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Abstract: We introduce new notions of superadditivity and convexity for games with coalitional externalities. We show parallel results to the classic ones for transferable utility games without externalities. In superadditive games the grand coalition is the most efficient organization of agents. The convexity of a game is equivalent to having non decreasing contributions to larger embedded coalitions. We also see that convex games can only have negative externalities.

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1 Introduction

Cooperative game theory provides tools to study situations in which the coalitions are the main actors. In a cooperative game the details of the underlying interaction among players are omitted to build a robust model. The focus is on what coalitions will emerge and how to share the benefits of the cooperation. Even if these games are as old as game theory itself[1] their applications to economics have not been as successful as the ones of their non-cooperative counterpart (Maskin, 2016). The fact that, traditionally externalities have been overlooked in the literature may be a reason. Indeed, externalities are present in most economic examples where coalitions are the fundamental elements. For instance, when firms merge in a cartel or after a takeover bid, the expected profit will depend on the potential merging carried out by the rest of firms in the market. Jelnov and Tauman (2009) use games with externalities to study the coalition formation in a Cournot market where there is a patent holder.

Thrall and Lucas (1963) introduced games in partition function form to describe situations in which coalitions generate externalities on one another. In this model, the main ingredient are not just coalitions but embedded coalitions, that consist of a coalition and a partition of the rest of agents. In this way, a coalition can have different values depending on what partition it is embedded in. More recently, many important contributions have been published, most of them focusing on the problem of how to share the benefits of the cooperation. For instance, Macho-Stadler et al. (2007), de Clippel and Serrano (2008), McQuillin (2009), and Dutta et al. (2010) address the issue of how to extend the Shapley value and Kóczy (2007) and Bloch and van den Nouweland (2014) propose generalizations of the core to games with externalities. Fewer papers have explored the properties of the game itself. Hafalir (2007) notes that extending the classic properties of superadditivity and convexity is not a trivial task. He shows that superadditivity, as defined by Maskin (2003) is not a sufficient condition for the efficiency of the grand coalition in situations with negative externalities. Abe (2016) proposes alternative definitions of superadditivity that do the work when externalities are either positive or negative. Hafalir (2007)

[1] Their origin dates back to Von Neumann and Morgenstern (1944).

also introduced a notion of convexity that guarantees that the grand coalition is the most efficient configuration. With a different purpose Abe (2019) introduced another notion of convexity, logically independent to the previous one. A different branch of the literature follows a non-cooperative approach to study situations with coalitional externalities. For instance, Ray and Vohra (1999) use an extensive form bargaining game to find out the coalition structures that are likely to arise.

Here, we rely on a partial order among embedded coalitions implicitly defined by de Clippel and Serrano (2008). Alonso-Meijide et al. (2017) analyze the set of embedded coalitions endowed with this partial order and show that it has a lattice structure. Then, it is very natural to interpret the supremum and the infimum of two embedded coalitions as their union and intersection, respectively. The supremum is obtained taking the union of the coalitions and the intersection of the partitions, more precisely their infimum in the lattice of partitions of a finite set. That is, the two coalitions whose worth is being evaluated are merged while the rest of agents form the partition obtained by keeping the divisions of the two original partitions. The infimum works just the other way around, intersection of coalitions and union of partitions, which results in only keeping the divisions in which the two partitions agree. These operations allow us to generalize the classic definitions of superadditivity and convexity to games with externalities in a natural way.

To start with, we see that our properties imply the superadditivity proposed by Maskin (2003) and the convexity studied by Hafalir (2007). Our main result is the characterization of convexity through a condition that requires the contributions to embedded coalitions to be non decreasing with respect to size. To define what is a contribution to an embedded coalition in a game with externalities we use the lattice structure again. Alonso-Meijide et al. (2019) introduce these contributions to build a super family of Shapley values that contains the ones proposed in the previous references. Some intermediate results that we use are interesting on their own. For instance, we show that a convex game can only have negative externalities. Which means that coalitions' worth decrease when the partition of the complement becomes coarser. Finally, we also obtain some interesting implications of our property with respect to certain core notions.

The rest of the paper is organized as follows. Section 2 presents some discrete mathematical terms that we will employ. Then, the partial order among embedded coalitions in which we ground our results is introduced. Finally, we revise some of the results of Alonso-Meijide et al. (2017) to adapt them to our framework. In Section 3 we introduce our notions of essential, superadditive, and convex game with externalities and discuss their implications. Next, we present some interesting lemmata followed by our main result. Section 4 concludes with some additional results on the cores of convex games with externalities. The proof of the main result is relegated to the Appendix.

2 A lattice of embedded coalitions

Let $(L, \leq)$ be a partially ordered set, with L being a finite set and $x, y \in L$.[2] The *supremum*, denoted by $x \vee y$, is the unique element of L such that $x, y \leq x \vee y$ and if $z \in L$ is such that $z \geq x, y$, then $z \geq x \vee y$. The *infimum*, denoted by $x \wedge y$, is the unique element of L such that $x \wedge y \leq x, y$ and if $z \in L$ is such that $z \leq x, y$, then $z \leq x \wedge y$.[3] A *finite lattice* is a finite partially ordered set in which every pair of elements have supremum and infimum. From now on, we assume that $(L, \leq)$ is a finite lattice. The *top*, denoted by $\hat{1}$, is the element of L such that $x \leq \hat{1}$ for every $x \in L$. Similarly, the *bottom*, denoted by $\hat{0}$, is the element of L such that $\hat{0} \leq x$ for every $x \in L$.

A key notion for our paper is the covering relation. We say that x is *covered* by y or y *covers* x if $x < y$ and there is no $z \in L$ such that $x < z < y$. A *chain* $\mathcal{C}$ (between x_0 and x_k) is an ordered subset of L, $\mathcal{C} = \{x_0, x_1, \ldots, x_k\}$ such that x_{l+1} covers x_l, for every $l = 0, \ldots, k-1$. If $x \leq y$, we denote by $[x, y]_L$ the set of elements $z \in L$ such that $x \leq z \leq y$. If no confusion arises, we may just write $[x, y]$. Notice that $[x, y]$ is also a lattice. A lattice satisfies the *Jordan-Dedekind condition* if all chains between a pair of elements have the same length. The *height* of $x \in L$ is the length of the chains between the bottom element and x. The *height of the lattice* is the length of every chain that joins the bottom and the top elements. We say that $(L, \leq)$ is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,

[2]We write $x = y$ if $x \leq y$ and $y \leq x$. Also, $x < y$ means that $x \leq y$ but $x \neq y$.

[3]The definition of supremum and infimum is extended to any finite subset of elements of L in the usual way.

for every $x, y, z \in L$. $(L, \leq)$ is *lower semimodular* if whenever $x \vee y$ covers both x and y, then both x and y cover $x \wedge y$, for every $x, y \in L$. $(L, \leq)$ is *semimodular* or *upper semimodular* if whenever both x and y cover $x \wedge y$, then $x \vee y$ covers both x and y, for every $x, y \in L$.

The classic notion of convexity (Shapley, 1971) is the supermodularity of the characteristic function, which is a real function on the Boolean lattice of subsets. In general, a real function on $(L, \leq)$, f, is said to be supermodular (submodular) if for every $x, y \in L$, $f(x) + f(y) \leq (\geq) f(x \wedge y) + f(x \vee y)$.

Let N be a finite set, $n = |N|$, $S \subseteq N$, and $i \in N$. We denote $S \cup \{i\}$ by $S \cup i$ and $S \setminus \{i\}$ by $S \setminus i$. The family of partitions of N is denoted by $\Pi(N)$. Let, $P \in \Pi(N)$. We denote by $|P|$ the number of non-empty elements of P, called blocks. The partition P_{-S} of $N \setminus S$ is given by $\{T \setminus S : T \in P\}$. The partition of singletons of S, $\{\{i\} : i \in S\}$, is denoted by $\lfloor S \rfloor$ and the partition of S in one block, $\{S\}$, is denoted by $\lceil S \rceil$. If $P \in \Pi(N \setminus i)$, we also denote $\{\{i\}\} \cup P$ by $\{i\} \cup P$. A well-known partial order on $\Pi(N)$ is the following:

$$P \preceq Q \text{ if and only if for every } S \in P \text{ there is some } T \in Q \text{ such that } S \subseteq T.$$

It is known that $(\Pi(N), \preceq)$ is a semimodular lattice. The height of an element, $P \in \Pi(N)$ is given by $r(P) = n - |P|$. If $P, Q \in \Pi(N)$, we denote by $P \bigwedge Q$ the infimum of P and Q; the supremum of P and Q is denoted by $P \bigvee Q$.

An *embedded coalition* of N is a pair $(S; P)$ with $S \subseteq N$ and $P \in \Pi(N \setminus S)$, i.e., $\{S\} \cup P \in \Pi(N)$. In particular, $(\emptyset; P)$ with $P \in \Pi(N)$ is also an (empty) embedded coalition. If all agents form the grand coalition we write $(N; \emptyset)$. That is, we consider that $\emptyset$ is the only partition in $\Pi(\emptyset)$. For simplicity we denote by $(S; N \setminus S)$ the embedded coalition $(S; \lceil N \setminus S \rceil)$, for every $S \subseteq N$. The family of all embedded coalitions of N is denoted by $\mathcal{EC}^N$.

Alonso-Meijide et al. (2017) studied the partial order outlined in de Clippel and Serrano (2008) over the set $(\mathcal{EC}^N \setminus \{(\emptyset; P) : P \in \Pi(N)\}) \cup \{\perp\}$, being $\perp$ a fictitious bottom element. Here we consider this partial order over the whole set $\mathcal{EC}^N$. It is convenient to extend some of the results in Alonso-Meijide et al. (2017) to this framework. Next, we introduce the partial order formally.

Definition 2.1. *Let* $(S;P),(T;Q)\in\mathcal{EC}^N$*. We define the* inclusion *among embedded coalitions as follows:*

$$(S;P)\sqsubseteq(T;Q) \text{ if and only if } S\subseteq T \text{ and } Q\preceq P_{-T}.^4 \tag{1}$$

Remark 2.1. *Let* $(S;P),(T;Q)\in\mathcal{EC}^N$*. Definition 2.1 can be rephrased as*

$$(S;P)\sqsubseteq(T;Q) \text{ if and only if } S\subseteq T \text{ and } Q\cup\lfloor T\setminus S\rfloor\preceq P. \tag{2}$$

Sometimes it will be convenient to use this formulation.

In the remainder of this section we describe some properties of the algebraic structure $\left(\mathcal{EC}^N,\sqsubseteq\right)$.

Proposition 2.1. *Let* $(S;P)$*,* $(T;Q)\in\mathcal{EC}^N$*. Then,*

1. $(S;P)\vee(T;Q)=(S\cup T;P_{-T}\bigwedge Q_{-S})$.
2. $(S;P)\wedge(T;Q)=(S\cap T;M)$*, with* $M=(P\cup\lfloor S\setminus T\rfloor)\bigvee(Q\cup\lfloor T\setminus S\rfloor)$.

Proof.

1. The first item can be proven in the same way as Item 1 of Proposition 1 in Alonso-Meijide et al. (2017).
2. Take $(S\cap T;M)$ with

$$M=(P\cup\lfloor S\setminus T\rfloor)\bigvee(Q\cup\lfloor T\setminus S\rfloor).$$

Then, $(S\cap T;M)\sqsubseteq(S;P)$ and $(S\cap T;M)\sqsubseteq(T;Q)$. Let $(R;M')\in\mathcal{EC}^N$ such that $(R;M')\sqsubseteq(S;P)$ and $(R;M')\sqsubseteq(T;Q)$ then, it is easy to see that $(R;M')\sqsubseteq(S\cap T;M)$.

$\square$

From Proposition 2.1 we conclude that $\left(\mathcal{EC}^N,\sqsubseteq\right)$ is a lattice. The bottom element of this structure is $(\emptyset;N)$ and the top is $(N;\emptyset)$. The next example illustrates that the lattice is not distributive.

[4] As usual, $(S;P)\sqsubset(T;Q)$ means that $(S;P)\sqsubseteq(T;Q)$ and $(S;P)\neq(T;Q)$.

Example 2.1. *Let* $N = \{1,2,3,4,5,6,7\}$*,* $(S;P) = (\{2,3\};\{\{1,4\},\{5,6\},\{7\}\})$*,* $(T;Q) = (\{1,2\};\{\{3,5\},\{4,6,7\}\})$*, and* $(L;M) = (\{1,3\};\{\{2\},\{4,5\},\{6,7\}\})$*. Then*

$$\begin{aligned}
&(T;Q) \wedge (L;M) = (\{1\};\{\{2\},\{3,4,5,6,7\}\}),\\
&(S;P) \vee ((T;Q) \wedge (L;M)) = (\{1,2,3\};\{\{4\},\{5,6\},\{7\}\}),\\
&(S;P) \vee (T;Q) = (\{1,2,3\};\lfloor N \setminus \{1,2,3\}\rfloor),\\
&(S;P) \vee (L;M) = (\{1,2,3\};\lfloor N \setminus \{1,2,3\}\rfloor), \quad \textit{and}\\
&((S;P) \vee (T;Q)) \wedge ((S;P) \vee (L;M)) = (\{1,2,3\};\lfloor N \setminus \{1,2,3\}\rfloor).
\end{aligned}$$

Then, $(S;P) \vee ((T;Q) \wedge (L;M)) \neq ((S;P) \vee (T;Q)) \wedge ((S;P) \vee (L;M))$*.*

Besides, $(S;P) \wedge ((T;Q) \vee (L;M)) \neq ((S;P) \wedge (T;Q)) \vee ((S;P) \wedge (L;M))$ *as we see next:*

$$\begin{aligned}
&(T;Q) \vee (L;M) = (\{1,2,3\};\lfloor 4,5\rfloor \cup \{6,7\}\}),\\
&(S;P) \wedge ((T;Q) \vee (L;M)) = (\{2,3\};\{\{1,4\},\{5,6,7\}\}),\\
&(S;P) \wedge (T;Q) = (\{2\};\lceil N \setminus \{2\}\rceil),\\
&(S;P) \wedge (L;M) = (\{3\};\lceil N \setminus \{2,3\}\rceil \cup \{2\}), \quad \textit{and}\\
&((S;P) \wedge (T;Q)) \vee ((S;P) \wedge (L;M)) = (\{2,3\};\lceil N \setminus \{2,3\}\rceil).
\end{aligned}$$

The next result implies that $\left(\mathcal{EC}^N, \sqsubseteq\right)$ is a graded lattice.

Proposition 2.2. *The lattice* $\left(\mathcal{EC}^N, \sqsubseteq\right)$ *satisfies the Jordan-Dedekind condition. Moreover, the height of any* $(S;P) \in \mathcal{EC}^N$ *is given by* $h(S;P) = |P| + 2|S| - 1$*.*

Proof. The result follows immediately if $|N| \leq 2$. Let us assume that $|N| \geq 3$ and $(S;P) \in \mathcal{EC}^N$. First, we prove that all chains joining $(\emptyset;N)$, the bottom element, and (S,P) have length $|P| + 2|S| - 1$. We proceed by induction on k, the length of such a chain.

If $k = 0$, then $(S;P) = (\emptyset;N)$ and $h(\emptyset;N) = 0 = |P| + 2|S| - 1$. Let us take $k = 1$. That is, we consider a chain of length $k = 1$ joining $(S;P)$ and $(\emptyset;N)$, this implies that $(S;P)$ covers $(\emptyset;N)$. Then, $(S;P) = (\emptyset;\{T, N \setminus T\})$ for some $T \notin \{\emptyset, N\}$, there is only one chain from the bottom element to $(S;P)$, and $h(S;P) = h(\emptyset;\{T, N \setminus T\}) = 1 + h(\emptyset;N) = |P| + 2|S| - 1$.

Suppose that the result holds for every $(S;P)$ such that there is a chain of length $k > 0$ from the bottom element to $(S;P)$. Let $(S;P) \in \mathcal{EC}^N$ such that there is a chain of length k joining $(S;P)$ and $(\emptyset;N)$. We distinguish two cases.

First, if $|P| \leq 1$, we have $|S| > 0$ because $k > 0$. Then, $(S;P)$ only covers embedded coalitions of type $(S \setminus i; P \cup \{i\})$, for every $i \in S$ and there is a chain of length $k-1$ from the bottom element to $(S \setminus i; P \cup \{i\})$. By the induction hypothesis, all chains from the bottom element to $(S \setminus i; P \cup \{i\})$ have length $k-1 = h(S \setminus i; P \cup \{i\})$. Since $(S;P)$ covers $(S \setminus i; P \cup \{i\})$,

$$k = h(S;P) = 1 + h(S \setminus i; P \cup \{i\}) = 1 + 1 + |P| + 2|S \setminus i| - 1 = |P| + 2|S| - 1.$$

Second, let us assume that $|P| > 1$ and take $P = \{P_1, \ldots, P_m\}$, with $m \geq 2$. Then, we can have $|S| = 0$ or $|S| > 0$. If $|S| = 0$, then $(S;P)$ only covers embedded coalitions of type $\left(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil\right)$ for every $j, l \in \{1, \ldots, m\}$ with $j \neq l$ and there is a chain of length $k-1$ from the bottom element to $(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$. By induction, all chains from the bottom element to $(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$ have length $k-1 = h(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$. Since $(S;P)$ covers $(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$,

$$k = h(S;P) = 1 + h(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil) = 1 + (|P| - 1) + 2|S| - 1 = |P| + 2|S| - 1.$$

Finally, if $|S| > 0$, $(S;P)$ covers embedded coalitions of two types, $(S \setminus i; P \cup \{i\})$, for every $i \in S$ and $(S; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$ for every $j, l \in \{1, \ldots, m\}$ with $j \neq l$. Using the induction hypothesis as before for each of the types of embedded coalitions we obtain that $k = h(S;P) = |P| + 2|S| - 1$.

To conclude, take $(S;P) \sqsubseteq (T;Q)$. Notice that any chain joining $(T;Q)$ and $(S;P)$ can be completed with a chain that joins $(S;P)$ with the bottom element. Since all chains that start at the bottom element have the same length, the chains from $(T;Q)$ to $(S;P)$ also have a common length. $\square$

Then, the height of the lattice is $h(N;\emptyset) = 2n - 1$. Notice that the height of every embedded coalition $(S;P) \in \mathcal{EC}^N$ can be described by means of the height of S in the Boolean lattice, $|S|$, and the height of $P \cup \lfloor S \rfloor$ in the partition lattice, $r(P \cup \lfloor S \rfloor)$ as follows:

$$h(S;P) = n - 1 - r(P \cup \lfloor S \rfloor) + |S|. \tag{3}$$

Since $(\Pi(N), \preceq)$ is a graded and semimodular lattice, the height is a submodular function. This fact and Equation (3) are used to prove the following result.

Proposition 2.3. *Let* $(S;P),(T;Q) \in \mathcal{EC}^N$. *Then,*

$$h((S;P) \vee (T;Q)) - h(T;Q) \geq h(S;P) - h((S;P) \wedge (T;Q)). \tag{4}$$

Proof. Let $(S;P),(T;Q) \in \mathcal{EC}^N$. First, recall that $(S;P) \vee (T;Q) = (S \cup T; P_{-T} \bigwedge Q_{-S})$ and $(S;P) \wedge (T;Q) = (S \cap T; (P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor))$. Using Equation (3), Inequality (4) is equivalent to

$$\begin{aligned} & r(Q \cup \lfloor T \rfloor) - r\left(\left(P_{-T} \bigwedge Q_{-S}\right) \cup \lfloor S \cup T \rfloor\right) \\ & \geq r\left((P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor) \cup \lfloor S \cap T \rfloor\right) - r(P \cup \lfloor S \rfloor). \end{aligned}$$

Taking $P \cup \lfloor S \rfloor$, $Q \cup \lfloor T \rfloor \in \Pi(N)$, it happens that

- $(P \cup \lfloor S \rfloor) \bigwedge (Q \cup \lfloor T \rfloor) = (P_{-T} \bigwedge Q_{-S}) \cup \lfloor S \cup T \rfloor$, and
- $(P \cup \lfloor S \rfloor) \bigvee (Q \cup \lfloor T \rfloor) = (P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor) \cup \lfloor S \cap T \rfloor$.

Using the fact that the height of an element on the the partition lattice is a submodular function and taking $P \cup \lfloor S \rfloor$, $Q \cup \lfloor T \rfloor \in \Pi(N)$, we obtain

$$\begin{aligned} & r(P \cup \lfloor S \rfloor) + r(Q \cup \lfloor T \rfloor) \\ & \geq r\left((P_{-T} \bigwedge Q_{-S}) \cup \lfloor S \cup T \rfloor\right) + r\left((P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor) \cup \lfloor S \cap T \rfloor\right) \end{aligned}$$

and the result follows. $\square$

As a consequence of Proposition 2.3 $\left(\mathcal{EC}^N, \sqsubseteq\right)$ is a lower semimodular lattice.

3 Superadditiviy and convexity

In this section, we extend some of the most important properties of classic games to situations with coalitional externalities. Let N be a finite set. A *game (with externalities)* with player set N is defined by a partition function $v : \mathcal{EC}^N \longrightarrow \mathbb{R}$

such that $v(\emptyset; P) = 0$, for every $P \in \Pi(N)$. We denote by $\mathcal{G}^N$ the class of all games with player set N. Any partition function v satisfying $v(S; P) = v(S; Q)$, for every $S \subseteq N$ and $P, Q \in \Pi(N \setminus S)$ is called a *classic game*. To begin with, we introduce the notion of superadditive game with externalities inspired by the inclusion relation studied in Section 2.

Definition 3.1. *Let $v \in \mathcal{G}^N$. We say that v is* superadditive *if and only if*

$$v((S; P) \vee (T; Q)) \geq v(S; P) + v(T; Q),$$

for every $(S; P),\ (T; Q) \in \mathcal{EC}^N$ such that $S \cap T = \emptyset$.

That is, for every pair of embedded coalitions whose intersection is an empty one, the worth of their supremum in $(\mathcal{EC}^N, \sqsubseteq)$ is greater or equal to the joint worths of the two embedded coalitions. Recall that $(S; P) \vee (T; Q) = (S \cup T; P_{-T} \bigwedge Q_{-S})$. In other words, if we evaluate the worths of two disjoint coalitions, each embedded in an arbitrary partition, this amount is weakly less than the worth of the union of the two coalitions embedded in the partition obtained by keeping all the divisions in the original partitions.

Definition 3.1 extends the classic notion of superadditivity of a game without externalities. The extension is not trivial because, as the next example shows there are superadditive games which are not classic games.

Example 3.1. *Let $N = \{1, 2, 3\}$ and consider the partition function v defined by*

$$v(N; \emptyset) = 8,\ v(\{1\}; \lceil 2, 3 \rceil) = 3,\ v(\{1\}; \lfloor 2, 3 \rfloor) = 0,$$
$$v(\{i\}; \lfloor N \setminus i \rfloor) = v(\{i\}; \lceil N \setminus i \rceil) = 2,\ \textit{for every } i \in N \setminus 1,\ \textit{and}$$
$$v(\{i, j\}; N \setminus \{i, j\}) = 5,\ \textit{for every } i, j \in N, i \neq j.$$

An important property of a game with externalities is the efficiency of the grand coalition. Let $v \in \mathcal{G}^N$. We say that v is *efficient* for the grand coalition if for every $P \in \Pi(N)$,

$$\sum_{S \in P} v(S; P_{-S}) \leq v(N; \emptyset).$$

It is easy to check that if a game is superadditive, then it is also efficient for the grand coalition. Hafalir (2007) points out that this fact does not happen with Maskin's definition of superaditivity (Maskin, 2003): $v \in \mathcal{G}^N$ is superadditive if for every $S, T \subseteq N$ with $S \cap T = \emptyset$ and $P \in \Pi(N \setminus (S \cup T))$,

$$v(S \cup T; P) \geq v(S; \lceil T \rceil \cup P) + v(T; \lceil S \rceil \cup P).$$

It is clear that any superadditive game in our sense is also a superadditive game in Maskin's sense, but the reverse does not hold.

Example 3.2. *Let* $N = \{1, 2, 3\}$ *and* $v \in \mathcal{G}^N$ *such that*

$$\begin{aligned}
&v(N; \emptyset) = 7,\ v(\{1\}; \lceil 2, 3 \rceil) = 3,\ v(\{1\}; \lfloor 2, 3 \rfloor) = 0,\\
&v(\{i\}; \lfloor N \setminus i \rfloor) = v(\{i\}; \lceil N \setminus i \rceil) = 2,\ \textit{for every } i \in N \setminus 1,\ \textit{and}\\
&v(\{i, j\}; N \setminus \{i, j\}) = 4,\ \textit{for every } i, j \in N, i \neq j.
\end{aligned}$$

Then, it is easy to check that v *is superadditive in Maskin's sense. However, it is not superadditive according to Definition 3.1 as we can see taking* $(S; P) = (\{2\}; \lceil 1, 3 \rceil)$ *and* $(T; Q) = (\{1\}; \lceil 2, 3 \rceil)$.

Next, we formulate our notion of convexity for games with externalities as the supermodularity of a function on the lattice $\left(\mathcal{EC}^N, \sqsubseteq\right)$.

Definition 3.2. *Let* $v \in \mathcal{G}^N$. *We say that* v *is* convex *if for every* $(S; P), (T; Q) \in \mathcal{EC}^N$ *it holds*

$$v((S; P) \vee (T; Q)) + v((S; P) \wedge (T; Q)) \geq v(S; P) + v(T; Q) \tag{5}$$

That is, for every pair of embedded coalitions, the sum of their worths is less than or equal to the sum of the worths of their supremum and infimum in $\left(\mathcal{EC}^N, \sqsubseteq\right)$. It is a very natural generalization of the classic definition (Shapley, 1971) if the supremum and infimum in $\left(\mathcal{EC}^N, \sqsubseteq\right)$ are understood as the union and intersection of embedded coalitions, respectively. As it happens when there are no externalities, any convex game is a superadditive game. In the literature there are several definitions of convexity for games with externalities. An important conceptual difference of our property is the fact that it applies to coalitions which are embedded in potentially different partitions. In a sense, we evaluate worths of coalitions that have different expectations about the organization of the complementary coalition.

Let us review the convexity notion of Hafalir (2007) and analyze its relationship with Definition 3.2. The game $v \in \mathcal{G}^N$ is *Hafalir convex* if and only if

$$v(S \cup T; P) + v(S \cap T; P \cup \lceil S \setminus T \rceil \cup \lceil T \setminus S \rceil) \geq v(S; P \cup \lceil T \setminus S \rceil) + v(T; P \cup \lceil S \setminus T \rceil)$$

for every $S, T \subseteq N$ and $P \in \Pi(N \setminus (S \cup T))$. Notice that for every $S, T \subseteq N$ and $P \in \Pi(N \setminus (S \cup T))$, we have $(S; P \cup \lceil T \setminus S \rceil) \vee (T; P \cup \lceil S \setminus T \rceil) = (S \cup T; P)$ and $(S; P \cup \lceil T \setminus S \rceil) \wedge (T; P \cup \lceil S \setminus T \rceil) = (S \cap T; P \cup \lceil S \setminus T \rceil \cup \lceil T \setminus S \rceil)$. This implies that our convexity implies Hafalir convexity. In the example below we show that the reverse implication does not hold.

Example 3.3. *Let $N = \{1, 2, 3, 4\}$ and $v \in \mathcal{G}^N$ be defined as follows:*

$$\begin{aligned}
&v(N; \emptyset) = 12;\ v(\{1,2,3\}; \{4\}) = 7,\ v(\{1,2,4\}; \{3\}) = 6,\ v(\{1,3,4\}; \{2\}) = 3,\\
&v(\{2,3,4\}; \{1\}) = 6,\ v(\{1,2\}; \lfloor 3,4 \rfloor) = 4,\ v(\{2,3\}; \lceil 1,4 \rceil) = 4,\\
&v(\{1,3\}; \lfloor 2,4 \rfloor) = 2,\ v(\{1\}; \lfloor 2,3,4 \rfloor) = 1,\ v(\{2\}; \lfloor 1,3,4 \rfloor) = 2,\\
&v(S; P) = 0,\ \textit{otherwise.}
\end{aligned}$$

This game is superadditive and Hafalir convex. But, it does not satisfy Inequality (5). For instance, if we take $(S; P) = (\{1, 2\}; \lfloor 3, 4 \rfloor)$, $(T; Q) = (\{2, 3\}; \lceil 1, 4 \rceil)$, then $(S; P) \vee (T; Q) = (\{1, 2, 3\}; \{4\})$, $(S; P) \wedge (T; Q) = (\{2\}; \{\{1, 4\}, \{3\}\})$, and

$$v(\{1,2,3\}; \{4\}) + v(\{2\}; \{\{1,4\}, \{3\}\}) = 7+0 < 4+4 = v(\{1,2\}; \lfloor 3,4 \rfloor) + v(\{2,3\}; \lceil 1,4 \rceil).$$

The next example illustrates that our notion of convexity is not obvious because there are convex games which are not classic games.

Example 3.4. *Let $N = \{1, 2, 3\}$ and $v \in \mathcal{G}^N$ defined as follows:*

$$\begin{aligned}
&v(N; \emptyset) = 15, \quad v(N \setminus i; \{i\}) = 10,\ \textit{for every } i \in N,\\
&v(N \setminus \{i,j\}; \lfloor i,j \rfloor) = 5,\ \textit{for every } i,j \in N, i \neq j,\\
&v(N \setminus \{i,j\}; \lceil i,j \rceil) = 4,\ \textit{for every } i,j \in N, i \neq j,\\
&v(\emptyset; N) = 0.
\end{aligned}$$

In order to present our main result we first have to generalize the contribution of an agent to a coalition to environments with externalities.[5] To that end, we use the lattice studied in Section 2. In classic games the contributions correspond to a link in the Boolean lattice of subsets $\left(2^N, \subseteq\right)$. Then, we consider that each link in the lattice $\left(\mathcal{EC}^N, \sqsubseteq\right)$ corresponds to a contribution to the embedded coalition on the top. Note that this leads to two kinds of contributions. The first is the movement of a player from being a singleton in the partition to join the coalition. The second is the movement of a block in the partition that splits in two. Next, we present these contributions that were introduced in Alonso-Meijide et al. (2019) and define what it means for a game to have non-decreasing contributions.

Let $v \in \mathcal{G}^N$ and $(S;P) \in \mathcal{EC}^N$ such that $\{i\} \in P$ for some $i \in N$. Then, we call *agent i's contribution* to the difference $v\left(S \cup i; P_{-i}\right) - v(S;P)$. Moreover, we say that agents' contributions are non-decreasing in v if

$$v(T \cup i; Q_{-i}) - v(T;Q) \geq v\left(S \cup i; P_{-i}\right) - v(S;P), \tag{6}$$

for every $i \in N$, $(S;P), (T;Q) \in \mathcal{EC}^N$ with $(S;P) \sqsubseteq (T;Q) \neq (N;\emptyset)$ and $\{i\} \in P$.

Let $v \in \mathcal{G}^N$, $(S;P) \in \mathcal{EC}^N$, and $P' \in \Pi(N \setminus S)$ covering P. Then, we call *external contribution* to the difference $v(S;P) - v(S;P')$.[6] Moreover, we say that external contributions are non-decreasing in v if

$$v(T;Q) - v(T;Q') \geq v(S;P) - v(S;P'), \tag{7}$$

for every $(S;P), (T;Q) \in \mathcal{EC}^N$ such that $(S;P) \sqsubseteq (T;Q) \neq (N;\emptyset)$, $P' \in \Pi(N \setminus S)$ covering P, $Q' \in \Pi(N \setminus T)$ covering Q, and $(S;P') \sqsubseteq (T;Q')$.

We state some auxiliary results that will be used to prove Theorem 3.1.

Lemma 3.1. *Let $v \in \mathcal{G}^N$ such that the external contributions are non-decreasing. Then, $v(S;P) \leq v(S;M)$, for every $(S;P), (S;M) \in \mathcal{EC}^N$ such that $(S;P) \sqsubseteq (S;M)$.*

That is, a game in which the external contributions are non-decreasing exhibits a monotonicity property in the sense that the worth of a coalition grows as the

[5] Many authors call this a marginal contribution.

[6] Notice that the external contribution is just the externality effect on the worth of coalition S when a coalition of $N \setminus S$ splits in two.

coalitions in the complement get more divided. In other words, it is a game with negative externalities (Hafalir, 2007).

Lemma 3.2. *Let $v \in \mathcal{G}^N$ such that the external contributions are non-decreasing. Then,*

$$v\left(T; P \bigwedge Q\right) + v\left(T; P \bigvee Q\right) \geq v(T;P) + v(T;Q), \tag{8}$$

for every $(T;P), (T;Q) \in \mathcal{EC}^N$.

That is, for a fixed coalition T the partition function of a game with non-decreasing external contributions is a supermodular function on $\Pi(N \setminus T)$.

Lemma 3.3. *Let $v \in \mathcal{G}^N$ such that agents' contributions are non-decreasing. Then,*

$$v(T;Q) - v(S; Q \cup \lfloor T \setminus S \rfloor) \geq v(T;P) - v(S; P \cup \lfloor T \setminus S \rfloor), \tag{9}$$

for every $S \subseteq T$ and $P, Q \in \Pi(N \setminus T)$, with $Q \preceq P$.

The above result states that when agents' contributions are non-decreasing in a game, the incorporation of several agents that were singletons in the partition is more beneficial for larger embedded coalitions.

Lemma 3.4. *Let $v \in \mathcal{G}^N$ such that agents' contributions are non-decreasing. Then,*

$$v(S \cup T; P) + v(S \cap T; P \cup \lfloor S \setminus T \rfloor \cup \lfloor T \setminus S \rfloor \cup P) \geq v(S; P \cup \lfloor T \setminus S \rfloor) + v(T; P \cup \lfloor S \setminus T \rfloor), \tag{10}$$

for every $S, T \subseteq N$ and $P \in \Pi(N \setminus (S \cup T))$.

Notice that Equation (10) is very similar to Hafalir convexity. The only difference is the fact that here we consider that agents who only participate in one of the two coalitions are singletons.

We are now ready to present our main result, which is a characterization of convexity by non-decreasing contributions. That is, we generalize the characterization of classic convex games by Shapley (1971) to environments with externalities.

Theorem 3.1. *Let $v \in \mathcal{G}^N$. The following three items are equivalent.*

i) v is a convex game.

ii) Let $(S;P),(T;Q) \in \mathcal{EC}^N \setminus \{(N;\emptyset)\}$ such that $(T;Q)$ covers $(S;P)$. Then,

1. *For every $i \in N$ with $\{i\} \in P$, we have*

$$v(T \cup i; Q_{-i}) - v(T;Q) \geq v\left(S \cup i; P_{-i}\right) - v(S;P) \tag{11}$$

2. *For every $P' \in \Pi(N \setminus S)$ covering P and $Q' \in \Pi(N \setminus T)$ covering Q such that $(T;Q')$ covers $(S;P')$, we have*

$$v(T;Q) - v(T;Q') \geq v(S;P) - v(S;P') \tag{12}$$

iii) v has non-decreasing agents' and external contributions.

Proof. First, we proof that *i)* implies *ii)*. Let $v \in \mathcal{G}^N$. Let us assume that v is a convex game. Take $(S;P),(T;Q) \in \mathcal{EC}^N$ such that $(S;P) \sqsubseteq (T;Q) \neq (N;\emptyset)$ and $(T;Q)$ covers $(S;P)$. If there is $\{i\} \in P$, then $\{i\} \in Q$ since $(S;P) \sqsubseteq (T;Q)$. Notice that $(T;Q) \bigvee (S \cup i; P_{-i}) = (T \cup i; Q_{-i})$ and $(T;Q) \bigwedge (S \cup i; P_{-i}) = (S;P)$. Applying Inequality (5) to $(T;Q)$ and $(S \cup i; P_{-i})$ and rearranging terms, we obtain

$$v\left(T \cup i; Q_{-i}\right) - v(T;Q) \geq v\left(S \cup i; P_{-i}\right) - v(S;P).$$

Let us take $P' \in \Pi(N \setminus S)$, $Q' \in \Pi(N \setminus T)$ such that P' covers P, Q' covers Q, and $(T;Q')$ covers $(S;P')$. Then, $(S;P)$ covers $(S;P')$ and $(T;Q)$ covers $(T;Q')$. Besides, $(S;P) \bigvee (T;Q') = (T;Q)$ and $(S;P) \bigwedge (T;Q') = (S;P')$. Then, applying Inequality (5) to $(S;P)$ and $(T;Q')$ and rearranging terms, we obtain

$$v(T;Q) - v\left(T;Q'\right) \geq v(S;P) - v\left(S;P'\right).$$

Second, we prove that *ii)* implies *iii)*. Let $(S;P),(T;Q) \in \mathcal{EC}^N$ with $(S;P) \sqsubseteq (T;Q) \neq (N;\emptyset)$. If $h(T;Q) - h(S;P) = 0$, then Inequalities (6) and (7) hold immediately because $(S;P) = (T;Q)$. If $h(T;Q) - h(S;P) = 1$, Inequalities (6) and (7) hold because v satisfies Inequalities (11) and (12). In the following, we assume that $h(T;Q) - h(S;P) > 1$. We divide the proof in two parts, the first to check Inequality (6) and the second to check Inequality (7). Figure 1 illustrates the scheme of the proof of the first part. Let us assume that $h(T;Q) - h(S;P) = k > 1$.

If there is some $\{i\} \in P$, then $\{i\} \in Q$. Take a chain $\lfloor T \setminus S \rfloor \cup Q = Q_0 \prec Q_1 \prec \ldots \prec Q_m = P$ with $m > 1$ in the lattice of partitions $(\Pi(N), \preceq)$. Notice that $\{i\} \in Q_j$, for every $j = 0, \ldots, m$. Note also that for every $j = 0, \ldots, m-1$, $(S; Q_j)$ covers $(S; Q_{j+1})$. Then, we can apply Inequality (11) to $(S; Q_{j+1})$ and $(S; Q_j)$ to get $v(S \cup i; (Q_j)_{-i}) - v(S; Q_j) \geq v(S \cup i; (Q_{j+1})_{-i}) - v(S; Q_{j+1})$, for every $j = 0, \ldots, m-1$. Thus,

$$\sum_{j=0}^{m-1} \left[v(S \cup i; (Q_j)_{-i}) - v(S; Q_j) \right] \geq \sum_{j=0}^{m-1} \left[v(S \cup i; (Q_{j+1})_{-i}) - v(S; Q_{j+1}) \right],$$

which yields

$$v(S \cup i; \lfloor T \setminus S \rfloor \cup Q_{-i}) - v(S; \lfloor T \setminus S \rfloor \cup Q) \geq v(S \cup i; P_{-i}) - v(S; P). \tag{13}$$

If $T \setminus S = \emptyset$, Inequality (13) is Inequality (6) and the proof is finished. If $T \setminus S \neq \emptyset$, let us assume that $T \setminus S = \{i_1, \ldots, i_r\}$ and take $R_j = \{i_1, \ldots, i_j\}$, for every $j = 1, \ldots, r$ and $R_0 = \emptyset$. Now, for every $j \in \{0, \ldots, r-1\}$, $(S \cup R_j \cup i; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q_{-i})$ covers $(S \cup R_j; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q)$. We apply Inequality (11) to $(S \cup R_j; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q) \sqsubseteq (S \cup R_j \cup i; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q_{-i})$ and $i_{j+1} \in T \setminus S$, obtaining

$$\begin{aligned} & v(S \cup R_{j+1} \cup i; \lfloor T \setminus (S \cup R_{j+1}) \rfloor \cup Q_{-i}) - v(S \cup R_j \cup i; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q_{-i}) \\ & \geq v(S \cup R_{j+1}; \lfloor T \setminus (S \cup R_{j+1}) \rfloor \cup Q) - v(S \cup R_j; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q). \end{aligned}$$

Adding up these r inequalities, we get

$$v(T \cup i; Q_{-i}) - v(S \cup i; \lfloor T \setminus S \rfloor \cup Q_{-i}) \geq v(T; Q) - v(S; \lfloor T \setminus S \rfloor \cup Q). \tag{14}$$

Adding up Inequalities (13) and (14), and rearranging terms, we obtain

$$v(T \cup i; Q_{-i}) - v(T; Q) \geq v(S \cup i; P_{-i}) - v(S; P).$$

Then, Inequality (6) holds.

We check that Inequality (7) also holds. Figure 2 illustrates the scheme of the proof. Let P' be a partition that covers P in $(\Pi(N \setminus S), \preceq)$, Q' be a partition that covers Q in $(\Pi(N \setminus T), \preceq)$, such that $(S; P') \sqsubseteq (T; Q')$. Take a pair of chains

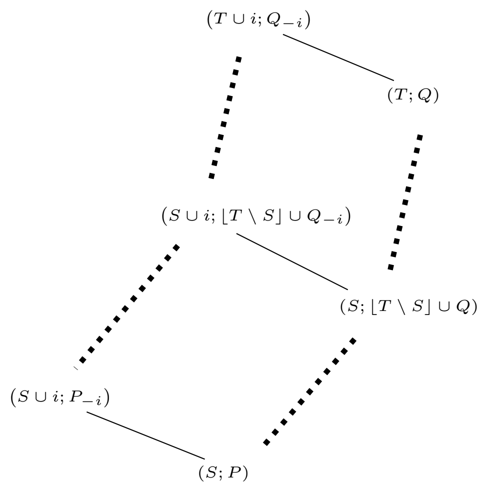

Figure 1: Inequality (6). Solid line: one link; dashed line: one or more links.

$Q_0 = \lfloor T \backslash S \rfloor \cup Q \prec Q_1 \prec \ldots \prec Q_m = P$ and $Q'_0 = \lfloor T \backslash S \rfloor \cup Q' \prec Q'_1 \prec \ldots \prec Q'_m = P'$ in the lattice of partitions $(\Pi(N), \preceq)$, such that Q'_j covers Q_j, for every $j = 0, \ldots, m$ with $m > 1$. Notice that both chains have the same length because P' covers P, Q' covers Q, $Q \prec P_{-T}$, and $Q' \prec P'_{-T}$. For every $j = 0, \ldots, m$, $(S; Q_j)$ covers $(S; Q'_j)$. Then we apply Inequality (12) to $(S; Q_{j+1}) \sqsubseteq (S; Q_j)$, Q'_{j+1}, and Q'_j, obtaining $v(S; Q_j) - v(S; Q'_j) \geq v(S; Q_{j+1}) - v(S; Q'_{j+1})$, for every $j = 0, \ldots, m-1$. Adding up these m inequalities, we get

$$v(S; \lfloor T \setminus S \rfloor \cup Q) - v(S; \lfloor T \setminus S \rfloor \cup Q') \geq v(S; P) - v(S; P'). \tag{15}$$

If $T \setminus S = \emptyset$, we finish the proof. If $T \setminus S \neq \emptyset$, we proceed as we did above in order to obtain Inequality (14) with $(S; \lfloor T \setminus S \rfloor \cup Q') \sqsubseteq (S; \lfloor T \setminus S \rfloor \cup Q)$ until we get $(T; Q')$ and $(T; Q)$. Hence,

$$v(T; Q) - v(S; \lfloor T \setminus S \rfloor \cup Q) \geq v(T; Q') - v(S; \lfloor T \setminus S \rfloor \cup Q') \tag{16}$$

Adding up Inequalities (15) and (16), we get $v(T; Q) - v(T; Q') \geq v(S; P) - v(S; P')$, concluding the proof.

Finally, we check that *iii)* implies *i)* using Lemma 3.3, Lemma 3.4, and Lemma 3.2. Let $(S; P), (T; Q) \in \mathcal{EC}^N$. If $(S; P) \sqsubseteq (T; Q)$ it is trivial to check Inequality (5).

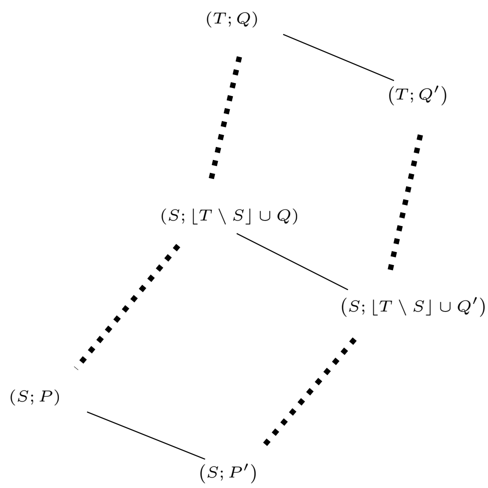

Figure 2: Inequality (7). Solid line: one link; dashed line: one or more links.

Let us assume $(S;P)$ and $(T;Q)$ are not comparable. We prove Inequality (5) using the disaggregation of the Hasse diagram among $(S;P)$, $(T;Q)$, $(S;P)\wedge(T;Q)$, and $(S;P)\vee(T;Q)$ depicted in Figure 3. Label **I** corresponds to a situation analized in Lemma 3.3, label **II** corresponds to a situation analized in Lemma 3.2, and label **III** corresponds to a situation analized in Lemma 3.4.

I.1 Apply Lemma 3.3 to $S\cap T\subseteq S$, $\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S})$, and $P\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S})$ because $\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S})\preceq P\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S})$. Then,

$$v(S;\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S}))+v(S\cap T;\lfloor S\setminus T\rfloor\cup(P\bigvee(Q_{-S}\cup\lfloor T\setminus S\rfloor))\geq v(S;P\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S}))+v(S\cap T;\lfloor S\setminus T\rfloor\cup\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S})). \tag{17}$$

I.2 Apply Lemma 3.3 to $S\cap T\subseteq T$, $\lfloor S\setminus T\rfloor\cup(P_{-T}\bigvee Q_{-S})$, and $Q\bigvee(\lfloor S\setminus T\rfloor\cup P_{-T})$ because $\lfloor S\setminus T\rfloor\cup(P_{-T}\bigvee Q_{-S})\preceq Q\bigvee(\lfloor S\setminus T\rfloor\cup P_{-T})$. Then,

$$v(T;\lfloor S\setminus T\rfloor\cup(P_{-T}\bigvee Q_{-S}))+v(S\cap T;\lfloor T\setminus S\rfloor\cup(Q\bigvee(P_{-T}\cup\lfloor S\setminus T\rfloor))\geq v(T;Q\bigvee(\lfloor S\setminus T\rfloor\cup P_{-T}))+v(S\cap T;\lfloor S\setminus T\rfloor\cup\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S})). \tag{18}$$

I.3 Apply Lemma 3.3 to $S\subseteq S\cup T$, P_{-T}, and $P_{-T}\bigvee Q_{-S}$ because $P_{-T}\preceq$

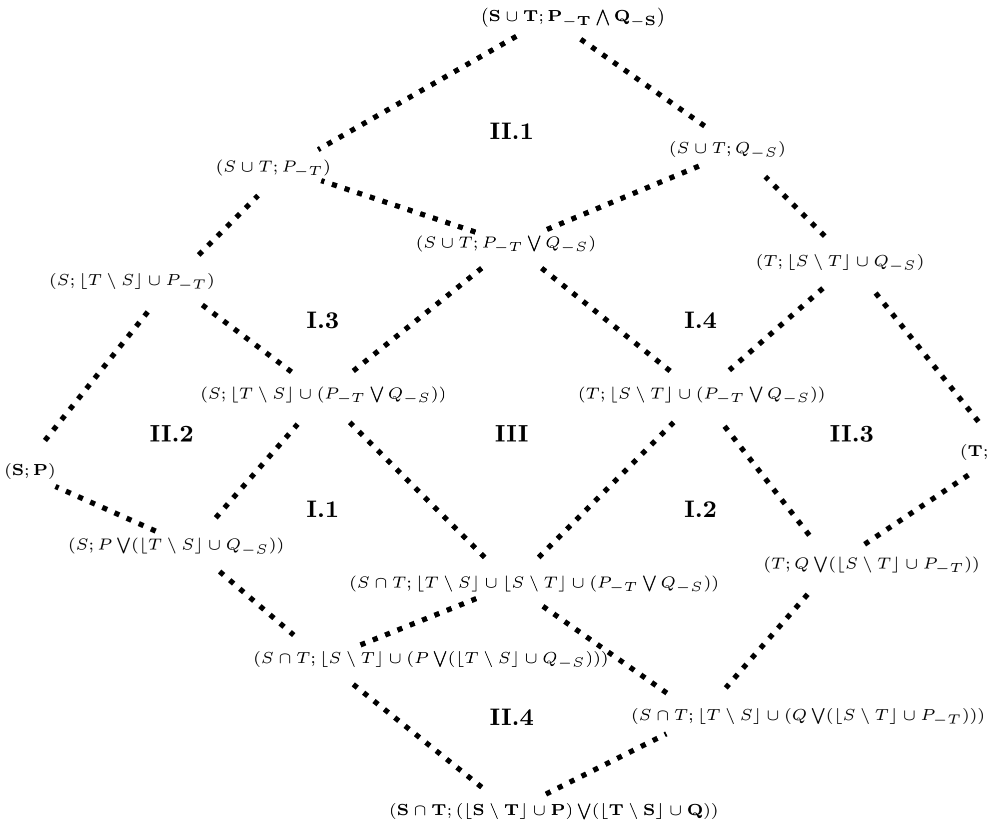

Figure 3: The structure of the proof.

$P_{-T} \bigvee Q_{-S}$. Then,

$$v(S \cup T; P_{-T}) + v(S; \lfloor T \setminus S \rfloor \cup (P_{-T} \bigvee Q_{-S})) \geq v(S \cup T; P_{-T} \bigvee Q_{-S}) + v(S; \lfloor T \setminus S \rfloor \cup P_{-T}). \tag{19}$$

I.4 Apply Lemma 3.3 to $T \subseteq S \cup T$, Q_{-S}, and $P_{-T} \bigvee Q_{-S}$ because $Q_{-S} \preceq P_{-T} \bigvee Q_{-S}$. Then,

$$v(S \cup T; Q_{-S}) + v(T; \lfloor S \setminus T \rfloor \cup (P_{-T} \bigvee Q_{-S})) \geq v(S \cup T; P_{-T} \bigvee Q_{-S}) + v(T; \lfloor S \setminus T \rfloor \cup Q_{-S}). \tag{20}$$

II.1 Apply Lemma 3.2 to $(S \cup T; P_{-T})$ and $(S \cup T; Q_{-S})$. Then,

$$v(S \cup T; P_{-T} \bigwedge Q_{-S}) + v(S \cup T; P_{-T} \bigvee Q_{-S}) \geq v(S \cup T; P_{-T}) + v(S \cup T; Q_{-S}). \tag{21}$$

II.2 Apply Lemma 3.2 to $(S;P)$ and $(S;\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S}))$. Then,

$$v(S;\lfloor T\setminus S\rfloor\cup P_{-T})+v(S;P\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S}))\geq v(S;P)+v(S;\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S})). \tag{22}$$

II.3 Apply Lemma 3.2 to $(T;Q)$ and $(T;\lfloor S\setminus T\rfloor\cup(P_{-T}\bigvee Q_{-S}))$. Then,

$$v(T;\lfloor S\setminus T\rfloor\cup Q_{-S})+v(T;Q\bigvee(\lfloor S\setminus T\rfloor\cup P_{-T}))\geq v(T;Q)+v(T;\lfloor S\setminus T\rfloor\cup(P_{-T}\bigvee Q_{-S})). \tag{23}$$

II.4 Apply Lemma 3.2 to $(S\cap T;\lfloor T\setminus S\rfloor\cup(Q\bigvee(P_{-T}\cup\lfloor S\setminus T\rfloor))$ and $(S\cap T;\lfloor S\setminus T\rfloor\cup(P\bigvee(Q_{-S}\cup\lfloor T\setminus S\rfloor))$. Then,

$$\begin{aligned}&v(S\cap T;\lfloor S\setminus T\rfloor\cup\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S}))+v(S\cap T;(\lfloor S\setminus T\rfloor\cup P)\bigvee(\lfloor T\setminus S\rfloor\cup Q))\geq\\&v(S\cap T;\lfloor T\setminus S\rfloor\cup(Q\bigvee(P_{-T}\cup\lfloor S\setminus T\rfloor))+v(S\cap T;\lfloor S\setminus T\rfloor\cup(P\bigvee(Q_{-S}\cup\lfloor T\setminus S\rfloor)).\end{aligned} \tag{24}$$

III Apply Lemma 3.4 to S, T, and $P_{-T}\bigvee Q_{-S}$. Then,

$$\begin{aligned}&v(S\cup T;P_{-T}\bigvee Q_{-S})+v(S\cap T;(\lfloor S\setminus T\rfloor\cup\lfloor T\setminus S\rfloor)\cup(P_{-T}\bigvee Q_{-S}))\geq\\&v(S;\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S}))+v\left(T;\lfloor S\setminus T\rfloor\cup(P_{-T}\bigvee Q_{-S})\right).\end{aligned} \tag{25}$$

Adding up Inequalities (19), (21), and (22), we obtain

$$v(S\cup T;P_{-T}\bigwedge Q_{-S})+v(S;P\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S}))\geq v(S\cup T;Q_{-S})+v(S;P). \tag{26}$$

Adding up Inequalities (17), (20), and (25), we obtain

$$\begin{aligned}&v(S\cup T;Q_{-S})+v(S\cap T;\lfloor S\setminus T\rfloor\cup(P\bigvee(Q_{-S}\cup\lfloor T\setminus S\rfloor))\geq\\&v(T;\lfloor S\setminus T\rfloor\cup Q_{-S})+v(S;P\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S})).\end{aligned} \tag{27}$$

Adding up Inequalities (18), (23), and (24), we obtain

$$\begin{aligned}&v(T;\lfloor S\setminus T\rfloor\cup Q_{-S})+v(S\cap T;(\lfloor S\setminus T\rfloor\cup P)\bigvee(\lfloor T\setminus S\rfloor\cup Q))\\&\geq v(T;Q)+v(S\cap T;\lfloor S\setminus T\rfloor\cup(P\bigvee(Q_{-S}\cup\lfloor T\setminus S\rfloor)).\end{aligned} \tag{28}$$

Finally, adding up Inequalities (26), (27), and (28), we obtain

$$v\left(S \cup T; P_{-T} \bigwedge Q_{-S}\right) + v\left(S \cap T; (\lfloor S \setminus T \rfloor \cup P) \bigvee (\lfloor T \setminus S \rfloor \cup Q)\right) \geq v(S;P) + v(T;Q).$$

Summarizing all the previous results, we have the characterization of convexity for games with externalities given in Theorem 3.1. $\square$

Observe that condition *ii)* is a weakening of *iii)* as it is only applied when the embedded coalition $(T;Q)$ covers $(S;P)$, in point *2.* it is also required that $(T;Q')$ covers $(S;P')$. This is parallel to the characterization of classic convex games where it is sufficient to check that the contributions are non-decreasing when one player is incorporated to the coalition. Hafalir (2007) also considered a weakening of his notion of convexity, which is obtained by requiring Inequality (10) only when $|T \setminus S| = |S \setminus T| = 1$. However, as he points out, this condition alone is not even sufficient for the efficiency of the grand coalition. Abe (2016) shows that for games with negative externalities, the weak version of Hafalir convexity is sufficient. From Theorem 3.1 we can also conclude that it is enough to check that contributions are non-decreasing to coalitions that are just one link away from one another to guarantee that the grand coalition is efficient.

4 Convexity and the core

In this section we include some comments on the core of the optimistic and the pessimistic games[7] associated to a convex game. Both of them are classic games. First we recall the notion of the core of a classic game. Let $w \in \mathcal{G}^N$ be a classic game. The *core* of w is given by

$$Core(w) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = w(N), \sum_{i \in S} x_i \geq w(S), \text{ for every } S \subseteq N \right\}.$$

In general, $Core(w)$ can be empty, but every convex classic game has a non-empty core. Besides, it is quite easy to describe its extreme points. Let the set of per-

[7]Which are essentially the α-core and β-core (Hart and Kurz, 1983). More recently Dutta et al. (2010), Bloch and van den Nouweland (2014), and Abe (2016) also use these games.

mutations of N be denoted by $\Theta(N)$, i.e., $\sigma \in \Theta(N)$ if and only if σ is a bijective mapping $\sigma : N \longrightarrow \{1, \ldots, n\}$. Let $\sigma \in \Theta(N)$ and $i \in N$. The set of *predecessors* of i is $Pr(\sigma, i) = \{j \in N : \sigma(j) < \sigma(i)\}$ and the set of *followers* of i is $F(\sigma, i) = \{j \in N : \sigma(j) > \sigma(i)\}$. The *vector of marginal contributions* with respect to σ is given by $m^{\sigma}(w) \in \mathbb{R}^n$ such that $m_i^{\sigma}(w) = w(Pr(\sigma, i) \cup i) - w(Pr(\sigma, i))$, for every $i \in N$. It is well known that if w is a classic convex game, then the vectors of marginal contributions are the vertices of the core, i.e., $Core(w) = conv\, \{m^{\sigma}(w) : \sigma \in \Theta(N)\}$.

Let $v \in \mathcal{G}^N$. The *optimistic game*, denoted by v_{max}, is the classic game defined by $v_{max}(S) = \max\{v(S; P) : \ P \in \Pi(N \setminus S)\}$, for every $S \subseteq N$. The *pessimistic game*, denoted by v_{min}, is the classic game defined by $v_{min}(S) = \min\{v(S; P) : \ P \in \Pi(N \setminus S)\}$, for every $S \subseteq N$. Notice that $v_{max}(S) \geq v_{min}(S)$, for every $S \subseteq N$ and $v_{max}(N) = v_{min}(N) = v(N; \emptyset)$. Then, $Core(v_{max}) \subseteq Core(v_{min})$. Abe (2016) proved that if v has negative externalities and satisfies the weak convexity condition, then $Core(v_{max})$ is non-empty as well as $Core(v_{min})$. Since a convex game according to Definition 3.2 satisfies the weak convexity condition, we already known that both $Core(v_{max})$ and $Core(v_{min})$ are non-empty sets when v is a convex game.

Definition 4.1. *Let $v \in \mathcal{G}^N$. For every $P \in \Pi(N)$, we define the classic game v^P by $v^P(S) = v(S; P_{-S})$, for every $S \subseteq N$.*

Notice that v^P is defined for every $S \subseteq N$ even if S is not a block in P. Besides, $v^P(S) = v^Q(S)$, for every $P, Q \in \Pi(N)$ and $S \subseteq N$ with $P_{-S} = Q_{-S}$. The optimistic game can then be defined by $v_{max}(S) = \max\{v^P(S) : \ P \in \Pi(N)\}$, analogously for the pessimistic game by $v_{min}(S) = \min\{v^P(S) : \ P \in \Pi(N)\}$, for every $S \subseteq N$.

Proposition 4.1. *Let $v \in \mathcal{G}^N$. Then,*

$$Core\left(v_{max}\right) = \bigcap_{P \in \Pi(N)} Core\left(v^P\right).$$

Proof. Let $x \in Core(v_{max})$ and $P \in \Pi(N)$. Then,

$$\sum_{i \in N} x_i = v(N; \emptyset) = v^P(N).$$

For every $S \subseteq N$,

$$\sum_{i \in S} x_i \geq v_{max}(S) \geq v^P(S).$$

Then, $x \in Core\left(v^P\right)$. Let $x \in \bigcap_{P \in \Pi(N)} Core\left(v^P\right)$. Let $(S, Q) \in \mathcal{EC}^N$ be such that $v_{max}(S) = v(S; Q)$. Take, for instance, $P = Q \cup \lceil S \rceil$. It is clear that $P_{-S} = Q$ and $v^P(S) = v(S; Q) = v_{max}(S)$. Since $x \in Core\left(v^P\right)$, we have

$$\sum_{i \in S} x_i \geq v^P(S) = v(S; Q) = v_{max}(S).$$

Thus, $x \in Core(v_{max})$. $\square$

Next, we characterize the extreme points of the core of v_{max} and the core of v_{min} of a convex game.

Theorem 4.1. *Let $v \in \mathcal{G}^N$ be a convex game.*

1. *Let $P, Q \in \Pi(N)$ such that $Q \preceq P$. Then, $Core\left(v^Q\right) \subseteq Core\left(v^P\right)$.*
2. *For every $S \subseteq N$, $v_{max}(S) = v^{\lfloor N \rfloor}(S) \quad$ and $\quad v_{min}(S) = v^{\lceil N \rceil}(S)$.*
3. *v_{max} is a convex classic game and*

$$Core\left(v_{max}\right) = Core\left(v^{\lfloor N \rfloor}\right) = conv\left\{m^\sigma\left(v^{\lfloor N \rfloor}\right) : \sigma \in \Theta(N)\right\},$$

with $m^\sigma\left(v^{\lfloor N \rfloor}\right) \in \mathbb{R}^N$ defined for every $i \in N$ by

$$\begin{aligned} m_i^\sigma\left(v^{\lfloor N \rfloor}\right) &= v^{\lfloor N \rfloor}(Pr(\sigma, i) \cup i) - v^{\lfloor N \rfloor}(Pr(\sigma, i)) \\ &= v(Pr(\sigma, i) \cup i; \lfloor F(\sigma, i) \rfloor) - v(Pr(\sigma, i); \lfloor F(\sigma, i) \cup i \rfloor) \end{aligned}$$

Proof. Let $v \in \mathcal{G}^N$ be a convex game.

1. Let $P, Q \in \Pi(N)$ such that $Q \preceq P$. Recall that, $v^P(N) = v^Q(N)$. Let $x \in Core\left(v^Q\right)$, then

$$x_S = \sum_{i \in S} x_i \geq v(S; Q_{-S}) \geq v(S; P_{-S}),$$

where the first inequality follows because $x \in Core(v^Q)$ and the second inequality because $(S; P_{-S}) \sqsubseteq (S; Q_{-S})$ and Lemma 3.1 holds. Then, $x \in Core(v^P)$.

2. Let $S \subseteq N$. Since v is convex and according to Lemma 3.1, we have $v(S;Q) \geq v(S;P)$, for every $(S;P),(S;Q) \in \mathcal{EC}^N$ with $(S;P) \sqsubseteq (S;Q)$. Notice that $(S; \lceil N \setminus S \rceil) \sqsubseteq (S;Q) \sqsubseteq (S; \lfloor N \setminus S \rfloor)$, for every $(S;Q) \in \mathcal{EC}^N$ and there is no $(S;M) \in \mathcal{EC}^N$ such that $(S;M) \sqsubset (S; \lceil N \setminus S \rceil) \sqsubseteq (S;Q)$ nor $(S;M') \in \mathcal{EC}^N$ with $(S;Q) \sqsubseteq (S; \lfloor N \setminus S \rfloor) \sqsubset (S;M')$. As a consequence of all this, $v_{max}(S) = \max \left\{ v^Q(S) : Q \in \Pi(N) \right\} = v^{\lfloor N \rfloor}(S)$ and $v_{min}(S) = \min \left\{ v^Q(S) : Q \in \Pi(N) \right\} = v^{\lceil N \rceil}(S)$.

3. First, we see that $v_{max} = v^{\lfloor N \rfloor}$ is a convex game. Let $i \in N$, $S \subseteq T \subseteq N \setminus i$. We prove that

$$v^{\lfloor N \rfloor}(T \cup i) - v^{\lfloor N \rfloor}(T) \geq v^{\lfloor N \rfloor}(S \cup i) - v^{\lfloor N \rfloor}(S). \tag{29}$$

Notice that $\{i\} \in \lfloor N \rfloor_{-S}$ and $(S; \lfloor N \setminus S \rfloor) \sqsubseteq (T; \lfloor N \setminus T \rfloor) \neq (N; \emptyset)$. According to Item *iii.1)* in Theorem 3.1, we have $v\left(T \cup i; \lfloor N \setminus (T \cup i) \rfloor\right) - v(T; \lfloor N \setminus T \rfloor) \geq v(S \cup i; \lfloor N \setminus (S \cup i) \rfloor) - v(S; \lfloor N \setminus S \rfloor)$, or equivalently, $v^{\lfloor N \rfloor}(T \cup i) - v^{\lfloor N \rfloor}(T) \geq v^{\lfloor N \rfloor}(S \cup i) - v^{\lfloor N \rfloor}(S)$. Thus, Inequality (29) holds and $v^{\lfloor N \rfloor}$ is a convex game.

□

As a consequence of Theorem 4.1, if v is convex the Externality-free value (de Clippel and Serrano, 2008) is the average of the extreme points of the core of v_{max} and it also belongs to the core of v_{min}. Notice that our definition of convexity is not enough to guarantee the convexity of the classic game v_{min}. We illustrate this using Example 3.4. In this case, we have

$$v_{min}(N) = 15, \ v_{min}(S) = 10, \text{ for every } S \subset N \text{ with } |S| = 2, \text{ and}$$
$$v_{min}(S) = 4, \text{ for every } S \subset N \text{ with } |S| = 1.$$

For instance, if we take $S = \{1\} \subseteq T = \{1,2\}$ and $i = 3$, we have

$$v_{min}(N) - v_{min}(T) = 15 - 10 = 5 < 6 = 10 - 4 = v_{min}(S \cup i) - v_{min}(S).$$

5 Final remarks

Finally, we compare our definition of superadditivy with *optimistic superadditivity* (optimistic-SA) as defined by Abe (2016). A game $v \in \mathcal{G}^N$ is optimistic-superadditive if v_{max} is a superadditive classic game. It is clear that any convex game according to Definition 3.2 is also optimistic-superadditive as a consequence of Theorem 4.1. Nevertheless, there are games with negative externalities that are optimistic-superadditive but not superadditive according to Definition 3.1. We illustrate this with the following example.

Example 5.1. *Let $N = \{1, 2, 3, 4\}$ and $v \in \mathcal{G}^N$ defined as follows:*

$$\begin{aligned}
&v(N; \emptyset) = 60,\ v(N \setminus i; \lfloor i \rfloor) = 45,\ \textit{for every } i \in N,\\
&v(\{i, j\}; \lceil h, k \rceil) = 29 \textit{ and } v(\{i, j\}; \lfloor h, k \rfloor) = 30,\ \textit{for every } \{i, j, h, k\} = N,\\
&v(\{i\}; P) = 15,\ \textit{for every } (\{i\}; P) \in \mathcal{EC}^N.
\end{aligned}$$

This game is an adaptation of Example 3.8 in Abe (2016). It is still superadditive in Maskin's sense, but it is not superadditive according to Definition 3.1 because, for instance,

$$v(\{1\}; \{\{2, 3\}, \{4\}\}) + v(\{4\}; \lceil N \setminus 4 \rceil) = 15 + 15 = 30 > v(\{1, 4\}; \lceil 2, 3 \rceil) = 29.$$

The optimistic game associated to it, given by

$$\begin{aligned}
&v_{max}(N) = 60,\ v_{max}(N \setminus i) = 45,\ \textit{for every } i \in N,\\
&v_{max}(S) = 30,\ \textit{if } |S| = 2,\\
&v_{max}(S) = 15,\ \textit{if } |S| = 1,
\end{aligned}$$

is a classic superadditive game.

Figure 4 illustrates the relationship between our concepts of superadditive and convex games with the concepts of efficient and optimistic-superadditive games in the framework of games with negative externalities.

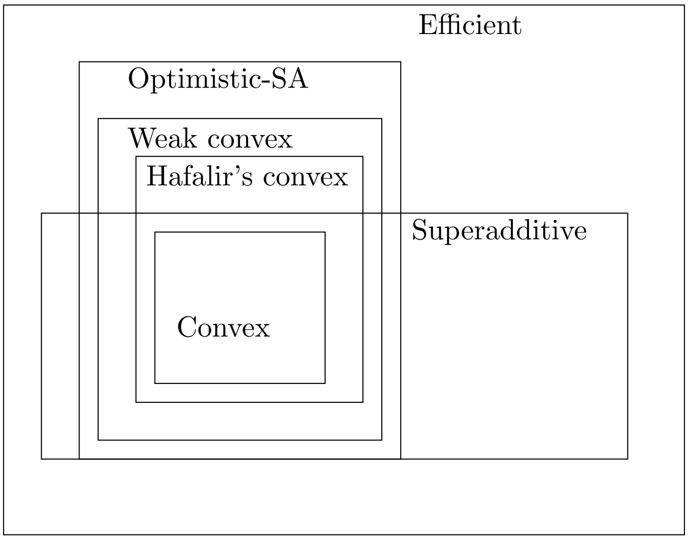

Figure 4: Relationship among several families of games with negative externalities.

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Appendix.

Proof of Lemma 3.1. Let $(S;P),(S;M) \in \mathcal{EC}^N$, such that $(S;P) \sqsubseteq (S;M)$ and $(S;P) \neq (S;M)$. If $S \in \{\emptyset, N\}$ or $P = M$, $v(S;P) = v(S;M)$ and the result follows immediately. Then, suppose that $S \notin \{\emptyset, N\}$ and $P \neq M$. Since $(S;P) \sqsubseteq (S;M) \neq (N;\emptyset)$ and $(S;P) \neq (S;M)$, $M \prec P$ holds. Take a chain $M = Q_0 \prec Q_1 \prec \cdots \prec Q_k = P$. Then, Q_r covers Q_{r-1}, for every $r = 1,\ldots,k$. Take the family of embedded coalitions $\{(\emptyset; \lfloor S \rfloor \cup Q_r) : \ r = 0,\ldots,k\}$. Let $r \in \{0,\ldots,k-1\}$. Then, $\lfloor S \rfloor \cup Q_{r+1}$ covers $\lfloor S \rfloor \cup Q_r$ and $(\emptyset; \lfloor S \rfloor \cup Q_r) \sqsubseteq (S;Q_r) \neq (N;\emptyset)$. Applying Inequality (7) to $(\emptyset; \lfloor S \rfloor \cup Q_r) \sqsubseteq (S;Q_r)$, $\lfloor S \rfloor \cup Q_{r+1}$, and Q_{r+1}, we obtain $v(S;Q_r) - v(S;Q_{r+1}) \geq v(\emptyset; \lfloor S \rfloor \cup Q_r) - v(\emptyset; \lfloor S \rfloor \cup Q_{r+1})$. Since $v(\emptyset; \lfloor S \rfloor \cup Q_{r+1}) = v(\emptyset; \lfloor S \rfloor \cup Q_r) = 0$, we get $v(S;Q_r) \geq v(S;Q_{r+1})$. Thus,

$$v(S;M) = v(S;Q_0) \geq v(S;Q_1) \geq \cdots \geq v(S;Q_{k-1}) \geq v(S;Q_k) = v(S;P).$$

$\square$

Proof of Lemma 3.3. Take $S \subseteq T$, $P,Q \in \Pi(N \setminus T)$ with $Q \preceq P$. We proceed by induction on $|T \setminus S|$. If $|T \setminus S| = 0$, Inequality (9) follows immediately. Let us assume that $|T \setminus S| = 1$, i.e., $T \setminus S = \{i\}$ for some $i \in N$. Then, $(S;\{i\} \cup P) \sqsubseteq (S;\{i\} \cup Q)$. Applying Inequality (6) to i, $(S;\{i\} \cup P)$, and $(S;\{i\} \cup Q)$ we get

$$v(S \cup i;Q) - v(S;\{i\} \cup Q) \geq v(S \cup i;P) - v(S;\{i\} \cup P).$$

Now, let us assume that the result holds for every $S \subseteq T$, $P,Q \in \Pi(N \setminus T)$ with $Q \preceq P$ and $|T \setminus S| < k$. Take $S \subseteq T$, $P,Q \in \Pi(N \setminus T)$ with $Q \preceq P$ and $|T \setminus S| = k$. Take $i \in T \setminus S$, $(T \setminus i;\{i\} \cup P)$, and $(T \setminus i;\{i\} \cup Q)$. It is clear that $T \setminus \{i\} \subseteq T$, $\{i\} \cup Q \preceq \{i\} \cup P$ and $|T \setminus (T \setminus \{i\})| = 1$. As we have just seen

$$v(T;Q) - v(T \setminus i;\{i\} \cup Q) \geq v(T;P) - v(T \setminus i;\{i\} \cup P). \tag{30}$$

Notice that $S \subseteq T \setminus i$. Take $P' = \{i\} \cup P$, and $Q' = \{i\} \cup Q$. Since $|T \setminus (S \cup i)| = k - 1 < k$, $\{i\} \in P'$, and $Q' \preceq P'$, applying the induction hypothesis we get

$$v(T \setminus i; Q') - v(S; \lfloor T \setminus S \rfloor \cup Q) \geq v(T \setminus i; P') - v(S; \lfloor T \setminus S \rfloor \cup P). \quad (31)$$

Adding up Inequalities (30) and (31) we get the result. □

Proof of Lemma 3.4. Let $S, T \subseteq N$, $P \in \Pi(N \setminus (S \cup T))$. If $S \in \{\emptyset, N\}$ or $T = \emptyset$, Inequality (10) follows immediately. Let us assume that both S and T are proper non-empty subsets of N. If $S \subseteq T$, Inequality (10) follows immediately. Then, let us assume that S and T are not comparable and $S \setminus T = \{i_1, \ldots, i_r\}$. Let $A_0 = S \cap T$ and $B_0 = T$. For each $j = 1, \ldots, r$, take

- $(A_j; P'_j) \in \mathcal{EC}^N$ given by $A_j = A_{j-1} \cup \{i_j\}$, $P'_j = P \cup \lfloor T \setminus S \rfloor \cup \lfloor S \setminus A_j \rfloor$, and
- $(B_j; Q'_j) \in \mathcal{EC}^N$ given by $B_j = B_{j-1} \cup \{i_j\}$, $Q'_j = P \cup \lfloor S \setminus B_j \rfloor$.

For every $j = 0, \ldots, r$, we have $(A_j; P'_j) \sqsubseteq (B_j; Q'_j)$. Thus, for every $j = 0, \ldots, r-1$, applying Inequality (6) to i_{j+1}, $(A_j; P'_j)$ and $(B_j; Q'_j)$, we obtain $v(B_j \cup \{i_{j+1}\}; Q'_{j+1}) - v(B_j; Q'_j) \geq v(A_j \cup \{i_{j+1}\}; P'_{j+1}) - v(A_j; P'_j)$. Adding up these r inequalities, we get

$$\sum_{j=0}^{r-1} [v(B_j \cup \{i_{j+1}\}; Q'_{j+1}) - v(B_j; Q'_j)] \geq \sum_{j=0}^{r-1} [v(A_j \cup \{i_{j+1}\}; P'_{j+1}) - v(A_j; P'_j)].$$

Hence,

$$v(S \cup T; P) - v(T; \lfloor S \setminus T \rfloor \cup P) \geq v(S; \lfloor T \setminus S \rfloor \cup P) - v(S \cap T; \lfloor T \setminus S \rfloor \cup \lfloor S \setminus T \rfloor \cup P),$$

concluding the proof. □

Proof of Lemma 3.2. Take $(T; P), (T; Q) \in \mathcal{EC}^N$. If $T \in \{N, \emptyset\} \cup \{N \setminus i : i \in N\}$ or $(T; P) \sqsubseteq (T; Q)$, Inequality (8) follows immediately. Let us assume that $(T; P)$ and $(T; Q)$ are not comparable, $0 < |T| < n - 1$, and w.l.o.g we assume $h(T; Q) \geq h(T; P)$. Then, $|Q| \geq |P|$ and $P \bigvee Q \notin \{P, Q\}$. Let $P \bigwedge Q = P_0 \prec P_1 \prec \ldots \prec P_k \prec P_{k+1} = P$, with $k \geq 1$, be a chain that joins $P \bigwedge Q$ and P, and

$P \bigwedge Q = Q_0 \prec Q_1 \prec \ldots \prec Q_r \prec Q_{r+1} = Q$, with $r \geq 1$, be a chain that joins $P \bigwedge Q$ and Q. Notice that P_j and Q_l are not comparable for every $j = 1, \ldots, k+1$, $l = 1, \ldots, r+1$. We distinguish four cases.

1. $h(T; P \bigwedge Q) - h(T; P) = 1$, $h(T; P \bigwedge Q) - h(T; Q) = 1$. That means both P and Q cover $P \bigwedge Q$. Figure 5 illustrates the situation. Since $\Pi(N \backslash T)$ is semimodular, $P \bigvee Q$ covers both P and Q. Then, $(T; P) \vee (T; Q) = (T; P \bigwedge Q)$ covers both $(T; P)$ and $(T; Q)$. Since $(\mathcal{EC}^N, \sqsubseteq)$ is lower semimodular, then $(T; P)$ and $(T; Q)$ both cover $(T; P) \wedge (T; Q) = (T; P \bigvee Q)$. Applying Inequality (7) to $(T; P)$, $(T; P \bigwedge Q)$, $(T; P \bigvee Q)$, and $(T; Q)$ we get Inequality (8).

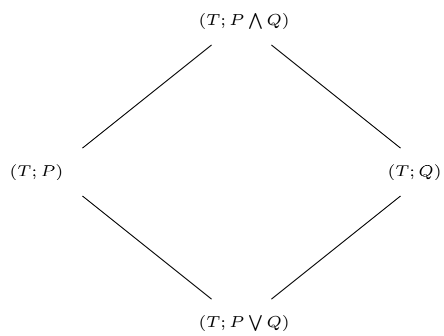

Figure 5: Case 1. solid line: one link.

2. $h(T; P \bigwedge Q) - h(T; Q) = 1$, but $h(T; P \bigwedge Q) - h(T; P) > 1$. Using Proposition 2.3 and the fact that $(T; P)$ and $(T; Q)$ are not comparable, $h(T; P) - h(T; P \bigvee Q) = 1$. Then, $P \bigvee Q$ covers P, Q covers $P \bigwedge Q$, but P does not cover $P \bigwedge Q$ and $P \neq P \bigwedge Q$. Figure 6 illustrates the situation. Take a chain $P \bigwedge Q = P_0 \prec P_1 \prec \ldots \prec P_k \prec P_{k+1} = P$, with $k \geq 1$. Notice that $Q \prec Q \bigvee P_1 \prec \ldots \prec Q \bigvee P_k \prec Q \bigvee P_{k+1} = Q \bigvee P$, with $k \geq 1$ is a chain from Q to $P \bigvee Q$. Let $j \in \{0, \ldots, k\}$. Then, $P_j \bigwedge Q = P \bigwedge Q$ and using Proposition 2.3 we have

$$1 = h\left(T; P \bigwedge Q\right) - h(T; Q) = h\left(T; P_j \bigwedge Q\right) - h(T; Q) \geq h(T; P_j) - h\left(T; P_j \bigvee Q\right).$$

Since $P_j \neq P_j \bigvee Q$, $h(T; P_j) - h(T; P_j \bigvee Q) = 1$. Besides, $(P_j \bigvee Q) \bigwedge P_{j+1} = P_j$. Using that $(\Pi(N), \preceq)$ is semimodular, $(P_j \bigvee Q) \bigvee P_{j+1} = P_{j+1} \bigvee Q$ covers both

$P_j \bigvee Q$ and P_{j+1}. Then, using Item 1, we have

$$v(T;P_j) + v\left(T;P_{j+1} \bigvee Q\right) \geq v(T;P_{j+1}) + v\left(T;P_j \bigvee Q\right).$$

Summing up the inequalities given above, we get

$$\sum_{j=0}^{k} \left[v(T;P_j) + v\left(T;P_{j+1} \bigvee Q\right)\right] \geq \sum_{j=0}^{k} \left[v(T;P_{j+1}) + v\left(T;P_j \bigvee Q\right)\right].$$

Rearranging this inequality, we obtain Inequality (8).

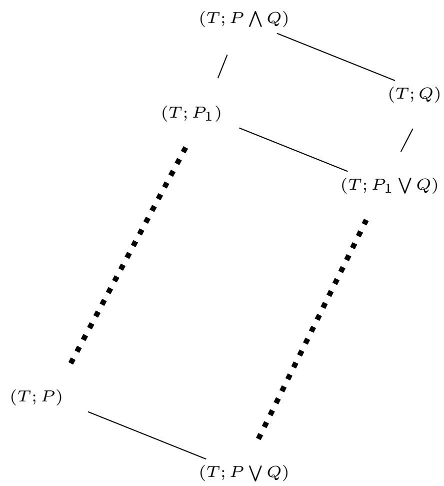

Figure 6: Case 2. solid line: one link; dashed line: more than one link.

3. $h(T;P) - h(T;P \bigvee Q) = 1$, but $h(T;P \bigwedge Q) - h(T;Q) > 1$. This means $P \bigvee Q$ covers P, but Q does not cover $P \bigwedge Q$. Figure 7 illustrates the situation. Since $h(T;Q) \geq h(T;P)$ and $h(T;P \bigwedge Q) - h(T;Q) > 1$, we have $h(T;P \bigwedge Q) - h(T;P) > 1$. Take $P \bigwedge Q = P_0 \prec Q_1 \prec \ldots \prec Q_r \prec Q_{r+1} = Q$, with $r \geq 1$, a chain that joins $P \bigwedge Q$ and Q. By the choice of Q_1, we have

- $(T;Q_1) \sqsubseteq (T;P \bigwedge Q)$, $(T;P) \sqsubseteq (T;P \bigwedge Q)$, $h(T;P \bigwedge Q) - h(T;Q_1) = 1$, and the fact that $(T;P)$ and $(T;Q_1)$ are not comparable imply that $(T;P \bigwedge Q) = (T;P \bigwedge Q_1)$.

- $(T;P \bigvee Q) \sqsubseteq (T;P \bigvee Q_1) \sqsubseteq (T;P)$ and $h(T;P) - h(T;P \bigvee Q) = 1$. Then, $P = P \bigvee Q_1$ or $P \bigvee Q_1 = P \bigvee Q$. If $P = P \bigvee Q_1$ we have $Q_1 \preceq P$ and

$(T;P) \sqsubseteq (T;Q_1)$, but this fact contradicts that $(T;P)$ and $(T;Q_1)$ are not comparable. Then, $(T;P \bigvee Q) = (T;P \bigvee Q_1)$.

As a consequence of all this, applying Item 2 to $(T;P)$ and $(T;Q_1)$, we obtain

$$v\left(T;P \bigwedge Q\right) + v\left(T;P \bigvee Q\right) \geq v(T;P) + v(T;Q_1) \tag{32}$$

Since v satisfies Inequality (7) and $Q_1 \prec Q$, using Lemma 3.1, we get $v(T;Q_1) \geq v(T;Q)$ and

$$v\left(T;P \bigwedge Q\right) + v\left(T;P \bigvee Q\right) \geq v(T;P) + v(T;Q),$$

concluding the proof of this item.

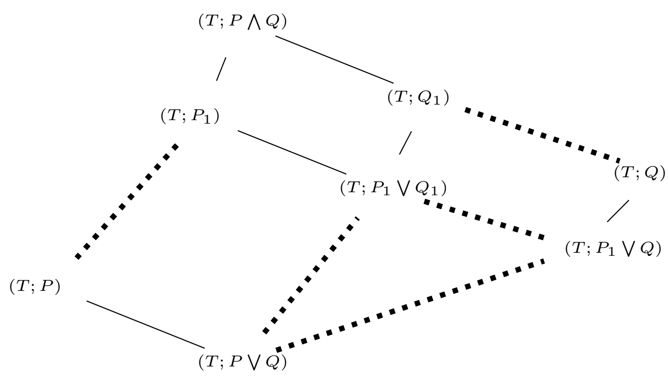

Figure 7: Case 3. solid line: one link; dashed line: one or more links.

4. $h(T;P) - h(T;P\bigvee Q) > 1$ and $h(T;P\bigwedge Q) - h(T;Q) > 1$. Then, $h(T;P\bigwedge Q) - h(T;P) \geq h(T;P\bigwedge Q) - h(T;Q) > 1$. That means $P\bigvee Q$ does not cover P nor does Q cover $P\bigwedge Q$. Figure 8 illustrates the situation. We proceed by induction on $h(T;P) - h(T;P\bigvee Q)$. The case $h(T;P) - h(T;P\bigvee Q) = 1$ corresponds to Item 3. Let us assume that the result holds if $1 \leq h(T;P) - h(T;P\bigvee Q) < l$. Take $(T;Q)$ and $(T;P)$ with $h(T;P) - h(T;P\bigvee Q) = l$. Take $P\bigwedge Q = P_0 \prec P_1 \prec \ldots \prec P_k \prec P_{k+1} = P$, with $k \geq 1$, a chain that joins $P\bigwedge Q$ and P and $P\bigwedge Q = Q_0 \prec Q_1 \prec \ldots \prec Q_r \prec Q_{r+1} = Q$, with $r \geq 1$, a chain that joins $P\bigwedge Q$ and Q. Applying Item 1 to $(T;P_1))$ and $(T;Q_1)$ because $(T;P_1) \vee (T;Q_1) = (T;P\bigwedge Q)$, $h(T;P\bigwedge Q) - h(T;P_1) = 1 = h(T;P\bigwedge Q) - h(T;Q_1)$, we get

$$v\left(T;P\bigwedge Q\right) + v\left(T;P_1\bigvee Q_1\right) \geq v(T;P_1) + v(T;Q_1) \tag{33}$$

Due to the choice of P_1 and Q_1, we have $(T;Q)\vee(T;P_1\bigvee Q_1) = (T;Q_1)$, $h(T;Q_1) - h(T;P_1\bigvee Q_1) = 1$, and $h(T;Q_1) - h(T;Q) \geq 1$. Then, applying Item 1 if $h(T;Q_1) - h(T;Q) = 1$ and applying Item 2 if $h(T;Q_1) - h(T;Q) > 1$ we get

$$v(T;Q_1) + v\left(T;P_1\bigvee Q\right) \geq v(T;Q) + v\left(T;P_1\bigvee Q_1\right). \tag{34}$$

In a similar way if we take $(T;P)$ and $(T;P_1\bigvee Q_1)$, we get

$$v(T;P_1) + v\left(T;P\bigvee Q_1\right) \geq v(T;P) + v\left(T;P_1\bigvee Q_1\right). \tag{35}$$

Finally, we take $(T;P_1\bigvee Q)$ and $(T;P\bigvee Q_1)$. Then, $(T;P_1\bigvee Q)\wedge(T;P\bigvee Q_1) = (T;P\bigvee Q)$ and $(T;P\bigvee Q_1)\vee(T;P_1\bigvee Q) \sqsubseteq (T;P_1\bigvee Q_1)$. Besides, $h(T;P\bigvee Q_1) - h(T;P\bigvee Q) = l-1 < l$. We apply the induction hypothesis and obtain

$$v\left(\left(T;P\bigvee Q_1\right)\vee\left(T;P_1\bigvee Q\right)\right) + v\left(T;P\bigvee Q\right) \geq v\left(T;P\bigvee Q_1\right) + v\left(T;P_1\bigvee Q\right). \tag{36}$$

Adding up Inequalities (33), (34), (35), and (36), and using Lemma 3.1 applied to $(T;P\bigvee Q_1)\vee(T;P_1\bigvee Q) \sqsubseteq (T;P_1\bigvee Q_1)$, we obtain

$$v\left(T;P\bigwedge Q\right) + v\left(T;P\bigvee Q\right) \geq v(T;P) + v(T;Q),$$

concluding the proof. $\square$

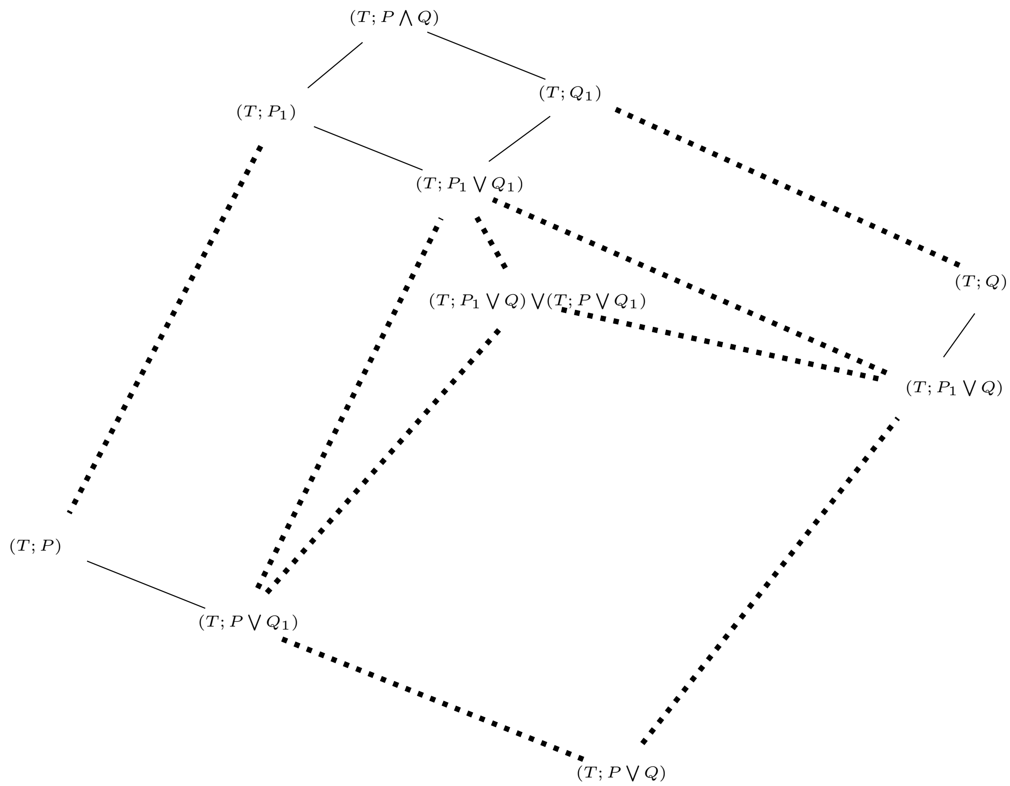

Figure 8: Case 4. solid line: one link; dashed line: more than one link.