \neq 1$. As a consequence, proportional weights assigned to utilities at time τ in a joint maximization problem do not remain constant, but evolve along time.

In this paper we propose a way to construct time-consistent cooperative solutions satisfying two common intertemporal individual rationality concepts: time-consistency (see [9]) and agreeability ([5, 4]). The agreeable dynamic bargaining solution introduced in the paper is, indeed, a *collusive equilibrium* ([10]) as defined in Definition 5.2 in [3] (see also [6]). Memory strategies are considered. In the trigger strategies, the punitive mode of play, the threat, consists of playing a Markovian (feedback) noncooperative Nash solution. Since this is itself an equilibrium, the threat proposed becomes credible.

Concerning the applications, depending on the economic problem, a benevolent social planner (a government or a supranational entity) could impose this solution, compelling the players to collude, because it is beneficial for all players at every moment. In the negotiation of environmental agreements (e.g. GHG emissions), the solution concept proposed shares the nice properties of the classical Nash bargaining solution together with being subgame perfect and time-consistent, guaranteeing in this way the stability of the coalition via the introduction of (credible) threats.

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The paper is organized as follows. Section 2 presents a brief review of the differential game model with nonconstant weights. Section 3 introduces the so-called agreeable dynamic bargaining solution. Section 4 studies the joint management in a nonrenewable model with general isoelastic utilities.

2. The t-cooperative equilibrium with nonconstant weights

Let N be the number of players, and $x \in X \subset \mathbf{R}$ the state variable. For each player $i \in \{1, \ldots, N\}$, let $c_i \in U_i \subset \mathbf{R}$ be the control (decision) variable, $c = (c_1, \ldots, c_N)$ the corresponding vector of decision rules of all players, $u_i(c_i)$ the instantaneous utility function, and ρ_i the discount rate. We will assume that $u_i(c_i)$ is continuously differentiable, increasing and strictly concave. The extension of the theoretical results in the paper to more general utility functions and multidimensional problems of the form $u(x, c)$ with $x \in X \subset \mathbf{R}^n$ and $c_i \in U_i \subset \mathbf{R}^{m_i}$ is straightforward, provided that the corresponding value functions are sufficiently smooth. A similar comment applies to the state equation below. The intertemporal utility function for player i at time t is

$$J_i(x_t; c_1, \ldots, c_N; t) = \int_t^{\infty} e^{-\rho_i(s-t)} u_i(c_i(s))\, ds\ , \quad \text{with} \tag{1}$$

$$\dot{x}(s) = g(x(s), c_1(s), \ldots, c_N(s))\ ,\ x(t) = x_t\ , \tag{2}$$

where $g(x(s), c(s)) = f(x(s)) - \sum_{i=1}^{N} c_i(s)$, with $f(x)$ a continuously differentiable and concave (possibly linear) production function, so (2) becomes

$$\dot{x}(s) = f(x(s)) - \sum_{i=1}^{N} c_i(s)\ ,\ x(t) = x_t\ . \tag{3}$$

In a cooperative setting, we aggregate preferences as

$$J^c(x_t, c, t) = \sum_{i=1}^{N} \lambda_i(x_t, t) J_i(x_t, c, t) = \sum_{i=1}^{N} \lambda_i(x, t) \int_t^{\infty} e^{-\rho_i(s-t)} u_i(c_i(s))\, ds\ . \tag{4}$$

Coefficients $\lambda_i(x_t, t) \geq 0$ represent the weight of agent i at state x_t.

The joint time preferences in (4) are time inconsistent. In order to find time-consistent solutions, the problem can be solved as a noncooperative sequential game with a continuum of "players" (each "player" is each coalition at time t). If $c^*(s) = \phi(x(s), s)$ is a continuously differentiable equilibrium rule for Problem (4) subject to (3), by denoting $x_t = x$, the corresponding value function is

$$V(x, t) \equiv \sum_{i=1}^{N} \lambda_i(x, t) V_i(x, t)\ , \quad \text{where} \quad V_i(x, t) \equiv \int_t^{\infty} e^{-\rho_i(s-t)} u_i(\phi_i(x(s), s))\, ds\ .$$

For $\epsilon > 0$ and $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_N)$, $\bar{c}_i \in U_i \subset \mathbf{R}$, let

$$c_\epsilon(s) = \begin{cases} \bar{c} & \text{if} \quad s \in [t, t+\epsilon)\ , \\ \phi(x(s), s) & \text{if} \quad s \geq t+\epsilon\ . \end{cases}$$

If the t-coalition (the coalition at time t) has the ability to precommit its behavior during the period $[t, t+\epsilon)$, the valuation along the perturbed control path c_ϵ is given by

$$V_{c_\epsilon}(x, t) = \sum_{i=1}^{N} \lambda_i(x, t) \left\{ \int_t^{t+\epsilon} e^{-\rho_i(s-t)} u_i(\bar{c}_i)\, ds + \int_{t+\epsilon}^{\infty} e^{-\rho_i(s-t)} u_i(\phi_i(x(s), s))\, ds \right\}\ .$$

If we expand $V_{c_\epsilon}(x, t)$ in ϵ, we obtain $V_{c_\epsilon}(x, t) = V(x, t) + P(x, \phi, \bar{c}, t)\epsilon + o(\epsilon)$.

Definition 1. *A decision rule $c^*(s) = \phi(x(s), s)$ is a t-cooperative equilibrium (t-CE) with nonconstant weights if function $P(x, \phi, \bar{c}, t)$ attains its maximum for $\bar{c} = \phi(x, t)$.*

In Theorem 1 in [7] it is proved that

$$P(x,\phi,\bar{c},t)=\lim_{\epsilon\to 0^+}\frac{V_{c_\epsilon}(x,t)-V(x,t)}{\epsilon}=\sum_{i=1}^{N}\lambda_i(x,t)[(u_i(\bar{c}_i)-u_i(\phi_i(x,t)))-\nabla_x V_i(x,t)\sum_{j=1}^{N}(\bar{c}_j-\phi_j(x,t))]\,.$$

Hence if, along the equilibrium rule: (i) function $V(x,t)=\sum_{i=1}^N \lambda_i(x,t)V_i(x,t)$, where $V_i(x,t)=\int_t^\infty e^{-\rho_i(s-t)}u_i(\phi_i(x(s),s))ds$ is finite (the integral converges); (ii) functions $\lambda_i(x,t)$ are continuously differentiable in all their arguments; (iii) value functions $V_i(x,t)$ are of class C^1; (iv) the solutions to

$$\phi_j(x,t)=\mathrm{argmax}_{\{c_j\}}\left\{\sum_{i=1}^{N}\lambda_i(x,t)\left(u_i(c_i)+\nabla_x V_i(x,t)\cdot\left(f(x)-\sum_{k=1}^{N}c_k\right)\right)\right\}\,, \tag{5}$$

for $j=1,\dots,N$, are (at least) continuous functions; (v) and there exists a unique absolutely continuous curve $x\colon[0,\infty]\to X$ solution to $\dot{x}(t)=f(x(t))-\sum_{i=1}^N\phi_i(x(t),t)$ with $x(0)=x_0$, then $(c_1,\dots,c_N)=(\phi_1(x,t),\dots,\phi_N(x,t))$ is a t-cooperative equilibrium with nonconstant weights. The equilibrium rule is completely characterized (provided that the corresponding value functions of the different players are continuously differentiable) by condition (5).

Remark 1. *The lack of transversality conditions can result in a multiplicity of solutions, a property that has been well documented in other problems with time inconsistent preferences (hyperbolic/nonconstant discounting) in discrete and continuous time.*

Remark 2. *Although in this paper we develop a theory for general weight functions, since there is no explicit dependence on time in functions u_i and f, it seems natural to restrict the attention to weights of the form $\lambda_i(x)$. In that case we consider just stationary convergent Markovian strategies, i.e. strategies $c_i=\phi_i(x)$ for which there exists $x_\infty<\infty$ and a neighbourhood U of x_∞ such that, for every $x_0\in U$, the solution to (3) along $c=\phi(x)=(\phi_1(x),\dots,\phi_N(x))$ converges to x_∞. For stationary convergent strategies, the integral in (1) converges and the value functions V_i (and hence V) are time-independent.*

3. The agreeable dynamic bargaining solution

The objective in this paper is to propose a method for constructing time-consistent agreeable cooperative solutions in differential games with asymmetric players. Such solutions can be seen as collusive equilibria. We assume that players can renegotiate agreements achieved at time t at every future moment $s>t$. More precisely, we assume that, at time t, players know both the state of the system and their future decision rule $c=\phi(x(s),s)$, $s>t$, as a reaction to their current decisions. Then, as in the classical Nash bargaining solution, they compare what they get cooperating at time t (or during the time interval $[t,t+\epsilon)$, with ϵ arbitrarily small) with what they receive otherwise (the threat point). As threat point $(W_1(x,t),\dots,W_N(x,t))$ we propose to use the noncooperative outcome given by a Markov Perfect Nash Equilibrium (MPNE, see e.g. [1] or [3]). When players at time t try to reach an agreement and to derive their corresponding actions, they take their decisions in the time interval $[t,t+\epsilon)$ as the maximizer of some "distance" between what they obtain in case of agreement and in case of disagreement. We assume that this distance is measured according to the generalized Nash welfare function with strictly positive bargaining powers $\eta_1,\dots,\eta_N$

$$\prod_{i=1}^{N}\left[V_i(x,c_1,\dots,c_N,t)-W_i(x,t)\right]^{\eta_i}\ .$$

For a decision rule $(c_1,\dots,c_N)=(\phi_1(x,t),\dots,\phi_N(x,t))$ and a set of threat value functions $W(x,t)=(W_1(x,t),\dots,W_N(x,t))$ satisfying $V_i(x,\phi_1(x,t),\dots,\phi_N(x,t),t)-W_i(x,t)\geq 0$, for all $i=1,\dots,N$, let $\Pi(x,t)=\prod_{i=1}^N\left[V_i(x,\phi_1(x,t),\dots,\phi_N(x,t),t)-W_i(x,t)\right]^{\eta_i}$. As in the previous section, take

$$c_\epsilon(s)=\begin{cases}\bar{c} & \text{if}\quad s\in[t,t+\epsilon)\,,\\ \phi(x(s),s) & \text{if}\quad s\geq t+\epsilon\,,\end{cases}$$

so that

$$V_{i,c_\epsilon}(x,c_\epsilon,t) = \int_t^{t+\epsilon} e^{-\rho_i(s-t)} u_i(\bar{c}_i)\, ds + \int_{t+\epsilon}^{\infty} e^{-\rho_i(s-t)} u_i(\phi_i(x(s),s))\, ds \ .$$

Then

$$\Pi_{c_\epsilon}(x,t) = \prod_{i=1}^{N} [V_{i,c_\epsilon}(x,c_\epsilon,t) - W_i(x,t)]^{\eta_i} = \Pi(x,t) + \Pi_1(x,\phi,\bar{c},t)\epsilon + o(\epsilon) \ ,$$

with

$$\Pi_1(x,\phi,\bar{c},t) = \lim_{\epsilon\to 0^+} \frac{\Pi_{c_\epsilon}(x,t) - \Pi(x,t)}{\epsilon} \ . \tag{6}$$

Definition 2. *The agreeable dynamic bargaining solution* $\phi^{db}(x,t) = (\phi_1^{db}(x,t),\dots,\phi_N^{db}(x,t))$*, with payments* $V^{db}(x,t) = (V_1^{db}(x,\phi^{db}(x,t),t),\dots,V_N^{db}(x,\phi_N^{db}(x,t),t))$*, where*

$$V_i^{db}(x,\phi^{db}(x,t),t) = \int_t^{\infty} e^{-\rho_i(s-t)} u_i(\phi_i^{db}(x(s),s))\, ds \ , \quad \textit{for} \quad i = 1,\dots,N \ ,$$

is given by

$$\phi^{db}(x,t) = \arg\max_{\{\bar{c}\}} \Pi_1(x,\phi^{db},\bar{c},t) \ ,$$

where the threat point $(W_1(x,t),\dots,W_N(x,t))$ *in case of disagreement at time* t*, with* $x(t) = x$*, is given as follows: For* $s \in [t,\infty)$*, players apply strategies* $\phi^n = (\phi_1^n,\dots,\phi_N^n)$ *such that*

$$W_i(x,t) = J_i(x,\phi^n,t) \geq J_i(x,\phi_{-i}^n,\sigma_i,t) \ , \quad \forall i,\sigma_i \ , \quad i = 1,\dots,N \ ,$$

where $\sigma_i \colon X \times [t,\infty) \to U_i \subset \mathbf{R}$ *is any possible admissible feedback law for player* i*.*

Alternatively, from (6), since for an agreeable dynamic bargaining solution $\Pi_{c_\epsilon}(x,t) - \Pi(x,t) \leq 0$, we can characterize these solutions as the values of c_ϵ such that

$$\lim_{\epsilon\to 0^+} \frac{\Pi_{c_\epsilon}(x,t) - \Pi(x,t)}{\epsilon} \leq 0 \tag{7}$$

The agreeable dynamic bargaining solution can be seen as the natural extension, to a setting with dynamic (continuous) bargaining, of the "classical" approach (as in [2]), by allowing for renegotiations at every future moment. This solution concept has two advantages. First, it is easy to implement, as we show below. In addition, if there are bargaining powers $\eta_1,\dots,\eta_N$ such that an agreeable dynamic bargaining solution exists, the related payoffs provide higher payments to both players at every moment in comparison with those given in case of noncooperation (MPNE). Therefore, the agreeable dynamic bargaining solution is agreeable in the sense of [5],[4]. The price to pay is that we are assuming that, in case of disagreement at time t, players will not cooperate forever. Hence, we are working within the setting of memory strategies, with trigger strategies characterized by moving to a noncooperative equilibrium in case of disagreement. This rather visceral reaction can be realistic in some settings, but becomes unrealistic in others. In addition, it is unclear what to do if there is multiplicity of MPNE, at least if they are not Pareto ranked. In any case, this is a common problem shared also with classical Nash bargaining theory (Nash equilibria are typically nonunique) and noncooperative differential games. For that reason, it is more justified to think of a particular MPNE as a reference point more than as a threat point.

The following proposition characterizes interior agreeable dynamic bargaining solutions.

Proposition 1. *Assume that an agreeable dynamic bargaining solution exists satisfying the constraints* $V_i^{db} - W_i > 0$*, for all* $i = 1,\dots,N$*. Then it is the solution to (5) with weight functions given by*

$$\lambda_i^{db}(x,t) = \frac{\eta_i}{V_i^{db}(x,\phi^{db}(x,t),t) - W_i(x,t)} \ .$$

Proof. From (6) and (7) we have

$$\Pi_1(x,\phi^{db},\bar{c},t) = \left.\frac{d\Pi_{c_\epsilon}}{d\epsilon}\right|_{\epsilon=0^+} = \sum_{i=1}^{N}\bar{\lambda}_i^{db}(x,t)\left.\frac{dV_{i,c_\epsilon}}{d\epsilon}\right|_{\epsilon=0^+} = \lim_{\epsilon\to 0^+}\frac{\sum_{i=1}^{N}\bar{\lambda}_i^{db}(x,t)\left[V_{i,c_\epsilon}(x,t)-V_i^{db}(x,t)\right]}{\epsilon} \le 0$$

where $\bar{\lambda}_i^{db}(x,t) = \frac{\eta_i\cdot\prod_{j=1}^{N}(V_j^{db}(x,\phi^{db}(x,t),t)-W_j(x,t))^{\eta_j}}{V_i^{db}(x,\phi^{db}(x,t),t)-W_i(x,t)}$. But the previous condition is the definition of a t-cooperative equilibrium with nonconstant weights (Definition 1) $\bar{\lambda}_i$, for $i=1,\ldots,N$. By defining

$$\lambda_i^{db}(x,t) = \frac{\bar{\lambda}_i}{\prod_{j=1}^{N}(V_j^{db}(x,\phi^{db}(x,t),t)-W_j(x,t))^{\eta_j}} = \frac{\eta_i}{V_i^{db}(x,\phi^{db}(x,t),t)-W_i(x,t)}\ ,$$

for $i=1,\ldots,N$, it becomes clear that the solution to (5) for weight functions $\bar{\lambda}_i$ and λ_i^{db} coincides. Hence the result follows. $\square$

Remark 3. *From Proposition 1 it is clear that the existence of interior solutions is related to the positivity of the nonconstant weights λ_i^{db}, for $i=1,\ldots,N$.*

4. An exhaustible resource model with general isoelastic utilities

As an example, we solve a model describing the joint management of an exhaustible natural resource if players have general isoelastic utilities with different marginal elasticities. The intertemporal utility function of Player i is given by

$$J_i = \int_t^{\infty} e^{-\rho_i(s-t)}u_i(c_i(s))\,ds \tag{8}$$

with

$$u_i(c_i) = \begin{cases} \dfrac{c_i^{1-\sigma_i}-1}{1-\sigma_i} & \text{if} \quad \sigma_i>0,\ \sigma_i\neq 1, \quad \text{for } i\in\{1,\ldots,J\} \\ \ln c_i & \text{if} \quad \sigma_i=1, \quad\quad\quad \text{for } i\in\{J+1,\ldots,N\} \end{cases}$$

The stock of the resource evolves according to

$$\dot{x}(s) = -\sum_{i=1}^{N}c_i(s)\ , \quad x(t)=x_t\ . \tag{9}$$

4.1. Noncooperative MPNE

The problem in which players do not cooperate and have constant discount rates has been studied in detail in the literature. For the sake of completeness, and to fix notation, we present next noncooperative MPNE in (stationary) linear strategies.

After some standard calculations, linear MPNE strategies $c_i^n = \phi_i^n(x) = A_i^n x$, $A_i^n \ge 0$, for $i=1,\ldots,N$, and the corresponding value functions $W_j(x) = \alpha_j^n x^{1-\sigma_j}+\beta_j^n$, for $j\in\{1,\ldots,J\}$, $\sigma_j\neq 1$, and $W_k(x) = \alpha_k^n\ln x+\beta_k^n$, for $k\in\{J+1,\ldots,N\}$, are given by

$$A_i^n = \begin{cases} \rho_i+(1-\sigma_i)\dfrac{\sum_{l=1}^{N}\rho_l}{1-\sum_{j=1}^{J}(1-\sigma_j)} & \text{for} \quad i\in\{1,\ldots,J\} \\ \rho_i & \text{for} \quad i\in\{J+1,\ldots,N\} \end{cases} \tag{10}$$

$$\alpha_i^n = \begin{cases} \dfrac{1}{1-\sigma_i}\left(\rho_i+(1-\sigma_i)\dfrac{\sum_{l=1}^{N}\rho_l}{1-\sum_{j=1}^{J}(1-\sigma_j)}\right)^{-\sigma_i} & \text{for} \quad i\in\{1,\ldots,J\} \\ 1/\rho_i & \text{for} \quad i\in\{J+1,\ldots,N\} \end{cases} \tag{11}$$

$$\beta_i^n = \begin{cases} -\left(\rho_i(1-\sigma_i)\right)^{-1} & \text{for} \quad i\in\{1,\ldots,J\} \\ \ln\rho_i/\rho_i-\displaystyle\sum_{l=1}^{N}\rho_l/\rho_i^2 & \text{for} \quad i\in\{J+1,\ldots,N\} \end{cases} \tag{12}$$

Note that, if

$$\rho_i > \frac{(1-\sigma_i)\sum_{l=1}^N \rho_l}{\sum_{j=1}^J (1-\sigma_j) - 1}\ , \text{ with } \sum_{j=1}^J (1-\sigma_j) - 1 \neq 0\ , \text{ for } i = 1, \dots, J\ ,$$

the extraction rates A_i^n of all agents are strictly positive, the value functions are well defined (integrals converge), and $\lim_{t\to\infty} x(t) = x_\infty = 0$.

4.2. An agreeable time-consistent cooperative solution

We divide the problem into two parts. First we analyze the existence of linear t-cooperative equilibria with nonconstant weighs. Later on, we show that the agreeable dynamic bargaining solution proposed in Definition 2 belongs to this set of cooperative solutions, and we compute it.

The following lemma characterizes the functional form of weight functions necessary for the existence of linear decision rules in the computation of the t-cooperative equilibria with nonconstant weights.

Lemma 1. *In Problem (8-9), if linear t-cooperative equilibria with nonconstant weights exist, then weight functions are of the form $\lambda_i(x) = \nu_i x^{\sigma_i} h(x)$ with $h(x)$ an arbitrary continuously differentiable function, i.e. $\lambda_i(x)/\lambda_l(x) = (A_i^{\sigma_i}/A_l^{\sigma_l})x^{\sigma_i - \sigma_l}$, for $i, l = 1, \dots, N$.*

Proof. From (5) we solve $\max_{\{c_1,\dots,c_N\}}\{\sum_{j=1}^J \lambda_j(x)\frac{c_j^{1-\sigma_j}-1}{1-\sigma_j} + \sum_{k=J+1}^N \lambda_k(x)\ln c_k - (\sum_{i=1}^N \lambda_i(x)V_i'(x))\cdot(\sum_{l=1}^N c_l)\}$. By applying the first order condition we obtain, for all $i = 1, \dots, N$,

$$c_i(x) = \phi_i(x) = \left(\frac{\lambda_i(x)}{\sum_{l=1}^N \lambda_l(x)V_l'(x)}\right)^{1/\sigma_i}\ . \tag{13}$$

If linear strategies exist, $\phi_i(x) = A_i x$, $A_i \geq 0$, for $i = 1, \dots, N$, then, from (13), we obtain $\lambda_i(x) = (\sum_{l=1}^N \lambda_l(x)V_l'(x))(A_i x)^{\sigma_i}$. As a result, weights must be of the form $\lambda_i(x) = \nu_i x_i^{\sigma} h(x)$, with $h(x) = \sum_{l=1}^N \lambda_l(x)V_l'(x)$ so $\lambda_i(x)/\lambda_l(x) = (A_i^{\sigma_i}/A_l^{\sigma_l})x^{\sigma_i-\sigma_l}$. □

A natural question arises in this context. Do linear strategies always exist for arbitrary weight functions of the form $\lambda_i(x) = \nu_i x^{\sigma_i} h(x)$? For this family of weight functions, from (13) we have $c_i = \nu^{1/\sigma_i}(\sum_{j=1}^N \nu_j x^{\sigma_j} V_j'(x))^{-1/\sigma_i}$. We need to prove that there are value functions $V_1(x), \dots, V_N(x)$ such that the quantity $\sum_{i=1}^N \nu_i x^{\sigma_i} V_i'(x)$ is constant. Since we are interested in the existence of linear strategies, we take, for $j \in \{1, \dots, J\}$, $V_j(x) = \alpha_j x^{1-\sigma_j} + \beta_j$, and for $k \in \{J+1, \dots, N\}$, $V_k(x) = \alpha_k \ln x + \beta_k$. Solutions obtained from other choices of functions $V_i(x)$, $i = 1, \dots, N$, will be nonlinear. Then $V_j'(x) = (1-\sigma_j)\alpha_j x^{-\sigma_j}$, $V_k'(x) = \alpha_k x^{-1}$ and $\sum_{i=1}^N \nu_i x^{\sigma_i} V_i'(x) = \sum_{j=1}^J \nu_j(1-\sigma_j)\alpha_j + \sum_{k=J+1}^N \nu_k\alpha_k$ so

$$c_i = \left(\frac{\nu_i}{\sum_{j=1}^J \nu_j(1-\sigma_j)\alpha_j + \sum_{k=J+1}^N \nu_k\alpha_k}\right)^{1/\sigma_i} x\ . \tag{14}$$

From (14) it becomes clear that, for the existence of linear strategies, it is necessary that coefficients α_i, $i = 1, \dots, N$, exist such that $\sum_{j=1}^J \nu_j(1-\sigma_j)\alpha_j + \sum_{k=J+1}^N \nu_k\alpha_k > 0$.

Next, for the above mentioned family of weight functions, the corresponding t-cooperative decision rules are presented.

Corollary 1. *In Problem (8)-(9), for $\lambda_i(x) = \nu_i x^{\sigma_i} h(x)$, $i = 1, \dots, N$, if t-cooperative equilibria with nonconstant weights exist, they are of the form $c_i(x) = A_i x$, for $A_i > 0$, with corresponding value functions $V_i(x) = \alpha_i x^{1-\sigma_i} + \beta_i$ for $i = 1, \dots, J$, and $V_i(x) = \alpha_i \ln x + \beta_i$ if $i = J+1, \dots, N$, where coefficients A_i, α_i and β_i are the solution to the equation system*

$$A_i = \left(\frac{\nu_i}{\sum_{j=1}^J \nu_j(1-\sigma_j)\alpha_j + \sum_{k=J+1}^N \nu_k\alpha_k}\right)^{1/\sigma_i}$$

$$\alpha_i(x) = \begin{cases} \dfrac{1}{1-\sigma_i}\left(\dfrac{(A_i)^{1-\sigma_i}}{\rho_i + (1-\sigma_i)\sum_{l=1}^{N} A_l}\right) & \textit{for} \quad i \in \{1,\ldots,J\} \\ 1/\rho_i & \textit{for} \quad i \in \{J+1,\ldots,N\} \end{cases}$$

$$\beta_i = \begin{cases} -\left((1-\sigma_i)\rho_i\right)^{-1} & \textit{for} \quad i \in \{1,\ldots,J\} \\ (\ln A_i)/\rho_i - \sum_{l=1}^{N} A_l/\rho_i^2 & \textit{for} \quad i \in \{J+1,\ldots,N\} \end{cases}$$

Proof. The solution to (9) is $x(s) = x_t \exp[-\sum_{i=1}^{N} A_i(s-t)]$. By proceeding as in the derivation of the MPNE,

$$V_i(x) = \begin{cases} \dfrac{1}{1-\sigma_i}\left(\dfrac{(A_j)^{1-\sigma_i}}{\rho_i + (1-\sigma_i)\sum_{l=1}^{N} A_l} x^{1-\sigma_i} - \dfrac{1}{\rho_i}\right) & \text{for} \quad i \in \{1,\ldots,J\} \\ \dfrac{1}{\rho_i}\ln x + \dfrac{1}{\rho_i}\ln A_i - \dfrac{\sum_{l=1}^{N} A_l}{\rho_i^2} & \text{for} \quad i \in \{J+1,\ldots,N\} \end{cases} . \tag{15}$$

The result follows from (14) and (15). □

Once we have computed both the MPNE and the time-consistent equilibria with nonconstant weights, we can use Proposition 1 for the calculation of the agreeable dynamic bargaining solution. Since just relative weights are relevant, we can take $h(x) = 1/x$. From equations (11)-(12) we have:

Proposition 2. *If agreeable dynamic bargaining solutions exist, they are given by Corollary 1 with weight functions* $\lambda_i^{db}(x)/\lambda_j^{db}(x) = (\nu_i^{db}/\nu_j^{db})x^{\sigma_i-\sigma_j}$ *(e.g.,* $\lambda_i^{db}(x) = \nu_i^{db}x^{\sigma_i-1}$ *for* $h(x) = 1/x$*) and coefficients*

$$\nu_i^{db} = \begin{cases} \eta_i/(\alpha_i^{db} - \alpha_i^{n}) & \textit{for} \quad i \in \{1,\ldots,J\} \\ \eta_i/(\beta_i^{db} - \beta_i^{n}) & \textit{for} \quad i \in \{J+1,\ldots,N\} \end{cases} \tag{16}$$

with α_i^n *and* β_i^n *given by (11)-(12), and* α_i^{db} *and* β_i^{db} *the values of* α_i *and* β_i *in Corollary 1 for* ν_i *given by (16). Agreeable dynamic bargaining solutions exist if* $\sum_{j=1}^{J}\nu_j^{db}(1-\sigma_j)\alpha_j^{db} + \sum_{k=J+1}^{N}\nu_k^{db}\alpha_k^{db} > 0$ *(where* $\sigma_j \neq 1$ *and* $\sigma_k = 1$*), so that* $A_i^{db} > 0$*, and* $\alpha_i^{db} > c\alpha_i^n$ *for* $i \in \{1,\ldots,J\}$*,* $\beta_i^{db} > \beta_i^n$ *for* $i \in \{J+1,\ldots,N\}$*.*

Proof. It follows from Proposition 1. □

The existence of agreeable dynamic bargaining solutions is not guaranteed for general utility functions, and conditions for $A_i^{db} > 0$, $\alpha_i^{db} > \alpha_i^n$ for $i \in \{1,\ldots,J\}$ and $\beta_i^{db} > \beta_i^n$ for $i \in \{J+1,\ldots,N\}$ are too cumbersome to be expressed analytically. In the case of logarithmic utility functions ($J = 0$), the corresponding weights are given by

$$\lambda_i^{db} = \nu_i^{db} = \frac{\eta_i}{\beta_i^{db} - \beta_i^n} . \tag{17}$$

4.3. Numerical illustrations

Although the equation systems involved in the calculation of the agreeable dynamic bargaining solutions are highly nonlinear, they can be easily solved numerically, so it is possible to perform sensibility analysis with respect to the different parameters of the model.

In Table 1, we take $\rho_1 = 0.07$, $\rho_2 = 0.13$, $\sigma_1 = \sigma_2 = \sigma \in [0.6, 0.15]$, $(\eta_1, \eta_2) = 1$, and $(\eta_1, \eta_1) = (1, \rho_1/\rho_2)$. An increase in the value of σ implies a reduction in the extraction levels for both players, being this reduction more severe for the player with the lower discount rate. As for the effects of the power indices, if η_2 decreases, the extraction of player 1 increases whereas it decreases for player 2. The intuition behind this result seems clear. A higher relative weight of player 1 with respect to player 2 (i.e. a lower value of η_2/η_1) implies assigning a higher relative value to the intertemporal utility of player 1, which is related to a higher extraction level (utilities increase in the extraction levels). Similar qualitative results are found for other values of the parameters.

Table 1: Equal marginal elasticities

			$\eta_1 = \eta_2 = 1$		$\eta_1 = 1,\ \eta_2 = \rho_1/\rho_2$	
σ	A_1^n	A_2^n	A_1^{db}	A_2^{db}	A_1^{db}	A_2^{db}
0.6	0.470000	0.530000	0.063417	0.107405	0.073037	0.089541
0.9	0.095000	0.155000	0.032804	0.084896	0.036033	0.078936
1.2	0.041429	0.101429	0.018062	0.072843	0.019515	0.070083
1.5	0.020000	0.080000	0.009427	0.065627	0.010102	0.064253

Table 2 uses the same values for ρ_1, ρ_2, η_1 and η_2 but now $\sigma_1 \in [0.9, 1.5]$ and $u_2(c_2) = \ln c_2$ (i.e. $\sigma_2 = 1$). we take $\rho_1 = 0.07$, $\rho_2 = 0.13$, $\sigma_1 = \sigma_2 = \sigma \in [0.6, 0.15]$, $(\eta_1, \eta_2) = 1$, and $(\eta_1, \eta_1) = (1, \rho_1/\rho_2)$.

Table 2: Different marginal elasticities

			$\eta_1 = \eta_2 = 1$		$\eta_1 = 1,\ \eta_2 = \rho_1/\rho_2$	
σ	A_1^n	A_2^n	A_1^{db}	A_2^{db}	A_1^{db}	A_2^{db}
0.9	0.092222	0.13	0.032729	0.077486	0.034897	0.073909
1.2	0.036667	0.13	0.016166	0.087367	0.018590	0.081626
1.5	0.003333	0.13	0.001673	0.112894	0.002251	0.109376

Finally, we solve the problem studied in [8]. For $N = 2$, $\rho_1 = 0.01$, $\rho_2 = 0.2$, $u_1(c_1) = \ln(c_1)^{0.01}$ and $u_2(c_2) = \ln(c_2)^{10}$, the t-cooperative equilibrium rule with equal weights, $\lambda_1 = \lambda_2$, was shown to be group inefficient, in the sense that joint payments are higher in the noncooperative MPNE than in the t-CE for every value of the resource stock $x > 0$. The agreeable dynamic bargaining solution for $\eta_1 = \eta_2 = 1$ solves this problem, as Table 3 illustrates. Similar results are found for $\eta_1 = 1$ and $\eta_2 = \rho_1/\rho_2$. In that case, as in the previous examples, a reduction in η_2 implies that A_1^{db} and V_1^{db} increase, whereas A_2^{db} and V_2^{db} decrease.

Table 3: $\mu_1 = 0.01$ and $\mu_2 = 10$, $\eta_1 = \eta_2 = 1$

ρ_1	ρ_2	A_1^n	A_2^n	A_1^{tc}	A_2^{tc}	A_1^{db}	A_2^{db}	$V_1^{tc} - V_1^n$	$V_2^{tc} - V_2^n$	$V_1^{db} - V_1^n$	$V_2^{db} - V_2^n$
0.01	0.07	0.01	0.07	0.000070	0.069513	0.001853	0.057032	-3.927100	20.263000	0.425600	13.824000
0.01	0.13	0.01	0.13	0.000128	0.128332	0.001246	0.113797	-3.201700	5.835000	0.413400	4.527000
0.01	0.2	0.01	0.2	0.000196	0.196078	0.000932	0.181370	-2.559200	2.441000	0.396400	2.036000

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