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Master in Pure and Applied Logic

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Gentzen Relations and Contextual Deduction-Detachment Theorems

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Abstract

This thesis studies Gentzen relations from the perspective of abstract algebraic logic and generalizes to Gentzen relations the results of [**28**] concerning the contextual deduction-detachment theorem (CDDT) and some of its variants.

We correct a statement in [**27**] that syntactically characterizes protoalgebraic Gentzen relations and we obtain a new version of it that is more suitable when working with sequents. Then, we generalize Raftery's bridge theorem of [**28**], connecting the CDDT and the algebraic property of having equationally semi-definable principal relative congruences (ESPRC), and we present an alternative proof of it based on the work of Blok and Pigozzi in [**6**] that shows more clearly the connection between a CDD-sequence and the equations that semi-define the principal relative congruences.

Keywords: Abstract algebraic logic, Gentzen relation, contextual deduction-detachment theorem, CDDT, equationally semi-definable principal relative congruences, ESPRC.

Resum

Aquest treball estudia les relacions Gentzen des de la perspectiva de la lògica algebraica abstracta i generalitza a relacions Gentzen els resultats de [**28**] que tracten del teorema contextual de la deducció-separació (CDDT) i d'algunes de les seves variants.

Corregim un enunciat de [**27**] que proporciona una caracterització sintàctica de les relacions Gentzen protoalgebraiques i obtenim una nova versió que resulta més útil quan es treballa amb seqüents. Després, generalitzem el teorema de Raftery de [**28**] que connecta el CDDT i la propietat algebraica de tenir congruències relatives principals semidefinibles per equacions (ESPRC), i presentem una demostració alternativa basada en el treball de Blok i Pigozzi a [**6**] que mostra més clarament la connexió entre una CDD-seqüència i les equacions que semidefineixen les congruències relatives principals.

Paraules clau: Lògica algebraica abstracta, relació Gentzen, teorema contextual de la deducció-separació, CDDT, congruències relatives principals semidefinibles per equacions, ESPRC.

Resumen

Este trabajo estudia las relaciones Gentzen desde la perspectiva de la lógica algebraica abstracta y generaliza a relaciones Gentzen los resultados de [**28**] acerca del teorema contextual de la deducción-separación (CDDT) y de algunas de sus variantes.

Corregimos un enunciado de [**27**] que caracteriza sintácticamente las relaciones Gentzen protoalgebraicas y obtenemos una nueva versión que resulta más útil cuando se trabaja con secuentes. Después, generalizamos el teorema de Raftery de [**28**] que conecta el CDDT y la propiedad algebraica de tener congruencias relativas principales semidefinibles por ecuaciones (ESPRC), y presentamos una demostración alternativa basada en el trabajo de Blok y Pigozzi en [**6**] que muestra más claramente la conexión entre una CDD-secuencia y las ecuaciones que semidefinen las congruencias relativas principales.

Palabras clave: Lógica algebraica abstracta, relación Gentzen, teorema contextual de la deducción-separación, CDDT, congruencias relativas principales semidefinibles por ecuaciones, ESPRC.

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Introduction

Abstract algebraic logic studies the connections between logic and universal algebra. More precisely, it seeks to provide a formal ground to the notion of the *algebraic counterpart* of any given logic, that is, a general procedure whereby classes of algebras, or of other objects of an algebraic nature, such as matrices, can be associated with logics in a way that is both natural and allows the tools of universal algebra to be applied to logic (and vice versa).

In its beginnings, abstract algebraic logic focused on *sentential logics*, i.e., on logics considered as consequence relations between (sets of) *formulas*. The notion of *algebraizability*, introduced by Blok and Pigozzi in [**4**], was one of the first major steps towards a general, formal definition of the algebraic counterpart of a sentential logic. Informally,

> A logic is algebraizable if there exists a class of algebras related to the logic in the same way as the class of Boolean algebras is related to classical propositional logic ([**15**, p. 38]).

Gentzen relations (also known as Gentzen systems) were introduced into abstract algebraic logic in the early 1990's by Torrens [**31**] and Rebagliato and Verdú [**29**], as a means of extending the theory of algebraizability to non-algebraizable logics that 'nevertheless have a clear algebraic character' ([**15**, p. 55]).

As an example, consider $\mathcal{C}\ell_{\wedge\vee}$, the conjunction-disjunction fragment of classical propositional logic. One would expect $\mathcal{C}\ell_{\wedge\vee}$ to be algebraizable and have the class of distributive lattices as its algebraic counterpart, but in fact it is well known that $\mathcal{C}\ell_{\wedge\vee}$ is not algebraizable (cf. [**14**, p. 118]). Despite this, Font and Verdú showed in [**16**] that there exists a Gentzen relation that is in some precise, strong sense equivalent to $\mathcal{C}\ell_{\wedge\vee}$ and has the distributive lattices as algebraic counterpart. Another paradigmatic example of how Gentzen relations can be used to understand the algebraic aspects of non-algebraizable logics is the implication-less fragment of intuitionistic propositional logic (cf. [**15**, p. 59]).

This thesis studies Gentzen relations from the perspective of abstract algebraic logic, focusing mainly on protoalgebraic ones, and generalizes to Gentzen relations big portions of Raftery's [**28**] concerning the contextual deduction-detachment theorem.

Chapter 1 deals with the basic preliminaries of logic and universal algebra that are needed to understand the rest of our work.

In Chapter 2 we generalize several results of abstract algebraic logic to Gentzen relations. In particular, we introduce the notion of direct product of matrices in the context of Gentzen relations, we correct a statement in Raftery's [**27**] that syntactically characterizes protoalgebraicity and we obtain a new version of it that we consider to be more akin to Gentzen relations.

Chapter 3 employs the tools presented in Chapter 2 to study the contextual deduction-detachment theorem (CDDT), among some of its variants, in the context of Gentzen relations. The CDDT was introduced by Raftery in [**28**] to extend some of the desirable features of the well-known deduction-detachment theorem (DDT) to logics that do not have a DDT. We generalize Raftery's bridge theorem connecting the CDDT and the algebraic property of having equationally semi-definable principal relative congruences (ESPRC), and we present an alternative proof of it based on the work of Blok and Pigozzi in [**6**] that shows more clearly the connection between a CDD-sequence and the equations that semi-define the principal relative congruences.

Finally, in Chapter 4 we give indications for future work regarding some lines of research that are natural continuations of this thesis.

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CHAPTER 1

Preliminaries

This chapter introduces the basic concepts and results, mainly from universal algebra and logic, upon which our thesis rests.

We work in Zermelo-Fraenkel set theory with the axiom of choice (ZFC). Besides elementary notions of *set theory*, which will not be defined, we assume that the reader is familiar with the basic concepts of *universal algebra* (homomorphisms, congruences, varieties...), *lattice theory* (algebraic lattices, distributivity, modularity...), *abstract algebraic logic* (closure systems, sentential logics, the Leibniz operator, protoalgebraicity...) and *order theory* (posets). Most of these concepts will be defined in order to introduce our notation for them.

One of the main topics of this thesis is the contextual deduction-detachment theorem in the context of Gentzen relations and the correspondence between it and the semi-definability of principal relative congruences via equations. For this reason it is convenient, though not strictly necessary, that the reader be familiar with the deduction-detachment theorem for sentential logics and its connection with the equational definability of principal relative congruences.

1.1. Basic notation

The set of the *natural numbers* $\{0, 1, 2, \dots\}$ is denoted by ω.

For any set A, its *power set* is denoted by $\mathcal{P}(A)$, and its *cardinality* by $|A|$.

If $f : A \to B$ is a function and $X \subseteq A$, then $f{\restriction}X$ denotes the *restriction* of f to X, i.e., the function $f{\restriction}X : X \to B$ given by $f{\restriction}X(x) := f(x)$ for all $x \in X$.

Any map $f : A \to B$ induces two functions on power sets: $f_{\mathcal{P}} : \mathcal{P}(A) \to \mathcal{P}(B)$, defined by $f_{\mathcal{P}}(X) := \{f(x) : x \in X\}$ for all $X \subseteq A$; and $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$, defined by $f^{-1}(Y) := \{a \in A : f(a) \in Y\}$ for all $Y \subseteq B$. Since the context will always avoid ambiguity, we denote $f_{\mathcal{P}}$ by f to simplify certain expressions.

If $\{A_i : i \in I\}$ is a family of sets, its *(generalized) Cartesian product* is denoted by $\prod_{i\in I} A_i$. The elements of $\prod_{i\in I} A_i$ are of the form $\langle a_i : i \in I\rangle$, with each $a_i \in A_i$. If $a \in \prod_{i\in I} A_i$, then for every $i \in I$ we denote by $a(i)$ the element of A_i such that $a = \langle a(i) : i \in I\rangle$. The Cartesian product of finitely many sets $A_1, \dots, A_n$ may be also denoted by $A_1 \times \cdots \times A_n$, and when $A_1 = \cdots = A_n = A$ for some set A, we may simply write A^n.

An *n-ary relation* on a set A, with $n \in \omega$, is a subset of A^n.

Given $n \in \omega$, we denote a *finite sequence* of the form $\langle a_1, \ldots, a_n \rangle$ by the symbol $\vec{a}_n$, so that the subindex is the length of the sequence. The *empty sequence* is identified with the *empty set*, and both are denoted by $\varnothing$. Note that $\vec{a}_0 = \varnothing$. As is usual in set theory, the set of all finite sequences of elements of a set A is denoted by $A^{<\omega}$, and, for any $s_1, s_2 \in A^{<\omega}$, the *concatenation* of s_1 and s_2 is written as $s_1{}^\frown s_2$. We use the symbol $\vec{A}$ to denote the collection of all sequences (of any length, finite or infinite) of elements of A. We shall mainly work with finite or countably infinite sequences.

We usually use sequences to refer to the elements contained in them. For example, by '$\vec{a}_m, \vec{b}_n, c$ are pairwise different' we actually mean 'the elements $a_1, \ldots, a_m$, $b_1, \ldots, b_n, c$ are pairwise different'. Also, we often abuse notation and write $b \in \vec{a}$ or say that b is in $\vec{a}$ to denote that the element b appears in the sequence $\vec{a}$.

The *ordered pair* formed by a and b (in that order) is identified with the sequence of length 2 whose first element is a and whose second element is b, and thus denoted by $\langle a, b \rangle$.

Given any function $f : A \to B$, we also denote by f, as no ambiguity will ever arise, the function $f : \vec{A} \to \vec{B}$ given by $f(\langle a_i : i \in I \rangle) := \langle f(a_i) : i \in I \rangle$ for every sequence $\langle a_i : i \in I \rangle \in \vec{A}$.

When defining a function $f : A \to B$, we often write $f(\vec{a}_n) := \vec{b}_n$, where $n \in \omega$, $\vec{a}_n \in \vec{A}$ and $\vec{b}_n \in \vec{B}$, as an abbreviation of $f(a_1) := b_1, \ldots, f(a_n) := b_n$.

The end of a proof of a claim is marked by $\square$, whereas the rest of the proofs end with the more visible symbol $\blacksquare$.

1.2. Universal algebra

For a comprehensive introduction to the topics of universal algebra with which we shall be concerned, the reader is referred to [**7**], [**8**] and [**22**].

1.2.1. Languages and algebras.

Definition 1.1. An *algebraic language* is a pair $\mathcal{L} = \langle L, \mathsf{ar} \rangle$, where L is a set and $\mathsf{ar} : L \to \omega$. The elements of L are called *symbols*, and for every $\lambda \in L$ the number $\mathsf{ar}(\lambda)$ is called the *arity* of λ. Given any $\lambda \in L$, we say that λ is an *n-ary* symbol, $n \in \omega$, if $\mathsf{ar}(\lambda) = n$. If $c \in L$ is a 0-ary symbol, we say that c is a *constant symbol*, and the n-ary symbols, for all $n > 0$, are called *function symbols*.

The words 'unary', 'binary' and 'ternary' mean '1-ary', '2-ary' and '3-ary', respectively.

DEFINITION 1.2. Let $\mathcal{L} = \langle L, \mathsf{ar} \rangle$ be an algebraic language. An *algebra of type* $\mathcal{L}$, or, briefly, an $\mathcal{L}$-*algebra*, is a pair $\boldsymbol{A} = \langle A, \langle \lambda^{\boldsymbol{A}} : \lambda \in L \rangle \rangle$, where A is a set, that satisfies:

(i) $A \neq \varnothing$.

(ii) For every constant symbol $c \in L$, we have $c^{\boldsymbol{A}} \in A$.

(iii) For every function symbol $f \in L$, we have $f^{\boldsymbol{A}} : A^{\mathsf{ar}(f)} \to A$.

The set A is called the *universe* of $\boldsymbol{A}$. If A is a singleton, $\boldsymbol{A}$ is said to be *trivial*.

When L is finite, say $L = \{\lambda_1, \dots, \lambda_n\}$, $n \in \omega$, we write just $\langle A, \lambda_1^{\boldsymbol{A}}, \dots, \lambda_n^{\boldsymbol{A}} \rangle$ in place of $\langle A, \langle \lambda^{\boldsymbol{A}} : \lambda \in L \rangle \rangle$, often omitting a separate specification of the language, and say that $\boldsymbol{A}$ is an algebra of type $\langle m_1, \dots, m_n \rangle$, where $m_i = \mathsf{ar}(\lambda_i)$ for all $i = 1, \dots, n$. Finally, when $f \in L$ is a binary function symbol, we frequently denote $f^{\boldsymbol{A}}(a, b)$ by $a f^{\boldsymbol{A}} b$, for all $a, b \in A$.

Algebras are denoted by bold Latin letters $\boldsymbol{A}, \boldsymbol{B}, \dots$, and the universe of an algebra is denoted by the corresponding regular Latin letter $A, B, \dots$ In order to reduce verbosity, results about algebras of an arbitrary type are stated without mentioning the algebraic language involved, except when this omission would be a source of ambiguity.

Henceforward we shall only consider classes of algebras *of the same type*, even when we do not explicitly specify it, so that, for example, by 'let K be a class of algebras' we shall always mean 'let K be a class of algebras of the same type'.

DEFINITION 1.3. Let $\mathcal{L} = \langle L, \mathsf{ar} \rangle$ be an algebraic language and $\boldsymbol{A}$ an $\mathcal{L}$-algebra. A subset $B \subseteq A$ is said to be $\mathcal{L}$-*closed* (in $\boldsymbol{A}$), or a *subuniverse* (of $\boldsymbol{A}$), if:

(i) For every constant symbol $c \in L$, we have $c^{\boldsymbol{A}} \in B$.

(ii) For every $n > 0$ and every n-ary function symbol $f \in L$, we have $f^{\boldsymbol{A}}(b_1, \dots, b_n) \in B$ for all $b_1, \dots, b_n \in B$.

Note that $\varnothing$ is a subuniverse of $\boldsymbol{A}$ iff L has no constant symbols.

The collection of all subuniverses of a given algebra $\boldsymbol{A}$ is denoted by $\mathbf{Sub}(\boldsymbol{A})$. It is easy to see that $\mathbf{Sub}(\boldsymbol{A})$ is closed under intersections of non-empty families, so, since $A \in \mathbf{Sub}(\boldsymbol{A})$, for every $X \subseteq A$ there exists the least (with respect to set inclusion) subuniverse of $\boldsymbol{A}$ containing X, denoted by $\mathrm{Sg}^{\boldsymbol{A}}(X)$.

DEFINITION 1.4. Let $\mathcal{L} = \langle L, \mathsf{ar} \rangle$ be an algebraic language and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $\mathcal{L}$-algebras. We say that $\boldsymbol{B}$ is a *subalgebra* of $\boldsymbol{A}$, and we write $\boldsymbol{B} \subseteq \boldsymbol{A}$, if:

(i) B is a subuniverse of $\boldsymbol{A}$.

(ii) For every constant symbol $c \in L$, we have $c^{\boldsymbol{B}} = c^{\boldsymbol{A}}$.

(iii) For every function symbol $f \in L$, we have $f^{\boldsymbol{B}} = f^{\boldsymbol{A}} {\restriction} B^{\mathsf{ar}(f)}$.

REMARK 1.5. A subuniverse B of an algebra $\boldsymbol{A}$ is the universe of a subalgebra of $\boldsymbol{A}$ iff $B \neq \varnothing$.

If $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, we define the *inclusion map* $j : B \to A$ by setting $j(b) := b$ for all $b \in B$.

DEFINITION 1.6. Let $\boldsymbol{A}$ be an algebra and $X \subseteq A$. We say that $\boldsymbol{A}$ is *generated* by X if $\mathrm{Sg}^{\boldsymbol{A}}(X) = A$, and we call X a set of *generators* (for $\boldsymbol{A}$).

When X is finite, say $X = \{g_1, \ldots, g_n\}$, $n \in \omega$, we drop the brackets and simply say that $\boldsymbol{A}$ is *(finitely) generated* by $g_1, \ldots, g_n$.

REMARK 1.7. Let $\mathcal{L} = \langle L, \mathsf{ar} \rangle$ be an algebraic language, and let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra generated by some $X \subseteq A$. If L has no constant symbols, then $X \neq \varnothing$.

PROPOSITION 1.8 (cf. [**22**, Thm. 1.9]). *Let $\mathcal{L} = \langle L, \mathsf{ar} \rangle$ be an algebraic language and $\boldsymbol{A}$ an $\mathcal{L}$-algebra. For every $X \subseteq A$, let*

$$E(X) := \{f^{\boldsymbol{A}}(a_1, \ldots, a_m) : m > 0, f \in L \textit{ is } m\textit{-ary and } a_1, \ldots, a_m \in X\}$$

and define X_n, for every $n \in \omega$, recursively as follows:

- $X_0 := X \cup \{c^{\boldsymbol{A}} : c$ *is a constant symbol of* $L\}$.
- $X_{n+1} := X_n \cup E(X_n)$.

Then, $\mathrm{Sg}^{\boldsymbol{A}}(X) = \bigcup_{n \in \omega} X_n$.

Note that Proposition 1.8 allows us to obtain results concerning generated algebras using induction.

1.2.2. Homomorphisms.

DEFINITION 1.9. Let $\mathcal{L} = \langle L, \mathsf{ar} \rangle$ be an algebraic language, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $\mathcal{L}$-algebras. A function $h : A \to B$ is said to be a *homomorphism* (from $\boldsymbol{A}$ to $\boldsymbol{B}$) if the following hold:

(i) For every constant symbol $c \in L$, we have $h(c^{\boldsymbol{A}}) = c^{\boldsymbol{B}}$.

(ii) For every $n > 0$, and every n-ary function symbol $f \in L$, we have $h(f^{\boldsymbol{A}}(a_1, \ldots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \ldots, h(a_n))$ for all $a_1, \ldots, a_n \in A$.

If in addition h is bijective, then we say that h is an *isomorphism* (between $\boldsymbol{A}$ and $\boldsymbol{B}$) and we write $h : \boldsymbol{A} \cong \boldsymbol{B} : h^{-1}$. In this case the algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic*, and we may simply write $\boldsymbol{A} \cong \boldsymbol{B}$ when the isomorphism h need not be specified.

The family of all homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$ is denoted by $\mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$. If $h \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$ and $\boldsymbol{A}=\boldsymbol{B}$, then h is said to be an *endomorphism* (of $\boldsymbol{A}$). We define $\mathrm{End}(\boldsymbol{A}) := \mathrm{Hom}(\boldsymbol{A},\boldsymbol{A})$.

It is well known that the composition of two homomorphisms is also a homomorphism. Also, if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $j : B \to A$ is the corresponding inclusion map, then $j \in \mathrm{Hom}(\boldsymbol{B},\boldsymbol{A})$. Another well-known algebraic result is that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are algebras of the same type and $h \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$, then $h(A)$, the *image* of h, is the universe of a subalgebra of $\boldsymbol{B}$, which we denote by $h(\boldsymbol{A})$. Therefore, $\boldsymbol{A} \cong h(\boldsymbol{A}) \subseteq \boldsymbol{B}$.

DEFINITION 1.10. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same type, and let $h \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$. The *kernel* of h, denoted by $\ker h$, is the binary relation on A given by $\ker h := \{\langle a,b\rangle \in A\times A : h(a)=h(b)\}$.

The following results can be easily obtained by induction, using Proposition 1.8:

PROPOSITION 1.11. *Let $\boldsymbol{B}$ be an algebra generated by some $X \subseteq B$, let $\boldsymbol{A}$ be an algebra of the same type as $\boldsymbol{B}$, and let $h \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$ be such that $X \subseteq h(A)$. Then, h is surjective.*

PROPOSITION 1.12. *Let $\boldsymbol{A}$ be an algebra generated by some $X \subseteq A$, and let $\boldsymbol{B}$ be an algebra of the same type as $\boldsymbol{A}$. If $g_1, g_2 \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$ are two homomorphisms such that $g_1(x) = g_2(x)$ for every $x \in X$, then $g_1 = g_2$.*

1.2.3. Free algebras and the algebra of formulas.

DEFINITION 1.13. Let $\mathcal{L}$ be an algebraic language, K a class of $\mathcal{L}$-algebras and $\boldsymbol{A}$ an $\mathcal{L}$-algebra generated by some $X \subseteq A$. We say that $\boldsymbol{A}$ is *free for* K *over* X if, for every algebra $\boldsymbol{B} \in \mathsf{K}$, every function $g : X \to B$ can be extended to a homomorphism $g \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$.

In this situation, we say that $\boldsymbol{A}$ is *freely generated by* X. And if K is the class of all $\mathcal{L}$-algebras, then we say that $\boldsymbol{A}$ is *absolutely free over* X.

DEFINITION 1.14. Let $\mathcal{L} := \langle L, \mathsf{ar}\rangle$ be an algebraic language and X a non-empty set such that $X \cap L = \varnothing$ and $(X\cup L)\cap (X\cup L)^{<\omega} = \varnothing$. The *set of $\mathcal{L}$-formulas over* X is the smallest subset $Fm_{\mathcal{L}}(X)$ of $(X\cup L)^{<\omega}$ satisfying:

(i) If $x \in X$, then $\langle x\rangle \in Fm_{\mathcal{L}}(X)$.
(ii) For every constant symbol $c \in L$, we have $\langle c\rangle \in Fm_{\mathcal{L}}(X)$.
(iii) For every $n > 0$, every n-ary function symbol $f \in L$ and all $\psi_1,\ldots,\psi_n \in Fm_{\mathcal{L}}(X)$, we have $\langle f\rangle^\frown\psi_1{}^\frown\cdots{}^\frown\psi_n \in Fm_{\mathcal{L}}(X)$.

As is customary, from now on we denote concatenation by juxtaposition and drop the (outermost) angle brackets when writing formulas (e.g., we write c and $f\psi_1\ldots\psi_n$ instead of $\langle c\rangle$ and $\langle f\rangle^\frown\psi_1{}^\frown\cdots{}^\frown\psi_n$, respectively).

Definition 1.15. Let $\mathcal{L}$ and X be as in Definition 1.14. The *algebra of $\mathcal{L}$-formulas over* X is the $\mathcal{L}$-algebra $\boldsymbol{Fm}_{\mathcal{L}}(X)$ with universe $Fm_{\mathcal{L}}(X)$ and such that:

(i) For every constant symbol $c \in L$, we have $c^{\boldsymbol{Fm}_{\mathcal{L}}(X)} = c$.
(ii) For every $n > 0$, every n-ary function symbol $f \in L$ and all $\psi_1, \dots, \psi_n \in Fm_{\mathcal{L}}(X)$, we have $f^{\boldsymbol{Fm}_{\mathcal{L}}(X)}(\psi_1, \dots, \psi_n) = f\psi_1 \dots \psi_n$.

The elements of X are called *variables*. Given $x \in X$ and $\varphi \in Fm_{\mathcal{L}}(X)$, we say that x *occurs* in φ if the sequence φ has x as one of its elements. If $\vec{x}$ is a sequence of variables, we write $\varphi(\vec{x})$ or $\varphi = \varphi(\vec{x})$ to indicate that the variables occurring in φ are all in $\vec{x}$. And if $\psi = \psi(\vec{x})$ for every $\psi \in \Gamma$, where $\Gamma \subseteq \boldsymbol{Fm}_{\mathcal{L}}(X)$, we write $\Gamma(\vec{x})$ or $\Gamma = \Gamma(\vec{x})$. When $\vec{x}$ is finite, say $\vec{x} = \langle x_1, \dots, x_n \rangle$, $n \in \omega$, we usually write $\varphi(x_1, \dots, x_n)$ and $\Gamma(x_1, \dots, x_n)$ for, respectively, $\varphi(\vec{x})$ and $\Gamma(\vec{x})$.

Until the end of this subsection, let $\boldsymbol{Fm}_{\mathcal{L}}(X)$ be as in Definition 1.15, for an algebraic language $\mathcal{L}$ and a set of variables X.

Theorem 1.16 (cf. [**7**, Thm. 10.8]). *$\boldsymbol{Fm}_{\mathcal{L}}(X)$ is absolutely free over X.*

Arguing inductively on the complexity of the formulas we can obtain:

Proposition 1.17. *Let $\boldsymbol{A}$ be $\mathcal{L}$-algebra. If $h, h' \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$ are such that $h(\vec{x}) = h'(\vec{x})$ for some sequence of variables $\vec{x}$, then $h(\varphi) = h'(\varphi)$ for every $\varphi(\vec{x}) \in \boldsymbol{Fm}_{\mathcal{L}}(X)$.*

Therefore, the following definition makes sense:

Definition 1.18. Let $\vec{x} := \langle x_i : i \in I \rangle$ be a sequence of variables, $\boldsymbol{A}$ an $\mathcal{L}$-algebra and $\vec{a} := \langle a_i : i \in I \rangle$ a sequence of elements of A of the same length as $\vec{x}$. For every formula $\varphi(\vec{x}) \in \boldsymbol{Fm}_{\mathcal{L}}(X)$, the *interpretation of φ in $\boldsymbol{A}$ with respect to $\vec{a}$*, denoted by $\varphi^{\boldsymbol{A}}(\vec{a})$, is the image of φ by any homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$ such that $h(\vec{x}) = \vec{a}$.

For any $\Gamma(\vec{x}) \subseteq \boldsymbol{Fm}_{\mathcal{L}}(X)$, we define $\Gamma^{\boldsymbol{A}}(\vec{a}) := \{\psi^{\boldsymbol{A}}(\vec{a}) : \psi(\vec{x}) \in \Gamma\}$.

As usual, we drop the superindex $\boldsymbol{A}$ when $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}(X)$. And when $\vec{a}$ is finite, say $\vec{a} = \langle a_1, \dots, a_n \rangle$ for some $n \in \omega$, we frequently write $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$ and $\Gamma^{\boldsymbol{A}}(a_1, \dots, a_n)$ for $\varphi^{\boldsymbol{A}}(\vec{a})$ and $\Gamma^{\boldsymbol{A}}(\vec{a})$, respectively.

Remark 1.19. The interpretation of a formula depends only on the interpretations of the variables that *actually* occur in it.

Another useful result that can be easily proved by induction is:

Lemma 1.20. *Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra, $B \subseteq A$, $\varphi(u_1, \dots, u_n) \in \boldsymbol{Fm}_{\mathcal{L}}(X)$ and $b_1, \dots, b_n \in B$. Then, $\varphi^{\boldsymbol{A}}(b_1, \dots, b_n) \in \mathrm{Sg}^{\boldsymbol{A}}(B)$, and if, additionally, B is the universe of a subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$, then $\varphi^{\boldsymbol{A}}(b_1, \dots, b_n) = \varphi^{\boldsymbol{B}}(b_1, \dots, b_n)$.*

An expression of the form $\varphi(\vec{y})$, where $\varphi \in Fm_{\mathcal{L}}(X)$ and $\vec{y}$ is a sequence of variables, is ambiguous: it can denote either the fact that all the variables occurring in φ are in $\vec{y}$, or the interpretation of φ in $\boldsymbol{Fm}_{\mathcal{L}}(X)$ with respect to $\vec{y}$. The next proposition, which can be easily proved by induction, shows that this ambiguity is not problematic. In order to properly state it, we write the superindex $\boldsymbol{Fm}_{\mathcal{L}}(X)$ whenever interpreting a formula in $\boldsymbol{Fm}_{\mathcal{L}}(X)$:

Proposition 1.21. *Let $\varphi(\vec{x}) \in Fm_{\mathcal{L}}(X)$, $\sigma \in \mathrm{End}(\boldsymbol{Fm}_{\mathcal{L}}(X))$ and $\psi := \sigma(\varphi)$. If a variable y occurs in $\sigma(\varphi) = \varphi^{\boldsymbol{Fm}_{\mathcal{L}}(X)}(\sigma(\vec{x}))$, then y occurs in $\sigma(x)$ for some $x \in \vec{x}$. In particular, if $\sigma(\vec{x}) = \vec{y}$, where $\vec{y}$ is a sequence of variables, then all the variables that occur in $\varphi^{\boldsymbol{Fm}_{\mathcal{L}}(X)}(\vec{y})$ are in $\vec{y}$.*

The algebra of formulas $\boldsymbol{Fm}_{\mathcal{L}}(X)$ is the 'template' from which all the $\mathcal{L}$-algebras generated by at most $|X|$-many elements are built, in the following sense:

Theorem 1.22. *Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra and $B \subseteq A$ such that $|B| \leq |X|$. Then, $\boldsymbol{A}$ is generated by B iff for all $a \in A$ there are $n \in \omega$ and $\varphi(u_1, \dots, u_n) \in \boldsymbol{Fm}_{\mathcal{L}}(X)$, $u_1, \dots, u_n \in X$, such that $a = \varphi^{\boldsymbol{A}}(b_1, \dots, b_n)$ for some $b_1, \dots, b_n \in B$.*

Proof. ($\Rightarrow$) Pick any surjective map $h : X \to B$. Note that $h : X \to A$, so by Theorem 1.16 and Proposition 1.11 we can extend h to a surjective homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$. Let $\varphi \in \boldsymbol{Fm}_{\mathcal{L}}(X)$ be such that $h(\varphi) = a$, and let $u_1, \dots, u_n$ be the variables that occur in φ, for suitable $n \in \omega$. Then,

$$a = h(\varphi) = \varphi^{\boldsymbol{A}}(h(u_1), \dots, h(u_n)),$$

and $h(u_i) \in B$ for all $i = 1, \dots, n$.

($\Leftarrow$) Let $a \in A$. By assumption, there is a formula $\varphi(u_1, \dots, u_n)$ such that $a = \varphi^{\boldsymbol{A}}(b_1, \dots, b_n)$ for some $b_1, \dots, b_n \in B$, so by Lemma 1.20 we have $a \in \mathrm{Sg}^{\boldsymbol{A}}(B)$. ■

Lemma 1.23 (cf. [**14**, Lem. 1.2]). *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $\mathcal{L}$-algebras, $g \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ a surjective homomorphism and $f \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{B})$. Then, there is some homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$ such that $f = g \circ h$.*

1.2.4. Congruences and quotient algebras. If A is a set, we define the following relations on A: $\boldsymbol{\Delta}_A := \{\langle a, a\rangle : a \in A\}$, called the *identity relation* (on A); and $\boldsymbol{\nabla}_A := A \times A$, called the *total relation* (on A).

Definition 1.24. Let $\mathcal{L} = \langle L, \mathsf{ar}\rangle$ be an algebraic language and $\boldsymbol{A}$ an $\mathcal{L}$-algebra. A set $\theta \subseteq A \times A$ is said to be a *congruence* of $\boldsymbol{A}$ if:

(i) θ is an equivalence relation.

(ii) θ satisfies the so-called *compatibility condition*: for every $n > 0$, every n-ary function symbol $f \in L$ and all $\vec{a}_n, \vec{b}_n \in \vec{A}$, if $\langle a_i, b_i\rangle \in \theta$ for all $1 \leq i \leq n$, then $\langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(b_1, \dots, b_n)\rangle \in \theta$.

If $a \in A$, the equivalence class of a with respect to θ is denoted by a/θ, and for every $F \subseteq A$ we define $F/\theta := \{a/\theta : a \in F\}$.

REMARK 1.25. Given an algebra $\boldsymbol{A}$, both $\boldsymbol{\Delta}_A$ and $\boldsymbol{\nabla}_A$ are congruences of $\boldsymbol{A}$. Therefore, $\boldsymbol{\Delta}_A$ is the least congruence of $\boldsymbol{A}$ and $\boldsymbol{\nabla}_A$ the largest one.

The collection of all congruences of an algebra $\boldsymbol{A}$ is denoted by $\mathrm{Co}(\boldsymbol{A})$. As with subuniverses, the intersection of any non-empty family of congruences of an algebra $\boldsymbol{A}$ is also a congruence of $\boldsymbol{A}$. So, since $\boldsymbol{\nabla}_{\boldsymbol{A}} \in \mathrm{Co}(\boldsymbol{A})$, for every $X \subseteq A \times A$ there exists the least congruence of $\boldsymbol{A}$ that includes X, which we denote by $\boldsymbol{\Theta}^{\boldsymbol{A}}(X)$, called the *congruence (of $\boldsymbol{A}$) generated by X*. We abbreviate $\boldsymbol{\Theta}^{\boldsymbol{A}}(\{\langle a, b\rangle\})$ to $\boldsymbol{\Theta}^{\boldsymbol{A}}(a, b)$. Congruences of this form, i.e., generated just by one element $\langle a, b\rangle$, are said to be *principal*.

Congruences play an important role in abstract algebra because they allow the formation of quotients that preserve the algebraic structure:

DEFINITION 1.26. Let $\mathcal{L} = \langle L, \mathsf{ar}\rangle$ be an algebraic language, $\boldsymbol{A}$ an $\mathcal{L}$-algebra and θ a congruence of $\boldsymbol{A}$. The *quotient algebra* $\boldsymbol{A}/\theta$ is the $\mathcal{L}$-algebra having A/θ as its universe and such that:

(i) For every constant symbol $c \in L$, we have $c^{\boldsymbol{A}/\theta} = c^{\boldsymbol{A}}/\theta$.
(ii) For every $n > 0$, every n-ary function symbol $f \in L$ and all elements $a_1/\theta, \dots, a_n/\theta \in A/\theta$, we have $f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta$.

Note that $f^{\boldsymbol{A}/\theta}$ is well defined by the compatibility condition.

Given an algebra $\boldsymbol{A}$ and a congruence $\theta \in \mathrm{Co}(\boldsymbol{A})$, the *natural projection* (with respect to θ) is the function $\pi_\theta : A \to A/\theta$ given by $\pi_\theta(a) := a/\theta$ for all $a \in A$. When the context avoids ambiguity, we may denote π_θ by π. It is well known that π_θ is a surjective homomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}/\theta$.

Besides $\boldsymbol{\Delta}_A$ and $\boldsymbol{\nabla}_A$, another well-known example of congruence of an algebra $\boldsymbol{A}$ is $\ker h$ for any $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$, where $\boldsymbol{B}$ is any algebra of the same type as $\boldsymbol{A}$. In this situation, the first isomorphism theorem says that $\boldsymbol{A}/\ker h \cong h(\boldsymbol{A})$.

Using induction, we can easily prove that formulas are interpreted in quotient algebras as expected:

PROPOSITION 1.27. *Let $\boldsymbol{Fm}_{\mathcal{L}}(X)$ be as in Definition 1.15, $\boldsymbol{A}$ an $\mathcal{L}$-algebra and $\theta \in \mathrm{Co}(\boldsymbol{A})$. For every formula $\varphi(\vec{x}_n) \in \boldsymbol{Fm}_{\mathcal{L}}(X)$, where $x_1, \dots, x_n \in X$, $n \in \omega$, and all elements $a_1, \dots, a_n \in A$, we have:*

$$\varphi^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = \varphi^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta.$$

And Theorem 1.22 yields:

PROPOSITION 1.28. *Let $\boldsymbol{A}$ be an algebra generated by some $B \subseteq A$, and let $\theta \in \mathrm{Co}(\boldsymbol{A})$. Then, $\boldsymbol{A}/\theta$ is generated by B/θ.*

DEFINITION 1.29. Let $\mathcal{L}$ be an algebraic language, K a class of $\mathcal{L}$-algebras and $\boldsymbol{A}$ any $\mathcal{L}$-algebra (not necessarily in K). We say that a congruence $\theta \in \mathrm{Co}(\boldsymbol{A})$ is a K-*relative congruence* of $\boldsymbol{A}$, or, briefly, a K-*congruence* of $\boldsymbol{A}$, if $\boldsymbol{A}/\theta \in \mathsf{K}$.

The collection of all K-congruences of $\boldsymbol{A}$ is denoted by $\mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$.

1.2.5. Direct products and ultraproducts.

DEFINITION 1.30. Let $\mathcal{L} := \langle L, \mathsf{ar}\rangle$ be an algebraic language, and $\{\boldsymbol{A}_i : i \in I\}$ a family of $\mathcal{L}$-algebras. The *direct product* of $\{\boldsymbol{A}_i : i \in I\}$, in symbols $\prod_{i\in I} \boldsymbol{A}_i$, is the $\mathcal{L}$-algebra $\boldsymbol{A}$ with universe $A := \prod_{i\in I} A_i$ and such that:

(i) For every constant symbol $c \in L$, we have $c^{\boldsymbol{A}} = \langle c^{\boldsymbol{A}_i} : i \in I\rangle$.

(ii) For every $n > 0$, every n-ary function symbol $f \in L$ and all elements $a_1, \dots, a_n \in A$, we have $f^{\boldsymbol{A}}(a_1, \dots, a_n) = \langle f^{\boldsymbol{A}_i}(a_1(i), \dots, a_n(i)) : i \in I\rangle$.

If $\boldsymbol{A} := \prod_{i\in I} \boldsymbol{A}_i$ is the direct product of some algebras $\{\boldsymbol{A}_i : i \in I\}$, for every $i \in I$ the *i-th projection* is the function $\pi_i : A \to A_i$ defined as $\pi_i(a) := a(i)$ for every $a \in A$, so that $a = \langle \pi_i(a) : i \in I\rangle$. It is well known that each π_i is a surjective homomorphism from $\prod_{i\in I} \boldsymbol{A}_i$ to $\boldsymbol{A}_i$.

Using induction, we can easily prove that formulas are interpreted in direct products as expected:

PROPOSITION 1.31. *Let $\boldsymbol{Fm}_{\mathcal{L}}(X)$ be as in Definition 1.15, and $\{\boldsymbol{A}_i : i \in I\}$ a family of $\mathcal{L}$-algebras. For every formula $\varphi(\vec{x}_n) \in \boldsymbol{Fm}_{\mathcal{L}}(X)$, where $x_1, \dots, x_n \in X$, $n \in \omega$, and every $a_1, \dots, a_n \in \prod_{i\in I} A_i$, we have:*

$$\varphi^{\prod_{i\in I} \boldsymbol{A}_i}(a_1, \dots, a_n) = \langle \varphi^{\boldsymbol{A}_i}(a_1(i), \dots, a_n(i)) : i \in I\rangle.$$

The following is a well-known result that will be needed in Subsection 3.1.2:

PROPOSITION 1.32. *Let $\mathcal{L}$ be an algebraic language and $\{\boldsymbol{A}_i : i \in I\}$ a family of $\mathcal{L}$-algebras. If the direct product $\boldsymbol{A} := \prod_{i\in I} \boldsymbol{A}_i$ is generated by some set $X \subseteq A$, then each $\boldsymbol{A}_i$ is generated by $\pi_i(X)$.*

DEFINITION 1.33. Let A be a set. A *filter* over A is a family $\mathcal{F} \subseteq \mathcal{P}(A)$ of subsets of A such that:

(i) $A \in \mathcal{F}$.

(ii) If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$, for all $X, Y \subseteq A$.

(iii) If $X \in \mathcal{F}$ and $X \subseteq Y$, then $Y \in \mathcal{F}$, for all $X, Y \subseteq A$.

If $\varnothing \notin \mathcal{F}$, i.e., if $\mathcal{F} \neq \mathcal{P}(A)$, then we say that $\mathcal{F}$ is *proper*.

DEFINITION 1.34. Let A be a set. An *ultrafilter* over A is a proper filter $\mathcal{U}$ over A such that, for all $X \subseteq A$, either $X \in \mathcal{U}$ or $A \setminus X \in \mathcal{U}$ (but not both, since $\mathcal{U}$ is proper).

Let $\mathcal{L} := \langle L, \mathsf{ar} \rangle$ be an algebraic language, $\{\boldsymbol{A}_i : i \in I\}$ a family of $\mathcal{L}$-algebras, $\boldsymbol{A} := \prod_{i \in I} \boldsymbol{A}_i$ and $\mathcal{U}$ and ultrafilter over I. Define the binary relation $\theta_{\mathcal{U}}$ on A by setting:

$$\langle a, b \rangle \in \theta_{\mathcal{U}} \iff \{i \in I : a(i) = b(i)\} \in \mathcal{U}$$

for all $a, b \in A$. It is easy to see that $\theta_{\mathcal{U}}$ is a congruence of $\boldsymbol{A}$ (cf. [**7**, Lem. 6.2]). Therefore, the following definition makes sense:

DEFINITION 1.35. Let $\mathcal{L} := \langle L, \mathsf{ar} \rangle$ be an algebraic language, $\{\boldsymbol{A}_i : i \in I\}$ a family of $\mathcal{L}$-algebras and $\mathcal{U}$ and ultrafilter over I. The *ultraproduct* of $\{\boldsymbol{A}_i : i \in I\}$, in symbols $\prod_{i \in I} \boldsymbol{A}_i / \mathcal{U}$, is the $\mathcal{L}$-algebra $(\prod_{i \in I} \boldsymbol{A}_i)/\theta_{\mathcal{U}}$.

For every $a \in \prod_{i \in I} A_i$, we denote $a/\theta_{\mathcal{U}}$ by $a/\mathcal{U}$.

1.2.6. Equations and quasiequations. Let $\mathcal{L}$ be an algebraic language and X a set of variables, as in Definition 1.15.

DEFINITION 1.36. An *equation* (over $\boldsymbol{Fm}_{\mathcal{L}}(X)$) is a pair of formulas $\langle \delta, \varepsilon \rangle$.

When working with equations, we write $\delta \approx \varepsilon$ interchangeably with $\langle \delta, \varepsilon \rangle$.

DEFINITION 1.37. Given an $\mathcal{L}$-algebra $\boldsymbol{A}$, a set of equations Θ and a homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$, we say that $\boldsymbol{A}$ *satisfies* Θ *with respect to* h, in symbols $\boldsymbol{A} \models \Theta[\![h]\!]$, if $h(\delta) = h(\varepsilon)$ for all equations $\delta \approx \varepsilon \in \Theta$.

If there are some $x_1, \ldots, x_n \in X$, $n \in \omega$, such that $\delta = \delta(x_1, \ldots, x_n)$ and $\varepsilon = \varepsilon(x_1, \ldots, x_n)$ for every equation $\delta \approx \varepsilon \in \Theta$, then we write $\boldsymbol{A} \models \Theta[\![a_1, \ldots, a_n]\!]$ or $\boldsymbol{A} \models \Theta[\![\vec{a}_n]\!]$ to denote that $\boldsymbol{A} \models \Theta[\![h]\!]$ holds for any $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$ such that $h(\vec{x}_n) = \vec{a}_n$, where $\vec{a}_n \in \vec{A}$.

When Θ is a singleton we omit the curly braces, so that, for example, we write $\boldsymbol{A} \models \delta \approx \varepsilon[\![h]\!]$ instead of $\boldsymbol{A} \models \{\delta \approx \varepsilon\}[\![h]\!]$.

DEFINITION 1.38. Given a class of $\mathcal{L}$-algebras K and a set of equations Θ, we say that Θ is *valid* in K, in symbols $\mathsf{K} \models \Theta$, if $\boldsymbol{A} \models \Theta[\![h]\!]$ for all $\boldsymbol{A} \in \mathsf{K}$ and all $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$.

When K or Θ are singletons we omit the curly braces, so that, for example, we write $\boldsymbol{A} \models \delta \approx \varepsilon$ instead of $\{\boldsymbol{A}\} \models \{\delta \approx \varepsilon\}$.

DEFINITION 1.39. Let Θ be a set of equations. The *equational models* of Θ is the class of $\mathcal{L}$-algebras $\mathrm{Mod}(\Theta) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an } \mathcal{L}\text{-algebra and } \boldsymbol{A} \models \Theta\}$.

REMARK 1.40. $\mathrm{Mod}(\Theta) \models \Theta$ for every set of equations Θ.

Definition 1.41. A class of $\mathcal{L}$-algebras K is said to be an *equational class* if there is a set of equations Θ such that $\mathsf{K} = \mathrm{Mod}(\Theta)$.

Definition 1.42. A *quasiequation* (over $\boldsymbol{Fm}_{\mathcal{L}}(X)$) is an expression of the form $\alpha_1 \approx \beta_1 \,\&\, \cdots \,\&\, \alpha_n \approx \beta_n \Rightarrow \delta \approx \varepsilon$, where $n \in \omega$ and $\alpha_1 \approx \beta_1, \dots, \alpha_n \approx \beta_n, \delta \approx \varepsilon$ are equations. We may abbreviate it to $\&_{1 \leq i \leq n}\, \alpha_i \approx \beta_i \Rightarrow \delta \approx \varepsilon$.

When $n = 0$, we identify the quasiequation $\Rightarrow \delta \approx \varepsilon$ with the equation $\delta \approx \varepsilon$, so that every equation is a quasiequation.

As with equations, we now define satisfaction and validity for quasiequations:

Definition 1.43. Given an $\mathcal{L}$-algebra $\boldsymbol{A}$, a set of quasiequations Θ and a homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$, we say that $\boldsymbol{A}$ *satisfies* Θ *with respect to* h, in symbols $\boldsymbol{A} \models \Theta[\![h]\!]$, if, for every quasiequation $\&_{1 \leq i \leq n}\alpha_i \approx \beta_i \Rightarrow \delta \approx \varepsilon \in \Theta$, if $\boldsymbol{A} \models \alpha_i \approx \beta_i[\![h]\!]$ for all $i = 1, \dots, n$, then $\boldsymbol{A} \models \delta \approx \varepsilon[\![h]\!]$.

If there are some $x_1, \dots, x_n \in X$, $n \in \omega$, such that for every quasiequation $\&_{1 \leq i \leq m}\alpha_i \approx \beta_i \Rightarrow \delta \approx \varepsilon \in \Theta$ we have $\eta = \eta(x_1, \dots, x_n)$ for every formula $\eta \in \{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m, \delta, \varepsilon\}$, then we write $\boldsymbol{A} \models \Theta[\![a_1, \dots, a_n]\!]$ or $\boldsymbol{A} \models \Theta[\![\vec{a}_n]\!]$ to denote that $\boldsymbol{A} \models \Theta[\![h]\!]$ holds for any $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$ such that $h(\vec{x}_n) = \vec{a}_n$, where $\vec{a}_n \in \vec{A}$.

When Θ is a singleton we omit the curly braces, so that, for example, we write $\boldsymbol{A} \models \&_{1 \leq i \leq n}\alpha_i \approx \beta_i \Rightarrow \delta \approx \varepsilon[\![h]\!]$ instead of $\boldsymbol{A} \models \{\&_{1 \leq i \leq n}\alpha_i \approx \beta_i \Rightarrow \delta \approx \varepsilon\}[\![h]\!]$.

Definition 1.44. Given a class of $\mathcal{L}$-algebras K and a set of quasiequations Θ, we say that Θ is *valid* in K, in symbols $\mathsf{K} \models \Theta$, if $\boldsymbol{A} \models \Theta[\![h]\!]$ for all $\boldsymbol{A} \in \mathsf{K}$ and all $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}(X), \boldsymbol{A})$.

When K or Θ are singletons we omit the curly braces, so that, for example, we write $\boldsymbol{A} \models \&_{1 \leq i \leq n}\alpha_i \approx \beta_i \Rightarrow \delta \approx \varepsilon$ instead of $\{\boldsymbol{A}\} \models \{\&_{1 \leq i \leq n}\alpha_i \approx \beta_i \Rightarrow \delta \approx \varepsilon\}$.

Definition 1.45. Let Θ be a set of quasiequations. The *quasiequational models* of Θ is the class of $\mathcal{L}$-algebras $\mathrm{Mod}(\Theta) := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an } \mathcal{L}\text{-algebra and } \boldsymbol{A} \models \Theta\}$.

Remark 1.46. $\mathrm{Mod}(\Theta) \models \Theta$ for every set of quasiequations Θ.

Definition 1.47. A class of $\mathcal{L}$-algebras K is said to be a *quasiequational class* if there is a set of quasiequations Θ such that $\mathsf{K} = \mathrm{Mod}(\Theta)$.

1.2.7. Varieties and quasivarieties.

Definition 1.48. Let K be a class of algebras. We define the following families of algebras:

- $\mathbb{I}(\mathsf{K})$: isomorphic copies of algebras of K.
- $\mathbb{H}(\mathsf{K})$: homomorphic images of algebras of K.

- $\mathbb{S}(\mathsf{K})$: subalgebras of algebras of K.
- $\mathbb{P}(\mathsf{K})$: direct products of non-empty families of algebras of K.
- $\mathbb{P}_{\mathsf{U}}(\mathsf{K})$: ultraproducts of non-empty families of algebras of K.

We call $\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathsf{U}}$ *class operators*, since they can be seen as functions that map classes of algebras to classes of algebras. When applying several class operators at once, we omit all the parenthesis save the innermost ones: for example, we write $\mathbb{HSP}(\mathsf{K})$ in place of $\mathbb{H}(\mathbb{S}(\mathbb{P}(\mathsf{K})))$.

REMARK 1.49. If K is a class of algebras, then $\mathsf{K} \subseteq \mathbb{I}(\mathsf{K})$, $\mathsf{K} \subseteq \mathbb{H}(\mathsf{K})$, $\mathsf{K} \subseteq \mathbb{S}(\mathsf{K})$, $\mathsf{K} \subseteq \mathbb{IP}(\mathsf{K})$ and $\mathbb{P}_{\mathsf{U}}(\mathsf{K}) \subseteq \mathbb{HP}(\mathsf{K})$.

If K is a class of algebras and $\mathbb{O}$ is a class operator, we say that K is *closed* under $\mathbb{O}$ if $\mathbb{O}(\mathsf{K}) \subseteq \mathsf{K}$.

DEFINITION 1.50. Let K be a non-empty class of algebras. We say that K is a *variety* if K is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. We say that K is a *quasivariety* if K contains a trivial algebra and is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_{\mathsf{U}}$.

REMARK 1.51. Every variety is a quasivariety.

THEOREM 1.52 (cf. [**7**, Thm. 11.9]). *A class of algebras K is a variety iff K is an equational class.*

THEOREM 1.53 (cf. [**7**, Thm. 2.25]). *A class of algebras K is a quasivariety iff K is a quasiequational class.*

1.3. Partially ordered sets

DEFINITION 1.54. Let A be a set. A *partial order* on A is a binary relation on A such that, for all $a, b, c \in A$, writing $a \leq b$ for $\langle a, b\rangle \in \leq$, we have:

(i) $a \leq a$. (*Reflexivity*)
(ii) $a \leq b$ and $b \leq c$ imply $a \leq c$. (*Transitivity*)
(iii) $a \leq b$ and $b \leq a$ imply $a = b$. (*Antisymmetry*)

DEFINITION 1.55. A *partially ordered set*, or a *poset*, is a pair $\langle A, \leq\rangle$, where A is a set and $\leq$ is a partial order on A.

DEFINITION 1.56. Let $\langle A, \leq\rangle$ be a poset, $X \subseteq A$ and $a \in A$. Then, a is said to be an *upper bound* of X if $x \leq a$ for all $x \in X$, and the *lowest upper bound*, or the *supremum*, of X if for any upper bound b of X we have $a \leq b$. Analogously, a is said to be a *lower bound* of X if $a \leq x$ for all $x \in X$, and the *greatest lower bound*, or the *infimum*, of X if for any lower bound b of X we have $b \leq a$.

It is straightforward to see that the supremum and infimum of X, if they exist, are unique. The supremum of X is denoted by $\sup X$, and the infimum by

$\inf X$. Whenever we write $\sup X$ (respectively, $\inf X$), we are assuming that the supremum (respectively, infimum) of X exists.

The following are well-known properties of the supremum and the infimum:

PROPOSITION 1.57. *Let* $\langle A, \leq \rangle$ *be a poset, and* $X, Y \subseteq A$. *Then:*

(i) *If* $X \subseteq Y$, *then* $\sup X \leq \sup Y$ *and* $\inf Y \leq \inf X$.
(ii) $\sup(X \cup Y) = \sup\{\sup X, \sup Y\}$.
(iii) $\inf(X \cup Y) = \inf\{\inf X, \inf Y\}$.

Let $\mathbb{P} := \langle P, \leq_P \rangle$ and $\mathbb{Q} := \langle Q, \leq_Q \rangle$ be two posets. A function $h : P \to Q$ is said to be *order-preserving* or *monotone* if $a \leq_P b$ implies $h(a) \leq_Q h(b)$ for all $a, b \in P$. If, moreover, h is bijective and its inverse h^{-1} is also order-preserving, then we call h an *order-isomorphism* and say that $\mathbb{P}$ and $\mathbb{Q}$ are *isomorphic*, in symbols $h : \mathbb{P} \cong \mathbb{Q} : h^{-1}$. When the map h need not be specified, we simply write $\mathbb{P} \cong \mathbb{Q}$.

DEFINITION 1.58. Let $\langle A, \leq \rangle$ be a poset. A subset $D \subseteq A$ is *upwards directed* if for every $a, b \in D$ there is a $c \in D$ such that $a \leq c$ and $b \leq c$.

1.4. Lattices

Lattices are ubiquitous both in universal algebra and in abstract algebraic logic. The reader is referred to [**20**] and [**7**, Ch. 1] for a comprehensive introduction to lattice theory, as we shall only present a few basic concepts and results.

There are two (equivalent) ways to define the notion of lattice: an order-theoretic approach and an algebraic approach. We shall need to see lattices both as posets and as algebras, so we state both definitions:

DEFINITION 1.59. From the order-theoretic perspective, a *lattice* is a poset $\langle L, \leq \rangle$ such that $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in L for all $a, b \in L$.

DEFINITION 1.60. From the (universal) algebraic perspective, a *lattice* is an algebra $\langle L, \wedge, \vee \rangle$ of type $\langle 2, 2 \rangle$ such that, for all $a, b, c \in L$:

(L1) $a \wedge a = a$ and $a \vee a = a$. (*Idempotency*)
(L2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$. (*Commutativity*)
(L3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$. (*Associativity*)
(L4) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$. (*Absorption*)

The operation $\wedge$ is called *meet*, and $\vee$ is called *join*.

It is well known that Definition 1.59 and Definition 1.60 are equivalent, in the following sense:

Theorem 1.61 (cf. [**20**, Thm. 3]).

(i) *If* $\mathbf{L} := \langle L, \leq \rangle$ *is a lattice, then* $\mathbf{L}^{alg} := \langle L, \wedge, \vee \rangle$*, where* $a \wedge b := \inf\{a, b\}$ *and* $a \vee b := \sup\{a, b\}$ *for all* $a, b \in L$*, is a lattice.*

(ii) *If* $\mathbf{L} := \langle L, \wedge, \vee \rangle$ *is a lattice, then* $\mathbf{L}^{ord} := \langle L, \leq \rangle$*, where* $a \leq b$ *iff* $a \wedge b = a$ *(equivalently,* $a \vee b = b$*) for all* $a, b \in L$*, is a lattice.*

(iii) *If* $\mathbf{L} := \langle L, \leq \rangle$ *is a lattice, then* $(\mathbf{L}^{alg})^{ord} = \mathbf{L}$*.*

(iv) *If* $\mathbf{L} := \langle L, \wedge, \vee \rangle$ *is a lattice, then* $(\mathbf{L}^{ord})^{alg} = \mathbf{L}$*.*

The partial order defined in Theorem 1.61(ii) is called the *lattice order* of $\mathbf{L}$.

If $\mathbf{L}_1, \mathbf{L}_2$ are lattices, then clearly every isomorphism between $\mathbf{L}_1$ and $\mathbf{L}_2$, considered as algebras, is an order-isomorphism between $\mathbf{L}_1$ and $\mathbf{L}_2$, considered as posets, and vice versa.

Given a lattice $\langle L, \leq \rangle$ and $X \subseteq L$, we define $\bigvee X := \sup X$ and $\bigwedge X := \inf X$. This notation is justified by Theorem 1.61.

Definition 1.62. A lattice $\langle L, \wedge, \vee \rangle$ is said to be *complete* if $\bigwedge X$ and $\bigvee X$ exist in L for every $X \subseteq L$.

Complete lattices play a major role in abstract algebraic logic due to their relation to closure systems (cf. Section 1.5 below). Note that if $\mathbf{L} := \langle L, \wedge, \vee \rangle$ is a complete lattice, then $\bigvee L$ and $\bigwedge L$ are, respectively, the maximum and minimum elements of $\mathbf{L}$ with respect to the lattice order.

Definition 1.63. Let $\langle L, \leq \rangle$ be a lattice. An element $a \in L$ is said to be *compact* if, for all $X \subseteq L$, if $\bigvee X$ exists and $a \leq \bigvee X$, then there is a finite $Y \subseteq X$ such that $\bigvee Y$ exists and $a \leq \bigvee Y$.

If a lattice $\langle L, \leq \rangle$ has a minimum element $\perp$ with respect to the lattice order, then clearly $\perp$ is a compact element.

Definition 1.64. A complete lattice $\langle L, \wedge, \vee \rangle$ is *algebraic* if for every $a \in L$ there is a (possibly infinite) set of compact elements $X \subseteq L$ such that $a = \bigvee X$.

Definition 1.65. Let $\mathbf{L} := \langle L, \wedge, \vee \rangle$ be a lattice, $\leq$ its lattice order and $a, b \in L$. If the set $\{c \in L : a \leq b \vee c\}$ has a minimum element (with respect to $\leq$), we call it the *dual relative pseudocomplement* of a and b (in that order), and denote it by $a \dot{-} b$.

If $a \dot{-} b$ exists for all $a, b \in L$, we say that $\mathbf{L}$ is *dually Brouwerian*.

Lemma 1.66 (cf. [**7**, Thm. 3.2]). *Let* $\langle L, \wedge, \vee \rangle$ *be a lattice and* $\leq$ *its lattice order. The following conditions are equivalent:*

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *for every* $a, b, c \in L$*.*

- *$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for every $a, b, c \in L$.*

DEFINITION 1.67. A *distributive lattice* is a lattice $\langle L, \wedge, \vee\rangle$ such that any of the equivalent conditions of Lemma 1.66 holds in L.

DEFINITION 1.68. A *modular lattice* is a lattice $\langle L, \wedge, \vee\rangle$ such that, for all $a, b, x \in L$, if $a \leq b$, where $\leq$ is the lattice order, then:

$$b \wedge (x \vee a) = (b \wedge x) \vee a.$$

PROPOSITION 1.69 (cf. [**7**, Thm. 3.4]). *Every distributive lattice is modular.*

Distributive and modular lattices will be used at the end of Chapter 3 to characterize certain particular cases of the local contextual deduction-detachment theorem.

1.4.1. Semilattices. We now turn our attention to join-semilattices, i.e., 'lattices' lacking the meet operation, which will become important in Chapter 3. As with lattices, we can define them as posets or as algebras:

DEFINITION 1.70. From the order-theoretic perspective, a *join-semilattice* is a poset $\langle S, \leq\rangle$ such that $\sup\{a, b\}$ exists in A for all $a, b \in S$.

DEFINITION 1.71. From the (universal) algebraic perspective, a *join-semilattice* is an algebra $\langle S, \vee\rangle$ of type $\langle 2\rangle$ such that, for all $a, b, c \in S$:

(S1) $a \vee a = a$. (*Idempotency*)
(S2) $a \vee b = b \vee a$. (*Commutativity*)
(S3) $a \vee (b \vee c) = (a \vee b) \vee c$. (*Associativity*)

The operation $\vee$ is called *join*.

It is straightforward to obtain a theorem for join-semilattices analogous to Theorem 1.61:

THEOREM 1.72.

(i) *If $\mathbf{S} := \langle S, \leq\rangle$ is a join-semilattice, then $\mathbf{S}^{alg} := \langle S, \vee\rangle$, where $a \vee b := \sup\{a, b\}$ for all $a, b \in S$, is a join-semilattice.*
(ii) *If $\mathbf{S} := \langle S, \vee\rangle$ is a join-semilattice, then $\mathbf{S}^{ord} := \langle S, \leq\rangle$, where $a \leq b$ iff $a \vee b = b$ for all $a, b \in S$, is a join-semilattice.*
(iii) *If $\mathbf{S} := \langle S, \leq\rangle$ is a join-semilattice, then $(\mathbf{S}^{alg})^{ord} = \mathbf{S}$.*
(iv) *If $\mathbf{S} := \langle S, \vee\rangle$ is a join-semilattice, then $(\mathbf{S}^{ord})^{alg} = \mathbf{S}$.*

As before, the order relation defined in Theorem 1.72(ii) is called the *lattice order* of $\mathbf{S}$, and given a join-semilattice $\langle S, \leq\rangle$ and $X \subseteq S$, we define $\bigvee X := \sup X$.

If $\mathbf{S}_1, \mathbf{S}_2$ are join-semilattices, then clearly every isomorphism between $\mathbf{S}_1$ and $\mathbf{S}_2$, considered as algebras, is an order-isomorphism between $\mathbf{S}_1$ and $\mathbf{S}_2$, considered as posets, and vice versa.

Given a lattice $\langle L, \wedge, \vee\rangle$ and $X \subseteq L$, we say that the elements of X form a join-semilattice if $\langle X, \vee\restriction(X \times X)\rangle$ is a join-semilattice.

PROPOSITION 1.73. *Let* $\mathbf{L} := \langle L, \wedge, \vee\rangle$ *be a complete lattice. The compact elements of* $\mathbf{L}$ *form a join-semilattice with a minimum element.*

PROOF. Let K be the set of compact elements of $\mathbf{L}$. Clearly, it suffices to prove that $k_1 \vee k_2 \in K$ for all $k_1, k_2 \in K$. So let $X \subseteq L$ be such that $\bigvee X$ exists and $k_1 \vee k_2 \leq \bigvee X$. Then, $k_1, k_2 \leq \bigvee X$, so there are finite $Y_1, Y_2 \subseteq X$ such that $k_1 \leq \bigvee Y_1$ and $k_2 \leq \bigvee Y_2$. Hence, $k_1 \vee k_2 \leq \bigvee Y_1 \vee \bigvee Y_2 = \bigvee Y_1 \cup Y_2$, so $k_1 \vee k_2 \in K$.

Finally, the minimum element of $\mathbf{L}$, namely $\bigwedge L$, is clearly compact, and thus it is also the minimum element of the join-semilattice. ■

The notions of compact element, of dual relative pseudocomplement and of being dually Brouwerian are defined for join-semilattices in ways analogous to those of Definition 1.63 and Definition 1.65, since these definitions do not depend on the meet operation.

The following lemmas, the second one of which is almost a triviality, will be needed in Chapter 3:

LEMMA 1.74 (cf. [**28**, Rk. 3.16]). *Let* $\mathbf{L}$ *be an algebraic lattice whose join-semilattice of compact elements* $\langle K, \vee\rangle$ *is dually Brouwerian. Let* $a, b \in K$*, so that* $a \dot{-} b$ *exists (and is in* K*). Then, for all* $c \in L$*, compact or not, we have:*

$$a \leq b \vee c \iff a \dot{-} b \leq c.$$

PROOF. ($\Rightarrow$) Assume $a \leq b \vee c$, and let $\{c_i : i \in I\} \subseteq K$ be such that $c = \bigvee_{i\in I} c_i$. Then, $a \leq b \vee \bigvee_{i\in I} c_i$, so, since a is compact, there is a finite $J \subseteq I$ such that $a \leq b \vee \bigvee_{j\in J} c_j$. Given that $a \dot{-} b$ exists in K and every c_j is compact, we get $a \dot{-} b \leq \bigvee_{j\in J} c_j \leq \bigvee_{i\in I} c_i = c$.

($\Leftarrow$) Assume $a \dot{-} b \leq c$. Since $a \leq b \vee (a \dot{-} b)$, the assumption yields $a \leq b \vee c$. ■

LEMMA 1.75. *Let* $\mathbf{L}_1, \mathbf{L}_2$ *be complete lattices, and, for* $i = 1, 2$*, let* $\mathbf{K}_i$ *be the join-semilattice of compact elements of* $\mathbf{L}_i$*. If* $\mathbf{L}_1 \cong \mathbf{L}_2$*, then* $\mathbf{K}_1$ *is dually Brouwerian iff* $\mathbf{K}_2$ *is dually Brouwerian.*

Even though the notions of distributivity and modularity depend both on the meet and the join operations, they can be generalized to join-semilattices:

Definition 1.76. Let $\mathbf{S} := \langle S, \vee \rangle$ be a join-semilattice with lattice order $\leq$, and let $a, b \in S$. We say that $D(a, b)$ *holds in* $\mathbf{S}$ if, for every $c, d \in S$, if $b \leq a \vee c$ and $b \leq a \vee d$, then there is some $e \in S$ such that $e \leq c$, $e \leq d$ and $b \leq a \vee e$.

Following Raftery [**28**, p. 300], who in turn follows Grätzer [**19**, p. 99], we say that $\mathbf{S}$ is *distributive* if $D(a, b)$ holds in $\mathbf{S}$ for every $a, b \in S$.

Definition 1.77. Let $\mathbf{S} := \langle S, \vee \rangle$ be a join-semilattice with lattice order $\leq$, and let $a, b \in S$. We say that $M(a, b)$ *holds in* $\mathbf{S}$ if, for every $c, d \in S$, if $b \leq a \vee c$ and $b \leq a \vee d$, then there is some $e \in S$ such that $e \leq c$, $e \leq a \vee d$ and $b \leq a \vee e$.

Following Raftery [**28**, p. 304], who in turn follows Czelakowski [**11**, p. 172], we say that $\mathbf{S}$ is *modular* if $M(a, b)$ holds in $\mathbf{S}$ for every $a, b \in S$.

Remark 1.78. Every distributive join-semilattice is modular.

Lemma 1.79 (cf. [**28**, Lem. 6.5] and [**11**, Lem. 2.7.4]). *Let $\mathbf{S} := \langle S, \vee \rangle$ be a join-semilattice generated as an algebra by some $X \subseteq S$ and with a minimum element $\perp$. If $D(a, b)$ (respectively, $M(a, b)$) holds in $\mathbf{S}$ for all $a, b \in X$, then $\mathbf{S}$ is distributive (respectively, modular).*

Lemma 1.80 (cf. [**28**, Lem. 6.6] and [**11**, Cor. 2.7.3]). *An algebraic lattice is distributive (respectively, modular) iff its compact elements form a distributive (respectively, modular) join-semilattice.*

1.5. Closure operators, closure systems and consequence relations

For a more detailed exposition of the contents of this section, the reader is referred to [**7**, Ch. 1, §5], [**8**, Ch. 2, §1] and [**14**, Ch. 1, §§1.2-1.5].

1.5.1. Closure operators and closure systems.

Definition 1.81. Let A be a set. A *closure operator* on A is a function $C : \mathcal{P}(A) \to \mathcal{P}(A)$ satisfying:

(i) $X \subseteq C(X)$ for every $X \subseteq A$.
(ii) $X \subseteq Y$ implies $C(X) \subseteq C(Y)$ for all $X, Y \subseteq A$.
(iii) $C(C(X)) = C(X)$ for every $X \subseteq A$.

A subset $X \subseteq A$ is said to be *closed* if $C(X) = X$.

When X is finite, say $X = \{a_1, \ldots, a_m\}$, we usually write $C(a_1, \ldots, a_m)$ in place of $C(\{a_1, \ldots, a_m\})$. Also, for every $X \cup \{b_1, \ldots, b_n\} \subseteq A$, we abbreviate $C(X \cup \{b_1, \ldots, b_n\})$ to $C(X, b_1, \ldots, b_n)$.

Definition 1.82. Let A be a set. A *closure system* on A is a collection of subsets $\mathcal{C} \subseteq \mathcal{P}(A)$ satisfying:

(i) $A \in \mathcal{C}$.
(ii) If $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{B} \neq \varnothing$, then $\bigcap \mathcal{B} \in \mathcal{C}$.

The elements of $\mathcal{C}$ are called *closed sets*.

We adopt the convention that $\bigcap \varnothing := A$.

The expression 'closed set' may seem ambiguous, but it is not due to the following bijective correspondence between the closure operators on a set A and the closure systems on A:

THEOREM 1.83 (cf. [**8**, Thm. 1.1]). *Let A be a set.*

(i) *If C is a closure operator on A, then $\mathcal{C}_C := \{X \subseteq A : C(X) = X\}$ is a closure system on A.*
(ii) *If $\mathcal{C}$ is a closure system on A, then the map $C_{\mathcal{C}} : \mathcal{P}(A) \to \mathcal{P}(A)$, given by*
$$C_{\mathcal{C}}(X) := \bigcap \{Y \in \mathcal{C} : X \subseteq Y\}$$
for all $X \subseteq A$, is a closure operator on A.
(iii) *If C is a closure operator on A, then $C_{\mathcal{C}_C} = C$.*
(iv) *If $\mathcal{C}$ is a closure system on A, then $\mathcal{C}_{C_{\mathcal{C}}} = \mathcal{C}$.*

If C is a closure operator on a set A, then $\mathcal{C}_C$ is called the *closure system associated with C*. And if $\mathcal{C}$ is a closure system on A, then $C_{\mathcal{C}}$ is called the *closure operator associated with $\mathcal{C}$*.

As stated in Section 1.4, closure systems are related to complete lattices, in the following sense:

THEOREM 1.84 (cf. [**14**, Prop. 1.28]). *Let $\mathcal{C}$ be a closure system on a set A. Then, $\langle \mathcal{C}, \subseteq \rangle$ is a complete lattice and, for every $\mathcal{B} := \{X_i : i \in I\} \subseteq \mathcal{C}$, we have $\bigwedge \mathcal{B} = \bigcap_{i \in I} X_i$ and $\bigvee \mathcal{B} = C_{\mathcal{C}}(\bigcup_{i \in I} X_i)$.*

DEFINITION 1.85. Let C be a closure operator on a set A. We say that C is *finitary* if
$$C(X) = \bigcup \{C(Y) : Y \subseteq X, Y \text{ is finite}\}$$
for all $X \subseteq A$.

DEFINITION 1.86. Let $\mathcal{C}$ be a closure system on a set A. We say that $\mathcal{C}$ is *inductive* if it is closed under unions of non-empty families that are upwards directed (with respect to set inclusion).

PROPOSITION 1.87 (Schmidt's Theorem, cf. [**14**, Thm. 1.42]). *A closure operator is finitary iff its associated closure system is inductive.*

DEFINITION 1.88. Let $\mathcal{C}$ be a closure system on A. A closed set X is said to be *finitely generated* if $X = C_{\mathcal{C}}(Y)$ for some finite $Y \subseteq X$.

The finitely generated elements of $\mathcal{C}$ are closely related to the compact elements of the lattice $\langle \mathcal{C}, \subseteq \rangle$:

PROPOSITION 1.89 (cf. [**7**, Thm. 5.5]). *Let $\mathcal{C}$ be a closure system on a set A.*

(i) *If $X \in \mathcal{C}$ is compact in $\langle \mathcal{C}, \subseteq \rangle$, then X is finitely generated.*
(ii) *If $C_{\mathcal{C}}$ is finitary, then $X \in \mathcal{C}$ is compact iff X is finitely generated.*

PROPOSITION 1.90. *Let $\mathcal{C}$ be an inductive closure system on a set A. Then, $\langle \mathcal{C}, \subseteq \rangle$ is an algebraic lattice.*

PROOF. By Theorem 1.84, $\langle \mathcal{C}, \subseteq \rangle$ is a complete lattice, so it suffices to show that every element of $\mathcal{C}$ is a join of compact elements.

Let C be the closure operator associated with $\mathcal{C}$, and let $X \in \mathcal{C}$. Then, $C(X) = X$, so, by Proposition 1.87, for every $x \in X$ there is a finite $Y_x \subseteq X$ such that $x \in C(Y_x)$, whence $X = \bigcup_{x \in X} C(Y_x)$. Therefore:

$$X = C(X) = C(\bigcup_{x \in X} C(Y_x)),$$

i.e., $X = \bigvee_{x \in X} C(Y_x)$, and each $C(Y_x)$ is compact by Proposition 1.89(ii). ■

Let $\boldsymbol{A}$ be an algebra and K a class of algebras of the same type as $\boldsymbol{A}$. In Section 1.2 we showed that $\mathbf{Sub}(\boldsymbol{A})$ and $\mathrm{Co}(\boldsymbol{A})$ are closure systems with associated closure operators $\mathrm{Sg}^{\boldsymbol{A}}$ and $\boldsymbol{\Theta}^{\boldsymbol{A}}$, respectively. The K-congruences of $\boldsymbol{A}$, however, do not in general form a closure system, but we have:

PROPOSITION 1.91. *Let $\boldsymbol{A}$ be an algebra and K a class of algebras of the same type as $\boldsymbol{A}$ closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and containing a trivial algebra. Then, $\mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system (on $A \times A$).*

SKETCH OF THE PROOF. Let $\boldsymbol{A}_t$ be a trivial algebra in K. From $\boldsymbol{A}/\boldsymbol{\nabla}_{\boldsymbol{A}} \cong \boldsymbol{A}_t$ we get $\boldsymbol{\nabla}_{\boldsymbol{A}} \in \mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$. Now let $\{\theta_i : i \in I\} \subseteq \mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$, $I \neq \varnothing$, and $\theta := \bigcap_{i \in I} \theta_i$. To show that $\boldsymbol{A}/\theta \in \mathsf{K}$, define $h : A/\theta \to \prod_{i \in I} A/\theta_i$ by setting $h(a/\theta) := \langle a/\theta_i : i \in I \rangle$ for all $a/\theta \in A/\theta$. It is easy to see that h is a homomorphism, and thus

$$\boldsymbol{A}/\theta \cong h(\boldsymbol{A}/\theta) \subseteq \prod_{i \in I} \boldsymbol{A}/\theta_i \in \mathsf{K},$$

so $\theta \in \mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$. ■

COROLLARY 1.92. *Let $\boldsymbol{A}$ be an algebra and K a quasivariety of algebras of the same type as $\boldsymbol{A}$. Then, $\mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system, and thus a complete lattice.*

When $\mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$ is a closure system, we denote its associated closure operator by $\boldsymbol{\Theta}_{\mathsf{K}}^{\boldsymbol{A}}$, and the meet and join operations of the complete lattice $\mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$ are denoted by $\wedge_{\mathsf{K}}^{\boldsymbol{A}}$ and $\vee_{\mathsf{K}}^{\boldsymbol{A}}$, respectively.

1.5.2. Consequence relations.

Definition 1.93. Let A be a set. A *consequence relation* (or *closure relation*) on A is a relation $\vdash \subseteq \mathcal{P}(A) \times A$ such that, writing $X \vdash a$ in place of $\langle X, a\rangle \in \vdash$, for all $X \cup Y \cup \{a\} \subseteq A$ we have:

(i) If $a \in X$, then $X \vdash a$. (*Identity*)
(ii) If $X \vdash b$ for all $b \in Y$ and $Y \vdash a$, then $X \vdash a$. (*Cut*)
(iii) If $X \vdash a$ and $X \subseteq Y$, then $Y \vdash a$. (*Monotonicity*)

Remark 1.94. The monotonicity condition of Definition 1.93 is redundant.

A result analogous to Theorem 1.83 can easily be obtained for consequence relations and closure operators:

Theorem 1.95. *Let A be a set.*

(i) *If C is a closure operator on A, then the relation $\vdash_C \subseteq \mathcal{P}(A) \times A$, given by*
$$X \vdash_C a \iff a \in C(X)$$
for all $X \cup \{a\} \subseteq A$, is a consequence relation on A.
(ii) *If $\vdash$ is a consequence relation on A, then the map $C_\vdash : \mathcal{P}(A) \to \mathcal{P}(A)$, given by*
$$C_\vdash(X) := \{a \in A : X \vdash a\}$$
for all $X \subseteq A$, is a closure operator on A.
(iii) *If C is a closure operator on A, then $C_{\vdash_C} = C$.*
(iv) *If $\vdash$ is a consequence relation on A, then $\vdash_{C_\vdash} = \vdash$.*

Therefore, the consequence relations on a set A are in bijective correspondence both with the closure operators and with the closure systems (on A). If C is a closure operator on A, then $\vdash_C$ is called the *consequence relation associated with C*. And if $\vdash$ is a consequence relation on A, then $C_\vdash$ is called the *closure operator associated with $\vdash$*.

Definition 1.96. Let $\vdash$ be a consequence relation on a set A. We say that $\vdash$ is *finitary* if $C_\vdash$ is finitary.

CHAPTER 2

Abstract Algebraic Logic for Gentzen Relations

Henceforward we shall work with a fixed but arbitrary algebraic language $\mathcal{L}$ and a countably infinite set of variables Var, which we take to be the *disjoint* union of the (countably infinite) sets of variables $\mathsf{Var}_x := \{x_1, x_2, \dots\}$, $\mathsf{Var}_y := \{y_1, y_2, \dots\}$ and $\mathsf{Var}_z := \{z_1, z_2, \dots\}$. The letters u, v, w, with and without indices, denote variables, and we define $x := x_1$, $y := y_1$ and $z := z_1$ to reduce verbosity. As usual, we drop the references to $\mathcal{L}$ and Var in regards to the algebra of formulas, so that, for example, $\boldsymbol{Fm}_{\mathcal{L}}(\mathsf{Var})$ is denoted by $\boldsymbol{Fm}$.

Even when we do not explicitly say so, all the algebras that we shall consider will be $\mathcal{L}$-algebras. In particular, the classes of algebras will always be classes of $\mathcal{L}$-algebras.

2.1. Sequents

DEFINITION 2.1. A *trace* is any non-empty set $\mathsf{tr} \subseteq \omega \times \omega$.

DEFINITION 2.2. Let A be a set and tr a trace. A tr-*sequent* of A is a pair of finite sequences $\langle \vec{a}_m, \vec{b}_n \rangle$ of elements of A, for some $m, n \in \omega$ such that $\langle m, n \rangle \in \mathsf{tr}$. For any $m, n \in \omega$, we abbreviate '$\{\langle m, n \rangle\}$-sequent' to '$\langle m, n \rangle$-sequent', and the sequent $\langle \vec{a}_m, \vec{b}_n \rangle$ is written as $a_1, \dots, a_m \rhd b_1, \dots, b_n$ or $\vec{a}_m \rhd \vec{b}_n$.

Sequents are denoted by lower-case Gothic letters, $\mathfrak{a}, \mathfrak{b}, \dots, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \dots$, and sets of sequents by upper-case Gothic letters, $\mathfrak{D}, \mathfrak{E}, \dots, \mathfrak{P}, \mathfrak{S}, \dots$, when working with $\boldsymbol{Fm}$ and with upper-case Latin letters, $F, G, H, \dots$, when working with other algebras. This distinction is made to notationally highlight the analogies that exist between several results for sentential logics and the corresponding ones for Gentzen relations.

The *type* of a sequent $\mathfrak{s} = \vec{a}_m \rhd \vec{b}_n$, denoted by $\mathsf{tp}(\mathfrak{s})$, is the pair $\langle m, n \rangle \in \omega \times \omega$. A type of the form $\langle m_0, m_1 \rangle$ is frequently denoted by $\hat{m}$, and $\Sigma(\hat{m})$ denotes $m_0 + m_1$. If $\mathfrak{P}$ is a set of sequents, we define $\mathsf{tp}(\mathfrak{P}) := \{\mathsf{tp}(\mathfrak{p}) : \mathfrak{p} \in \mathfrak{P}\}$. When writing the type of a sequent as a subindex or a superindex, we drop the angle brackets for readability; for example, if $\mathsf{tp}(\mathfrak{s}) = \langle m, n \rangle$, we use $\mathfrak{x}_{m,n}$ interchangeably with $\mathfrak{x}_{\mathsf{tp}(s)}$.

REMARK 2.3. There is only one sequent of type $\langle 0, 0 \rangle$, namely $\varnothing \rhd \varnothing$.

The set of all tr-sequents of A is denoted by $\mathsf{tr}\text{-}\mathsf{Seq}(A)$, and when $A = Fm$ we simply write $\mathsf{tr}\text{-}\mathsf{Seq}$. When tr is a singleton, say $\mathsf{tr} = \{\langle m, n\rangle\}$, we write just $\langle m, n\rangle\text{-}\mathsf{Seq}$ in place of $\{\langle m, n\rangle\}\text{-}\mathsf{Seq}$. Also, $\mathsf{Seq}(A)$ abbreviates $(\omega \times \omega)\text{-}\mathsf{Seq}(A)$, and Seq means $\mathsf{Seq}(Fm)$.

Given $u \in \mathsf{Var}$ and $\mathfrak{s} \in \mathsf{Seq}$, say $\mathfrak{s} = \vec{\varphi}_m \rhd \vec{\psi}_n$, we say that u *occurs* in $\mathfrak{s}$ if u occurs in any of the formulas $\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n$ that make up $\mathfrak{s}$. If $\vec{u}$ is a sequence of variables, we write $\mathfrak{s}(\vec{u})$ or $\mathfrak{s} = \mathfrak{s}(\vec{u})$ to indicate that the variables occurring in $\mathfrak{s}$ are all in $\vec{u}$. And if $\mathfrak{r} = \mathfrak{r}(\vec{u})$ for every $\mathfrak{r} \in \mathfrak{P}$, where $\mathfrak{P} \subseteq \mathsf{Seq}$, we write $\mathfrak{P}(\vec{u})$ or $\mathfrak{P} = \mathfrak{P}(\vec{u})$. When $\vec{u}$ is finite, say $\vec{u} = \langle u_1, \dots, u_n\rangle$, $n \in \omega$, we usually write $\mathfrak{s}(u_1, \dots, u_n)$ and $\mathfrak{P}(u_1, \dots, u_n)$ for, respectively, $\mathfrak{s}(\vec{u})$ and $\mathfrak{P}(\vec{u})$.

By Fm_n, where $n \in \omega$, we mean the set of all formulas in which all the variables that occur are among $\vec{x}_n = \langle x_1, \dots, x_n\rangle$, and we define $\mathsf{tr}\text{-}\mathsf{Seq}_n := \mathsf{tr}\text{-}\mathsf{Seq}(Fm_n)$ for all traces tr. In this situation we say that n is a *context*. The following facts about Fm_n are straightforward to prove:

PROPOSITION 2.4.

(i) *For every $n > 0$, Fm_n is the universe of a subalgebra $\boldsymbol{Fm}_n$ of $\boldsymbol{Fm}$.*

(ii) *If $Fm_0 \neq \varnothing$, then Fm_0 is the universe of a subalgebra $\boldsymbol{Fm}_0$ of $\boldsymbol{Fm}$.*

(iii) *If Fm_m,Fm_n are non-empty and $m \leq n$, then $\boldsymbol{Fm}_m \subseteq \boldsymbol{Fm}_n$.*

(iv) *If $Fm_n \neq \varnothing$, then $Fm_n = \mathrm{Sg}^{\boldsymbol{Fm}_n}(x_1, \dots, x_n)$.*

(v) *If $Fm_n \neq \varnothing$, then $\boldsymbol{Fm}_n$ is absolutely free over $\{x_1, \dots, x_n\}$.*

Since our notation will always avoid any ambiguity, given a trace tr and a function $f : A \to B$, we also denote by f the function $f : \mathsf{tr}\text{-}\mathsf{Seq}(A) \to \mathsf{tr}\text{-}\mathsf{Seq}(B)$ defined by $f(\vec{a}_m \rhd \vec{b}_n) := f(\vec{a}_n) \rhd f(\vec{b}_m)$ for every sequent $\vec{a}_m \rhd \vec{b}_n \in \mathsf{Seq}(A)$.[1]

DEFINITION 2.5. Let $\boldsymbol{A}$ be an algebra, $\vec{u}$ a sequence of variables and $\vec{a} \in \vec{A}$ a sequence of elements of A of the same length as $\vec{u}$. For every sequent $\mathfrak{s}(\vec{u}) \in \mathsf{Seq}$, say $\mathfrak{s} = \varphi_1(\vec{u}), \dots, \varphi_m(\vec{u}) \rhd \psi_1(\vec{u}), \dots, \psi_n(\vec{u})$, the *interpretation* of $\mathfrak{s}$ (with respect to $\boldsymbol{A}$ and $\vec{a}$), written as $\mathfrak{s}^{\boldsymbol{A}}(\vec{a})$, is the sequent $\varphi_1^{\boldsymbol{A}}(\vec{a}), \dots, \varphi_m^{\boldsymbol{A}}(\vec{a}) \rhd \psi_1^{\boldsymbol{A}}(\vec{a}), \dots, \psi_n^{\boldsymbol{A}}(\vec{a})$.

For any $\mathfrak{P}(\vec{u}) \subseteq \mathsf{Seq}$, we define $\mathfrak{P}^{\boldsymbol{A}}(\vec{a}) := \{\mathfrak{r}^{\boldsymbol{A}}(\vec{a}) : \mathfrak{r}(\vec{u}) \in \mathfrak{P}\}$.

As usual, we drop the superindex $\boldsymbol{A}$ when $\boldsymbol{A} = \boldsymbol{Fm}$. And when $\vec{a}$ is finite, say $\vec{a} = \langle a_1, \dots, a_n\rangle$ for some $n \in \omega$, we frequently write $\mathfrak{s}^{\boldsymbol{A}}(a_1, \dots, a_n)$ and $\mathfrak{P}^{\boldsymbol{A}}(a_1, \dots, a_n)$ for $\mathfrak{s}^{\boldsymbol{A}}(\vec{a})$ and $\mathfrak{P}^{\boldsymbol{A}}(\vec{a})$, respectively.

We shall often interpret sequents with respect to *sequents*, in the following sense: if $\mathfrak{s}(\vec{u}_{m+n}) \in \mathsf{Seq}$ is a sequent, $\boldsymbol{A}$ an algebra and $\mathfrak{a} \in \langle m, n\rangle\text{-}\mathsf{Seq}(A)$, say $\mathfrak{a} = a_1, \dots, a_m \rhd a_{m+1}, \dots, a_{m+n}$, then we define $\mathfrak{s}^{\boldsymbol{A}}(\mathfrak{a}) := \mathfrak{s}^{\boldsymbol{A}}(a_1, \dots, a_{m+n})$.

[1] Cf. page 2 for the meaning of $f(\vec{a}_m)$.

As an immediate consequence of Proposition 1.17, we have:

PROPOSITION 2.6. *Let $\boldsymbol{A}$ be an algebra, $h, h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ and $\vec{u}$ a sequence of variables. If $h(\vec{u}) = h'(\vec{u})$, then $h(\mathfrak{s}) = h'(\mathfrak{s})$ for any $\mathfrak{s}(\vec{u}) \in \mathsf{Seq}$.*

When we have a sequent $u_1, \ldots, u_m \rhd u_{m+1}, \ldots, u_{m+n}$ such that $u_i \in \mathsf{Var}$ for $i = 1, \ldots, m+n$ and we want to define a homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, for an algebra $\boldsymbol{A}$, we often write $h(u_1, \ldots, u_m \rhd u_{m+1}, \ldots, u_{m+n}) := \mathfrak{a}$, where $\mathfrak{a} \in \langle m, n\rangle\text{-}\mathsf{Seq}(A)$, say $\mathfrak{a} = a_1, \ldots, a_m \rhd a_{m+1}, \ldots, a_{m+n}$, as an abbreviation of $h(u_1) := a_1, \ldots, h(u_{m+n}) := a_{m+n}$.

Having to always write a sequent separating the elements of the two sequences that constitute it would hinder or obscure many of the proofs to come. For example, in order to prove certain results we shall eventually need to transform a sequent $a_1, \ldots, a_m \rhd a_{m+1}, \ldots, a_{m+n}$ into another sequent $b_1, \ldots, b_m \rhd b_{m+1}, \ldots, b_{m+n}$ *step-by-step*, i.e., considering all the intermediate sequents

$$b_1, \ldots, b_i, a_{i+1}, \ldots, a_m \rhd a_{m+1}, \ldots, a_{m+n} \tag{2.1}$$

and

$$b_1, \ldots, b_m \rhd b_{m+1}, \ldots, b_{m+j}, a_{m+j+1}, \ldots, a_{m+n} \tag{2.2}$$

for $i = 0, \ldots, m$ and $j = 0, \ldots, n$. The presence of the symbol '$\rhd$' makes it impossible to write a single general form for such intermediate sequents. Hence, it is convenient to define a more homogeneous notation, one in which the two sequences that make up a sequent are not separated from each other.

Given any $m, n \in \omega$, any algebra $\boldsymbol{A}$ and any elements $a_1, \ldots, a_t \in A$, with $t \geq m+n$, we define

$$\langle a_1, \ldots, a_t\rangle_{m,n} := \underbrace{a_1, \ldots, a_m}_{m \text{ elements}} \rhd \underbrace{a_{t-n+1}, \ldots, a_t}_{n \text{ elements}}, \tag{2.3}$$

that is, $\langle a_1, \ldots, a_t\rangle_{m,n}$ is the $\langle m, n\rangle$-sequent (the subindex always indicates the type of the sequent) constituted by the first m and the last n elements of $a_1, \ldots, a_t$. For instance, we have:

$$\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7\rangle_{2,3} = a_1, a_2 \rhd a_5, a_6, a_7.$$

Note that we can now refer to all the intermediate sequents of (2.1) and (2.2) with just one expression, namely:

$$\langle b_1, \ldots, b_i, a_{i+1}, \ldots, a_{m+n}\rangle_{m,n}$$

for $i = 0, \ldots, m+n$.

The reason why in (2.3) we allow t to be strictly greater than $m+n$ and decide to take the elements of $a_1, \ldots, a_t$ both from the left and from the right is to accommodate cases where we need to specify a general form for a sequent with some distinguished elements that must always appear on the same side of the symbol '$\rhd$'.

An example will make this clear: let $m, n \in \omega$, $n > 0$ for simplicity, and suppose that we need to consider the following $\langle m, n \rangle$-sequents, where $l := \max\{m-1, 0\}$:

$$\begin{gathered} c, a_1, \dots, a_l \rhd b_1, \dots, b_n \\ a_1, c, a_2, \dots, a_l \rhd b_1, \dots, b_n \\ \vdots \\ a_1, \dots, a_{l-1}, c, a_l \rhd b_1, \dots, b_n \\ a_1, \dots, a_l, c \rhd b_1, \dots, b_n \end{gathered} \tag{2.4}$$

In this situation the notation that uses the symbol '$\rhd$' would not be convenient, since we cannot write (2.4) as

$$a_1, \dots, a_{i-1}, c, a_i, \dots, a_l \rhd b_1, \dots, b_n \tag{2.5}$$

because, if $m = 0$, then (2.5) simplifies to $c \rhd b_1, \dots, b_n$, which is not an $\langle m, n \rangle$-sequent. Hence, our new notation must be capable of handling cases in which we intend to denote an $\langle m, n \rangle$-sequent using more than $(m+n)$-many elements. Moreover, in (2.4) we want c to always occur among the a_i's, that is, on the left of the symbol '$\rhd$', so we want an expression like $\langle c, b_1, \dots, b_n \rangle_{0,n}$ to denote $\varnothing \rhd b_1, \dots, b_n$, and not $\varnothing \rhd c, b_1, \dots, b_{n-1}$. Thus, the n elements $b_1, \dots, b_n$ must be taken from the right of the sequence $a_1, \dots, a_{i-1}, c, a_i, \dots, a_l, b_1, \dots, b_n$ (an analogous example shows that the m elements $a_1, \dots, a_{i-1}, c, a_i, \dots, a_l$ must be taken from the left). Using our new notation, (2.4) can be compactly written as

$$\langle a_1, \dots, a_{i-1}, c, a_i, \dots, a_l, b_1, \dots, b_n \rangle_{m,n}, \quad i = 1, \dots, m,$$

which always denotes an $\langle m, n \rangle$-sequent with c on the left of '$\rhd$'.

2.2. Substitutions

DEFINITION 2.7. A *substitution* is any map $\sigma \in \mathrm{End}(\boldsymbol{Fm})$.

As an immediate consequence of Proposition 1.12, we have:

PROPOSITION 2.8. *Let $\sigma, \sigma' \in \mathrm{End}(\boldsymbol{Fm})$ be two substitutions. If $\sigma(u) = \sigma'(u)$ for every variable u, then $\sigma = \sigma'$.*

If σ is a substitution and $\varphi(\vec{u})$ a formula, by Proposition 1.21 we know that if a variable occurs in $\sigma(\varphi)$, then that variable occurs in $\sigma(u)$ for some $u \in \vec{u}$. This yields:

PROPOSITION 2.9. *Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be a substitution. If $\sigma(\mathsf{Var}) \subseteq V$ for some $V \subseteq \mathsf{Var}$, then all the variables occurring in $\sigma(\varphi)$ are in V, for every $\varphi \in Fm$.*

As a first use of substitutions, let us generalize Theorem 1.22 to sequents:

PROPOSITION 2.10. *Let $\boldsymbol{A}$ be an algebra generated by some $B \subseteq A$ such that $|B| \leq |\mathsf{Var}|$. Then, for every $\mathfrak{a} \in \mathsf{Seq}(A)$ there is some $n \in \omega$ and some $\mathfrak{s}(\vec{x}_n) \in \mathsf{Seq}$ such that $\mathfrak{a} = \mathfrak{s}^{\boldsymbol{A}}(\vec{b}_n)$ for some $b_1, \ldots, b_n \in B$.*

PROOF. Since $|\mathsf{Var}| = |\mathsf{Var}_x|$, pick any injective function $f : B \to \mathsf{Var}_x$.

Let $\langle m, n \rangle := \mathsf{tp}(\mathfrak{a})$, so that we may write $\mathfrak{a} = a_1, \ldots, a_m \rhd a_{m+1}, \ldots, a_{m+n}$ for some $a_1, \ldots, a_{m+n} \in A$. By Theorem 1.22, for each $i = 1, \ldots, m+n$ there is a formula $\varphi_i(u_{i,1}, \ldots, u_{i,l_i})$, $l_i \in \omega$, such that $a_i = \varphi_i^{\boldsymbol{A}}(b_{i,1}, \ldots, b_{i,l_i})$ for some elements $b_{i,1}, \ldots, b_{i,l_i} \in B$. Let $U := \{u_{1,1}, \ldots, u_{m+n,l_{m+n}}\}$.

If $U = \varnothing$, then we can simply take

$$\mathfrak{s} := \varphi_1, \ldots, \varphi_m \rhd \varphi_{m+1}, \ldots, \varphi_{m+n},$$

so assume $U \neq \varnothing$. Note that this implies $B \neq \varnothing$, so fix any $b \in B$.

Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be such that $\sigma(u_{i,j}) := f(b_{i,j})$ for every $u_{i,j} \in U$. Let $n := \max\{m \in \omega : (\exists u \in U)\, \sigma(u) = x_m\}$.

For any $1 \leq i \leq m+n$, let $\psi_i := \sigma(\varphi_i)$. By Proposition 1.21 and the definition of n, we have $\psi_i = \psi_i(\vec{x}_n)$ for each $i = 1, \ldots, m+n$. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that

$$h(x_k) := \begin{cases} b_{i,j} & \text{if } x_k = f(b_{i,j}) \text{ for some } b_{i,j} \\ b & \text{otherwise} \end{cases}$$

for all $k = 1, \ldots, n$. Since f is injective, h is well defined. Let $b_k := h(x_k)$, $k = 1, \ldots, n$, and let $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $h'(u_{i,j}) := b_{i,j}$ for every $i = 1, \ldots, m+n$ and every $j = 1, \ldots, l_i$.

Since $f(b_{i,j}) \in \vec{x}_n$ by the definition of n, for all $k = 1, \ldots, n$ we have:

$$h(\sigma(u_{i,j})) = h(f(b_{i,j})) = b_{i,j} = h'(u_{i,j}).$$

So, by Proposition 2.6, for every $i = 1, \ldots, m+n$ we get:

$$\psi_i^{\boldsymbol{A}}(\vec{b}_n) = h(\psi_i) = h(\sigma(\varphi_i)) = h'(\varphi_i) = \varphi_i^{\boldsymbol{A}}(b_{i,1}, \ldots, b_{i,l_i}) = a_i.$$

Therefore, we can take $\mathfrak{s} := \psi_1, \ldots, \psi_m \rhd \psi_{m+1}, \ldots, \psi_{m+n}$. ■

COROLLARY 2.11. *Let $\boldsymbol{A}$ be an algebra generated by some $b_1, \ldots, b_m \in A$, with $m \in \omega$. Then, for every $\mathfrak{a} \in \mathsf{Seq}(A)$ there is some $\mathfrak{s}(\vec{x}_m) \in \mathsf{Seq}$ such that $\mathfrak{a} = \mathfrak{s}^{\boldsymbol{A}}(\vec{b}_m)$.*

PROOF. Let $B := \{b_1, \ldots, b_m\}$. By Proposition 2.10, there is some $n \in \omega$ and some $\mathfrak{r}(\vec{x}_n) \in \mathsf{Seq}$ such that $\mathfrak{a} = \mathfrak{r}^{\boldsymbol{A}}(b'_1, \ldots, b'_n)$ for some $b'_1, \ldots, b'_n \in B$. For every $i = 1, \ldots, n$, let $g(i) \in \{1, \ldots, m\}$ be such that $b'_i = b_{g(i)}$.

Pick a substitution $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ such that $\sigma(x_i) := x_{g(i)}$ for all $i = 1, \ldots, n$. Let $\mathfrak{s} := \sigma(\mathfrak{r})$. Note that, by Proposition 1.21, $\mathfrak{s} = \mathfrak{s}(\vec{x}_m)$.

Let $h, h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $h'(x_i) := b'_i$ for all $i = 1, \ldots, n$ and $h(x_j) := b_j$ for all $j = 1, \ldots, m$. For every $i = 1, \ldots, n$, we have

$$h(\sigma(x_i)) = h(x_{g(i)}) = b_{g(i)} = b'_i = h'(x_i),$$

and thus, by Proposition 2.6 we get

$$\mathfrak{s}^{\boldsymbol{A}}(\vec{b}_m) = h(\mathfrak{s}) = h(\sigma(\mathfrak{r})) = h'(\mathfrak{r}) = \mathfrak{r}^{\boldsymbol{A}}(b'_1, \ldots, b'_n) = \mathsf{a}. \qquad \blacksquare$$

The following corollary of Theorem 1.22, analogous to Corollary 2.11, will be needed in Chapter 3:

COROLLARY 2.12. *Let $\boldsymbol{A}$ be an algebra generated by some $b_1, \ldots, b_m \in A$, with $m \in \omega$. Then, for every $a \in A$ there is a $\varphi(\vec{x}_m) \in Fm$ such that $a = \varphi^{\boldsymbol{A}}(\vec{b}_m)$.*

INDICATION FOR THE PROOF. Consider the sequent $\varnothing \rhd a \in \mathsf{Seq}(A)$ and use Corollary 2.11. $\blacksquare$

2.3. Gentzen relations

DEFINITION 2.13. An $\mathcal{L}$-*Gentzen relation* with trace tr is a pair $\mathbf{G} = \langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$, where $\mathcal{L}$ is an algebraic language and $\vdash_{\mathbf{G}}$ is a consequence relation on $\mathsf{tr}\text{-}\mathsf{Seq}$ satisfying the so-called *structurality* condition, i.e., a relation $\vdash_{\mathbf{G}} \subseteq \mathcal{P}(\mathsf{tr}\text{-}\mathsf{Seq}) \times \mathsf{tr}\text{-}\mathsf{Seq}$ such that, writing $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ in place of $\langle \mathfrak{P}, \mathfrak{s} \rangle \in \vdash_{\mathbf{G}}$, for all $\mathfrak{P} \cup \mathfrak{P}' \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ we have:

(i) If $\mathfrak{s} \in \mathfrak{P}$, then $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$. (*Identity*)

(ii) If $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{r}$ for all $\mathfrak{r} \in \mathfrak{P}'$ and $\mathfrak{P}' \vdash_{\mathbf{G}} \mathfrak{s}$, then $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$. (*Cut*)

(iii) If $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$, then $\sigma(\mathfrak{P}) \vdash_{\mathbf{G}} \sigma(\mathfrak{s})$ for every $\sigma \in \mathrm{End}(\boldsymbol{Fm})$. (*Structurality*)

The following condition follows immediately from (i) and (ii):

(iv) If $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ and $\mathfrak{P} \subseteq \mathfrak{P}'$, then $\mathfrak{P}' \vdash_{\mathbf{G}} \mathfrak{s}$. (*Monotonicity*)

The trace of $\mathbf{G}$ is denoted by $\mathsf{tr}(\mathbf{G})$.

REMARK 2.14. A Gentzen relation $\mathbf{G}$ with trace tr is finitary iff for every $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ such that $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ there is some finite $\mathfrak{P}_0 \subseteq \mathfrak{P}$ satisfying $\mathfrak{P}_0 \vdash_{\mathbf{G}} \mathfrak{s}$.

Since we shall only work with the (arbitrary) algebraic language $\mathcal{L}$ that we have already fixed, we abbreviate '$\mathcal{L}$-Gentzen relation' to 'Gentzen relation'.

Given $\mathfrak{P}, \mathfrak{P}' \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, we write $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{P}'$ to mean that $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ holds for all $\mathfrak{s} \in \mathfrak{P}'$. And when $\mathfrak{P} = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_n\}$, we write $\mathfrak{s}_1, \ldots, \mathfrak{s}_n \vdash_{\mathbf{G}} \mathfrak{r}$ instead of $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_n\} \vdash_{\mathbf{G}} \mathfrak{r}$. Finally, by $\mathfrak{P} \dashv\vdash_{\mathbf{G}} \mathfrak{P}'$ we mean that both $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{P}'$ and $\mathfrak{P}' \vdash_{\mathbf{G}} \mathfrak{P}$ are the case.

If $\mathbf{G} = \langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is a Gentzen relation, the closure operator associated with $\vdash_{\mathbf{G}}$ is denoted by $\mathrm{Cn}_{\mathbf{G}}$, and the corresponding closure system by $\mathcal{T}h(\mathbf{G})$. The elements of $\mathcal{T}h(\mathbf{G})$ are called $\mathbf{G}$-*theories*, and $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{P})$ is the $\mathbf{G}$-*theory generated by* $\mathfrak{P}$, for every $\mathfrak{P} \subseteq \mathsf{tr}(\mathbf{G})\text{-}\mathsf{Seq}$.

The following lemma will be needed in Chapter 3:

LEMMA 2.15. *Let* $\mathbf{G}$ *be a Gentzen relation with trace* tr*,* $\boldsymbol{A}$ *an algebra,* $\vec{u} := \langle u_i : i \in I \rangle$ *a sequence of variables such that* $|\mathsf{Var}| = |\mathsf{Var} \setminus \{u_i : i \in I\}|$ *and* $\vec{a} := \langle a_i : i \in I \rangle$ *a sequence of elements of* A *of the same length as* $\vec{u}$*. Then, for all* $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ *such that* $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ *and all* $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$*, there are* $\mathfrak{P}' \cup \{\mathfrak{s}'\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ *and* $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ *such that:*

(i) $\mathfrak{P}' \vdash_{\mathbf{G}} \mathfrak{s}'$.
(ii) $h'(\mathfrak{P}') = h(\mathfrak{P})$.
(iii) $h'(\mathfrak{s}') = h(\mathfrak{s})$.
(iv) $h'(u_i) = a_i$ *for all* $i \in I$.
(v) u_i *does not occur in* $\mathfrak{P}' \cup \{\mathfrak{s}'\}$ *for any* $i \in I$.

PROOF. Pick any bijection $\sigma : \mathsf{Var} \to \mathsf{Var} \setminus \{u_i : i \in I\}$, and extend it to a substitution $\sigma \in \mathrm{End}(\boldsymbol{Fm})$. Let $\mathfrak{P}' := \sigma(\mathfrak{P})$ and $\mathfrak{s}' := \sigma(\mathfrak{s})$. Note that (i) holds by structurality, and (v) by Proposition 2.9.

Define $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ by setting $h'(u_i) := a_i$ for all $i \in I$ and $h'(v) := h(\sigma^{-1}(v))$ for every variable v not in $\vec{u}$. Clearly, (iv) holds. If w is a variable that occurs in $\mathfrak{s}$, we have:

$$h'(\sigma(w)) = h(\sigma^{-1}(\sigma(w))) = h(w),$$

so (iii) holds by Proposition 2.6. The same reasoning shows that $h'(\sigma(\mathfrak{p})) = h(\mathfrak{p})$ for every $\mathfrak{p} \in \mathfrak{P}$, so (ii) also holds. ■

Gentzen relations, under the name of 'Gentzen systems', are ubiquitous in proof theory, where they have proved to be useful in the study of formal properties of mathematical proofs (cf. [**32**, Chs. 3-4]). We now define two kinds of Gentzen relations of particular importance in abstract algebraic logic.

2.3.1. Sentential logics. Abstract algebraic logic has mainly been concerned with structural consequence relations between sets of *formulas* and *formulas*, i.e., with the study of the so-called *sentential logics*. If we adopt the (rather natural) convention of identifying each $\langle 0, 1 \rangle$-sequent $\varnothing \rhd a$ with the element a, then Gentzen relations are a generalization of sentential logics:

DEFINITION 2.16. A *sentential logic* is a Gentzen relation with trace $\{\langle 0, 1 \rangle\}$.

Two prominent examples of sentential logics are, of course, *classical propositional logic* and *intuitionistic propositional logic.*

2.3.2. Blok and Pigozzi's k-dimensional deductive systems. In [**6**], Blok and Pigozzi introduced the notion of a k-dimensional deductive system (or k-deductive system for short), where $k > 0$, motivated 'by a desire to find a general framework in which both deductive systems [sentential logics] [...] and equational logic [cf. Section 2.8] can be treated in a uniform way' ([**6**, p. 26]).

Intuitively, k-deductive systems are 'sentential logics' in which every formula is actually a sequence of k-many formulas, i.e., they are essentially consequence relations on Fm^k. Hence, they can easily be seen as Gentzen relations:

DEFINITION 2.17. Let $k > 0$. A *k-dimensional deductive system*, or *k-deductive system* for short, is a Gentzen relation with trace $\{\langle 0, k\rangle\}$.

Of course, every sentential logic is a 1-dimensional deductive system, as in [**6**].

2.4. Congruences and compatibility

Given an algebra $\boldsymbol{A}$, a trace $\mathsf{tr} \subseteq \omega \times \omega$ and a congruence $\theta \in \mathrm{Co}(\boldsymbol{A})$, the natural projection $\pi_\theta : A \to A/\theta$ induces a (surjective) function $\pi_\theta : \mathsf{tr}\text{-}\mathsf{Seq}(A) \to \mathsf{tr}\text{-}\mathsf{Seq}(A/\theta)$ given by

$$\pi_\theta(a_1, \dots, a_m \rhd b_1, \dots, b_n) := \pi_\theta(a_1), \dots, \pi_\theta(a_m) \rhd \pi_\theta(b_1), \dots, \pi_\theta(b_n)$$

for every $\langle m, n\rangle \in \mathsf{tr}$ and every $\vec{a}_m \rhd \vec{b}_n \in \mathsf{tr}\text{-}\mathsf{Seq}(A)$. As stated in Subsection 1.2.4, when the context avoids ambiguity we write just π in place of π_θ.

Following Raftery [**27**, p. 929], for all $\mathfrak{a} \in \mathsf{tr}\text{-}\mathsf{Seq}(A)$ we usually write $\mathfrak{a}/\theta$ for $\pi_\theta(\mathfrak{a})$. Also, for every $F \cup \{\mathfrak{a}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(A)$ we define

$$F/\theta := \pi_\theta(F) = \{\mathfrak{b}/\theta : \mathfrak{b} \in F\}$$

and

$$[\mathfrak{a}]_\theta := \{\mathfrak{b} \in \mathsf{tr}\text{-}\mathsf{Seq}(A) : \mathfrak{a}/\theta = \mathfrak{b}/\theta\}.$$

Note that $\mathfrak{b} \in [\mathfrak{a}]_\theta$ implies $\mathsf{tp}(\mathfrak{b}) = \mathsf{tp}(\mathfrak{a})$.

In the context of sentential logics, i.e., when $\mathsf{tr} = \{\langle 0, 1\rangle\}$, we have $\mathfrak{a} = \varnothing \rhd a$ for some $a \in A$, and thus, provided that we identify each sequent $\varnothing \rhd b \in \mathsf{tr}\text{-}\mathsf{Seq}(A)$ with the element $b \in A$, we obtain $[\mathfrak{a}]_\theta = \pi(a) = \mathfrak{a}/\theta$. In general, however, $[\mathfrak{a}]_\theta$ and $\mathfrak{a}/\theta$ are fundamentally different objects (the latter being a sequent while the former a set of sequents), and both are needed to generalize some well-known important facts concerning the compatibility of congruences (e.g., Proposition 2.20 below).

REMARK 2.18. Let $\boldsymbol{A}$ be an algebra, $\mathfrak{a}, \mathfrak{b} \in \mathsf{Seq}(A)$ and $\theta_1, \theta_2 \in \mathrm{Co}(\boldsymbol{A})$. If $\mathfrak{a}/\theta_1 = \mathfrak{b}/\theta_1$ and $\theta_1 \subseteq \theta_2$, then $\mathfrak{a}/\theta_2 = \mathfrak{b}/\theta_2$.

DEFINITION 2.19. A congruence θ of an algebra $\boldsymbol{A}$ is said to be *compatible* with a set of sequents $F \subseteq \mathsf{Seq}(A)$ if $\pi^{-1}(\pi(F)) \subseteq F$.

PROPOSITION 2.20. *Let $\boldsymbol{A}$ be an algebra, $F \subseteq \mathsf{Seq}(A)$ and $\theta \in \mathrm{Co}(\boldsymbol{A})$. The following conditions are equivalent:*

(i) *θ is compatible with F.*
(ii) $F = \pi^{-1}(\pi(F))$.
(iii) $F = \bigcup_{\mathfrak{a} \in F} [\mathfrak{a}]_\theta$.
(iv) *For every $\mathfrak{a}, \mathfrak{b} \in \mathsf{Seq}(A)$, if $\mathfrak{a} \in F$ and $\mathfrak{a}/\theta = \mathfrak{b}/\theta$, then $\mathfrak{b} \in F$.*
(v) *For every $\mathfrak{a} \in \mathsf{Seq}(A)$, $\mathfrak{a} \in F$ iff $\mathfrak{a}/\theta \in F/\theta$.*

PROOF. (i) $\Rightarrow$ (ii) Clear, as $F \subseteq \pi^{-1}(\pi(F))$.

(ii) $\Rightarrow$ (iii)

$$\begin{aligned} \mathfrak{b} \in F &\overset{\text{(ii)}}{\iff} \mathfrak{b} \in \pi^{-1}(\pi(F)) \\ &\iff \mathfrak{b}/\theta \in \pi(F) \\ &\iff (\exists \mathfrak{a} \in F)\ \mathfrak{b}/\theta = \mathfrak{a}/\theta \\ &\iff (\exists \mathfrak{a} \in F)\ \mathfrak{b} \in [\mathfrak{a}]_\theta \\ &\iff \mathfrak{b} \in \bigcup_{\mathfrak{a} \in F} [\mathfrak{a}]_\theta \end{aligned}$$

(iii) $\Rightarrow$ (iv) Since $\mathfrak{b}/\theta = \mathfrak{a}/\theta$, we have $\mathfrak{b} \in [\mathfrak{a}]_\theta$, so $\mathfrak{b} \in \bigcup_{\mathfrak{a} \in F} [\mathfrak{a}]_\theta \overset{\text{(iii)}}{=} F$.

(iv) $\Rightarrow$ (v): If $\mathfrak{a} \in F$, then clearly $\mathfrak{a}/\theta \in F/\theta$. Conversely, if $\mathfrak{a}/\theta \in F/\theta$, then there is some $\mathfrak{b} \in F$ such that $\mathfrak{a}/\theta = \mathfrak{b}/\theta$, so $\mathfrak{a} \in F$ by (iv).

(v) $\Rightarrow$ (i) $\mathfrak{a} \in \pi^{-1}(\pi(F)) \iff \mathfrak{a}/\theta \in \pi(F) \iff \mathfrak{a}/\theta \in F/\theta \overset{\text{(v)}}{\iff} \mathfrak{a} \in F$. ■

With respect to sets of sequents, homomorphisms behave as expected, in the sense that we have:

PROPOSITION 2.21. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras, $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$, $F \subseteq \mathsf{Seq}(A)$ and $G \subseteq \mathsf{Seq}(B)$. Then:*

(i) $F \subseteq h^{-1}(h(F))$.
(ii) *$F = h^{-1}(h(F))$ iff $\ker h$ is compatible with F.*
(iii) $h(h^{-1}(G)) \subseteq G$.
(iv) *If h is surjective, then $h(h^{-1}(G)) = G$.*

PROOF.

(i) Clear, as $h(F) \subseteq h(F)$.

(ii) Assume $F = h^{-1}(h(F))$. Let $\mathfrak{a}, \mathfrak{b} \in \mathsf{Seq}(A)$ be such that $\mathfrak{a} \in F$ and $\mathfrak{a}/\ker h = \mathfrak{b}/\ker h$. Then, $h(\mathfrak{b}) = h(\mathfrak{a}) \in h(F)$, so $\mathfrak{b} \in h^{-1}(h(F)) \subseteq F$. By Proposition 2.20(iv), $\ker h$ is compatible with F.

Conversely, assume $\ker h$ is compatible with F. By (i), it suffices to show that $h^{-1}(h(F)) \subseteq F$, so let $\mathfrak{a} \in h^{-1}(h(F))$. Then, $h(\mathfrak{a}) \in h(F)$, so there is some $\mathfrak{b} \in F$ such that $h(\mathfrak{a}) = h(\mathfrak{b})$, whence $\mathfrak{a}/\ker h = \mathfrak{b}/\ker h$, so $\mathfrak{a} \in F$ by Proposition 2.20(iv).

(iii) If $\mathfrak{b} \in h(h^{-1}(G))$, there is some $\mathfrak{a} \in h^{-1}(G)$ such that $h(\mathfrak{a}) = \mathfrak{b}$, so $\mathfrak{b} \in G$.

(iv) By (iii), it suffices to show that $G \subseteq h(h^{-1}(G))$, so let $\mathfrak{b} \in G$. Since h is surjective, there is some $\mathfrak{a} \in A$ such that $h(\mathfrak{a}) = \mathfrak{b}$. Then, $\mathfrak{a} \in h^{-1}(G)$, so $\mathfrak{b} \in h(h^{-1}(G))$. ∎

LEMMA 2.22. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras and $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$. Then, $\ker h$ is compatible with $h^{-1}(X)$ for every $X \subseteq \mathsf{Seq}(B)$.*

PROOF. Let $\mathfrak{s}, \mathfrak{r} \in \mathsf{Seq}(B)$ be such that $\mathfrak{s} \in h^{-1}(X)$ and $\mathfrak{s}/\ker h = \mathfrak{r}/\ker h$. Then, $h(\mathfrak{r}) = h(\mathfrak{s}) \in X$, so $\mathfrak{r} \in h^{-1}(X)$, whence $\ker h$ is compatible with $h^{-1}(X)$ by Proposition 2.20(iv). □

The following definition, inspired by [**27**, p. 929], will simplify many statements to come:

DEFINITION 2.23. Let tr be a trace and $\boldsymbol{A}$ an algebra. For any $n \in \omega$, an *n-ary tr-valued polynomial function of $\boldsymbol{A}$* is a function $p : A^n \to \mathsf{tr}\text{-}\mathsf{Seq}(A)$ for which there is some sequent $\mathfrak{s}(\vec{u}_n, \vec{v}_m) \in \mathsf{tr}\text{-}\mathsf{Seq}$, where $m \in \omega$ and $\vec{u}_n, \vec{v}_m$ are pairwise different variables, and some elements $\vec{b}_m \in \vec{A}$, such that

$$p(a_1, \ldots, a_n) = \mathfrak{s}^{\boldsymbol{A}}(\vec{a}_n, \vec{b}_m)$$

for all $\vec{a}_n \in A^n$. When tr is a singleton, say $\mathsf{tr} = \{\langle m, n\rangle\}$, we abbreviate '$\{\langle m, n\rangle\}$-valued' to '$\langle m, n\rangle$-valued'.

PROPOSITION 2.24. *Let $\boldsymbol{A}$ be an algebra, tr a trace and $n \in \omega$. There is at least one n-ary tr-valued polynomial function of $\boldsymbol{A}$.*

PROOF. Since traces are non-empty, fix any $\langle r, s\rangle \in \mathsf{tr}$. Define the sequent:

$$\mathfrak{x} = x_1, \ldots, x_r \rhd x_{r+1}, \ldots, x_{r+s}.$$

If $r + s \leq n$, then $\mathfrak{x} = \mathfrak{x}(\vec{x}_n)$, so the function $p : A^n \to \mathsf{tr}\text{-}\mathsf{Seq}(A)$ given by $p(\vec{a}_n) := \mathfrak{x}^{\boldsymbol{A}}(\vec{a}_n)$ for all $\vec{a}_n \in A^n$ is an n-ary tr-valued polynomial function of $\boldsymbol{A}$.

Now suppose that $r + s > n$. Let $m := r + s - n$ and pick any $\vec{b}_m \in A^m$. The function $p : A^n \to \mathsf{tr}\text{-}\mathsf{Seq}(A)$ defined by setting $p(\vec{a}_n) := \mathfrak{r}^{\boldsymbol{A}}(\vec{a}_n, \vec{b}_m)$ for all $\vec{a}_n \in A^n$ is an n-ary tr-valued polynomial function of $\boldsymbol{A}$. ■

The following is a well-known fact that can be easily proved by induction on the complexity of the formulas:

PROPOSITION 2.25. *Let $\boldsymbol{A}$ be an algebra, $\theta \in \mathrm{Co}(\boldsymbol{A})$ and $a, b \in A$. If $\langle a, b\rangle \in \theta$, then $\langle \mu^{\boldsymbol{A}}(a, \vec{c}_l), \mu^{\boldsymbol{A}}(b, \vec{c}_l)\rangle \in \theta$ for every formula $\mu(u, \vec{v}_l)$ and all elements $\vec{c}_l \in \vec{A}$, where $l \in \omega$ and $u, \vec{v}_l$ are pairwise different variables.*

COROLLARY 2.26. *Let $\boldsymbol{A}$ be an algebra, $\theta \in \mathrm{Co}(\boldsymbol{A})$ and $a, b \in A$. If $\langle a, b\rangle \in \theta$, then $p(a)/\theta = p(b)/\theta$ for every unary tr-valued polynomial function of $\boldsymbol{A}$.*

PROOF. Let $\mathfrak{s}(u, \vec{v}_l) \in \mathsf{Seq}$, $l \in \omega$, $\vec{d}_l \in \vec{A}$ and $u, \vec{v}_l$ pairwise different variables be such that $p(c) = \mathfrak{s}^{\boldsymbol{A}}(c, \vec{d}_l)$ for all $c \in A$. Then, apply Proposition 2.25 to every formula in $\mathfrak{s}$. ■

Our goal now is to prove that, for any algebra $\boldsymbol{A}$ and any $F \subseteq \mathsf{Seq}(A)$, the largest (with respect to set inclusion) congruence of $\boldsymbol{A}$ compatible with F always exists. Even though we could argue algebraically, generalizing [**14**, Thm. 4.20] to show that the congruences of $\boldsymbol{A}$ compatible with F form a complete sublattice of the lattice $\mathrm{Co}(\boldsymbol{A})$, we take the more direct path of Raftery's [**27**, p. 929].

We need the following result, which appears without proof as [**27**, Lem. 11.6]:

THEOREM 2.27. *Let tr be a trace, $\boldsymbol{A}$ an algebra, $F \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(A)$ a set of sequents of $\boldsymbol{A}$ and $\theta \in \mathrm{Co}(\boldsymbol{A})$ a congruence of $\boldsymbol{A}$. The following are equivalent:*

(i) *θ is compatible with F.*

(ii) *If $\langle a, b\rangle \in \theta$, then $p(a) \in F$ iff $p(b) \in F$ for every unary tr-valued polynomial function p of $\boldsymbol{A}$.*

PROOF. (i) ⇒ (ii) By Corollary 2.26, $p(a)/\theta = p(b)/\theta$, so (ii) follows from (i) and Proposition 2.20(iv).

(ii) ⇒ (i) Let $\mathfrak{a}, \mathfrak{b} \in \mathsf{tr}\text{-}\mathsf{Seq}(A)$ be such that $\mathfrak{a}/\theta = \mathfrak{b}/\theta$ and $\mathfrak{a} \in F$. By Proposition 2.20(iv), we need to prove that $\mathfrak{b} \in F$.

Let $\langle m, n\rangle := \mathsf{tp}(\mathfrak{a}) = \mathsf{tp}(\mathfrak{b})$, so that we may write $\mathfrak{a} = \langle a_1, \dots, a_{m+n}\rangle_{m,n}$ and $\mathfrak{b} = \langle b_1, \dots, b_{m+n}\rangle_{m,n}$ for some elements $a_1, \dots, a_{m+n}, b_1, \dots, b_{m+n} \in A$. For every $i = 1, \dots, m + n$, define the function $p_i : A \to \mathsf{tr}\text{-}\mathsf{Seq}(A)$ by setting

$$p_i(c) := \langle b_1, \dots, b_{i-1}, c, a_{i+1}, \dots, a_{m+n}\rangle_{m,n}$$

for all $c \in A$. Clearly, each p_i is a unary tr-valued polynomial function of $\boldsymbol{A}$. Note that $p_i(b_i) = p_{i+1}(a_{i+1})$ for every $1 \leq i < m + n$. Thus, since $\langle a_i, b_i\rangle \in \theta$ for all

$i = 1, \ldots, m+n$, we have:

$$\begin{aligned} p_1(a_1) \in F &\overset{\text{(ii)}}{\iff} p_1(b_1) \in F \iff p_2(a_2) \in F \\ &\overset{\text{(ii)}}{\iff} p_2(b_2) \in F \iff p_3(a_3) \in F \\ &\ \ \vdots \\ &\overset{\text{(ii)}}{\iff} p_{m+n}(b_{m+n}) \in F. \end{aligned}$$

We know $p_1(a_1) = \mathfrak{a} \in F$, so we obtain $p_{m+n}(b_{m+n}) = \mathfrak{b} \in F$. ∎

Corollary 2.28. *Given a trace* tr*, an algebra* $\boldsymbol{A}$ *and* $F \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(A)$*, let* $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ *be the set of all pairs* $\langle a, b\rangle \in A \times A$ *such that:*

$p(a) \in F \iff p(b) \in F$ *for every unary tr-valued polynomial function* p *of* $\boldsymbol{A}$.

Then, $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ *is the largest congruence of* $\boldsymbol{A}$ *compatible with* F.

Proof. By Theorem 2.27, every congruence of $\boldsymbol{A}$ compatible with F is included in $\boldsymbol{\Omega}^{\mathbf{A}}(F)$, so all we need to check is that $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is a congruence of $\boldsymbol{A}$.

Clearly, $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is an equivalence relation. Let $f \in \boldsymbol{\mathcal{L}}$ be an m-ary function symbol, $m > 0$, and let $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$ be such that $\langle a_i, b_i\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$ for $i = 1, \ldots, m$. Let p be a unary tr-valued polynomial function of $\boldsymbol{A}$ and define, for every $1 \leq i \leq m$, a function $q_i : A \to \mathsf{tr}\text{-}\mathsf{Seq}(A)$ by setting, for all $c \in A$:

$$q_i(c) := p(f^{\boldsymbol{A}}(b_1, \ldots, b_{i-1}, c, a_{i+1}, \ldots, a_m)).$$

Note that every q_i is a unary tr-valued polynomial function of $\boldsymbol{A}$, because if $\mathfrak{s}(u, \vec{v}_l)$ and $\vec{d}_l \in \vec{A}$, for some $l \in \omega$ and some pairwise different variables $u, \vec{v}_l$, are such that $p(c) = \mathfrak{s}^{\boldsymbol{A}}(c, \vec{d}_l)$ for every $c \in A$, then

$$q_i(c) = \mathfrak{r}^{\boldsymbol{A}}(c, b_1, \ldots, b_{i-1}, a_{i+1}, \ldots, a_m, \vec{d}_n),$$

where $\mathfrak{r}(w_i, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_m, \vec{v}_l) := \mathfrak{s}(fw_1 \ldots w_m, \vec{v}_l)$, with the variables $\vec{w}_m$ chosen to be pairwise different from $u, \vec{v}_l$.

Also, note that $q_i(b_i) = q_{i+1}(a_{i+1})$ for all $1 \leq i < m$.

Therefore, since $\langle a_i, b_i\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$ for $i = 1, \ldots, m$, we get:

$$\begin{aligned} q_1(a_1) \in F &\iff q_1(b_1) \in F \iff q_2(a_2) \in F \\ &\iff q_2(b_2) \in F \iff q_3(a_3) \in F \\ &\ \ \vdots \\ &\iff q_m(b_m) \in F. \end{aligned}$$

Hence, $q_1(a_1) = p(f^{\boldsymbol{A}}(a_1, \ldots, a_m))$ belongs to F iff $q_m(b_m) = p(f^{\boldsymbol{A}}(b_1, \ldots, b_m))$ belongs to F. As p was arbitrary, we get $\langle f^{\boldsymbol{A}}(a_1, \ldots, a_m), f^{\boldsymbol{A}}(b_1, \ldots, b_m)\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$. Thus, $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is a congruence. ■

Definition 2.29. Let $\boldsymbol{A}$ be an algebra. The *Leibniz operator* (on $\boldsymbol{A}$) is the function $\boldsymbol{\Omega}^{\mathbf{A}} : \mathcal{P}(\mathsf{Seq}(A)) \to \mathrm{Co}(\boldsymbol{A})$ that maps each $F \subseteq \mathsf{Seq}(A)$ to the largest congruence of $\boldsymbol{A}$ compatible with F, which exists by Corollary 2.28. The congruence $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is called the *Leibniz congruence* of F.

As usual, when $\boldsymbol{A} = \boldsymbol{Fm}$ we write just $\boldsymbol{\Omega}$ in place of $\boldsymbol{\Omega}^{\boldsymbol{Fm}}$.

The Leibniz operator has been extensively studied for sentential logics, and it has led to the classification of a large amount of them in the so-called *Leibniz hierarchy* (cf. [**14**, Ch. 6]). According to [**15**, §3.1] and [**14**, §4.2], the notion of the Leibniz congruence was first introduced by Łos in [**21**] for the algebra of formulas, and in general by Wójcicki in [**33**]. It was named after Leibniz by Blok and Pigozzi in their monograph [**4**], arguing that $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is the first-order analogue of Leibniz's (second-order) principle of the identity of indiscernibles (cf. [**4**, §1.4]).

Combining Theorem 2.27 and Corollary 2.28, we obtain:

Proposition 2.30. *Let $\boldsymbol{A}$ be an algebra, $\theta \in \mathrm{Co}(\boldsymbol{A})$ and $F \subseteq \mathsf{Seq}(A)$. Then, θ is compatible with F iff $\theta \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F)$.*

In Section 2.7 we shall make use of the following generalization of Theorem 2.27 to provide a syntactic characterization of protoalgebraic Gentzen relations:

Lemma 2.31. *Let tr be a trace, $\boldsymbol{A}$ an algebra, $\theta \in \mathrm{Co}(\boldsymbol{A})$ a congruence and $F \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(A)$. The following conditions are equivalent:*

(i) *θ is compatible with F.*

(ii) *For every $n \in \omega$ and every n-ary tr-valued polynomial function p of $\boldsymbol{A}$, we have*
$$p(\vec{a}_n) \in F \iff p(\vec{b}_n) \in F$$
for every $\vec{a}_n, \vec{b}_n \in \vec{A}$ such that $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \theta$.

Proof. (i) ⇒ (ii) For every $1 \leq i \leq n$, define the map $p_i : A \to \mathsf{tr}\text{-}\mathsf{Seq}(A)$ as:
$$p_i(c) := p(b_1, \ldots, b_{i-1}, c, a_{i+1}, \ldots, a_n)$$
for all $c \in A$. Clearly, each p_i is a unary tr-valued polynomial function of $\boldsymbol{A}$ and $p_i(b_i) = p_{i+1}(a_{i+1})$ for all $1 \leq i < n$. By Theorem 2.27, we have:
$$\begin{aligned} p_1(a_1) \in F &\iff p_1(b_1) \in F \iff p_2(a_2) \in F \\ &\iff p_2(b_2) \in F \iff p_3(a_3) \in F \\ &\quad\vdots \end{aligned}$$

$$\iff p_n(b_n) \in F$$

Thus, $p_1(a_1) = p(\vec{a}_n)$ belongs to F iff $p_n(b_n) = p(\vec{b}_n)$ belongs to F.

(ii) $\Rightarrow$ (i) By Theorem 2.27. ■

2.5. Filters

DEFINITION 2.32. Let G be a Gentzen relation with trace tr, and let $\boldsymbol{A}$ be an algebra. A set $F \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(A)$ is said to be a G*-filter of* $\boldsymbol{A}$ if, for all $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ such that $\mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{s}$ and all $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h(\mathfrak{P}) \subseteq F$, we have $h(\mathfrak{s}) \in F$.

For every algebra $\boldsymbol{A}$, the set of all G-filters of $\boldsymbol{A}$ is denoted by $\mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$. It is clear that $\mathsf{tr}\text{-}\mathsf{Seq}(A) \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$, and it is easy to see that the intersection of a non-empty family of G-filters of $\boldsymbol{A}$ is also a G-filter of $\boldsymbol{A}$. Hence, $\mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$ is a closure system on $\mathsf{tr}\text{-}\mathsf{Seq}(A)$, whose associated closure operator is denoted by $\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{A}}$. By Theorem 1.84, $\langle \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A}), \subseteq \rangle$ is a complete lattice, whose meet and join operations are denoted by $\wedge^{\boldsymbol{A}}$ and $\vee^{\boldsymbol{A}}$, respectively (as usual, when $\boldsymbol{A} = \boldsymbol{Fm}$ we drop the superindices).

In the case of finitary Gentzen relations, the filters of an algebra form an algebraic lattice whose compact elements are exactly the finitely generated ones:

PROPOSITION 2.33. *Let* G *be a finitary Gentzen relation and* $\boldsymbol{A}$ *an algebra. Then,* $\langle \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A}), \subseteq \rangle$ *is an algebraic lattice and* $F \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$ *is compact iff* F *is finitely generated.*

PROOF. By Proposition 1.90, Proposition 1.87 and Proposition 1.89(ii), it suffices to show that $\mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$ is inductive. So let $\{F_i : i \in I\} \subseteq \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$ be a non-empty upwards directed family, and let $F := \bigcup_{i\in I} F_i$. We need to prove that $F \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$.

Let $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}(\mathsf{G})\text{-}\mathsf{Seq}$ and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $\mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{s}$ and $h(\mathfrak{P}) \subseteq F$. Since G is finitary, there is a finite $\mathfrak{P}_0 \subseteq \mathfrak{P}$ such that $\mathfrak{P}_0 \vdash_{\mathsf{G}} \mathfrak{s}$. Pick a finite $J \subseteq I$ for which $h(\mathfrak{P}_0) \subseteq \bigcup_{j\in J} F_j$. Given that $\{F_i : i \in I\}$ is upwards directed, there is some $k \in I$ such that $\bigcup_{j\in J} F_j \subseteq F_k$, and therefore $h(\mathfrak{s}) \in F_k \subseteq F$ because F_k is a G-filter of $\boldsymbol{A}$. ■

The G-filters of the formula algebra are simply the G-theories:

PROPOSITION 2.34. *Let* G *be a Gentzen relation with trace* tr*. Then, for every* $\mathfrak{P} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ *we have* $\mathrm{Fg}_{\mathsf{G}}(\mathfrak{P}) = \mathrm{Cn}_{\mathsf{G}}(\mathfrak{P})$*, and thus* $\mathcal{F}i_{\mathsf{G}}(\boldsymbol{Fm}) = \mathcal{T}h(\mathsf{G})$.

PROOF. We prove that $\mathrm{Cn}_{\mathsf{G}}(\mathfrak{P})$ is the least G-filter of $\boldsymbol{Fm}$ containing $\mathfrak{P}$.

Clearly, $\mathfrak{P} \subseteq \mathrm{Cn}_{\mathsf{G}}(\mathfrak{P})$.

Let $\mathfrak{S} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ and $h \in \mathrm{End}(\boldsymbol{Fm})$ be such that $\mathfrak{S} \vdash_{\mathbf{G}} \mathfrak{s}$ and $h(\mathfrak{S}) \subseteq \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P})$. Note that h is a substitution, so $h(\mathfrak{S}) \vdash_{\mathbf{G}} h(\mathfrak{s})$ by structurality, and thus $h(\mathfrak{s}) \in \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P})$ by cut. Hence, $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{P}) \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{Fm})$.

Let $G \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{Fm})$ be such that $\mathfrak{P} \subseteq G$. If $\mathfrak{r} \in \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P})$, i.e., if $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{r}$, then $\mathfrak{r} \in G$ because the identity function on Fm is a homomorphism. ∎

Corollary 2.35. *Let* $\mathbf{G}$ *be a Gentzen relation with trace* tr*, and let* $n \in \omega$ *be such that* $Fm_n \neq \varnothing$*, so that* Fm_n *is the universe of a subalgebra* $\boldsymbol{Fm}_n$ *of* $\boldsymbol{Fm}$*. Then, for every* $\mathfrak{P} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_n$ *we have:*

$$\mathrm{Fg}_{\mathbf{G}}^{\boldsymbol{Fm}_n}(\mathfrak{P}) = \mathrm{Fg}_{\mathbf{G}}(\mathfrak{P}) \cap \mathsf{tr}\text{-}\mathsf{Seq}_n.$$

Proof. Let $F := \mathrm{Fg}_{\mathbf{G}}(\mathfrak{P}) \cap \mathsf{tr}\text{-}\mathsf{Seq}_n$. We prove that F is the least $\mathbf{G}$-filter of $\boldsymbol{Fm}_n$ containing $\mathfrak{P}$.

By Proposition 2.34, $F = \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P}) \cap \mathsf{tr}\text{-}\mathsf{Seq}_n$, and thus $\mathfrak{P} \subseteq F$.

Let $\mathfrak{S} \cup \mathfrak{s} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{Fm}_n)$ be such that $\mathfrak{S} \vdash_{\mathbf{G}} \mathfrak{s}$ and $h(\mathfrak{S}) \subseteq F$. Then, $h(\mathfrak{S}) \subseteq \mathrm{Fg}_{\mathbf{G}}(\mathfrak{P})$, so $h(\mathfrak{s}) \in \mathrm{Fg}_{\mathbf{G}}(\mathfrak{P})$, and thus $h(\mathfrak{s}) \in F$. Hence, $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{Fm}_n)$.

Let $G \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{Fm}_n)$ be such that $\mathfrak{P} \subseteq G$. If $\mathfrak{r} \in F$, then $\mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}_n$ and $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{r}$ by Proposition 2.34. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{Fm}_n)$ be such that $h(\vec{x}_n) := \vec{x}_n$ and $h(u)$ is any element of $\boldsymbol{Fm}_n$. Then, $h(\mathfrak{P}) = \mathfrak{P} \subseteq G$, so $\mathfrak{r} = h(\mathfrak{r}) \in G$ because G is a $\mathbf{G}$-filter of $\boldsymbol{Fm}_n$. ∎

As regards the behaviour of homomorphisms with respect to filters, we have the following important result:

Proposition 2.36. *Let* $\mathbf{G}$ *be a Gentzen relation with trace* tr*. Let* $\boldsymbol{A}, \boldsymbol{B}$ *be any algebras,* $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$*,* $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$ *and* $G \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{B})$*. Then:*

(i) $h^{-1}(G) \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$.

(ii) *If* h *is surjective and* $\ker h$ *is compatible with* F*, then* $h(F) \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{B})$.

Proof.

(i) Assume $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ holds for some $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, and let $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $g(\mathfrak{P}) \subseteq h^{-1}(G)$. Then, $h(g(\mathfrak{P})) \subseteq G$, so, since $h \circ g$ is a homomorphism and $G \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{B})$, we obtain $h(g(\mathfrak{s})) \in G$, whence $g(\mathfrak{s}) \in h^{-1}(G)$. Thus, $h^{-1}(G)$ is a $\mathbf{G}$-filter of $\boldsymbol{A}$.

(ii) Assume $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ for some $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, and let $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{B})$ be such that $g(\mathfrak{P}) \subseteq h(F)$. Then, $h^{-1}(g(\mathfrak{P})) \subseteq h^{-1}(h(F)) = F$ by Proposition 2.21(ii). By Lemma 1.23, there is an $f \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $g = h \circ f$. Thus, $f(\mathfrak{P}) \subseteq h^{-1}(h(f(\mathfrak{P}))) = h^{-1}(g(\mathfrak{P})) \subseteq F$, whence

$f(\mathfrak{s}) \in F$ because $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$. Hence, $g(\mathfrak{s}) = h(f(\mathfrak{s})) \in h(F)$, so $h(F)$ is a $\mathbf{G}$-filter of $\boldsymbol{B}$. ∎

The 'quotients' of the filters of an algebra are filters of the quotient algebra, in the following sense:

PROPOSITION 2.37. *Let* $\mathbf{G}$ *be a Gentzen relation,* $\boldsymbol{A}$ *an algebra,* $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$ *and* $\theta \in \mathrm{Co}(\boldsymbol{A})$ *compatible with* F*. Then,* $F/\theta \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A}/\theta)$*.*

PROOF. Let $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}(\mathbf{G})\text{-}\mathsf{Seq}$ be such that $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ and fix a homomorphism $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}/\theta)$ satisfying $h(\mathfrak{P}) \subseteq F/\theta$. We need to prove that $h(\mathfrak{s}) \in F/\theta$.

By Lemma 1.23, there is some $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h = \pi_\theta \circ g$, so $\pi_\theta(g(\mathfrak{P})) \subseteq F/\theta = \pi_\theta(F)$. Thus, $g(\mathfrak{P}) \subseteq \pi_\theta^{-1}(\pi_\theta(F)) = F$, where the last equality is given by Proposition 2.21(ii) because $\ker \pi_\theta = \theta$. Since $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$, we obtain $g(\mathfrak{s}) \in F$, whence $h(\mathfrak{s}) = \pi_\theta(g(\mathfrak{s})) \subseteq \pi_\theta(F) = F/\theta$. ∎

2.6. Matrices

DEFINITION 2.38. An $\mathcal{L}$*-matrix* is a pair $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an algebra and $F \subseteq \mathsf{Seq}(A)$. The set F is called the set of *designated elements*. $\langle \boldsymbol{A}, F\rangle$ is said to be *finite* if A is finite, and *finitely generated* if $\boldsymbol{A}$ is finitely generated.

When $F \subseteq \langle 0, 1\rangle\text{-}\mathsf{Seq}(A)$, we say that $\langle \boldsymbol{A}, F\rangle$ is a *sentential matrix*.

REMARK 2.39. Since we identify each $\langle 0, 1\rangle$-sequent $\varnothing \rhd a$ with the element a, sentential matrices are just the matrices usually considered in abstract algebraic logic (i.e., when working with sentential logics).

DEFINITION 2.40. Let $\langle \boldsymbol{A}, F\rangle$ be an $\mathcal{L}$-matrix. A *submatrix* of $\langle \boldsymbol{A}, F\rangle$ is an $\mathcal{L}$-matrix $\langle \boldsymbol{B}, \mathsf{Seq}(B) \cap F\rangle$, where $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$.

We shall seldom work with arbitrary $\mathcal{L}$-matrices. Instead, we shall consider matrices that serve as 'models' for a given Gentzen relation, in the following sense:

DEFINITION 2.41. Let $\mathbf{G}$ be a Gentzen relation with trace tr. A $\mathbf{G}$*-matrix*, or a *(matrix) model* of $\mathbf{G}$, is an $\mathcal{L}$-matrix $\langle \boldsymbol{A}, F\rangle$ such that $F \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(A)$ and, for every $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ and every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, if $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ and $h(\mathfrak{P}) \subseteq F$, then $h(\mathfrak{s}) \in F$.

Combining Definition 2.32 and Definition 2.41, we get:

PROPOSITION 2.42. *Let* $\mathbf{G}$ *be a Gentzen relation. An* $\mathcal{L}$*-matrix* $\langle \boldsymbol{A}, F\rangle$ *is a* $\mathbf{G}$*-matrix iff* $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$*.*

When working with a $\mathbf{G}$-matrix $\langle \boldsymbol{A}, F\rangle$, where $\mathbf{G}$ is a given Gentzen relation, we will sometimes be interested in the $\mathbf{G}$-filters of $\boldsymbol{A}$ that include the $\mathbf{G}$-filter F, which we denote by $\mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})^F$. It is straightforward to check that $\mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})^F$ is a sublattice of $\mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$.

Proposition 2.43. *Let* $\mathbf{G}$ *be a Gentzen relation. The class of all matrix models of* $\mathbf{G}$ *is closed under taking submatrices.*

Proof. Let $\mathsf{tr} := \mathsf{tr}(\mathbf{G})$ and let $\langle \boldsymbol{A}, F\rangle$ be a $\mathbf{G}$-matrix.

Let $\langle \boldsymbol{B}, \mathsf{Seq}(B)\cap F\rangle$ be a submatrix of $\langle \boldsymbol{A}, F\rangle$, $\mathfrak{P}\cup\{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ such that $\mathfrak{P}\vdash_{\mathbf{G}} \mathfrak{s}$, and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{B})$ such that $h(\mathfrak{P}) \subseteq \mathsf{Seq}(B)\cap F$. Let $j \in \mathrm{Hom}(\boldsymbol{B}, \boldsymbol{A})$ be the inclusion map, so that $j\circ h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ and $(j\circ h)(\mathfrak{P}) \subseteq F$. Since F is a $\mathbf{G}$-filter, $h(\mathfrak{s}) = (j\circ h)(\mathfrak{s}) \in F$, so $h(\mathfrak{s}) \in \mathsf{Seq}(B)\cap F$. Thus, $\langle \boldsymbol{B}, \mathsf{Seq}(B)\cap F\rangle$ is a $\mathbf{G}$-matrix. ■

Definition 2.44. Let $\mathbf{G}$ be a Gentzen relation and $\langle \boldsymbol{A}, F\rangle$ a $\mathbf{G}$-matrix. A *contraction* of $\langle \boldsymbol{A}, F\rangle$ is a $\mathbf{G}$-matrix $\langle \boldsymbol{A}, G\rangle$ such that $G \supseteq F$.

2.6.1. Homomorphisms of matrices.

Definition 2.45. Let $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, G\rangle$ be $\mathcal{L}$-matrices. A *(matrix) homomorphism* from $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{B}, G\rangle$ is a homomorphism $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ satisfying $h(F) \subseteq G$. We write $h : \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ to denote that h is a matrix homomorphism from $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{B}, G\rangle$.

If, moreover, $h^{-1}(G) \subseteq F$, we say that h is *strict*.

Remark 2.46. If $h : \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is strict, then $F = h^{-1}(G)$.

2.6.2. Gentzen relations defined by matrices. If M is a class of $\mathcal{L}$-matrices, then it is straightforward to check that

$$\begin{aligned}\mathfrak{P}\vdash_{\mathsf{M}} \mathfrak{r} \iff &\text{for all } \langle \boldsymbol{A}, F\rangle \in \mathsf{M} \text{ and all } h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}),\\ &h(\mathfrak{P}) \subseteq F \text{ implies } h(\mathfrak{r}) \in F\end{aligned}$$

defines a Gentzen relation $\mathbf{G}_{\mathsf{M}} := \langle \mathcal{L}, \vdash_{\mathsf{M}}\rangle$ with trace $\omega\times\omega$. We call it the *Gentzen relation defined by* M. When M is a singleton, say $\mathsf{M} = \{\mathcal{M}\}$, we write $\mathbf{G}_{\mathcal{M}}$ and $\vdash_{\mathcal{M}}$ for $\mathbf{G}_{\mathsf{M}}$ and $\vdash_{\mathsf{M}}$, respectively, and call $\mathbf{G}_{\mathcal{M}}$ the *Gentzen relation defined by* $\mathcal{M}$.

Remark 2.47. An $\mathcal{L}$-matrix $\mathcal{M}$ is a model of a Gentzen relation $\mathbf{G}$ iff $\vdash_{\mathbf{G}} \subseteq \vdash_{\mathcal{M}}$.

2.6.3. Direct products of matrices. Let $\mathsf{M} := \{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$ be a non-empty family of $\mathcal{L}$-matrices. In the context of sentential logics, the direct product of M is defined to be the $\mathcal{L}$-matrix $\langle \boldsymbol{A}, F\rangle := \langle \prod_{i\in I}\boldsymbol{A}_i, \prod_{i\in I} F_i\rangle$, but this definition makes no sense when working with sequents because $\prod_{i\in I} F_i \not\subseteq \mathsf{Seq}(A)$. Nevertheless, we can generalize the notion of direct product of matrices to sequents.

Let $\boldsymbol{A} := \prod_{i\in I} \boldsymbol{A}_i$, and let $\mathsf{tr} := \bigcup_{i\in I} \mathsf{tp}(F_i)$ be the collection of all the types of the sequents in $\bigcup_{i\in I} F_i$. For every $\langle m, n\rangle \in \mathsf{tr}$, define

$$F_{m,n} := \prod_{i\in I}(F_i \cap \langle m, n\rangle\text{-}\mathsf{Seq}(A_i)).$$

If $a \in F_{m,n}$, then $a = \langle \mathfrak{a}_i : i \in I\rangle$, where $\mathfrak{a}_i \in F_i$ and $\mathsf{tp}(\mathfrak{a}_i) = \langle m, n\rangle$ for each $i \in I$, so that we may write:

$$\mathfrak{a}_i = a_{i,1}, \ldots, a_{i,m} \rhd a_{i,m+1}, \ldots, a_{i,m+n}$$

for some elements $a_{i,1}, \ldots, a_{i,m+n} \in A_i$. Define the following sequent:

$$\tilde{\mathfrak{a}} := \langle a_{i,1} : i \in I\rangle, \ldots, \langle a_{i,m} : i \in I\rangle \rhd \langle a_{i,m+1} : i \in I\rangle, \ldots, \langle a_{i,m+n} : i \in I\rangle.$$

Finally, let $\tilde{F}_{m,n} := \{\tilde{\mathfrak{a}} : a \in F_{m,n}\}$ and $F := \bigcup_{\langle m,n\rangle\in\mathsf{tr}} \tilde{F}_{m,n}$. Since $\tilde{\mathfrak{a}} \in \langle m, n\rangle\text{-}\mathsf{Seq}(A)$ for every $\langle m, n\rangle \in \mathsf{tr}$ and every $a \in F_{m,n}$, we have $F \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(A)$.

REMARK 2.48. A sequent $\mathfrak{a} \in \mathsf{Seq}(A)$ of type $\langle m, n\rangle \in \mathsf{tr}$ is in F iff for every $i \in I$ there is an $\langle m, n\rangle$-sequent $\langle a_{i,1}, \ldots, a_{i,m+n}\rangle_{m,n} \in F_i$ and we have:

$$\mathfrak{a} = \langle\langle a_{i,1} : i \in I\rangle, \ldots, \langle a_{i,m+n} : i \in I\rangle\rangle_{m,n}.$$

This construction, whose notation we shall use until the end of the current subsection, leads to the following:

DEFINITION 2.49. Let $\mathsf{M} := \{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$ be a non-empty family of $\mathcal{L}$-matrices. The *direct product* of M, in symbols $\prod \mathsf{M}$ or $\prod_{i\in I}\langle \boldsymbol{A}_i, F_i\rangle$, is the $\mathcal{L}$-matrix $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A} := \prod_{i\in I} \boldsymbol{A}_i$ and F is built as before.

In this situation, we write $F := \triangle_{i\in I} F_i$.

REMARK 2.50. If $\mathsf{tr} \subseteq \{\langle 0, 1\rangle\}$ in the context of Definition 2.49, then we recover the usual direct product of sentential matrices (provided, of course, that we identify each $\langle 0, 1\rangle$-sequent $\varnothing \rhd a$ with the element a).

Therefore, Definition 2.49 is really a generalization of the direct product for sentential matrices. Now we need to check that it preserves the property of being a matrix model (for a given Gentzen relation), as otherwise it would be a rather useless construction.

THEOREM 2.51. *Let* $\mathbf{G}$ *be a Gentzen relation and* $\mathsf{M} := \{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$ *a non-empty family of* $\mathbf{G}$*-matrices. Then,* $\langle \boldsymbol{A}, F\rangle := \prod \mathsf{M}$ *is also a* $\mathbf{G}$*-matrix.*

PROOF. Note that, since the matrices in M are all $\mathbf{G}$-matrices, the trace obtained in the construction of $\prod \mathsf{M}$ is included in $\mathsf{tr} := \mathsf{tr}(\mathbf{G})$.

Let $\mathfrak{P}\cup\{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s}$ and $h(\mathfrak{P}) \subseteq F$. We need to prove that $h(\mathfrak{s}) \in F$.

Fix any $\mathfrak{p} \in \mathfrak{P}$, and let $\langle m, n\rangle := \mathsf{tp}(\mathfrak{p})$. Since $h(\mathfrak{p}) \in F$, by Remark 2.48 we know that for every $i \in I$ there is a sequent $\langle a_{i,1}, \ldots, a_{i,m+n}\rangle_{m,n} \in F_i$ and

$$h(\mathfrak{p}) = \langle\langle a_{i,1} : i \in I\rangle, \ldots, \langle a_{i,m+n} : i \in I\rangle\rangle_{m,n}.$$

Thus, $(\pi_i \circ h)(\mathfrak{p}) = \langle a_{i,1}, \ldots, a_{i,m+n}\rangle_{m,n} \in F_i$, where $\pi_i : A \to A_i$ is the i-th projection. As $\mathfrak{p}$ was an arbitrary element of $\mathfrak{P}$, we obtain $(\pi_i \circ h)(\mathfrak{P}) \subseteq F_i$, whence $(\pi \circ h)(\mathfrak{s}) \in F_i$ for every $i \in I$ because F_i is a G-filter of $\boldsymbol{A}_i$.

Let $\langle r, s\rangle := \mathsf{tp}(\mathfrak{s})$. If, for every $i \in I$, we write $(\pi_i \circ h)(\mathfrak{s}) = \langle b_{i,1}, \ldots, b_{i,r+s}\rangle_{r,s}$ for some elements $b_{i,1}, \ldots, b_{i,r+s} \in A_i$, we obtain

$$h(\mathfrak{s}) = \langle\langle b_{i,1} : i \in I\rangle, \ldots, \langle b_{i,r+s} : i \in I\rangle\rangle_{r,s},$$

whence $h(\mathfrak{s}) \in F$ by Remark 2.48. ■

The following lemma will be used in Subsection 3.1.2.

LEMMA 2.52. *Let G be a Gentzen relation with trace tr, $\{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$ a family of G-matrices and $\langle \boldsymbol{A}, F\rangle := \prod_{i\in I}\langle \boldsymbol{A}_i, F_i\rangle$. For every $\mathfrak{s}(\vec{u}_l) \in \mathsf{tr}\text{-}\mathsf{Seq}$, $l \in \omega$, and every $a_1, \ldots, a_l \in A$, we have:*

$$\mathfrak{s}^{\boldsymbol{A}}(\vec{a}_l) \in F \iff \mathfrak{s}^{\boldsymbol{A}_i}(\vec{a}_l(i)) \in F_i \textit{ for all } i \in I, \tag{2.6}$$

where $\vec{a}_l(i) := \langle a_1(i), \ldots, a_l(i)\rangle$.

PROOF. Let $\langle m, n\rangle := \mathsf{tp}(\mathfrak{s})$, so that we may write $\mathfrak{s} = \langle \varphi_1, \ldots, \varphi_{m+n}\rangle_{m,n}$ for some formulas $\varphi_1, \ldots, \varphi_{m+n} \in Fm$. For every $i \in I$, we have:

$$\mathfrak{s}^{\boldsymbol{A}_i}(\vec{a}_l(i)) = \langle \varphi_1^{\boldsymbol{A}_i}(\vec{a}_l(i)), \ldots, \varphi_{m+n}^{\boldsymbol{A}_i}(\vec{a}_l(i))\rangle_{m,n}. \tag{2.7}$$

And, as a consequence of Proposition 1.31:

$$\mathfrak{s}^{\boldsymbol{A}}(\vec{a}_l) = \langle\langle \varphi_1^{\boldsymbol{A}_i}(\vec{a}_l(i)) : i \in I\rangle, \ldots, \langle \varphi_{m+n}^{\boldsymbol{A}_i}(\vec{a}_l(i)) : i \in I\rangle\rangle_{m,n}. \tag{2.8}$$

By Remark 2.48, we know that $\mathfrak{s}^{\boldsymbol{A}}(\vec{a}_l) \in F$ iff for every $i \in I$ there are some elements $b_{i,1}, \ldots, b_{i,m+n} \in A_i$ satisfying $\langle b_{i,1}, \ldots, b_{i,m+n}\rangle_{m,n} \in F_i$ and such that

$$\mathfrak{s}^{\boldsymbol{A}}(\vec{a}_l) = \langle\langle b_{i,1} : i \in I\rangle, \ldots, \langle b_{i,m+n} : i \in I\rangle\rangle_{m,n},$$

whence (2.6) follows immediately using (2.7) and (2.8). ■

Even though we shall only work with direct products of matrices to prove a minor result (cf. Proposition 3.14), we have decided to introduce them in order to present a construction that is notably more convoluted for Gentzen relations than it is so for sentential logics.

2.7. Protoalgebraic Gentzen relations

Definition 2.53. Following Raftery's [**27**, Def. 13.3], we say that a Gentzen relation G is *protoalgebraic* if the Leibniz operator $\mathbf{\Omega}$ is monotone on the lattice of G-theories.

Protoalgebraic logics have been extensively studied since they were first introduced, for finitary sentential logics, by Blok and Pigozzi in [**3**]. Under a very different form, they were independently presented by Czelakowski, who named them 'non-pathological', in [**10**] (this article is a continuation of [**9**], which already contains the key elements of Czelakowski's definition of protoalgebraicity). Both definitions were proved to be essentially equivalent in [**5**], where Blok and Pigozzi generalized the notion of being protoalgebraic to finitary k-deductive systems using a definition analogous to Definition 2.53. Comprehensive introductions to the subject of protoalgebraic logics can be found in [**11**, Part I] and [**14**, Sec. 6.2].

Many different characterizations of protoalgebraicity are currently well known. We shall generalize some of them to arbitrary Gentzen relations, as well as proving a new, useful version of a syntactic one and correcting [**27**, Thm. 13.4].

2.7.1. Blok and Pigozzi's original definition. The original definition due to Blok and Pigozzi can be easily adapted to our framework. To properly state it (as a characterization, of course), we first need:

Definition 2.54. Let G be a Gentzen relation with trace tr, $\mathfrak{T}$ a G-theory and $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$. We say that $\mathfrak{s}$ and $\mathfrak{r}$ are $\mathfrak{T}$-*indiscernible* if $\mathfrak{s}/\mathbf{\Omega}(\mathfrak{T}) = \mathfrak{r}/\mathbf{\Omega}(\mathfrak{T})$, and that they are $\mathfrak{T}$-*interderivable* if both $\mathfrak{T}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r}$ and $\mathfrak{T}, \mathfrak{r} \vdash_{\mathsf{G}} \mathfrak{s}$ are the case.

Theorem 2.55. *A Gentzen relation* G *with trace* tr *is protoalgebraic iff for every* G*-theory* $\mathfrak{T}$ *and every* $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$*, if* $\mathfrak{s}$ *and* $\mathfrak{r}$ *are* $\mathfrak{T}$*-indiscernible, then they are* $\mathfrak{T}$*-interderivable.*

Proof. ($\Rightarrow$) Let $\mathfrak{T}' := \mathrm{Cn}_{\mathsf{G}}(\mathfrak{T}, \mathfrak{s})$. Since $\mathfrak{T} \subseteq \mathfrak{T}'$, we have $\mathbf{\Omega}(\mathfrak{T}) \subseteq \mathbf{\Omega}(\mathfrak{T}')$ by protoalgebraicity, so from $\mathfrak{s}/\mathbf{\Omega}(\mathfrak{T}) = \mathfrak{r}/\mathbf{\Omega}(\mathfrak{T})$ we obtain $\mathfrak{s}/\mathbf{\Omega}(\mathfrak{T}') = \mathfrak{r}/\mathbf{\Omega}(\mathfrak{T}')$ by Remark 2.18. As $\mathfrak{s} \in \mathfrak{T}'$, compatibility yields $\mathfrak{r} \in \mathfrak{T}'$ by Proposition 2.20(iv), i.e., $\mathfrak{T}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r}$. By symmetry, exchanging the roles of $\mathfrak{s}$ and $\mathfrak{r}$ shows $\mathfrak{T}, \mathfrak{r} \vdash_{\mathsf{G}} \mathfrak{s}$.

($\Leftarrow$) Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathcal{T}h(\mathsf{G})$ be such that $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. To prove $\mathbf{\Omega}(\mathfrak{T}_1) \subseteq \mathbf{\Omega}(\mathfrak{T}_2)$ it suffices to show that $\mathbf{\Omega}(\mathfrak{T}_1)$ is compatible with $\mathfrak{T}_2$, so let $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$ be such that $\mathfrak{s} \in \mathfrak{T}_2$ and $\mathfrak{s}/\mathbf{\Omega}(\mathfrak{T}_1) = \mathfrak{r}/\mathbf{\Omega}(\mathfrak{T}_1)$. Since $\mathrm{Cn}_{\mathsf{G}}(\mathfrak{T}_1, \mathfrak{s}) \subseteq \mathfrak{T}_2$ and, by assumption, $\mathfrak{T}_1, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r}$, we get $\mathfrak{r} \in \mathfrak{T}_2$. By Proposition 2.20(iv), $\mathbf{\Omega}(\mathfrak{T}_1)$ is compatible with $\mathfrak{T}_2$. ∎

2.7.2. Syntactic characterizations. We are now going to characterize protoalgebraic Gentzen relations in the spirit of the initial definition given in [**3**] by

Czelakowski, namely, as those for which there is a set of sequents satisfying a form of reflexivity and *modus ponens*. This characterization has proven to be quite useful for sentential logics, because it 'facilitates the classification of a large number of logics as protoalgebraic' ([**14**, p. 325]). In [**5**, Thm. 13.2], Blok and Pigozzi generalized it to finitary k-deductive systems, but their statement contained a mistake that was independently noticed by Pałasińska in [**25**, §3.1], by Pynko in [**26**, §5] and by Elgueta and Jansana in [**13**]. For a detailed analysis of the mistake, the reader is referred to the recently[2] published [**24**], in which Pałasińska states and proves a corrected version ([**24**, Thm. 6]) of Blok and Pigozzi's theorem.

First, we generalize [**24**, Thm. 6] to Gentzen relations. Then, we correct an attempt in Raftery's [**27**] of obtaining a similar result. Finally, at the end of the subsection we present a new generalization, which we consider to be closer in spirit to Gentzen relations.

With few changes, Pałasińska's [**24**, Thm. 6] can be easily seen to hold for Gentzen relations:

THEOREM 2.56. *A Gentzen relation* G *with trace* tr *is protoalgebraic iff for every* $\langle m, n\rangle \in \mathsf{tr}$ *there is a set* $\mathfrak{E}_{m,n}(x, y, \vec{z}_t) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, *where* $t := \max\{m+n-1, 0\}$, *satisfying the following conditions:*

($\mathrm{R_{BP}}$) $\vdash_{\mathsf{G}} \mathfrak{E}_{m,n}(x, x, \vec{z}_t)$.

($\mathrm{MP_{BP}}$) $\langle z_1, \ldots, z_{i-1}, x, z_i, \ldots, z_t\rangle_{m,n}, \mathfrak{E}_{m,n}(x, y, \vec{z}_t) \vdash_{\mathsf{G}} \langle z_1, \ldots, z_{i-1}, y, z_i, \ldots, z_t\rangle_{m,n}$ *for all* $i = 1, \ldots, m+n$.

Moreover, if G *is finitary then all the sets* $\mathfrak{E}_{m,n}$ *can be taken finite.*

PROOF. ($\Leftarrow$) Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathcal{T}h(\mathsf{G})$ be such that $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. We need to prove that $\boldsymbol{\Omega}(\mathfrak{T}_1) \subseteq \boldsymbol{\Omega}(\mathfrak{T}_2)$, so fix any $\langle \alpha, \beta\rangle \in \boldsymbol{\Omega}(\mathfrak{T}_1)$.

Let $p : Fm \to \mathsf{tr}\text{-}\mathsf{Seq}$ be a unary tr-valued polynomial function of $\boldsymbol{Fm}$, and let $l \in \omega$, $\mathfrak{s}(w, \vec{v}_l) \in \mathsf{tr}\text{-}\mathsf{Seq}$ and $\varphi_1, \ldots, \varphi_l \in Fm$ be such that $p(\psi) = \mathfrak{s}(\psi, \vec{\varphi}_l)$ for all $\psi \in Fm$. Let $\langle m, n\rangle := \mathsf{tp}(\mathfrak{s})$, so that we may write

$$\mathfrak{s} = \langle \chi_1(w, \vec{v}_l), \ldots, \chi_{m+n}(w, \vec{v}_l)\rangle_{m,n}$$

for some formulas $\chi_1, \ldots, \chi_{m+n}$.

Assume $p(\alpha) \in \mathfrak{T}_2$. For every $i = 1, \ldots, m+n$, define $\alpha_i := \chi_i(\alpha, \vec{\varphi}_l)$ and $\beta_i := \chi_i(\beta, \vec{\varphi}_l)$. By ($\mathrm{R_{BP}}$), we have:

$$\mathfrak{E}_{m,n}(\alpha_i, \alpha_i, \beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_{m+n}) \in \mathfrak{T}_1 \tag{2.9}$$

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for all $1 \leq i \leq m + n$. By Proposition 2.25, $\langle \alpha_i, \beta_i \rangle \in \boldsymbol{\Omega}(\mathfrak{T}_1)$, so applying Theorem 2.27 to (2.9) yields:

$$\mathfrak{E}_{m,n}(\alpha_i, \beta_i, \beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_{m+n}) \in \mathfrak{T}_1 \subseteq \mathfrak{T}_2. \tag{2.10}$$

Hence, if $\langle \beta_1, \ldots, \beta_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_{m+n} \rangle_{m,n} \in \mathfrak{T}_2$, from (2.10) and (MP$_{\mathrm{BP}}$) we get $\langle \beta_1, \ldots, \beta_{i-1}, \beta_i, \alpha_{i+1}, \ldots, \alpha_{m+n} \rangle_{m,n} \in \mathfrak{T}_2$, for all $i = 1, \ldots, m + n$. Since

$$\langle \alpha_1, \ldots, \alpha_{m+n} \rangle_{m,n} = p(\alpha) \in \mathfrak{T}_2,$$

iterating this step $(m + n)$-many times we obtain $p(\beta) = \langle \beta_1, \ldots, \beta_{m+n} \rangle_{m,n} \in \mathfrak{T}_2$.

This proves that $p(\alpha) \in \mathfrak{T}_2$ implies $p(\beta) \in \mathfrak{T}_2$. Since we also have $\langle \beta, \alpha \rangle \in \boldsymbol{\Omega}(\mathfrak{T}_1)$, the same argument, exchanging the α's with the β's, proves that $p(\beta) \in \mathfrak{T}_2$ implies $p(\alpha) \in \mathfrak{T}_2$. Therefore, $\langle \alpha, \beta \rangle \in \boldsymbol{\Omega}(\mathfrak{T}_2)$.

($\Rightarrow$) Fix any $\langle m, n \rangle \in \mathsf{tr}$ and let $t := \max\{m + n - 1, 0\}$.

Define a substitution $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ as follows: if $t = 0$, let $\sigma(u) := x$ for every variable u, and otherwise let σ be any substitution mapping $\mathsf{Var} \setminus \{x, y, z_1, \ldots, z_t\}$ onto $\{z_1, \ldots, z_t\}$ and such that $\sigma(\vec{z}_t) := \vec{z}_t$ and $\sigma(x) := \sigma(y) := x$.

Now define another substitution $\sigma' \in \mathrm{End}(\boldsymbol{Fm})$ by setting $\sigma'(y) := y$ and $\sigma'(u) := \sigma(u)$ for every variable $u \neq y$. By Proposition 2.9, the variables occurring in $\sigma'(\varphi)$ are all among $x, y, \vec{z}_t$ for every $\varphi \in Fm$.

Let $\mathfrak{T}_{m,n} := \{\mathfrak{s} \in \mathsf{tr}\text{-}\mathsf{Seq} : \varnothing \vdash_{\mathbf{G}} \sigma(\mathfrak{s})\}$.

CLAIM 2.56.1. $\mathfrak{T}_{m,n} \in \mathcal{T}h(\mathbf{G})$.

PROOF. Let $\mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$ be such that $\mathfrak{T}_{m,n} \vdash_{\mathbf{G}} \mathfrak{r}$. Then, $\sigma(\mathfrak{T}_{m,n}) \vdash_{\mathbf{G}} \sigma(\mathfrak{r})$ by structurality. By the definition of $\mathfrak{T}_{m,n}$, we have $\vdash_{\mathbf{G}} \sigma(\mathfrak{T}_{m,n})$, so $\vdash_{\mathbf{G}} \sigma(\mathfrak{r})$ by cut. Thus, $\mathfrak{r} \in \mathfrak{T}_{m,n}$. $\square$

CLAIM 2.56.2. $\langle x, y \rangle \in \boldsymbol{\Omega}(\mathfrak{T}_{m,n})$.

PROOF. Let p be a unary tr-valued polynomial function of $\boldsymbol{Fm}$. Let $\mathfrak{s}(u, \vec{w}_l) \in \mathsf{tr}\text{-}\mathsf{Seq}$, $l \in \omega$, and $\varphi_1, \ldots, \varphi_l \in Fm$ be such that $p(\psi) = \mathfrak{s}(\psi, \vec{\varphi}_l)$ for all $\psi \in Fm$. We have

$$p(x) \in \mathfrak{T}_{m,n} \iff \mathfrak{s}(x, \vec{\varphi}_l) \in \mathfrak{T}_{m,n} \iff \vdash_{\mathbf{G}} \sigma(\mathfrak{s}(x, \vec{\varphi}_l)) \iff \vdash_{\mathbf{G}} \mathfrak{s}(x, \sigma(\vec{\varphi}_l))$$

and

$$p(y) \in \mathfrak{T}_{m,n} \iff \mathfrak{s}(y, \vec{\varphi}_l) \in \mathfrak{T}_{m,n} \iff \vdash_{\mathbf{G}} \sigma(\mathfrak{s}(y, \vec{\varphi}_l)) \iff \vdash_{\mathbf{G}} \mathfrak{s}(x, \sigma(\vec{\varphi}_l)),$$

so $\langle x, y \rangle \in \boldsymbol{\Omega}(\mathfrak{T}_{m,n})$. $\square$

Let us see that the set $\mathfrak{E}_{m,n}(x, y, \vec{z}_t) := \sigma'(\mathfrak{T}_{m,n})$ satisfies (R$_{\mathrm{BP}}$) and (MP$_{\mathrm{BP}}$).

For $i = 1, \dots, m+n$, let $\mathfrak{S}_i := \mathrm{Cn}_{\mathbf{G}}(\mathfrak{T}_{m,n}, \langle z_1, \dots, z_{i-1}, x, z_i, \dots, z_t\rangle_{m,n})$. Since $\mathbf{G}$ is protoalgebraic and $\mathfrak{T}_{m,n} \subseteq \mathfrak{S}_i$, the claims yield $\langle x, y\rangle \in \mathbf{\Omega}(\mathfrak{S}_i)$. Hence, by Theorem 2.27 we get $\langle z_1, \dots, z_{i-1}, y, z_i, \dots, z_t\rangle_{m,n} \in \mathfrak{S}_i$, i.e.:

$$\langle z_1, \dots, z_{i-1}, x, z_i, \dots, z_t\rangle_{m,n}, \mathfrak{T}_{m,n} \vdash_{\mathbf{G}} \langle z_1, \dots, z_{i-1}, y, z_i, \dots, z_t\rangle_{m,n}. \tag{2.11}$$

By structurality, applying σ' to both sides of (2.11) yields ($\mathrm{MP}_{\mathrm{BP}}$).

As regards (R_{BP}), we first need the following:

Claim 2.56.3. *$\sigma \circ \sigma' = \sigma' \circ \sigma$.*

Proof. If $t = 0$, then $\sigma(\sigma'(u)) = x = \sigma'(x) = \sigma'(\sigma(u))$ for every $u \in \mathsf{Var}$.

Suppose $t > 0$. Since neither x nor $\vec{z}_t$ are changed by σ or σ', we have $\sigma(\sigma'(u)) = \sigma'(\sigma(u))$ if u is among $x, \vec{z}_t$. For the variable y, we have:

$$\sigma(\sigma'(y)) = \sigma(y) = x = \sigma'(x) = \sigma'(\sigma(y)).$$

Finally, if u is not in $x, y, \vec{z}_t$, then $\sigma'(u) = \sigma(u) \in \{z_1, \dots, z_t\}$, so:

$$\sigma(\sigma'(u)) = \sigma'(u) = \sigma(u) = \sigma(\sigma(u)) = \sigma'(\sigma(u)).$$

In any case we have $\sigma(\sigma'(u)) = \sigma'(\sigma(u))$ for every $u \in \mathsf{Var}$, so the claim holds by Proposition 2.8. □

By the definition of $\mathfrak{T}_{m,n}$, we have $\vdash_{\mathbf{G}} \sigma(\mathfrak{T}_{m,n})$, so by structurality we get $\vdash_{\mathbf{G}} \sigma'(\sigma(\mathfrak{T}_{m,n}))$, which, as we have just seen, is equivalent to $\vdash_{\mathbf{G}} \sigma(\sigma'(\mathfrak{T}_{m,n}))$, i.e., $\vdash_{\mathbf{G}} \sigma(\mathfrak{E}_{m,n}(x, y, \vec{z}_t))$, and this is ($\mathrm{R}_{\mathrm{BP}}$).

Finally, if $\mathbf{G}$ is finitary, then there is some finite $\mathfrak{F}_{m,n} \subseteq \mathfrak{E}_{m,n}$ such that ($\mathrm{MP}_{\mathrm{BP}}$) holds with $\mathfrak{F}_{m,n}$ in place of $\mathfrak{E}_{m,n}$, and clearly (R_{BP}) holds for $\mathfrak{F}_{m,n}$ as well. ■

A variation of Theorem 2.56 appears in [**18**, Thm. 2.17] for multi-dimensional Gentzen relations. For Gentzen relations, Raftery adapts Pałasińska's [**23**, Thm. 5.12] and presents it without proof as [**27**, Thm. 13.4], but his statement contains two errors that we shall now discuss.

In what follows, concatenation of sequences is denoted by a comma, and sequences of length one are written without the angle brackets. For example, an expression of the form $\vec{a}, b, \vec{c} \rhd \vec{d}, e, \vec{f}$ denotes the sequent $\vec{a}^\frown\langle b\rangle^\frown\vec{c} \rhd \vec{d}^\frown\langle e\rangle^\frown\vec{f}$. We adopt this convention here to facilitate the comparison of the corrected version of Raftery's statement with the original one.

Raftery states that a Gentzen relation $\mathbf{G}$ with trace tr is protoalgebraic iff for every $\langle m, n\rangle \in \mathsf{tr}$ there is $\mathfrak{E}_{m,n}(x, y, z, w, \vec{r}, \vec{s}, \vec{t}, \vec{u}) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, where $x, y, z, w, \vec{r}, \vec{s}, \vec{t}, \vec{u}$

are pairwise different variables,[3] $\vec{r}$ and $\vec{t}$ have both length $\max\{m-1,0\}$, and $\vec{s}$ and $\vec{u}$ have both length $\max\{n-1,0\}$, such that:

($\mathrm{R}^*_{\mathrm{R}}$) $\vdash_{\mathsf{G}} \mathfrak{E}_{m,n}(x,y,x,y,\vec{r},\vec{s},\vec{t},\vec{u})$.

($\mathrm{MP}^*_{\mathrm{R}}$) $\vec{r},x,\vec{s} \rhd \vec{t},y,\vec{u},\ \ \mathfrak{E}_{m,n}(x,y,z,w,\vec{r},\vec{s},\vec{t},\vec{u}) \vdash_{\mathsf{G}} \vec{r},z,\vec{s} \rhd \vec{t},w,\vec{u}$.

In order for ($\mathrm{MP}^*_{\mathrm{R}}$) to make sense, the types of $\vec{r},x,\vec{s} \rhd \vec{t},y,\vec{u}$ and $\vec{r},z,\vec{s} \rhd \vec{t},w,\vec{u}$, which are shown in Table 2.1, must be in tr. The only type we assume to be in tr

CASE	TYPE
$m=0,\ n=0$	$\langle 1,1\rangle$
$m=0,\ n>0$	$\langle n,n\rangle$
$m>0,\ n=0$	$\langle m,m\rangle$
$m>0,\ n>0$	$\langle m+n-1, m+n-1\rangle$

TABLE 2.1. Possible types of $\vec{r},x,\vec{s} \rhd \vec{t},y,\vec{u}$ and $\vec{r},z,\vec{s} \rhd \vec{t},w,\vec{u}$.

is $\langle m,n\rangle$, so either those two sequents are both of type $\langle m,n\rangle$ or we must assume that, for each $\langle m,n\rangle \in \mathsf{tr}$, the corresponding type of the ones depicted in Table 2.1 is in tr. This assumption would overcomplicate the statement of the theorem and, besides, it is clear that Raftery does not assume it, since the only type he ever mentions is $\langle m,n\rangle$.

Therefore, the sequents $\vec{r},x,\vec{s} \rhd \vec{t},y,\vec{u}$ and $\vec{r},z,\vec{s} \rhd \vec{t},w,\vec{u}$ must be of type $\langle m,n\rangle$, which, as Table 2.1 indicates, is not the case in general. This is due to the fact that the lengths of the sequences $\vec{r},\vec{s},\vec{t},\vec{u}$ have not been chosen adequately: note that $\vec{r}^\frown\vec{s}$ and $\vec{t}^\frown\vec{u}$ have both either length $\max\{m+n-1,0\}$, if $m=0$ or $n=0$, or length $m+n-2$ otherwise, when their lengths should always be $\max\{m-1,0\}$ and $\max\{n-1,0\}$, respectively.

But the problem would not be entirely solved even if those lengths were correct, because Raftery's notation forces the variables x,y (respectively, z,w) to occur in $\vec{r},x,\vec{s} \rhd \vec{t},y,\vec{u}$ (respectively, $\vec{r},z,\vec{s} \rhd \vec{t},w,\vec{u}$), and thus makes it impossible for these two sequents to be of type $\langle m,n\rangle$ whenever $m=0$ or $n=0$: for example, if $m=0$ and $n>0$, then $\vec{r}^\frown\vec{s}$ has length 0 and $\vec{r},x,\vec{s} \rhd \vec{t},y,\vec{u}$ simplifies to $x \rhd \vec{t},y,\vec{u}$, which is not of type $\langle m,n\rangle$. In order to allow for the empty sequence $\varnothing$ to occur at either side of the symbol '$\rhd$', we must use a more flexible notation, one capable of picking *exactly* m elements from $\vec{r},x,\vec{s}$ and *exactly* n elements from $\vec{t},y,\vec{u}$. Hence, we can solve the problem if we use the notation $\langle\vec{r},x,\vec{s},\vec{t},y,\vec{u}\rangle_{m,n}$ that we introduced in Section 2.1, which always denotes an $\langle m,n\rangle$-sequent.

[3]We use Raftery's denotation for those variables to facilitate the comparison with his result.

Having discussed the two problems with Raftery's statement, we can use a proof analogous to that of Theorem 2.56 to obtain a corrected version of it:

THEOREM 2.57 (Correction of [**27**, Thm. 13.4]). *A Gentzen relation* G *with trace* tr *is protoalgebraic iff for every type* $\langle m, n\rangle \in \mathsf{tr}$ *there is a set of sequents* $\mathfrak{E}_{m,n}(x, y, z, w, \vec{r}, \vec{s}, \vec{t}, \vec{u}) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$*, where* $x, y, z, w, \vec{r}, \vec{s}, \vec{t}, \vec{u}$ *are pairwise different variables,* $\vec{r}{}^\frown\vec{s}$ *has length* $\max\{m-1, 0\}$ *and* $\vec{t}{}^\frown\vec{u}$ *has length* $\max\{n-1, 0\}$*, satisfying the following conditions:*

($\mathrm{R_R}$) $\vdash_{\mathsf{G}} \mathfrak{E}_{m,n}(x, y, x, y, \vec{r}, \vec{s}, \vec{t}, \vec{u})$.

($\mathrm{MP_R}$) $\langle \vec{r}, x, \vec{s}, \vec{t}, y, \vec{u}\rangle_{m,n}, \mathfrak{E}_{m,n}(x, y, z, w, \vec{r}, \vec{s}, \vec{t}, \vec{u}) \vdash_{\mathsf{G}} \langle \vec{r}, z, \vec{s}, \vec{t}, w, \vec{u}\rangle_{m,n}$.

Moreover, if G *is finitary then all the sets* $\mathfrak{E}_{m,n}$ *can be taken finite.*

The proof of Theorem 2.56 can also be easily adapted to obtain a variant of it with a simpler set $\mathfrak{E}_{m,n}$:

THEOREM 2.58. *A Gentzen relation* G *with trace* tr *is protoalgebraic iff for every* $\langle m, n\rangle \in \mathsf{tr}$ *there is a set* $\mathfrak{E}_{m,n}(x, y) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ *such that:*

($\mathrm{R^*_{BP}}$) $\vdash_{\mathsf{G}} \mathfrak{E}_{m,n}(x, x)$.

($\mathrm{MP^*_{BP}}$) $\langle z_1, \dots, z_{i-1}, x, z_i, \dots, z_t\rangle_{m,n}, \mathfrak{E}_{m,n}(x, y) \vdash_{\mathsf{G}} \langle z_1, \dots, z_{i-1}, y, z_i, \dots, z_t\rangle_{m,n}$ *for all* $i = 1, \dots, m+n$*, where* $t := \max\{m+n-1, 0\}$.

Moreover, if G *is finitary then all the sets* $\mathfrak{E}_{m,n}$ *can be taken finite.*

INDICATION FOR THE PROOF. For the '⇒' part, make σ map $\mathsf{Var} \setminus \{x, y, \vec{z}_t\}$ onto $\{x, y\}$ and let $\sigma'(u) := x$ for every variable $u \in \{z_1, \dots, z_t\}$. Everything else is just as in the proof of Theorem 2.56, with some minor, obvious changes. ∎

Of course, the price to pay for Theorem 2.58 is that variables that do not occur in $\mathfrak{E}_{m,n}$, namely $z_1, \dots, z_t$, will in general occur in the sequents that appear in ($\mathrm{MP^*_{BP}}$) alongside $\mathfrak{E}_{m,n}$.

In the context of Theorem 2.56, it is clear that the variables x, y have a more prominent role than the ones in $\vec{z}_t$, since the latter are just there in order to make it possible to build $\langle m, n\rangle$-sequents 'around' x and y. Following a usual convention in logic, we say that the variables $\vec{z}_t$ are the *parameters* of the set $\mathfrak{E}_{m,n}$. A natural question, then, is: can we get rid of the parameters? In a sense, we have already done so in Theorem 2.58, but in reality we have just swiped them under the carpet because they reappear in condition ($\mathrm{MP^*_{BP}}$). Nevertheless, we can prove a characterization of protoalgebraicity similar to that of Theorem 2.56 but in which no parameters are needed.

For this, we need to see certain sequents as variables:

DEFINITION 2.59. Let tr be a trace. For every $\langle m, n\rangle \in \mathsf{tr}$, the $\langle m, n\rangle$-*Gentzen variables* are defined as follows:

- $\mathfrak{x}_{m,n} := x_1, \dots, x_m \rhd x_{m+1}, \dots, x_{m+n}$.
- $\mathfrak{y}_{m,n} := y_1, \dots, y_m \rhd y_{m+1}, \dots, y_{m+n}$.
- $\mathfrak{z}_{m,n} := z_1, \dots, z_m \rhd z_{m+1}, \dots, z_{m+n}$.

Note that, for every sequent $\mathfrak{s}(\vec{u}_m, \vec{v}_n)$, where $\vec{u}_m, \vec{v}_n$ are pairwise different, we have $\mathfrak{s}(\mathfrak{x}_{m,n}) = \mathfrak{s}(\vec{x}_{m+n})$, $\mathfrak{s}(\mathfrak{y}_{m,n}) = \mathfrak{s}(\vec{y}_{m+n})$ and $\mathfrak{s}(\mathfrak{z}_{m,n}) = \mathfrak{s}(\vec{z}_{m+n})$.

DEFINITION 2.60. Let G be a Gentzen relation with trace tr. For all $\langle m, n\rangle \in \mathsf{tr}$, let $\sigma_{m,n} \in \mathrm{End}(\boldsymbol{Fm})$ be such that $\sigma_{m,n}(\mathfrak{y}_{m,n}) := \mathfrak{x}_{m,n}$ and $\sigma_{m,n}(u) := u$ for every variable u not in $\vec{y}_{m+n}$. The $\langle m, n\rangle$-*fundamental set* is defined as follows:

$$\mathfrak{S}^{m,n}_{\mathsf{G}} := \sigma^{-1}_{m,n}(\mathrm{Cn}_{\mathsf{G}}(\varnothing)) = \{\mathfrak{t} \in \mathsf{tr}\text{-}\mathsf{Seq} : \varnothing \vdash_{\mathsf{G}} \sigma_{m,n}(\mathfrak{t})\}.$$

We call $\sigma_{m,n}$ the $\langle m, n\rangle$-*fundamental substitution.*

According to [**14**, p. 324], the concept of fundamental set 'was introduced by Blok and Pigozzi, but it was Hermann who fully exploited its crucial role in the theory of protoalgebraic [sentential] logics'. The notion of Gentzen variables, i.e., sequents that play the same role played by ordinary variables in sentential logics, already appears in [**18**, p. 59] under the name of 'sequent-variables'; for k-deductive systems, the idea is present in [**6**, p. 26].

The fundamental sets will help us prove our characterization of protoalgebraicity. Let us see some of their properties:

PROPOSITION 2.61. *Let G be a Gentzen relation with trace tr. For every type $\langle m, n\rangle \in \mathsf{tr}$, we have:*

(i) $\mathfrak{S}^{m,n}_{\mathsf{G}} \in \mathcal{T}h(\mathsf{G})$.
(ii) $\mathfrak{x}_{m,n}/\boldsymbol{\Omega}(\mathfrak{S}^{m,n}_{\mathsf{G}}) = \mathfrak{y}_{m,n}/\boldsymbol{\Omega}(\mathfrak{S}^{m,n}_{\mathsf{G}})$.
(iii) *If G is protoalgebraic, then* $\mathfrak{x}_{m,n}, \mathfrak{S}^{m,n}_{\mathsf{G}} \vdash_{\mathsf{G}} \mathfrak{y}_{m,n}$.

PROOF.

(i) By Proposition 2.34 and Proposition 2.36(i).
(ii) Let p be any tr-valued polynomial function of $\boldsymbol{Fm}$, and let $\mathfrak{s}(u, \vec{v}_l) \in \mathsf{tr}\text{-}\mathsf{Seq}$ and $\varphi_1, \dots, \varphi_l \in Fm$, with $l \in \omega$, be such that $p(\psi) = \mathfrak{s}(\psi, \vec{\varphi}_l)$ for all $\psi \in Fm$. We have

$$\mathfrak{s}(x_i, \vec{\varphi}_l) \in \mathfrak{S}^{m,n}_{G} \iff \vdash_{\mathsf{G}} \sigma_{m,n}(\mathfrak{s}(x_i, \vec{\varphi}_l)) \iff \vdash_{\mathsf{G}} \mathfrak{s}(x_i, \sigma_{m,n}(\vec{\varphi}_l))$$

and

$$\mathfrak{s}(y_i, \vec{\varphi}_l) \in \mathfrak{S}^{m,n}_{G} \iff \vdash_{\mathsf{G}} \sigma_{m,n}(\mathfrak{s}(y_i, \vec{\varphi}_l)) \iff \vdash_{\mathsf{G}} \mathfrak{s}(x_i, \sigma_{m,n}(\vec{\varphi}_l)).$$

Hence, $\langle x_i, y_i\rangle \in \boldsymbol{\Omega}(\mathfrak{S}^{m,n}_{G})$ for $i = 1, \dots, m+n$, so (ii) holds.

(iii) By (ii) and Theorem 2.55. ∎

Finally, our version of the syntactic characterization, the proof of which generalizes the one of [**14**, Thm. 6.7]:

THEOREM 2.62. *A Gentzen relation* G *with trace* tr *is protoalgebraic iff for every* $\langle m, n\rangle \in \mathsf{tr}$ *there is a set* $\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n}) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ *for which the following conditions hold:*

(R) $\vdash_{\mathsf{G}} \mathfrak{E}_{m,n}(\mathfrak{x}_{m,n}, \mathfrak{x}_{m,n})$.

(MP) $\mathfrak{x}_{m,n}, \mathfrak{E}_{m,n}(\mathfrak{x}_{m,n}, \mathfrak{y}_{m,n}) \vdash_{\mathsf{G}} \mathfrak{y}_{m,n}$.

Moreover, if G *is finitary then all the sets* $\mathfrak{E}_{m,n}$ *can be taken finite.*

PROOF. ($\Leftarrow$) Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathcal{T}h(\mathsf{G})$ be such that $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. In order to prove that $\boldsymbol{\Omega}(\mathfrak{T}_1) \subseteq \boldsymbol{\Omega}(\mathfrak{T}_2)$ it suffices to show that $\boldsymbol{\Omega}(\mathfrak{T}_1)$ is compatible with $\mathfrak{T}_2$, so let $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$ be such that $\mathfrak{s}/\boldsymbol{\Omega}(\mathfrak{T}_1) = \mathfrak{r}/\boldsymbol{\Omega}(\mathfrak{T}_1)$ and $\mathfrak{s} \in \mathfrak{T}_2$. By Proposition 2.20(iv), we need to prove that $\mathfrak{r} \in \mathfrak{T}_2$.

Let $\langle m, n\rangle := \mathsf{tp}(\mathfrak{s}) = \mathsf{tp}(\mathfrak{r})$. If $\langle m, n\rangle = \langle 0, 0\rangle$ then $\mathfrak{s} = \mathfrak{r}$, so $\mathfrak{r} \in \mathfrak{T}_2$ and we are done. Thus, assume $\langle m, n\rangle \neq \langle 0, 0\rangle$. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be any substitution such that $\sigma(\mathfrak{x}_{m,n}) := \mathfrak{s}$ and $\sigma(\mathfrak{y}_{m,n}) := \mathfrak{r}$. By structurality, applying σ to both sides of (MP) yields

$$\mathfrak{s}, \mathfrak{E}_{m,n}(\mathfrak{s}, \mathfrak{r}) \vdash_{\mathsf{G}} \mathfrak{r},$$

so we shall get $\mathfrak{r} \in \mathfrak{T}_2$ by proving $\mathfrak{E}_{m,n}(\mathfrak{s}, \mathfrak{r}) \subseteq \mathfrak{T}_2$, i.e., $\sigma(\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n})) \subseteq \mathfrak{T}_2$.

To this end, let $\mathfrak{t} \in \mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n})$. By (R) we have $\vdash_{\mathsf{G}} \mathfrak{t}(\vec{x}_{m+n}, \vec{x}_{m+n})$, so by structurality we get $\vdash_{\mathsf{G}} \sigma(\mathfrak{t}(\vec{x}_{m+n}, \vec{x}_{m+n}))$, i.e., $\vdash_{\mathsf{G}} \mathfrak{t}(\mathfrak{s}, \mathfrak{s})$. As $\mathfrak{T}_1$ is a theory, $\mathfrak{t}(\mathfrak{s}, \mathfrak{s}) \in \mathfrak{T}_1$. Given that $\boldsymbol{\Omega}(\mathfrak{T}_1)$ is compatible with $\mathfrak{T}_1$ and that $\mathfrak{s}/\boldsymbol{\Omega}(\mathfrak{T}_1) = \mathfrak{r}/\boldsymbol{\Omega}(\mathfrak{T}_1)$, by Lemma 2.31 we get $\sigma(\mathfrak{t}) = \mathfrak{t}(\mathfrak{s}, \mathfrak{r}) \in \mathfrak{T}_1 \subseteq \mathfrak{T}_2$. Hence, $\mathfrak{E}_{m,n}(\mathfrak{s}, \mathfrak{r}) \subseteq \mathfrak{T}_2$.

($\Rightarrow$) Fix any $\langle m, n\rangle \in \mathsf{tr}$. If $\langle m, n\rangle = \langle 0, 0\rangle$, then $\mathfrak{x}_{m,n} = \mathfrak{y}_{m,n} = \varnothing \rhd \varnothing$ and the set $\mathfrak{E}_{0,0} := \varnothing$ clearly satisfies (R) and (MP), regardless of protoalgebraicity.

Assume now that $\langle m, n\rangle \neq \langle 0, 0\rangle$. Then, the variable x_1 occurs among $\vec{x}_{m+n}$. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be given by $\sigma(\vec{x}_{m+n}) := \vec{x}_{m+n}$, $\sigma(\vec{y}_{m+n}) := \vec{y}_{m+n}$ and $\sigma(u) := x_1$ for all variables u not appearing among $\vec{x}_{m+n}, \vec{y}_{m+n}$. By Proposition 2.9, all the variables occurring in $\sigma(\varphi)$, where φ is any formula, are among $\vec{x}_{m+n}, \vec{y}_{m+n}$. Define the set $\mathfrak{E}_{m,n}$ as follows:

$$\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n}) := \sigma(\mathfrak{S}_G^{m,n}).$$

Let us see that $\mathfrak{E}_{m,n}$ satisfies conditions (R) and (MP).

By Proposition 2.61(iii) we have

$$\mathfrak{x}_{m,n}, \mathfrak{S}_G^{m,n} \vdash_{\mathsf{G}} \mathfrak{y}_{m,n}, \tag{2.12}$$

whence (MP) follows by structurality, applying σ to both sides of (2.12).

As regards (R), note that, by Proposition 2.8, we have $\sigma \circ \sigma_{m,n} = \sigma_{m,n} \circ \sigma$ because $(\sigma \circ \sigma_{m,n})(u) = (\sigma_{m,n} \circ \sigma)(u)$ for every $u \in \mathsf{Var}$. This is clear for the variables $\vec{x}_{m,n}$, as they are not changed by σ or $\sigma_{m,n}$. For the variables $\vec{y}_{m+n}$, we have:

$$\sigma(\sigma_{m,n}(\vec{y}_{m+n})) = \sigma(\vec{x}_{m+n}) = \vec{x}_{m+n} = \sigma_{m,n}(\vec{y}_{m+n}) = \sigma_{m,n}(\sigma(\vec{y}_{m+n})).$$

And for every variable u not appearing in $\vec{x}_{m+n}, \vec{y}_{m+n}$:

$$\sigma_{m,n}(\sigma(u)) = \sigma_{m,n}(x_1) = x_1 = \sigma(u) = \sigma(\sigma_{m,n}(u)).$$

By Definition 2.60, $\vdash_{\mathsf{G}} \sigma_{m,n}(\mathfrak{S}_G^{m,n})$, so by structurality we get $\vdash_{\mathsf{G}} \sigma(\sigma_{m,n}(\mathfrak{S}_G^{m,n}))$, which, as we have just seen, is equivalent to $\vdash_{\mathsf{G}} \sigma_{m,n}(\sigma(\mathfrak{S}_G^{m,n}))$, i.e., $\vdash_{\mathsf{G}} \sigma_{m,n}(\mathfrak{E}_{m,n})$, and this is (R).

Finally, if G is finitary, then there is some finite $\mathfrak{F}_{m,n} \subseteq \mathfrak{E}_{m,n}$ such that (MP) holds with $\mathfrak{F}_{m,n}$ in place of $\mathfrak{E}_{m,n}$, and clearly (R) holds for $\mathfrak{F}_{m,n}$ as well. ■

Both Theorem 2.56 and Theorem 2.62 characterize protoalgebraic Gentzen relations. The former employs $m+n+1$ variables for each type $\langle m, n\rangle$, of which $\max\{m+n-1, 0\}$ are parameters. The latter uses almost twice as many (ordinary) variables per type, $2(m+n)$, but no parameters, and, moreover, the variables $\vec{x}_{m+n}, \vec{y}_{m+n}$ actually appear as two $\langle m, n\rangle$-Gentzen variables, since none of them is ever treated individually, but rather as forming part of the sequent $\mathfrak{x}_{m,n}$ or $\mathfrak{y}_{m,n}$. This is the reason why the *modus ponens* of Theorem 2.62 is much simpler than that of Theorem 2.56 (which is actually a conjunction of several *modi ponentes*), hence easier to work with, and also why Theorem 2.62 bears a closer resemblance to the syntactic characterization of protoalgebraic sentential logics (cf. [**14**, Thm. 6.7]) than Theorem 2.56.

2.7.3. Algebraic characterizations. We now prove some characterizations of protoalgebraicity of an algebraic flavour, including the generalization to Gentzen relations of the so-called correspondence theorem.

The first one, concerning the behaviour of the Leibniz operator in *arbitrary* algebras, is a consequence of Theorem 2.62:

THEOREM 2.63. *A Gentzen relation* G *with trace* tr *is protoalgebraic iff for every algebra* $\boldsymbol{A}$ *the Leibniz operator* $\boldsymbol{\Omega}^{\mathbf{A}}$ *is monotone on* $\mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$.

PROOF. ($\Rightarrow$) Let $F, G \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$ be such that $F \subseteq G$. To show that $\boldsymbol{\Omega}^{\mathbf{A}}(F) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(G)$ it suffices to prove that $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is compatible with G, so let $\mathfrak{a}, \mathfrak{b} \in \mathsf{Seq}(A)$ be such that $\mathfrak{a} \in G$ and $\mathfrak{a}/\boldsymbol{\Omega}^{\mathbf{A}}(F) = \mathfrak{b}/\boldsymbol{\Omega}^{\mathbf{A}}(F)$. Let $\langle m, n\rangle := \mathsf{tp}(\mathfrak{a}) = \mathsf{tp}(\mathfrak{b})$. By Proposition 2.20(iv), we need to prove $\mathfrak{b} \in G$.

If $\langle m,n\rangle = \langle 0,0\rangle$ then $\mathfrak{b} = \mathfrak{a} \in G$, so assume $\langle m,n\rangle \neq \langle 0,0\rangle$. By Theorem 2.62, there is a set $\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n}) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ for which (R) and (MP) hold.

Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $h(\vec{x}_{m+n}) := \mathfrak{a}$ and $h(\vec{y}_{m+n}) := \mathfrak{b}$. Then, since $h(\mathfrak{x}_{m,n}) = \mathfrak{a} \in G$ and $\mathfrak{E}^{\boldsymbol{A}}(\mathfrak{a},\mathfrak{a}) \subseteq F$ by (R), Lemma 2.31 yields

$$h(\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n})) = \mathfrak{E}^{\boldsymbol{A}}(\mathfrak{a},\mathfrak{b}) \subseteq F \subseteq G,$$

whence $h(\mathfrak{y}_{m,n}) = \mathfrak{b} \in G$ by (MP).

($\Leftarrow$) By Definition 2.53 and Proposition 2.34, taking $\boldsymbol{A} := \boldsymbol{Fm}$. ■

The following lemma provides another algebraic characterization of protoalgebraicity, which will mainly be used in the proof of Theorem 2.67 below.

DEFINITION 2.64 (cf. [**5**, Def. 7.3]). A Gentzen relation $\mathbf{G}$ is said to have the *compatibility property* if the following holds for every algebra $\boldsymbol{A}$, every $\theta \in \mathrm{Co}(\boldsymbol{A})$ and every $\mathbf{G}$-filters F, G of $\boldsymbol{A}$: if $F \subseteq G$ and θ is compatible with F, then θ is compatible with G.

LEMMA 2.65. *A Gentzen relation* $\mathbf{G}$ *is protoalgebraic iff* $\mathbf{G}$ *has the compatibility property.*

PROOF. ($\Rightarrow$) Let $\boldsymbol{A}$ be an algebra and $F, G \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$ be such that $F \subseteq G$. If $\theta \in \mathrm{Co}(\boldsymbol{A})$ is compatible with F, then $\theta \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F)$. By Theorem 2.63, $\boldsymbol{\Omega}^{\mathbf{A}}(F) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(G)$, so $\theta \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(G)$, whence θ is compatible with G by Proposition 2.30.

($\Leftarrow$) Let $\boldsymbol{A}$ be an algebra and $F, G \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$ be such that $F \subseteq G$. Since $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is compatible with F, the assumption yields that $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is compatible with G, so $\boldsymbol{\Omega}^{\mathbf{A}}(F) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(G)$. Hence, $\mathbf{G}$ is protoalgebraic by Theorem 2.63. ■

We are now prepared for the generalization of the correspondence theorem to Gentzen relations:

DEFINITION 2.66. A Gentzen relation $\mathbf{G}$ is said to have the *correspondence property* if, for every $\mathbf{G}$-matrices $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{B}, G\rangle$ and every strict surjective homomorphism $h : \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$, we have $F' = h^{-1}(h(F'))$ for every $F' \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})^F$.

THEOREM 2.67 (Correspondence Theorem). *Let* $\mathbf{G}$ *be a Gentzen relation with trace* tr*. The following are equivalent:*

(i) $\mathbf{G}$ *has the correspondence property.*

(ii) $\mathbf{G}$ *is protoalgebraic.*

(iii) *For all algebras* $\boldsymbol{A}, \boldsymbol{B}$*, all* $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$*, all* $G \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{B})$ *and all surjective homomorphisms* $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$*, we have:*

$$h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathbf{G}}(h(F)) \vee^{\boldsymbol{B}} G) = F \vee^{\boldsymbol{A}} h^{-1}(G). \tag{2.13}$$

(iv) *If* $h : \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ *is a strict surjective homomorphism, where* $\langle \boldsymbol{A}, F\rangle$ *and* $\langle \boldsymbol{B}, G\rangle$ *are* G*-matrices, then:*

$$h : \langle \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{F}, \subseteq\rangle \cong \langle \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})^{G}, \subseteq\rangle : h^{-1}.$$

Proof. (i) $\Rightarrow$ (ii) Let $\boldsymbol{A}$ be an algebra, $F, G \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$ such that $F \subseteq G$ and $\theta \in \mathrm{Co}(\boldsymbol{A})$ such that θ is compatible with F. By Proposition 2.37 and Proposition 2.42, $\langle \boldsymbol{A}/\theta, F/\theta\rangle$ is a G-matrix, so the natural projection $\pi_\theta : \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a strict surjective homomorphism from $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{A}/\theta, F/\theta\rangle$ because $\pi_\theta^{-1}(F/\theta) \subseteq F$ by Definition 2.19. Since $F \subseteq G$, (i) yields $\pi_\theta^{-1}(\pi_\theta(G)) = G$, so θ is compatible with G. By Lemma 2.65, G is protoalgebraic.

(ii) $\Rightarrow$ (iii) We need to prove that $h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(h(F)) \vee^{\boldsymbol{B}} G)$ is the least G-filter of $\boldsymbol{A}$ that contains $F \cup h^{-1}(G)$.

By Proposition 2.36(i), $h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(h(F)) \vee^{\boldsymbol{B}} G)$ is a G-filter of $\boldsymbol{A}$.

Since $h(F) \cup G \subseteq \mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(h(F) \vee^{\boldsymbol{B}} G)$, we have:

$$F \cup h^{-1}(G) \subseteq h^{-1}(h(F) \cup G) \subseteq h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(h(F)) \vee^{\boldsymbol{B}} G).$$

Finally, let $H \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$ be such that $F \cup h^{-1}(G) \subseteq H$. By Proposition 2.36(i), we have $h^{-1}(G) \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$, and by Lemma 2.22 $\ker h$ is compatible with $h^{-1}(G)$, so $\ker h$ is compatible with H by Lemma 2.65. Thus, $h(H) \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})$ by Proposition 2.36(ii), so the surjectivity of h yields $\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(h(F)) \vee^{\boldsymbol{B}} G \subseteq h(H)$ and, therefore,

$$h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(h(F)) \vee^{\boldsymbol{B}} G) \subseteq h^{-1}(h(H)) = H,$$

where the equality is due to Proposition 2.21(ii).

(iii) $\Rightarrow$ (i) Let $h : \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ be a strict surjective homomorphism between two G-matrices, and let $F' \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{F}$. By Proposition 2.42, $G \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})$, so (iii) yields

$$h^{-1}(h(F')) \subseteq h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(h(F')) \vee^{\boldsymbol{B}} G) = F' \vee^{\boldsymbol{A}} h^{-1}(G) = F' \subseteq h^{-1}(h(F')),$$

where the last equality is due to the strictness of h.

(i) $\Rightarrow$ (iv) Let $H : \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{F} \to \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})^{G}$ be defined by $H(F') := h(F')$ for all $F' \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{F}$. By (i) and Proposition 2.21(ii), $\ker h$ is compatible with F' for all $F' \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{F}$. So, since h is surjective, $h(F') \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})$ by Proposition 2.36(ii). Moreover, the surjectivity and the strictness of h yield $G = h(h^{-1}(G)) = h(F) \subseteq h(F')$. Hence, H is well defined and by (i) we have:

$$h^{-1}(H(F')) = F' \tag{2.14}$$

for all $F' \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{F}$.

If $F_1, F_2 \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^F$ are such that $H(F_1) = H(F_2)$, then

$$F_1 = h^{-1}(H(F_1)) = h^{-1}(H(F_2)) = F_2$$

by (2.14), so H is injective.

Let $G' \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})^G$. By Proposition 2.36(i), $h^{-1}(G')$ is a G-filter of $\boldsymbol{A}$. Also, $F = h^{-1}(G) \subseteq h^{-1}(G')$, so $h^{-1}(G') \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^F$. As $H(h^{-1}(G)) = h(h^{-1}(G)) = G$ because h is surjective, H is surjective.

If $F_1, F_2 \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^F$ are such that $F_1 \subseteq F_2$, then clearly $H(F_1) \subseteq H(F_2)$, so H is order-preserving.

Finally, $h^{-1} : \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})^G \to \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^F$ is the inverse of H by (2.14) and, if $h^{-1}(G_1) \subseteq h^{-1}(G_2)$ for some $G_1, G_2 \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})^G$, then the surjectivity of h yields $G_1 = h(h^{-1}(G_1)) \subseteq h(h^{-1}(G_2)) = G_2$, so h^{-1} is also order-preserving.

(iv) $\Rightarrow$ (i) Let $h : \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ be a strict surjective homomorphism, where $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, G\rangle$ are G-matrices. If $F' \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^F$, then $h^{-1}(h(F')) = F'$ holds by (iv). ∎

2.8. Relative equational Gentzen relations

Recall (cf. Definition 1.36) that an equation is a pair of formulas.

Technically, a $\langle 1,1\rangle$-sequent of a set A is a pair of the form $\langle\langle a\rangle, \langle b\rangle\rangle$ for some $a, b \in A$, but we identify $\langle 1,1\rangle$-sequents and pairs, so that, in particular, we make no distinction among equations, pairs of formulas and $\langle 1,1\rangle$-sequents of Fm.

Definition 2.68. Let K be a class of algebras. The *equational Gentzen relation relative to* K, denoted by $\mathsf{EQ}(\mathsf{K})$, is the Gentzen relation $\langle \mathcal{L}, \vdash_{\mathsf{EQ}(\mathsf{K})}\rangle$ with trace $\{\langle 1,1\rangle\}$ defined as follows, for every $\mathfrak{C} \cup \{\delta \rhd \varepsilon\} \subseteq \langle 1,1\rangle\text{-}\mathsf{Seq}$:

$$\begin{aligned}\mathfrak{C} \vdash_{\mathsf{EQ}(\mathsf{K})} \delta \rhd \varepsilon \iff & \text{for all } \boldsymbol{A} \in \mathsf{K} \text{ and all } h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}),\\ & \text{if } \boldsymbol{A} \models \mathfrak{C}[\![h]\!], \text{ then } \boldsymbol{A} \models \delta \approx \varepsilon[\![h]\!].\end{aligned}$$

It is straightforward to check that $\mathsf{EQ}(\mathsf{K})$ is indeed a Gentzen relation.

We frequently write $\mathfrak{C} \models_{\mathsf{K}} \delta \approx \varepsilon$ in place of $\mathfrak{C} \vdash_{\mathsf{EQ}(\mathsf{K})} \delta \rhd \varepsilon$.

Under the identification of $\langle 1,1\rangle$-sequents and pairs, we have:

Proposition 2.69 (cf. [**14**, Thm. 1.76]). *If* K *is a quasivariety, then*

$$\mathcal{F}i_{\mathsf{EQ}(\mathsf{K})}(\boldsymbol{A}) = \mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$$

for every algebra $\boldsymbol{A}$ *(so* $\mathrm{Fg}^{\boldsymbol{A}}_{\mathsf{EQ}(\mathsf{K})}(X) = \boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(X)$ *for every* $X \subseteq A \times A$*).*

As a consequence, we obtain:

Proposition 2.70. *Let* K *be a quasivariety. For every algebra* $\boldsymbol{A}$ *and every* $F \in \mathcal{F}i_{\mathsf{EQ(K)}}(\boldsymbol{A})$*, we have* $\boldsymbol{\Omega}^{\mathbf{A}}(F) = F$*.*

Proof. By Proposition 2.69, $F \in \mathrm{Co}(\boldsymbol{A})$.

Let $\alpha \rhd \beta, \gamma \rhd \delta \in \langle 1,1\rangle\text{-}\mathsf{Seq}(A)$ be such that $\alpha \rhd \beta/F = \gamma \rhd \delta/F$ and with $\alpha \rhd \beta \in F$. Then, $\langle \alpha, \beta\rangle, \langle \alpha, \gamma\rangle, \langle \beta, \delta\rangle \in F$. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $h(x_1) := \alpha$, $h(x_2) := \beta$, $h(y_1) := \gamma$ and $h(y_2) := \delta$. Since

$$x_1 \approx x_2, x_1 \approx y_1, x_2 \approx y_2 \models_{\mathsf{K}} y_1 \approx y_2$$

and F is an $\mathsf{EQ(K)}$-filter of $\boldsymbol{A}$, we get $\gamma \rhd \delta \in F$. By Proposition 2.20(iv), F is compatible with F.

Finally, let $\theta \in \mathrm{Co}(\boldsymbol{A})$ be compatible with F, and let $\langle \alpha, \beta\rangle \in \theta$. Define the unary $\langle 1,1\rangle$-valued polynomial function p of $\boldsymbol{A}$ by setting $p(\gamma) := \alpha \rhd \gamma$ for every $\gamma \in A$. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $h(x) := \alpha$. Since F is an $\mathsf{EQ(K)}$-filter of $\boldsymbol{A}$ and $\models_{\mathsf{K}} x \approx x$, we have $p(\alpha) = \alpha \rhd \alpha \in F$. By Theorem 2.27, $p(\beta) = \alpha \rhd \beta \in F$.

Therefore, F is the largest congruence of $\boldsymbol{A}$ compatible with F. ■

The $\mathsf{EQ(K)}$-theories are the K-congruences of the formula algebra:

Proposition 2.71. *Let* K *be a quasivariety, and let* $\mathfrak{E} \cup \{\delta \approx \varepsilon\} \subseteq \langle 1,1\rangle\text{-}\mathsf{Seq}$*. Then:*

$$\mathfrak{E} \models_{\mathsf{K}} \delta \approx \varepsilon \iff \langle \delta, \varepsilon\rangle \in \boldsymbol{\Theta}_{\mathsf{K}}(\mathfrak{E}).$$

Proof.

$$\begin{aligned}
\mathfrak{E} \models_{\mathsf{K}} \delta \approx \varepsilon &\iff \mathfrak{E} \vdash_{\mathsf{EQ(K)}} \delta \rhd \varepsilon \\
&\iff \delta \rhd \varepsilon \in \mathrm{Cn}_{\mathsf{EQ(K)}}(\mathfrak{E}) \\
&\iff \delta \rhd \varepsilon \in \mathrm{Fg}_{\mathsf{EQ(K)}}(\mathfrak{E}) && \text{(by Proposition 2.34)} \\
&\iff \langle \delta, \varepsilon\rangle \in \boldsymbol{\Theta}_{\mathsf{K}}(\mathfrak{E}) && \text{(by Proposition 2.69)}
\end{aligned}$$ ■

Corollary 2.72. *Let* K *be a quasivariety, and let* $\mathfrak{E} \cup \{\delta \approx \varepsilon\} \subseteq \langle 1,1\rangle\text{-}\mathsf{Seq}_n$ *for some* $n \in \omega$*. Then:*

$$\mathfrak{E} \models_{\mathsf{K}} \delta \approx \varepsilon \iff \langle \delta, \varepsilon\rangle \in \boldsymbol{\Theta}_{\mathsf{K}}^{\boldsymbol{Fm}_n}(\mathfrak{E}).$$

Proof.

$$\begin{aligned}
\mathfrak{E} \models_{\mathsf{K}} \delta \approx \varepsilon &\iff \langle \delta, \varepsilon\rangle \in \boldsymbol{\Theta}_{\mathsf{K}}(\mathfrak{E}) && \text{(by Proposition 2.71)} \\
&\iff \langle \delta, \varepsilon\rangle \in \mathrm{Fg}_{\mathsf{EQ(K)}}(\mathfrak{E}) && \text{(by Proposition 2.69)} \\
&\iff \langle \delta, \varepsilon\rangle \in \mathrm{Fg}_{\mathsf{EQ(K)}}^{\boldsymbol{Fm}_n}(\mathfrak{E}) && \text{(by Corollary 2.35)} \\
&\iff \langle \delta, \varepsilon\rangle \in \boldsymbol{\Theta}_{\mathsf{K}}^{\boldsymbol{Fm}_n}(\mathfrak{E}) && \text{(by Proposition 2.69)}
\end{aligned}$$ ■

When K is a quasivariety (in particular, a variety), $\mathsf{EQ}(\mathsf{K})$ is protoalgebraic and finitary:

PROPOSITION 2.73. *If* K *is a quasivariety, then* $\mathsf{EQ}(\mathsf{K})$ *is protoalgebraic.*

PROOF. By Proposition 2.70 and Theorem 2.63. ∎

PROPOSITION 2.74 (cf. [**30**, Lem. 0.6]). *Let* K *be a class of algebras. If* K *is closed under ultraproducts, then* $\mathsf{EQ}(\mathsf{K})$ *is finitary.*

COROLLARY 2.75. *If* K *is a quasivariety, then* $\mathsf{EQ}(\mathsf{K})$ *is finitary.*

2.9. Equivalent Gentzen relations

One of the main goals of abstract algebraic logic is to determine, given a logic, the class of algebras (or other structures heavily based on algebras, like matrices) that can most naturally be associated with that logic, in the sense of there being so nice a correspondence between the two, that the tools of universal algebra can be used to study the logic (and vice versa).

In [**4**], Blok and Pigozzi gave a mathematical definition of what it means for a (finitary) sentential logic to be 'algebraizable', i.e., to be, in some precise sense (cf. Definition 2.88 below), equivalent to the equational sentential logic relative to a class of algebras. They broadened the concept in [**6**] to encompass (finitary) k-deductive systems, by defining algebraizability as a particular case of a notion of equivalence between k-deductive systems. In [**30**], Rebagliato and Verdú generalized the notions of equivalence and algebraizability to finitary Gentzen relations. Our presentation closely follows that of [**27**, §6], which is essentially the same as Rebagliato and Verdú's but without finitarity assumptions.

DEFINITION 2.76. Let tr and tr' be traces. A *transformer from* tr*-sequents to* tr'*-sequents* is a function $\tau : \mathsf{tr}\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}'\text{-}\mathsf{Seq})$ that commutes with substitutions, in the sense that $\tau(\sigma(\mathfrak{s})) = \sigma(\tau(\mathfrak{s}))$ for every $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ and every $\mathfrak{s} \in \mathsf{tr}\text{-}\mathsf{Seq}$. Such a transformer is said to be *finitary* if the set $\tau(\mathfrak{s})$ is finite for all $\mathfrak{s} \in \mathsf{tr}\text{-}\mathsf{Seq}$.

When the traces tr and tr' need not be mentioned, we simply say that τ is a transformer.

In abstract algebraic logic transformers are sometimes defined without the commutativity requirement, and then those that satisfy it are called 'structural transformers' (cf. [**14**, Defs. 3.1,3.2]). We have decided not to make this distinction because we shall only work with 'transformers' that commute with substitutions.

If τ is a transformer from tr-sequents to tr'-sequents, for some traces tr and tr', then τ induces a function $\tau : \mathcal{P}(\mathsf{tr}\text{-}\mathsf{Seq}) \to \mathcal{P}(\mathsf{tr}'\text{-}\mathsf{Seq})$ given by

$$\tau(\mathfrak{P}) := \bigcup\{\tau(\mathfrak{s}) : \mathfrak{s} \in \mathfrak{P}\}$$

for all $\mathfrak{P} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$.

LEMMA 2.77. *Transformers commute with arbitrary unions, i.e., if* $\mathsf{tr}, \mathsf{tr}'$ *are traces,* τ *is a transformer from* tr*-sequents to* tr'*-sequents and* $\{\mathfrak{P}_i : i \in I\}$ *is a family of sets of* tr*-sequents, then:*

$$\tau(\bigcup_{i\in I} \mathfrak{P}_i) = \bigcup_{i\in I} \tau(\mathfrak{P}_i).$$

PROOF.

$$\tau(\bigcup_{i\in I} \mathfrak{P}_i) = \bigcup\{\tau(\mathfrak{s}) : \mathfrak{s} \in \bigcup_{i\in I} \mathfrak{P}_i\} = \bigcup_{i\in I}\bigcup\{\tau(\mathfrak{s}) : \mathfrak{s} \in \mathfrak{P}_i\} = \bigcup_{i\in I} \tau(\mathfrak{P}_i).$$

■

LEMMA 2.78. *Let* $\mathsf{tr}, \mathsf{tr}'$ *be traces,* τ *a transformer from* tr*-sequents to* tr'*-sequents and* $\mathfrak{s} \in \mathsf{tr}\text{-}\mathsf{Seq}$*. For every* $u \in \mathsf{Var}$*, if* u *occurs in* $\tau(\mathfrak{s})$*, then* u *occurs in* $\mathfrak{s}$*.*

PROOF. Suppose u does not occur in $\mathfrak{s}$. Let v be a variable different than u, and let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be the substitution given by $\sigma(u) := v$ and $\sigma(w) := w$ for every variable $w \neq u$. By Proposition 2.9, u does not occur in $\sigma(\mathfrak{r})$ for any $\mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$. By the assumption, $\sigma(\mathfrak{s}) = \mathfrak{s}$, so we have:

$$\tau(\mathfrak{s}) = \tau(\sigma(\mathfrak{s})) = \sigma(\tau(\mathfrak{s})).$$

But this is impossible, since u occurs in $\tau(\mathfrak{s})$ and not in $\sigma(\tau(\mathfrak{s}))$. Hence, u does occur in $\mathfrak{s}$. ■

COROLLARY 2.79. *In the conditions of Lemma 2.78, if no variable occurs in* $\mathfrak{s}$*, then no variable occurs in* $\tau(\mathfrak{s})$*.*

REMARK 2.80. The converse of Lemma 2.78 does not hold, so transformers do not preserve variables (they simply do not add new ones). For example, let $\mathsf{tr} := \{\langle 0, 1\rangle\}$, $\mathsf{tr}' := \{\langle 0, 0\rangle\}$ and define $\tau : \mathsf{tr}\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}'\text{-}\mathsf{Seq})$ by setting $\tau(\mathfrak{s}) := \{\varnothing \rhd \varnothing\}$ for all $\mathfrak{s} \in \mathsf{tr}$. Clearly, τ is a transformer from tr-sequents to tr'-sequents, and $\tau(\varnothing \rhd x_1) = \{\varnothing \rhd \varnothing\}$ provides a counterexample to the converse of Lemma 2.78.

Raftery defines in [**27**, Def. 5.2] the notion of 'definable transformer' as any function $\tau : \mathsf{tr}\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}'\text{-}\mathsf{Seq})$, where $\mathsf{tr}, \mathsf{tr}'$ are traces, such that for every $\langle m, n\rangle \in \mathsf{tr}$ there is a set $\tau_{m,n}(\vec{x}_{m+n}) \subseteq \mathsf{tr}'\text{-}\mathsf{Seq}$ satisfying $\tau(\mathfrak{s}) = \tau_{m,n}(\mathfrak{s})$ for all $\mathfrak{s} \in \langle m, n\rangle\text{-}\mathsf{Seq}$. He then proves that this condition is equivalent to the commutativity with substitutions, so in our terminology we have:

PROPOSITION 2.81 (cf. [**27**, Thm. 5.4]). *Let* tr *and* tr' *be traces. A function* $\tau : \mathsf{tr}\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}'\text{-}\mathsf{Seq})$ *is a transformer from* tr*-sequents to* tr'*-sequents iff for* *every* $\langle m,n\rangle \in \mathsf{tr}$ *there is a set of* tr'*-sequents* $\tau_{m,n}(\vec{x}_{m+n}) \subseteq \mathsf{tr}'\text{-}\mathsf{Seq}$ *such that* $\tau(\mathfrak{s}) = \tau_{m,n}(\mathfrak{s})$ *for every* $\mathfrak{s} \in \langle m,n\rangle\text{-}\mathsf{Seq}$.

DEFINITION 2.82. Let $\mathbf{G}$ and $\mathbf{G}'$ be Gentzen relations with traces tr and tr', respectively. We say that $\mathbf{G}$ and $\mathbf{G}'$ are *equivalent* if there are transformers $\tau : \mathsf{tr}\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}'\text{-}\mathsf{Seq})$ and $\rho : \mathsf{tr}'\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}\text{-}\mathsf{Seq})$ such that, for all $\mathfrak{P} \cup \{\mathfrak{p}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ and all $\mathfrak{S} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}'\text{-}\mathsf{Seq}$, the following hold:

(ALG1) $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{p} \iff \tau(\mathfrak{P}) \vdash_{\mathbf{G}'} \tau(\mathfrak{p})$.
(ALG2) $\mathfrak{s} \dashv\vdash_{\mathbf{G}'} \tau(\rho(\mathfrak{s}))$.
(ALG3) $\mathfrak{S} \vdash_{\mathbf{G}'} \mathfrak{s} \iff \rho(\mathfrak{S}) \vdash_{\mathbf{G}} \rho(\mathfrak{s})$.
(ALG4) $\mathfrak{p} \dashv\vdash_{\mathbf{G}} \rho(\tau(\mathfrak{p}))$.

In this situation, we say that $\mathbf{G}$ is equivalent to $\mathbf{G}'$ (in that order) with respect to the transformers τ and ρ (in that order), and we denote this by $\tau : \mathbf{G} \cong \mathbf{G}' : \rho$.

REMARK 2.83. $\tau : \mathbf{G} \cong \mathbf{G}' : \rho$ implies $\rho : \mathbf{G}' \cong \mathbf{G} : \tau$. Therefore, we may simply say that $\mathbf{G}$ and $\mathbf{G}'$ are equivalent, and write $\mathbf{G} \cong \mathbf{G}'$ or $\mathbf{G}' \cong \mathbf{G}$, when the transformers establishing the equivalence need not be mentioned.

REMARK 2.84. Let $\mathbf{G}$ and $\mathbf{G}'$ be Gentzen relations with traces tr and tr', respectively. Suppose that the following hold:

(i) $\langle 0,0\rangle \in \mathsf{tr}$.
(ii) $\langle 0,0\rangle \notin \mathsf{tr}'$.
(iii) Our current algebraic language has no constant symbols.

Then, we must have $\tau(\varnothing \rhd \varnothing) = \varnothing$ for every transformer τ from tr-sequents to tr'-sequents. This is not a problem because we have not imposed that transformers must map sequents to *non-empty* sets of sequents. If one wishes to do so, then there are two main ways in the literature of dealing with $\tau(\varnothing \rhd \varnothing)$. Either one defines transformers as in Definition 2.76 but with the only additional condition that their images be non-empty, or one adds that condition and, additionally, allows at most one variable, say x, to occur in $\tau(\varnothing \rhd \varnothing)$. The first option makes it impossible for $\mathbf{G}$ and $\mathbf{G}'$ to be equivalent if the three conditions (i)-(iii) stated above hold, because then one must have $\tau(\varnothing \rhd \varnothing) = \{\varnothing \rhd \varnothing\} \not\subseteq \mathsf{tr}'\text{-}\mathsf{Seq}$. The second option allows $\mathbf{G} \cong \mathbf{G}'$, hence is more general, but invalidates Lemma 2.78 and introduces a distinction between $\varnothing \rhd \varnothing$ and the other tr-sequents.

Both Blok and Pigozzi in [**6**] and Rebagliato and Verdú in [**30**] choose the second option. Although Raftery never explicitly states that the images of a transformer must be non-empty, the footnote in [**27**, p. 910] makes it clear that, at least in [**27**, §5], he is working with transformers whose images are non-empty. Later, in [**27**, §9], when defining algebraizability, he does allow empty images.

We have decided to allow transformers with possibly empty images to gain a little simplicity in some cases. The changes required should we wish to restrict ourselves to working with transformers whose images are non-empty are all minor and obvious. In fact, 'the only effect of prohibiting empty transformers is that inconsistent Gentzen relations become non-algebraizable rather than algebraizable by a class of trivial algebras' ([**27**, p. 922]).

The four conditions stated in Definition 2.82 exhibit the symmetry between τ and ρ, but they are actually redundant:

PROPOSITION 2.85. *Let* G, G' *be Gentzen relations with traces* tr *and* tr'*, respectively. The following are equivalent:*

(i) $\tau : \mathsf{G} \cong \mathsf{G}' : \rho$ *for some transformers* τ *and* ρ.
(ii) G *and* G' *satisfy conditions* (ALG1) *and* (ALG2) *with respect to some transformers* τ *and* ρ.
(iii) G *and* G' *satisfy conditions* (ALG3) *and* (ALG4) *with respect to some transformers* τ *and* ρ.

Moreover, the transformers τ, ρ *of any of the previous conditions work for the rest.*

PROOF. (i) $\Rightarrow$ (ii) By Definition 2.82.

(ii) $\Rightarrow$ (iii) Let $\mathfrak{S} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}'\text{-}\mathsf{Seq}$. We have

$$\mathfrak{S} \vdash_{\mathsf{G}'} \mathfrak{s} \overset{\text{(ALG2)}}{\iff} \tau(\rho(\mathfrak{S})) \vdash_{\mathsf{G}'} \tau(\rho(\mathfrak{s})) \overset{\text{(ALG1)}}{\iff} \rho(\mathfrak{S}) \vdash_{\mathsf{G}} \rho(\mathfrak{s}),$$

so (ALG3) holds.

Now let $\mathfrak{p} \in \mathsf{tr}$. We have:

$$\mathfrak{p} \dashv\vdash_{\mathsf{G}} \rho(\tau(\mathfrak{p})) \overset{\text{(ALG1)}}{\iff} \tau(\mathfrak{p}) \dashv\vdash_{\mathsf{G}'} \tau(\rho(\tau(\mathfrak{p}))) \overset{\text{(ALG2)}}{\iff} \tau(\mathfrak{p}) \dashv\vdash_{\mathsf{G}'} \tau(\mathfrak{p}).$$

Therefore, (ALG4) also holds.

(iii) $\Rightarrow$ (i) By assumption, (ALG3) and (ALG4) hold for τ and ρ, and an argument analogous to the previous one shows that (ALG1) and (ALG2) also hold for τ and ρ. ∎

THEOREM 2.86 (cf. [**27**, Thm. 6.6]). *Let* G, G' *be Gentzen relations such that* $\tau : \mathsf{G} \cong \mathsf{G}' : \rho$ *for some transformers* τ *and* ρ*. Then:*

$$\rho^{-1} : \langle \mathcal{T}h(\mathsf{G}), \subseteq \rangle \cong \langle \mathcal{T}h(\mathsf{G}'), \subseteq \rangle : \tau^{-1}$$

If τ and τ' are two transformers from tr-sequents to tr'-sequents, where tr and tr' are traces, then we write $\tau' \subseteq \tau$ to denote that $\tau'(\mathfrak{s}) \subseteq \tau(\mathfrak{s})$ is the case for every $\mathfrak{s} \in \mathsf{tr}\text{-}\mathsf{Seq}$.

PROPOSITION 2.87. *Let* $\mathbf{G}, \mathbf{G}'$ *be Gentzen relations such that* $\tau : \mathbf{G} \cong \mathbf{G}' : \rho$ *for some transformers* τ *and* ρ. *If* $\mathbf{G}$ *is finitary, then there is a finitary transformer* $\tau' \subseteq \tau$ *such that* $\tau' : \mathbf{G} \cong \mathbf{G}' : \rho$.

PROOF. Let $\mathsf{tr} := \mathsf{tr}(\mathbf{G})$ and $\mathsf{tr}' := \mathsf{tr}(\mathbf{G}')$. For any $\langle m, n\rangle \in \mathsf{tr}$, let $\tau_{m,n}$ be as in Proposition 2.81. Fix any $\langle m, n\rangle \in \mathsf{tr}$. By (ALG4), $\rho(\tau(\mathfrak{x}_{m,n})) \vdash_{\mathbf{G}} \mathfrak{x}_{m,n}$, so, since $\mathbf{G}$ is finitary, there is a finite

$$\mathfrak{P}_0 \subseteq \rho(\tau(\mathfrak{x}_{m,n})) = \rho(\tau_{m,n}(\mathfrak{x}_{m,n})) = \bigcup\{\rho(\mathfrak{p}) : \mathfrak{p} \in \tau_{m,n}(\mathfrak{x}_{m,n})\}$$

such that $\mathfrak{P}_0 \vdash_{\mathbf{G}} \mathfrak{x}_{m,n}$, whence there is a finite $\tau'_{m,n}(\vec{x}_{m+n}) \subseteq \tau_{m,n}(\mathfrak{x}_{m,n})$ such that

$$\rho(\tau'_{m,n}(\mathfrak{x}_{m,n})) \vdash_{\mathbf{G}} \mathfrak{x}_{m,n}. \tag{2.15}$$

Define $\tau' : \mathsf{tr}\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}'\text{-}\mathsf{Seq})$ by setting $\tau'(\mathfrak{s}) := \tau'_{m,n}(\mathfrak{s})$ for all $\langle m, n\rangle \in \mathsf{tr}$ and all $\mathfrak{s} \in \langle m, n\rangle\text{-}\mathsf{Seq}$. By Proposition 2.81, τ' is a transformer from tr-sequents to tr'-sequents, and it is finitary by construction. It remains to see $\tau' : \mathbf{G} \cong \mathbf{G}' : \rho$.

Since $\tau : \mathbf{G} \cong \mathbf{G}' : \rho$, condition (ALG3) clearly holds for τ' and ρ. To see that (ALG4) also holds for τ' and ρ, let $\langle m, n\rangle \in \mathsf{tr}$ and $\mathfrak{s} \in \langle m, n\rangle\text{-}\mathsf{Seq}$. As (ALG4) holds for τ and ρ, we have $\mathfrak{s} \vdash_{\mathbf{G}} \rho(\tau(\mathfrak{s}))$, whence $\mathfrak{s} \vdash_{\mathbf{G}} \rho(\tau'(\mathfrak{s}))$ because

$$\tau'(\mathfrak{s}) = \tau'_{m,n}(\mathfrak{s}) \subseteq \tau_{m,n}(\mathfrak{s}) = \tau(\mathfrak{s}),$$

and $\rho(\tau'(\mathfrak{s})) \vdash_{\mathbf{G}} \mathfrak{s}$ follows from (2.15) and structurality.

Therefore, $\tau' : \mathbf{G} \cong \mathbf{G}' : \rho$ by Proposition 2.85(iii). ■

2.9.1. Algebraizable Gentzen relations. We present now the notion of algebraizability of a Gentzen relation as a special case of equivalence between Gentzen relations. The definition is the one first introduced by Rebagliato and Verdú in [**30**] but without finitarity assumptions, as Raftery's [**27**, Def. 9.1].

DEFINITION 2.88. A Gentzen relation $\mathbf{G}$ is said to be *algebraizable* if there exists a class of algebras K such that $\mathbf{G} \cong \mathsf{EQ}(\mathsf{K})$. If, moreover, K is a quasivariety, then we say that $\mathbf{G}$ is *elementarily algebraizable*, and if K is a variety we say that $\mathbf{G}$ is *strongly algebraizable*.

As expected, algebraic Gentzen relations are protoalgebraic:

THEOREM 2.89. *Let* $\mathbf{G}$ *be a Gentzen relation with trace* tr. *If* $\mathbf{G}$ *is algebraizable, then* $\mathbf{G}$ *is protoalgebraic.*

PROOF. Let K be a class of algebras such that $\tau : \mathbf{G} \cong \mathsf{EQ}(\mathsf{K}) : \rho$, for suitable transformers τ, ρ. We prove that $\mathbf{G}$ is protoalgebraic using Theorem 2.62, so let $\langle m, n\rangle \in \mathsf{tr}$ and define:

$$\mathfrak{E}_{m,n} := \bigcup_{1 \leq i \leq m+n} \rho(x_i \approx y_i).$$

Note that each $\rho(x_i \approx y_i)$ is a set of tr-sequents in the variables x_i, y_i, and thus $\mathfrak{E}_{m,n} = \mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n})$. Let us see that $\mathfrak{E}_{m,n}$ satisfies conditions (R) and (MP).

For every $i = 1, \ldots, m+n$, we have $\models_{\mathsf{K}} x_i \approx x_i$, so (ALG3) yields $\vdash_{\mathsf{G}} \rho(x_i \approx x_i)$, whence $\vdash_{\mathsf{G}} \mathfrak{E}_{m,n}(\mathfrak{x}_{m,n}, \mathfrak{x}_{m,n})$. Thus, $\mathfrak{E}_{m,n}$ satisfies (R).

As regards (MP), we clearly have

$$\tau(\mathfrak{x}_{m,n}), x_1 \approx y_1, \ldots, x_{m+n} \approx y_{m+n} \models_{\mathsf{K}} \tau(\mathfrak{y}_{m,n}),$$

so (ALG3) and (ALG4) yield

$$\mathfrak{x}_{m,n}, \rho(x_1 \approx y_1), \ldots, \rho(x_{m+n} \approx y_{m+n}) \vdash_{\mathsf{G}} \mathfrak{y}_{m,n},$$

i.e.,

$$\mathfrak{x}_{m,n}, \mathfrak{E}(\mathfrak{x}_{m,n}, \mathfrak{y}_{m,n}) \vdash_{\mathsf{G}} \mathfrak{y}_{m,n}.$$

Thus, $\mathfrak{E}_{m,n}$ satisfies (MP). ■

CHAPTER 3

Contextual Deduction-Detachment Theorems

The contextual deduction-detachment theorem was introduced by Raftery in [**28**] to extend some of the desirable features of the deduction-detachment theorem to logics that do not have it. We are now going to employ the tools presented in Chapter 2 to study the contextual deduction-detachment theorem (CDDT), among some of its variants, in the context of Gentzen relations.

Recall (cf. Section 2.1) that a context is just a natural number $n \in \omega$, that Fm_n is the set of all formulas in which all the variables that occur are among $x_1, \ldots, x_n$, and that we defined $\mathsf{tr}\text{-}\mathsf{Seq}_n := \mathsf{tr}\text{-}\mathsf{Seq}(Fm_n)$ for all traces tr.

3.1. The CDDT for Gentzen relations

Definition 3.1. A Gentzen relation $\mathbf{G}$ with trace tr is said to have the *contextual deduction-detachment theorem* (*CDDT*) if for every context $n \in \omega$ and every $\hat{m}_1, \hat{m}_2 \in \mathsf{tr}$ there is a set $\mathfrak{D}[n, \hat{m}_1, \hat{m}_2](\vec{x}_t) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_t$, where $t = n + \Sigma(\hat{m}_1) + \Sigma(\hat{m}_2)$, such that, for every $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_n$, with $\mathsf{tp}(\mathfrak{s}) = \hat{m}_1$ and $\mathsf{tp}(\mathfrak{r}) = \hat{m}_2$, the following holds:

$$\mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r} \iff \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{D}[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}).$$

In this situation, the sequence $\langle \{\mathfrak{D}[n, \hat{m}_1, \hat{m}_2] : \hat{m}_1, \hat{m}_2 \in \mathsf{tr}\} : n \in \omega \rangle$ is said to be a *CDD-sequence* for $\mathbf{G}$, and we denote it by $\langle \mathfrak{D}[n] : n \in \omega \rangle$ when tr is a singleton.

Moreover, if $\mathfrak{D}[n, \hat{m}_1, \hat{m}_2] = \mathfrak{D}[0, \hat{m}_1, \hat{m}_2]$ for every $n \in \omega$ and every $\hat{m}_1, \hat{m}_2 \in \mathsf{tr}$, we say that $\mathbf{G}$ has the *deduction-detachment theorem* (*DDT*).

It was already known in the 1920's that classical propositional logic has the DDT (cf. [**14**, p. 163]), and since then the DDT and many variants of it have been extensively studied for a wide variety of sentential logics. Blok and Pigozzi introduced the DDT in [**6**] for finitary k-deductive systems. In [**30**, §3.1], Rebagliato and Verdú generalized the DDT to Gentzen relations. The CDDT, as we have said, was first introduced by Raftery in [**28**], for sentential logics.

Remark 3.2. In the context of Definition 3.1, if $\mathbf{G}$ is a sentential logic, then $\mathsf{tr}(\mathbf{G}) = \{\langle 0, 1 \rangle\}$ and we recover Raftery's original definition [**28**, Def. 3.1] by identifying each singleton $\{\mathfrak{D}[n]\}$ with its unique element, which Raftery denotes by Σ_n.

In the context of Definition 3.1, we call the left-to-right implication the *contextual deduction theorem*, and the right-to-left *contextual detachment*.

PROPOSITION 3.3. *Let* $\langle\{\mathfrak{D}[n,\hat{m}_1,\hat{m}_2] : \hat{m}_1,\hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$ *be a CDD-sequence for a Gentzen relation* $\mathbf{G}$ *with trace* tr. *For every* $n \in \omega$ *and every* $\mathfrak{s},\mathfrak{t} \in \mathsf{tr}\text{-}\mathsf{Seq}_n$ *with* $\mathsf{tp}(\mathfrak{s}) = \hat{m}_1$ *and* $\mathsf{tp}(\mathfrak{t}) = \hat{m}_2$*, the following hold:*

(i) $\vdash_{\mathbf{G}} \mathfrak{D}[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{s})$.
(ii) $\mathfrak{s},\mathfrak{D}[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t}) \vdash_{\mathbf{G}} \mathfrak{t}$.
(iii) $\mathfrak{D}[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t}) \vdash_{\mathbf{G}} \mathfrak{D}[n+1,\hat{m}_1,\hat{m}_2](\vec{x}_{n+1},\mathfrak{s},\mathfrak{t})$.

PROOF.

(i) Apply the contextual deduction theorem to $\mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{s}$.
(ii) Apply contextual detachment to
$$\mathfrak{D}[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t}) \vdash_{\mathbf{G}} \mathfrak{D}[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t}).$$
(iii) Since $\mathsf{tr}\text{-}\mathsf{Seq}_n \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_{n+1}$, applying the contextual deduction theorem to (ii) yields (iii). ■

Point (ii) of Proposition 3.3 allows us to obtain a uniqueness result for CDD-sequences, in the following sense:

THEOREM 3.4. *Let* $\langle\{\mathfrak{D}[n,\hat{m}_1,\hat{m}_2] : \hat{m}_1,\hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$ *and* $\langle\{\mathfrak{D}'[n,\hat{m}_1,\hat{m}_2] : \hat{m}_1,\hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$ *be two CDD-sequences for a Gentzen relation* $\mathbf{G}$ *with trace* tr. *For every context* $n \in \omega$ *and every* $\mathfrak{s},\mathfrak{t} \in \mathsf{tr}\text{-}\mathsf{Seq}_n$, *with* $\mathsf{tp}(\mathfrak{s}) = \hat{m}_1$ *and* $\mathsf{tp}(\mathfrak{t}) = \hat{m}_2$, *we have:*
$$\mathfrak{D}[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t}) \dashv\vdash_{\mathbf{G}} \mathfrak{D}'[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t}).$$

PROOF. From Proposition 3.3(ii) we have $\mathfrak{s},\mathfrak{D}[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t}) \vdash_{\mathbf{G}} \mathfrak{t}$, so
$$\mathfrak{D}[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t}) \vdash_{\mathbf{G}} \mathfrak{D}'[n,\hat{m}_1,\hat{m}_2](\vec{x}_n,\mathfrak{s},\mathfrak{t})$$
holds because $\langle\{\mathfrak{D}'[n,\hat{m}_1,\hat{m}_2] : \hat{m}_1,\hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$ is a CDD-sequence. By symmetry, the converse also holds. ■

Therefore, CDD-sequences are unique up to 'trace-wise' interderivability, a fact that justifies our use of the determinate article in the expression 'to have *the* CDDT' in Definition 3.1.

Even though in principle a CDD-sequence allows us to move *only one* sequent to the other side of the symbol '$\vdash_{\mathbf{G}}$', we can built from it a family of sets that allow us to move several sequents at once:

THEOREM 3.5. *Let* $\langle\{\mathfrak{D}[n,\hat{m}_1,\hat{m}_2] : \hat{m}_1,\hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$ *be a CDD-sequence for a Gentzen relation* $\mathbf{G}$ *with trace* tr*. Then, for every* $k \in \omega$*, every context* $p \in \omega$ *and every* $\hat{m}_1,\ldots,\hat{m}_{k+1},\hat{n} \in \mathsf{tr}$*, there is a set*

$$\mathfrak{D}^*_k[p,\hat{m}_1,\ldots,\hat{m}_{k+1},\hat{n}](\vec{x}_t) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_t,$$

where $t = p+\Sigma(\hat{m}_1)+\cdots+\Sigma(\hat{m}_{k+1})+\Sigma(\hat{n})$*, such that, for all* $\mathfrak{P}\cup\{\mathfrak{s}_1,\ldots,\mathfrak{s}_{k+1},\mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_p$ *with* $\mathsf{tp}(\mathfrak{r}) = \hat{n}$ *and* $\mathsf{tp}(\mathfrak{s}_i) = \hat{m}_i$ *for* $i = 1,\ldots,k+1$*, we have:*

$$\mathfrak{P},\mathfrak{s}_1,\ldots,\mathfrak{s}_{k+1} \vdash_{\mathbf{G}} \mathfrak{r} \iff \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{D}^*_k[p,\hat{m}_1,\ldots,\hat{m}_{k+1},\hat{n}](\vec{x}_p,\mathfrak{s}_1,\ldots,\mathfrak{s}_{k+1},\mathfrak{r}).$$

PROOF. By induction on k. If $k = 0$ the statement of the theorem is just the defining property of any CDD-sequence for $\mathbf{G}$, so we can take $\mathfrak{D}^*_0[p,\hat{m}_1,\hat{n}] := \mathfrak{D}[p,\hat{m}_1,\hat{n}]$. Assuming that the theorem holds for k (IH), let us consider the case of $k+1$.

Let $t = p+\Sigma(\hat{m}_1)+\cdots+\Sigma(\hat{m}_{k+2})+\Sigma(\hat{n})$ and $\mathfrak{E} := \mathfrak{D}^*_k[p,\hat{m}_2,\ldots,\hat{m}_{k+2},\hat{n}]$. Note that $\mathfrak{E} = \mathfrak{E}(\vec{x}_{t-\Sigma(\hat{m}_1)})$ by IH.

Define:

$$\mathfrak{D}^*_{k+1}[p,\hat{m}_1,\ldots,\hat{m}_{k+2},\hat{n}] := \bigcup_{\mathfrak{t}\in\mathfrak{E}} \mathfrak{D}[p,\hat{m}_1,\mathsf{tp}(\mathfrak{t})](\vec{x}_{p+\Sigma(\hat{m}_1)},\mathfrak{t}(\vec{x}_p,x_{p+\Sigma(\hat{m}_1)+1},\ldots,x_t)).$$

Note that $\mathfrak{D}^*_{k+1}[p,\hat{m}_1,\ldots,\hat{m}_{k+2},\hat{n}] = \mathfrak{D}^*_{k+1}[p,\hat{m}_1,\ldots,\hat{m}_{k+2},\hat{n}](\vec{x}_t)$.

We have:

$$\begin{aligned}
&\mathfrak{P},\mathfrak{s}_1,\ldots,\mathfrak{s}_{k+2} \vdash_{\mathbf{G}} \mathfrak{r}\\
&\quad\overset{\text{IH}}{\iff} \mathfrak{P},\mathfrak{s}_1 \vdash_{\mathbf{G}} \underbrace{\overbrace{\mathfrak{D}^*_k[p,\hat{m}_2,\ldots,\hat{m}_{k+2},\hat{n}]}^{\mathfrak{E}}(\vec{x}_p,\mathfrak{s}_2,\ldots,\mathfrak{s}_{k+2},\mathfrak{r})}_{\subseteq\, \mathsf{tr}\text{-}\mathsf{Seq}_p}\\
&\quad\iff \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{D}[p,\hat{m}_1,\mathsf{tp}(\mathfrak{t})](\vec{x}_p,\mathfrak{s}_1,\mathfrak{t}) \text{ for all } \mathfrak{t} \in \mathfrak{E}(\vec{x}_p,\mathfrak{s}_2,\ldots,\mathfrak{s}_{k+2},\mathfrak{r})\\
&\quad\iff \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{D}[p,\hat{m}_1,\mathsf{tp}(\mathfrak{t})](\vec{x}_p,\mathfrak{s}_1,\mathfrak{t}(\vec{x}_p,\mathfrak{s}_2,\ldots,\mathfrak{s}_{k+2},\mathfrak{r})) \text{ for all } \mathfrak{t} \in \mathfrak{E}\\
&\quad\iff \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{D}^*_{k+1}[p,\hat{m}_1,\ldots,\hat{m}_{k+2},\hat{n}](\vec{x}_p,\mathfrak{s}_1,\ldots,\mathfrak{s}_{k+2},\mathfrak{r}).
\end{aligned}$$

■

When working with Gentzen relations having the CDDT, we may use the sets whose existence the previous theorem guarantees without defining them again.

Theorem 3.5 holds as well in arbitrary finitely generated algebras, in the following form:

THEOREM 3.6. *Let* $\langle\{\mathfrak{D}[n,\hat{m}_1,\hat{m}_2] : \hat{m}_1,\hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$ *be a CDD-sequence for a Gentzen relation* $\mathbf{G}$ *with trace* tr*,* $\boldsymbol{A}$ *an algebra finitely generated by some* $g_1,\ldots,g_p \in A$*,* $\mathfrak{a}_1,\ldots,\mathfrak{a}_{k+1} \in \mathsf{tr}\text{-}\mathsf{Seq}_p(A)$ *and* $\hat{m}_i := \mathsf{tp}(\mathfrak{a}_i)$ *for* $i = 1,\ldots,k+1$*. Then:*

(i) *For any* $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$ *and any* $\mathfrak{b} \in \mathsf{tr\text{-}Seq}_p(A)$*, we have:*

$$\mathfrak{b} \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}) \iff \mathfrak{D}^*_k[\hat{n}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F,$$

where $\hat{n} := \mathsf{tp}(\mathfrak{b})$ *and* $\mathfrak{D}^*_k[\hat{n}] := \mathfrak{D}^*_k[p, \hat{m}_1, \ldots, \hat{m}_{k+1}, \hat{n}]$*.*

(ii) *If* $\mathbf{G}$ *is finitary, then for each* $\mathfrak{b} \in \mathsf{tr\text{-}Seq}_p(A)$ *there is a finite set* $\mathfrak{D} \subseteq \mathfrak{D}^*_k[p, \hat{m}_1, \ldots, \hat{m}_{k+1}, \mathsf{tp}(\mathfrak{b})]$ *such that, for any* $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$*, we have:*

$$\mathfrak{b} \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}) \iff \mathfrak{D}^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F.$$

Proof. By Corollary 2.11, let $\mathfrak{s}_i \in \mathsf{tr\text{-}Seq}_p$ be such that $\mathfrak{a}_i = \mathfrak{s}^{\boldsymbol{A}}_i(\vec{g}_p)$, for $i = 1, \ldots, k+1$. Also, for every $\mathfrak{r} \in \mathsf{tr\text{-}Seq}(A)$ let $\hat{n}_{\mathfrak{r}} := \mathsf{tp}(\mathfrak{r})$.

Note that, by Theorem 3.5, for every $\mathfrak{r} \in \mathsf{tr\text{-}Seq}$ we have:

$$\mathfrak{D}^*_k[p, \hat{m}_1, \ldots, \hat{m}_{k+1}, \hat{n}_{\mathfrak{r}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \mathfrak{r}), \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1} \vdash_{\mathbf{G}} \mathfrak{r}. \tag{3.1}$$

As $p, \hat{m}_1, \ldots, \hat{m}_{k+1}$ will remain fixed throughout this proof, to improve readability let us denote the sets $\mathfrak{D}^*_k[p, \hat{m}_1, \ldots, \hat{m}_{k+1}, \hat{n}_{\mathfrak{r}}]$ by $\mathfrak{D}^*_k[\hat{n}_{\mathfrak{r}}]$.

(i) Define

$$G := \{\mathfrak{b} \in \mathsf{tr\text{-}Seq}(A) : \mathfrak{D}^*_k[\hat{n}_{\mathfrak{b}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F\}.$$

We prove that $G = \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1})$, whence (i) clearly follows.

Claim 3.6.1. $F \subseteq G$.

Proof. Let $\mathfrak{b} \in F$ and, by Corollary 2.11, let $\mathfrak{r} \in \mathsf{tr\text{-}Seq}_p$ be such that $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p)$. Since $\mathfrak{r}, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1} \vdash_{\mathbf{G}} \mathfrak{r}$, by Theorem 3.5 we have

$$\mathfrak{r} \vdash_{\mathbf{G}} \mathfrak{D}^*_k[\hat{n}_{\mathfrak{b}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \mathfrak{r}),$$

whence $\mathfrak{D}^*_k[\hat{n}_{\mathfrak{b}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F$ because $\mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p) = \mathfrak{b} \in F$ and F is a $\mathbf{G}$-filter. So $\mathfrak{b} \in G$. □

Claim 3.6.2. $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}\} \subseteq G$.

Proof. For every $i = 1, \ldots, k+1$ we have $\mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1} \vdash_{\mathbf{G}} \mathfrak{s}_i$, so

$$\vdash_{\mathbf{G}} \mathfrak{D}^*_k[\hat{m}_i](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \mathfrak{s}_i)$$

by Theorem 3.5, and thus $\mathfrak{D}^*_k[\hat{m}_i]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{a}_i) \subseteq F$, so $\mathfrak{a}_i \in G$. □

Claim 3.6.3. *For all* $H \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$ *such that* $H \supseteq F \cup \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}\}$*, we have* $H \supseteq G$*.*

PROOF. Let $\mathfrak{b} \in G$ and, by Corollary 2.11, let $\mathfrak{r} \in \mathsf{tr\text{-}Seq}_p$ be such that $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p)$. By the definition of G we have $\mathfrak{D}_k^*[\hat{n}_{\mathfrak{r}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F \subseteq H$, so, since H also contains $\{\mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}\}$ and H is a G-filter, (3.1) implies $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p) \in H$. $\square$

CLAIM 3.6.4. *$G \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$.*

PROOF. Let $\mathfrak{P} \cup \{\mathfrak{q}\} \subseteq \mathsf{tr\text{-}Seq}$ be such that $\mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{q}$, and let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $h(\mathfrak{P}) \subseteq G$. We need to prove that $h(\mathfrak{q}) \in G$.

By Lemma 2.15 we may assume, without loss of generality, that the variables in Var_x do not occur in $\mathfrak{P} \cup \{\mathfrak{q}\}$ and, moreover, that $h(\vec{x}_p) = \vec{g}_p$. Note that this implies $h(\mathfrak{s}_i) = \mathfrak{a}_i$ for all $1 \leq i \leq k+1$.

For every variable $u \notin \mathsf{Var}_x$ we have $h(u) \in A$, so by Corollary 2.12 there is some formula $\eta_u(\vec{x}_p)$ such that $h(u) = \eta_u^{\boldsymbol{A}}(\vec{g}_p) = h(\eta_u)$. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be any substitution mapping u to η_u for every variable $u \notin \mathsf{Var}_x$.

Arguing inductively, let us see that $h(\sigma(\varphi)) = h(\varphi)$ for every formula φ in which none of the variables in Var_x occurs. We have just seen that this holds for all $u \notin \mathsf{Var}_x$, and clearly $h(\sigma(c)) = h(c)$ for every constant c. Thus, let f be an n-ary function symbol, $n > 0$, and suppose $\varphi = f\psi_1 \dots \psi_n$, where none of the variables in Var_x occurs in $\psi_1, \dots, \psi_n$ and $h(\sigma(\psi_i)) = h(\psi_i)$ for $i = 1, \dots, n$ (IH). Then, since $h \circ \sigma : \boldsymbol{Fm} \to \boldsymbol{A}$ is a homomorphism,

$$\begin{aligned} h(\sigma(f\psi_1 \dots \psi_n)) &= f^{\boldsymbol{A}}(h(\sigma(\psi_1)), \dots, h(\sigma(\psi_n)) \\ &\overset{\mathrm{IH}}{=} f^{\boldsymbol{A}}(h(\psi_1), \dots, h(\psi_n)) \\ &= h(f\psi_1 \dots \psi_n), \end{aligned}$$

so we are done. In particular:

(a) $h(\sigma(\mathfrak{p})) = h(\mathfrak{p}) \in G$ for all $\mathfrak{p} \in \mathfrak{P}$.
(b) $h(\sigma(\mathfrak{q})) = h(\mathfrak{q})$.

By Theorem 3.5, we have:

$$\mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{D}_k^*[\hat{n}_{\mathfrak{p}}](\vec{x}_p, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{p})) \vdash_{\mathsf{G}} \sigma(\mathfrak{P}). \tag{3.2}$$

Applying cut to (3.2) and $\sigma(\mathfrak{P}) \vdash_{\mathsf{G}} \sigma(\mathfrak{q})$, which holds by structurality, we obtain:

$$\mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{D}_k^*[\hat{n}_{\mathfrak{p}}](\vec{x}_p, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{p})) \vdash_{\mathsf{G}} \sigma(\mathfrak{q}).$$

So, again by Theorem 3.5, we have:

$$\bigcup_{\mathfrak{p}\in\mathfrak{P}} \mathfrak{D}^*_k[\hat{n}_{\mathfrak{p}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{p})) \vdash_{\mathbf{G}} \mathfrak{D}^*_k[\hat{n}_{\mathfrak{q}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{q})). \tag{3.3}$$

From (a) we deduce that, for every $\mathfrak{p} \in \mathfrak{P}$,

$$h(\mathfrak{D}^*_k[\hat{n}_{\mathfrak{p}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{p}))) = \mathfrak{D}^*_k[\hat{n}_{\mathfrak{p}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, h(\mathfrak{p})) \subseteq F,$$

so, since $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$, (3.3) and (b) imply:

$$F \supseteq h(\mathfrak{D}^*_k[\hat{n}_{\mathfrak{q}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{q}))) = \mathfrak{D}^*_k[\hat{n}_{\mathfrak{q}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, h(\mathfrak{q})).$$

Therefore, $h(\mathfrak{q}) \in G$. □

From the previous claims it follows that $G = \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1})$, so we are done.

(ii) Let $H := \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1})$ and, by Corollary 2.11, let $\mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}_p$ be such that $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p)$. If $\mathbf{G}$ is finitary, from (3.1) we know there is a finite set $\mathfrak{D} \subseteq \mathfrak{D}^*_k[\hat{n}_{\mathfrak{r}}]$ such that

$$\mathfrak{D}(\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \mathfrak{r}), \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1} \vdash_{\mathbf{G}} \mathfrak{r}. \tag{3.4}$$

If $\mathfrak{D}^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F$, then

$$\mathfrak{D}^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \cup \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}\} \subseteq H,$$

so, since $H \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$, (3.4) implies $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p) \in H$.

Conversely, assume $\mathfrak{b} \in H$. By (i), $\mathfrak{D}^*_k[\hat{n}_{\mathfrak{r}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F$, so in particular $\mathfrak{D}^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F$. ■

Items (i) and (ii) of Proposition 3.3 resemble conditions (R) and (MP) of Theorem 2.62, and in fact the existence of a CDD-sequence guarantees protoalgebraicity. However, we shall later see that the converse does not hold.

Proposition 3.7. *Every Gentzen relation* $\mathbf{G}$ *with the CDDT is protoalgebraic.*

Proof. Let $\mathsf{tr} := \mathsf{tr}(\mathbf{G})$ and let $\langle\{\mathfrak{D}[n, \hat{m}_1, \hat{m}_2] : \hat{m}_1, \hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$ be a CDD-sequence for $\mathbf{G}$. We use Theorem 2.62 to prove that $\mathbf{G}$ is protoalgebraic, so fix any $\langle m, n\rangle \in \mathsf{tr}$. If $\langle m, n\rangle = \langle 0, 0\rangle$, then we can take $\mathfrak{E}_{0,0} := \varnothing$, which clearly satisfies conditions (R) and (MP) of Theorem 2.62, so let us assume $\langle m, n\rangle \neq \langle 0, 0\rangle$.

Let $\mathfrak{s} := \langle x_1, \ldots, x_{m+n}\rangle_{m,n}$ and $\mathfrak{r} := \langle x_{m+n+1}, \ldots, x_{2(m+n)}\rangle_{m,n}$. Note that all the variables occurring in $\mathfrak{s}$ or $\mathfrak{r}$ are among $x_1, \ldots, x_{2(m+n)}$. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be the substitution given by $\sigma(\mathfrak{r}) := \langle y_1, \ldots, y_{m+n}\rangle_{m,n}$ and $\sigma(u) := u$ for every variable u not occurring in $\mathfrak{r}$. Note that $\sigma(\mathfrak{s}) = \mathfrak{s} = \mathfrak{x}_{m,n}$ and $\sigma(\mathfrak{r}) = \mathfrak{y}_{m,n}$.

Let $\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n}) := \mathfrak{D}[0, \langle m, n\rangle, \langle m, n\rangle](\mathfrak{x}_{m,n}, \mathfrak{y}_{m,n})$. Points (i) and (ii) of Proposition 3.3 yield, respectively, conditions (R) and (MP) of Theorem 2.62 for $\mathfrak{E}_{m,n}$. ■

3.1.1. The finite model property. Now we generalize [**28**, Thm. 3.6, Cor. 3.7] to show that if a Gentzen relation having the CDDT has the finite model property, then it has the strong finite model property.

DEFINITION 3.8. Let G be a Gentzen relation, $\mathsf{tr} := \mathsf{tr}(\mathsf{G})$ and M a class of G-matrices. We say that G is *(weakly) complete* with respect to M if $\vdash_{\mathsf{M}} \mathfrak{s}$ implies $\vdash_{\mathsf{G}} \mathfrak{s}$ for every $\mathfrak{s} \in \mathsf{tr}\text{-}\mathsf{Seq}$.

We say that G is *strongly complete* with respect to M if $\mathfrak{P} \vdash_{\mathsf{M}} \mathfrak{s}$ implies $\mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{s}$ for all $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, with $\mathfrak{P}$ finite.

DEFINITION 3.9. A Gentzen relation G is said to have the *[strong] finite model property* if it is [strongly] complete with respect to the class of its finite matrix models.

THEOREM 3.10. *Let G be a Gentzen relation with trace tr and M a class of G-matrices closed under taking contractions and finitely generated submatrices. If G has the CDDT and is complete with respect to M, then G is strongly complete with respect to M.*

PROOF. Let $\langle\{\mathfrak{D}[p, \hat{m}, \hat{n}] : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ be a CDD-sequence for G. We show that G is strongly complete with respect to M by contraposition, so assume that $\mathfrak{s}_1, \ldots, \mathfrak{s}_l \nvdash_{\mathsf{G}} \mathfrak{r}$ is the case for some $\mathfrak{s}_1, \ldots, \mathfrak{s}_l, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$, with $l \in \omega$. If $l = 0$ there is nothing to prove because G is assumed to be complete with respect to M, so assume $l > 0$.

For every $j = 1, \ldots, l$, let $\hat{m}_j := \mathsf{tp}(\mathfrak{s}_j)$, let $\hat{n} := \mathsf{tp}(\mathfrak{r})$ and let $p > 0$ be such that $\mathfrak{s}_1, \ldots, \mathfrak{s}_l, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}_p$. By Theorem 3.5, there is a $\mathfrak{t} \in \mathfrak{D}^*_{l-1}[p, \hat{m}_1, \ldots, \hat{m}_l, \hat{n}]$ such that $\nvdash_{\mathsf{G}} \mathfrak{t}(\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_l, \mathfrak{r})$. By completeness, there is some G-matrix $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ and some $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $\mathfrak{t}^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_l, \mathfrak{b}) \notin F$, where $\vec{g}_p := h(\vec{x}_p)$, $\mathfrak{a}_j := h(\mathfrak{s}_j) = \mathfrak{s}_j^{\boldsymbol{A}}(\vec{g}_p)$ for $j = 1, \ldots, l$ and $\mathfrak{b} := h(\mathfrak{r}) = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p)$.

Let $B := \mathrm{Sg}^{\boldsymbol{A}}(g_1, \ldots, g_p)$. Since $p > 0$, B is the universe of a subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$. By Lemma 1.20, we have $\mathfrak{a}_1, \ldots, \mathfrak{a}_l, \mathfrak{b} \in \mathsf{Seq}(B)$. Let $G := \mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_l)$, $H := (\mathsf{Seq}(B) \cap F) \vee^{\boldsymbol{B}} G$ and $\mathcal{M} := \langle \boldsymbol{B}, H\rangle$. Note that $\mathcal{M}$ is a contraction of a finitely generated submatrix of $\langle \boldsymbol{A}, F\rangle$, so $\mathcal{M} \in \mathsf{M}$. By Proposition 2.43 and Proposition 2.42, $\mathsf{Seq}(B) \cap F$ is a G-filter of $\boldsymbol{B}$. By Lemma 1.20,

$$\mathfrak{t}^{\boldsymbol{B}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_l, \mathfrak{b}) = \mathfrak{t}^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_l, \mathfrak{b}) \notin \mathsf{Seq}(B) \cap F,$$

so we get $\mathfrak{b} \notin H$ by Theorem 3.6(i).

Therefore, if $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{B})$ is such that $h'(\vec{x}_p) := \vec{g}_p$, by Lemma 1.20 we have $h(\mathfrak{s}_j) = \mathfrak{a}_j \in H$ for $j = 1, \dots, l$ but $h(\mathfrak{r}) = \mathfrak{b} \notin H$.

Conclusion: $\mathfrak{s}_1, \dots, \mathfrak{s}_l \nvdash_{\mathsf{M}} \mathfrak{r}$. ■

COROLLARY 3.11. *If a Gentzen relation* $\mathbf{G}$ *having the CDDT has the finite model property, then it has the strong finite model property.*

PROOF. By Proposition 2.43, the class of finite matrix models of $\mathbf{G}$ is closed under taking submatrices, and it is clearly also closed under contractions, so $\mathbf{G}$ has the strong finite model property by Theorem 3.10. ■

3.1.2. Factor-determined principal filters. Recall (cf. Subsection 2.6.3) that if we have a family of $\mathbf{G}$-matrices $\{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$, where $\mathbf{G}$ is a given Gentzen relation, then $\bigtriangleup_{i\in I} F_i$ denotes the $\mathbf{G}$-filter F such that $\langle \boldsymbol{A}, F\rangle = \prod_{i\in I}\langle \boldsymbol{A}_i, F_i\rangle$, where $\boldsymbol{A} := \prod_{i\in I} \boldsymbol{A}_i$.

In this situation, for each sequent $\mathfrak{a} \in \mathsf{Seq}(A)$ of the direct product $\prod_{i\in I} \boldsymbol{A}_i$, say $\mathfrak{a} := \langle\langle a_{i,1} : i \in I\rangle, \dots, \langle a_{i,m+n} : i \in I\rangle\rangle_{m,n}$, let us define

$$\mathfrak{a}(i) := \pi_i(\mathfrak{a}) = \langle a_{i,1}, \dots, a_{i,m+n}\rangle_{m,n} \in \mathsf{Seq}(A_i)$$

for all $i \in I$.

DEFINITION 3.12 (cf. [**11**, p. 140]). A Gentzen relation $\mathbf{G}$ has *factor-determined n-principal filters on direct products (n-FDPF)*, where $n > 0$, if for every family of $\mathbf{G}$-matrices $\{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$, setting $\langle \boldsymbol{A}, F\rangle := \prod_{i\in I}\langle \boldsymbol{A}_i, F_i\rangle$ we have:

$$\mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \dots, \mathfrak{a}_n) = \bigtriangleup_{i\in I} \mathrm{Fg}^{\boldsymbol{A}_i}_{\mathbf{G}}(F_i, \mathfrak{a}_1(i), \dots, \mathfrak{a}_n(i))$$

for all $\mathfrak{a}_1, \dots, \mathfrak{a}_n \in \mathsf{tr}\text{-}\mathsf{Seq}(A)$.

In [**11**, §2.4], Czelakowski proved that every protoalgebraic sentential logic having a variant of the DDT known as the *parametrised DDT* ([**14**, p. 336]) has n-FDPF for every $n \geq 1$. We can obtain a similar result for the CDDT and a weakening of n-FDPF.

DEFINITION 3.13. A Gentzen relation $\mathbf{G}$ is said to have *factor-determined principal filters on finitely generated direct products (FDPF_{fg})* if, for every finitely generated direct product $\langle \boldsymbol{A}, F\rangle$ of a family of $\mathbf{G}$-matrices $\{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$, every $n > 0$ and all $\mathfrak{a}_1, \dots, \mathfrak{a}_n \in \mathsf{tr}\text{-}\mathsf{Seq}(A)$, we have:

$$\mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \dots, \mathfrak{a}_n) = \bigtriangleup_{i\in I} \mathrm{Fg}^{\boldsymbol{A}_i}_{\mathbf{G}}(F_i, \mathfrak{a}_1(i), \dots, \mathfrak{a}_n(i)).$$

PROPOSITION 3.14. *Let* $\mathbf{G}$ *be a Gentzen relation with trace* tr. *If* $\mathbf{G}$ *has the CDDT, then* $\mathbf{G}$ *has FDPF_{fg}.*

PROOF. Let $\{\langle \boldsymbol{A}_i, F_i\rangle : i \in I\}$ be a family of $\mathbf{G}$-matrices, and suppose that their direct product $\langle \boldsymbol{A}, F\rangle := \prod_{i\in I}\langle \boldsymbol{A}_i, F_i\rangle$ is finitely generated by some elements $g_1, \dots, g_l \in A$, $l \in \omega$. By Proposition 1.32, each $\boldsymbol{A}_i$ is finitely generated by the elements in $\vec{g}_l(i) := \langle g_1(i), \dots, g_l(i)\rangle$.

Fix any $k > 0$ and any sequents $\mathfrak{a}_1, \dots, \mathfrak{a}_k \in \mathsf{tr\text{-}Seq}(A)$. Let $\hat{m}_j := \mathsf{tp}(\mathfrak{a}_j)$ for $j = 1, \dots, k$ and $G_i := \mathrm{Fg}^{\boldsymbol{A}_i}_{\mathbf{G}}(F_i, \mathfrak{a}_1(i), \dots, \mathfrak{a}_k(i))$ for every $i \in I$. Pick any sequent $\mathfrak{b} \in \mathsf{tr\text{-}Seq}(A)$ and let $\hat{n} := \mathsf{tp}(\mathfrak{b})$, so that we may write

$$\mathfrak{b} = \langle\langle b_{i,1} : i \in I\rangle, \dots, \langle b_{i,\Sigma(\hat{n})} : i \in I\rangle\rangle_{\hat{n}}$$

for some elements $b_{i,1}, \dots, b_{i,\Sigma(\hat{n})} \in A_i$ and all $i \in I$. We need to prove:

$$\mathfrak{b} \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \dots, \mathfrak{a}_k) \iff \mathfrak{b} \in \bigtriangleup_{i\in I} G_i. \tag{3.5}$$

Since $\mathbf{G}$ has the CDDT, let $\mathfrak{E} := \mathfrak{D}^*_{k-1}[l, \hat{m}_1, \dots, \hat{m}_k, \hat{n}]$. We have:

$$\begin{aligned}
&\mathfrak{b} \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \dots, \mathfrak{a}_k) \\
&\quad\iff \mathfrak{E}^{\boldsymbol{A}}(\vec{g}_l, \mathfrak{a}_1, \dots, \mathfrak{a}_k, \mathfrak{b}) \subseteq F && \text{(Theorem 3.6(i))} \\
&\quad\iff (\forall i \in I)\ \mathfrak{E}^{\boldsymbol{A}_i}(\vec{g}_l(i), \mathfrak{a}_1(i), \dots, \mathfrak{a}_k(i), \mathfrak{b}(i)) \subseteq F_i && \text{(Lemma 2.52)} \\
&\quad\iff (\forall i \in I)\ \mathfrak{b}(i) \in \mathrm{Fg}^{\boldsymbol{A}_i}_{\mathbf{G}}(F_i, \mathfrak{a}_1(i), \dots, \mathfrak{a}_k(i)) && \text{(Theorem 3.6(i))} \\
&\quad\iff (\forall i \in I)\ \mathfrak{b}(i) \in G_i \\
&\quad\iff (\forall i \in I)\ \langle b_{i,1}, \dots, b_{i,\Sigma(\hat{n})}\rangle_{\hat{n}} \in G_i.
\end{aligned} \tag{3.6}$$

We know (cf. Remark 2.48) that $\mathfrak{b} \in \bigtriangleup_{i\in I} G_i$ iff for every $i \in I$ there is an $\hat{n}$-sequent $\langle c_{i,1}, \dots, c_{i,\Sigma(\hat{n})}\rangle_{\hat{n}} \in G_i$ and $\mathfrak{b} = \langle\langle c_{i,1} : i \in I\rangle, \dots, \langle c_{i,\Sigma(\hat{n})} : i \in I\rangle\rangle_{\hat{n}}$. Therefore, (3.5) follows from (3.6). ■

3.2. A Bridge between the CDDT and ESPRC

As stated in Section 2.9, one of the main goals of abstract algebraic logic is to find connections between logic and (abstract) algebra in such a way that the tools of universal algebra can be used in the study of logic (and vice versa). Results that establish a correspondence between a property of a logic (or a class of logics) and a property of its algebraic counterpart are known as *bridge theorems*, a term coined by Andréka, Németi and Sain in [**1**], where they indicate that this technique of solving logical problems by algebraic means can be traced back to Leibniz and Pascal.

The purpose of this section is to prove a bridge theorem connecting the CDDT with having equationally semi-definable principal relative congruences (cf. Definition 3.16 below), first obtained by Raftery in [**28**] for sentential logics.

DEFINITION 3.15. A quasivariety K is said to have *equationally definable principal relative congruences* (*EDPRC*) if there is a finite set $\Psi \subseteq \langle 1,1\rangle\text{-}\mathsf{Seq}_4$ of equations in 4 variables such that, if $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$, then:

$$\langle c, d\rangle \in \boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \iff \boldsymbol{A} \models \eta[\![a, b, c, d]\!] \text{ for all } \eta \in \Psi.$$

If, moreover, K is a variety, we say that K has *equationally definable principal congruences* (*EDPC*), since then $\mathrm{Co}_{\mathsf{K}}(\boldsymbol{A}) = \mathrm{Co}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{K}$.

According to [**30**, p. 12], the notion of EDPC was first introduced by Fried, Grätzer and Quackenbush in [**17**]. In [**28**, Def. 8.5], Raftery generalized the property of having EDPRC to:

DEFINITION 3.16. A quasivariety K is said to have *equationally semi-definable principal relative congruences* (*ESPRC*) if, for every $n \in \omega$, there is a finite set $\Psi_n \subseteq \langle 1,1\rangle\text{-}\mathsf{Seq}_{n+4}$ of equations in $n+4$ variables such that, if $\boldsymbol{A} \in \mathsf{K}$ is generated by some elements $g_1, \ldots, g_n \in A$ and $a, b, c, d \in A$, then:

$$\langle c, d\rangle \in \boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \iff \boldsymbol{A} \models \eta[\![\vec{g}_n, a, b, c, d]\!] \text{ for all } \eta \in \Psi_n.$$

If, moreover, K is a variety, we say that K has *equationally semi-definable principal congruences* (*ESPC*).

REMARK 3.17. In the context of Definition 3.16, if $\Psi_n = \Psi_0$ for all $n \in \omega$, then K has EDPRC.

Having EDPRC implies having ESPRC. The converse, however, does not hold, as is shown in [**28**, Exmpl. 9.5].

One of the most well-known bridge theorems is the one connecting the DDT with having EDPRC (cf. [**14**, pp. 163-75]). This result, due to Blok and Pigozzi, was obtained in [**2**] for strongly algebraizable finitary sentential logics (thus, for EDPC), and was later generalized in [**6**, Thm. 5.5] to elementarily algebraizable finitary k-deductive systems. Rebagliato and Verdú generalized it further to elementarily algebraizable finitary Gentzen relations in [**30**, Cor. 3.12].

Raftery proved an analogous bridge theorem for elementarily algebraizable finitary sentential logics connecting the CDDT and having ESPRC, [**28**, Thm. 9.2]. His proof, however, employs a strategy quite different than the one followed by Blok and Pigozzi in [**6**]: their proof is more syntactic and clearly shows the correspondence between the sets witnessing the DDT and the equations defining the principal relative congruences, while Raftery's is more algebraic and conceals the correspondence between the CDD-sequence and the equations, a fact that Raftery himself acknowledges and the reason why he goes on to explicitly stating it in a separate result, namely [**28**, Thm. 9.4].

We shall now generalize Raftery's bridge theorem to elementarily algebraizable finitary Gentzen relations, first presenting an alternative proof that follows Blok and Pigozzi's strategy and then generalizing the one given by Raftery.

3.2.1. Tools for building the bridge. Both proofs of the bridge theorem that we shall present rely on two results: the first (Lemma 3.18) is a special case of the correspondence theorem (Theorem 2.67), and the second (Theorem 3.21) shows that finitary Gentzen relations that have the CDDT have CDD-sequences made up of finite sets, and also an equivalence between having the CDDT and an algebraic property of the compact filters of finitely generated algebras.

LEMMA 3.18. *Let* K *be a quasivariety,* $\boldsymbol{A}$ *an algebra,* $\boldsymbol{B} \in \mathsf{K}$ *and* $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ *a surjective homomorphism. For every* $a, b \in A$*, we have:*

$$h^{-1}(\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(a), h(b))) = \boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \vee^{\boldsymbol{A}}_{\mathsf{K}} \ker h.$$

PROOF. By Proposition 2.73, $\mathsf{EQ}(\mathsf{K})$ is protoalgebraic, so the correspondence theorem (Theorem 2.67) yields:

$$h^{-1}(\underbrace{\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b))) \vee^{\boldsymbol{B}}_{\mathsf{K}} \boldsymbol{\Delta}_{\boldsymbol{B}}}_{\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a,b)))}) = \boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \vee^{\boldsymbol{A}}_{\mathsf{K}} h^{-1}(\boldsymbol{\Delta}_{\boldsymbol{B}}).$$

Hence, given that $h^{-1}(\boldsymbol{\Delta}_{\boldsymbol{B}}) = \ker h$, it suffices to prove:

$$\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b))) = \boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(a), h(b)).$$

From $\langle a, b\rangle \in h^{-1}(\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(a), h(b)))$ we get $\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \subseteq h^{-1}(\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(a), h(b)))$ by Proposition 2.69 and Proposition 2.36(i), i.e., $h(\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b)) \subseteq \boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(a), h(b))$, and thus $\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b))) \subseteq \boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(a), h(b))$.

For the other inclusion, note that $\langle a, b\rangle \in \boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b)$ implies $\langle h(a), h(b)\rangle \in h(\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b))$, whence $\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(a), h(b)) \subseteq \boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(h(\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b)))$. ■

For the proof of the other main result of this subsection, we shall need two lemmas:

LEMMA 3.19. *Let* G *be a Gentzen relation with trace* tr*,* $\boldsymbol{A}, \boldsymbol{B}$ *algebras,* $X \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(A)$*,* $Y \subseteq \mathsf{tr}\text{-}\mathsf{Seq}(B)$ *and* $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ *a surjective homomorphism such that* $\ker h$ *is compatible with* $\mathrm{Fg}^{\boldsymbol{A}}_{\mathsf{G}}(h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(Y)) \cup X)$*. Then:*

$$h(\mathrm{Fg}^{\boldsymbol{A}}_{\mathsf{G}}(h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(Y)) \cup X)) = \mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(Y \cup h(X)).$$

PROOF. Let $F := \mathrm{Fg}^{\boldsymbol{A}}_{\mathsf{G}}(h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathsf{G}}(Y)) \cup X)$. We need to prove that $h(F)$ is the least G-filter of $\boldsymbol{B}$ containing $Y \cup h(X)$.

By Proposition 2.36(ii), $h(F) \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{B})$. From $h^{-1}(Y) \subseteq h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathbf{G}}(Y)) \subseteq F$ we get $Y \subseteq h(F)$ by the surjectivity of h, and from $X \subseteq F$ we get $h(X) \subseteq h(F)$, so $Y \cup h(X) \subseteq h(F)$.

Let G be any $\mathbf{G}$-filter of $\boldsymbol{B}$ such that $Y \cup h(X) \subseteq G$. Then, $\mathrm{Fg}^{\boldsymbol{B}}_{\mathbf{G}}(Y) \cup h(X) \subseteq G$, whence $h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathbf{G}}(Y) \cup h(X)) = h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathbf{G}}(Y)) \cup h^{-1}(h(X)) \subseteq h^{-1}(G)$. Since $X \subseteq h^{-1}(h(X))$, we obtain $h^{-1}(\mathrm{Fg}^{\boldsymbol{B}}_{\mathbf{G}}(Y)) \cup X \subseteq h^{-1}(G)$. We know $h^{-1}(G)$ is a $\mathbf{G}$-filter of $\boldsymbol{A}$ by Proposition 2.36(i), so $F \subseteq h^{-1}(G)$, and thus $h(F) \subseteq h(h^{-1}(G)) \subseteq G$. ∎

LEMMA 3.20. *Let* $\mathbf{G}$ *be a Gentzen relation with trace* tr*,* $\boldsymbol{A}$ *an algebra and* $F \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$ *such that* $F \neq \varnothing$*. If* F *is finitely generated, then there are some* $\mathfrak{a}_1, \dots, \mathfrak{a}_{n+1}$*,* $n \in \omega$*, such that* $F = \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(\mathfrak{a}_1, \dots, \mathfrak{a}_{n+1})$*.*

PROOF. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_m \in \mathsf{tr}\text{-}\mathsf{Seq}(A)$, $m \in \omega$, be such that $F = \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(\mathfrak{a}_1, \dots, \mathfrak{a}_m)$. If $m > 0$ there is nothing to prove, so assume $m = 0$. Then, $\bigcap \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A}) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(\varnothing) = F \neq \varnothing$, so pick any $\mathfrak{a} \in \bigcap \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$. Clearly, $F = \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(\mathfrak{a})$. ∎

THEOREM 3.21. *Let* $\mathbf{G}$ *be a finitary Gentzen relation with trace* tr*. The following are equivalent:*

(i) $\mathbf{G}$ *has the CDDT.*

(ii) $\mathbf{G}$ *is protoalgebraic and, for every finitely generated algebra* $\boldsymbol{A}$*, the compact* $\mathbf{G}$*-filters of* $\boldsymbol{A}$ *form a dually Brouwerian join-semilattice.*

(iii) $\mathbf{G}$ *has a CDD-sequence* $\langle \{\mathfrak{D}[n, \hat{m}_1, \hat{m}_2] : \hat{m}_1, \hat{m}_2 \in \mathsf{tr}\} : n \in \omega \rangle$ *in which every set* $\mathfrak{D}[n, \hat{m}_1, \hat{m}_2]$ *is finite.*

PROOF. By Proposition 3.7 we may assume, without loss of generality, that $\mathbf{G}$ is protoalgebraic. So, for every $\langle m, n \rangle \in \mathsf{tr}$, let $\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n}) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ be as in Theorem 2.62.

(i) $\Rightarrow$ (ii) By Proposition 1.73, the compact elements of $\mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$ form a join-semilattice. Let $\mathfrak{D} := \langle \{\mathfrak{D}[n, \hat{m}_1, \hat{m}_2] : \hat{m}_1, \hat{m}_2 \in \mathsf{tr}\} : n \in \omega \rangle$ be a CDD-sequence for $\mathbf{G}$. Let $g_1, \dots, g_l \in A$, $l \in \omega$, be such that $A = \mathrm{Sg}^{\boldsymbol{A}}(g_1, \dots, g_l)$, and let G, H be compact $\mathbf{G}$-filters of $\boldsymbol{A}$. We need to prove that $G \dot{-} H$ exists and is compact. We distinguish three cases.

Case 1: $G = \varnothing$. Clearly, $G \dot{-} H = \varnothing = G$.

Case 2: $G \neq \varnothing$ and $H = \varnothing$. Clearly, $G \dot{-} H = G$.

Case 3: $G \neq \varnothing$ and $H \neq \varnothing$. By Proposition 2.33 and Lemma 3.20, there are $\mathfrak{a}_1, \dots, \mathfrak{a}_{p+1}, \mathfrak{b}_1, \dots, \mathfrak{b}_{q+1} \in \mathsf{tr}\text{-}\mathsf{Seq}(A)$, $p, q \in \omega$, such that $G = \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(\mathfrak{b}_1, \dots, \mathfrak{b}_{q+1})$ and $H = \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(\mathfrak{a}_1, \dots, \mathfrak{a}_{p+1})$. Let $\hat{m}_i := \mathsf{tp}(\mathfrak{a}_i)$ for $i = 1, \dots, p+1$ and $\hat{n}_j := \mathsf{tp}(\mathfrak{b}_j)$ for $j = 1, \dots, q+1$.

For every $1 \leq j \leq q+1$, by Theorem 3.6(ii) there is a finite set

$$\mathfrak{E}_j \subseteq \mathfrak{D}_p^*[l, \hat{m}_1, \ldots, \hat{m}_{p+1}, \hat{n}_j]$$

such that, for every G-filter F of $\boldsymbol{A}$:

$$\mathfrak{b}_j \in \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{A}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{p+1}) \text{ iff } \mathfrak{E}_j^{\boldsymbol{A}}(\vec{g}_l, \mathfrak{a}_1, \ldots, \mathfrak{a}_{p+1}, \mathfrak{b}_j) \subseteq F. \tag{3.7}$$

Let $X := \bigcup_{j=1}^{q+1} \mathfrak{E}_j^{\boldsymbol{A}}(\vec{g}_l, \mathfrak{a}_1, \ldots, \mathfrak{a}_{p+1}, \mathfrak{b}_j)$. Being a finite union of finite sets, X is finite, so $\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{A}}(X)$ is compact by Proposition 2.33. For every G-filter F of $\boldsymbol{A}$, (3.7) yields:

$$\begin{aligned} G \subseteq F \vee^{\boldsymbol{A}} H &\iff \mathfrak{b}_1, \ldots, \mathfrak{b}_{q+1} \in F \vee^{\boldsymbol{A}} H \\ &\iff \mathfrak{b}_1, \ldots, \mathfrak{b}_{q+1} \in \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{A}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{p+1}) \\ &\iff X \subseteq F \\ &\iff \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{A}}(X) \subseteq F. \end{aligned}$$

Therefore, $G \dot{-} H = \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{A}}(X)$.

(ii) ⇒ (iii) Fix a context $p \in \omega$ and $\hat{m}, \hat{n} \in \mathsf{tr}$. Let $t := p + \Sigma(\hat{m}) + \Sigma(\hat{n})$. We need to find a finite set $\mathfrak{D} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_t$ such that

$$\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r} \iff \mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{D}(\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \tag{3.8}$$

for all $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_p$ with $\mathsf{tp}(\mathfrak{s}) = \hat{m}$ and $\mathsf{tp}(\mathfrak{r}) = \hat{n}$. This holds vacuously if $\mathsf{tr}\text{-}\mathsf{Seq}_p = \varnothing$ regardless of the choice of $\mathfrak{D}$ (in particular, for $\mathfrak{D} := \varnothing$), and if $t = 0$ we can take $\mathfrak{D} := \varnothing$ because then $\mathfrak{s} = \mathfrak{r} = \varnothing \rhd \varnothing$. Hence, let us assume both that $\mathsf{tr}\text{-}\mathsf{Seq}_p \neq \varnothing$ and that $t > 0$, whence Fm_p and Fm_t are universes of subalgebras $\boldsymbol{Fm}_p$ and $\boldsymbol{Fm}_t$ of $\boldsymbol{Fm}$, respectively.

Define the sequents $\mathfrak{x}_1 := \langle x_{p+1}, \ldots, x_{p+\Sigma(\hat{m})} \rangle_{\hat{m}}$ and $\mathfrak{x}_2 := \langle x_{p+\Sigma(\hat{m})+1}, \ldots, x_t \rangle_{\hat{n}}$. Note that $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathsf{tr}\text{-}\mathsf{Seq}(Fm_t)$. Since G is finitary and $\boldsymbol{Fm}_t$ is finitely generated, by Proposition 2.33 and (ii) the element $\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_2) \dot{-} \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_1)$ exists in the join-semilattice of compact G-filters of $\boldsymbol{Fm}_t$. Thus, by Proposition 2.33 there is a finite $\mathfrak{D}(\vec{x}_t) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_t$ such that:

$$\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{D}) = \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_2) \dot{-} \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_1).$$

By Lemma 1.74, for any G-filter F of $\boldsymbol{Fm}_t$, compact or not, we have:

$$\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_2) \subseteq F \vee^{\boldsymbol{Fm}_t} \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_1) \iff \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{D}) \subseteq F \iff \mathfrak{D} \subseteq F. \tag{3.9}$$

Let $h \in \mathrm{Hom}(\boldsymbol{Fm}_t, \boldsymbol{Fm}_p)$ be given by $h(\vec{x}_p) := \vec{x}_p$, $h(\mathfrak{x}_1) := \mathfrak{s}$, and $h(\mathfrak{x}_2) := \mathfrak{r}$. By Proposition 1.11, h is surjective.

By Proposition 2.36(i), both $h^{-1}(\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P}))$ and $h^{-1}(\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P}, \mathfrak{s}))$ are G-filters of $\boldsymbol{Fm}_t$, and $\ker h$ is compatible with $h^{-1}(\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P}))$ by Lemma 2.22.

Hence, by protoalgebraicity and Lemma 2.65, $\ker h$ is compatible with

$$H := h^{-1}(\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P})) \vee^{\boldsymbol{Fm}_t} \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_1) = \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(h^{-1}(\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P})), \mathfrak{x}_1).$$

Therefore, $h(H) = \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P}, \mathfrak{s})$ by Lemma 3.19.

Claim 3.21.1. $\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{t} \iff \mathfrak{t} \in h(H)$.

Proof. ($\Rightarrow$) Let $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{Fm}_p)$ be such that $g(\vec{x}_p) := \vec{x}_p$. Then, $g(\mathfrak{P} \cup \{\mathfrak{s}\}) = \mathfrak{P} \cup \{\mathfrak{s}\} \subseteq h(H)$, so $g(\mathfrak{t}) = \mathfrak{t} \in h(H)$ because $h(H) \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{Fm}_p)$.

($\Leftarrow$) By Corollary 2.35 we have $\mathfrak{t} \in \textsf{tr-Seq}(Fm_p) \cap \mathrm{Fg}_{\mathsf{G}}(\mathfrak{P}, \mathfrak{s})$, whence $\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{t}$ by Proposition 2.34. □

We have:

$$\begin{aligned}
&\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{t} \\
&\qquad \iff \mathfrak{t} \in h(H) && \text{(Claim 3.21.1)} \\
&\qquad \iff h(\mathfrak{x}_2) \in h(H) \\
&\qquad \iff \mathfrak{x}_2 \in H && \text{(Proposition 2.20(v))} \\
&\qquad \iff \mathfrak{x}_2 \in h^{-1}(\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P})) \vee^{\boldsymbol{Fm}_t} \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_1) && \text{(definition of } H\text{)} \\
&\qquad \iff \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_2) \subseteq h^{-1}(\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P})) \vee^{\boldsymbol{Fm}_t} \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_t}(\mathfrak{x}_1) && (H \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{Fm}_t)) \\
&\qquad \iff \mathfrak{D} \subseteq h^{-1}(\mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P})) && \text{(by (3.9))} \\
&\qquad \iff h(\mathfrak{D}) \subseteq \mathrm{Fg}_{\mathsf{G}}^{\boldsymbol{Fm}_p}(\mathfrak{P}) \\
&\qquad \iff h(\mathfrak{D}) \subseteq \textsf{tr-Seq}(Fm_p) \cap \mathrm{Fg}_{\mathsf{G}}(\mathfrak{P}) && \text{(Corollary 2.35)} \\
&\qquad \iff h(\mathfrak{D}) \subseteq \mathrm{Fg}_{\mathsf{G}}(\mathfrak{P}) && (h(\mathfrak{D}) \subseteq \textsf{tr-Seq}(Fm_p)) \\
&\qquad \iff \mathfrak{P} \vdash_{\mathsf{G}} h(\mathfrak{D}) && \text{(Proposition 2.34)} \\
&\qquad \iff \mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{D}(\vec{x}_p, \mathfrak{s}, \mathfrak{t}) && \text{(definition of } h\text{)}
\end{aligned}$$

Conclusion: (3.8) holds.

(iii) $\Rightarrow$ (i) Clear. ■

We are finally ready to present the two proofs that an elementarily algebraizable finitary Gentzen relation G has the CDDT iff K has ESPRC, where K is a quasivariety satisfying $\mathsf{G} \cong \mathsf{EQ}(\mathsf{K})$.

3.2.2. Blok and Pigozzi's strategy. Following Blok and Pigozzi's proofs of [**6**, Thms. 5.4,5.5], where they connect the DDT with having EDPRC, we show that an elementarily algebraizable finitary Gentzen relation G has the CDDT iff $\mathsf{EQ}(\mathsf{K})$ has the CDDT, where K is a quasivariety such that $\mathsf{G} \cong \mathsf{EQ}(\mathsf{K})$. We then show that $\mathsf{EQ}(\mathsf{K})$ has the CDDT iff K has ESPRC.

By Proposition 2.87 and Remark 2.83, if two *finitary* Gentzen relations G, G' are equivalent then there are *finitary* transformers τ, ρ such that $\tau : \mathsf{G} \cong \mathsf{G}' : \rho$. This allows us to prove that the CDDT is preserved by equivalence between finitary Gentzen relations:

THEOREM 3.22. *Let G and G' be finitary equivalent Gentzen relations with traces tr and tr', respectively. Then, G has the CDDT iff G' has the CDDT.*

PROOF. Let $\tau : \mathsf{tr}\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}'\text{-}\mathsf{Seq})$ and $\rho : \mathsf{tr}'\text{-}\mathsf{Seq} \to \mathcal{P}(\mathsf{tr}\text{-}\mathsf{Seq})$ be finitary transformers such that $\tau : \mathsf{G} \cong \mathsf{G}' : \rho$. Note that, by Remark 2.83, it suffices to prove that if G has the CDDT then G' has the CDDT.

Assume G has the CDDT, and let $\langle \{\mathfrak{D}[p, \hat{m}, \hat{n}] : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega \rangle$ be a CDD-sequence for G.

Let us build a CDD-sequence $\langle \{\mathfrak{E}[p, \hat{m}, \hat{n}] : \hat{m}, \hat{n} \in \mathsf{tr}'\} : p \in \omega \rangle$ for G'. Fix any context $p \in \omega$ and any $\hat{m}, \hat{n} \in \mathsf{tr}'$. Since ρ is finitary, by Proposition 2.81 there are finite sets of tr-sequents

$$\rho_{\hat{m}} := \{\mathfrak{m}_1, \ldots, \mathfrak{m}_l\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_{\Sigma(\hat{m})} \quad \text{and} \quad \rho_{\hat{n}} := \{\mathfrak{n}_1, \ldots, \mathfrak{n}_t\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_{\Sigma(\hat{n})},$$

with $l, t \in \omega$, such that $\rho(\mathfrak{s}) = \rho_{\hat{m}}(\mathfrak{s})$ and $\rho(\mathfrak{r}) = \rho_{\hat{n}}(\mathfrak{r})$ for every $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}'\text{-}\mathsf{Seq}$ with $\mathsf{tp}(\mathfrak{s}) = \hat{m}$ and $\mathsf{tp}(\mathfrak{r}) = \hat{n}$.

For every $j = 1, \ldots, t$ let $q_j := p + \Sigma(\mathsf{tp}(\mathfrak{m}_1)) + \cdots + \Sigma(\mathsf{tp}(\mathfrak{m}_l)) + \Sigma(\mathsf{tp}(\mathfrak{n}_j))$. Define the set $\mathfrak{E}[p, \hat{m}, \hat{n}] \subseteq \mathsf{tr}'\text{-}\mathsf{Seq}$ as follows: if $l = 0$, i.e., $\rho_{\hat{m}} = \varnothing$, let

$$\mathfrak{E}[p, \hat{m}, \hat{n}] := \{\langle x_{p+\Sigma(\hat{m})+1}, \ldots, x_{p+\Sigma(\hat{m})+\Sigma(\hat{n})} \rangle_{\hat{n}}\},$$

and otherwise let

$$\mathfrak{E}[p, \hat{m}, \hat{n}] := \tau\Big(\bigcup_{j=1}^{t} \mathfrak{D}^*_{l-1}[p, \mathsf{tp}(\mathfrak{m}_1), \ldots, \mathsf{tp}(\mathfrak{m}_l), \mathsf{tp}(\mathfrak{n}_j)](\vec{x}_p, \tilde{\mathfrak{m}}_1, \ldots, \tilde{\mathfrak{m}}_l, \tilde{\mathfrak{n}}_j)\Big),$$

where $\tilde{\mathfrak{m}}_i := \mathfrak{m}_i(x_{p+1}, \ldots, x_{p+\Sigma(\hat{m})})$ and $\tilde{\mathfrak{n}}_j := \mathfrak{n}_j(x_{p+\Sigma(\hat{m})+1}, \ldots, x_{p+\Sigma(\hat{m})+\Sigma(\hat{n})})$ for $i = 1, \ldots, l$ and $j = 1, \ldots, t$.

CLAIM 3.22.1. $\mathfrak{E}[p, \hat{m}, \hat{n}] = \mathfrak{E}[p, \hat{m}, \hat{n}](\vec{x}_{p+\Sigma(\hat{m})+\Sigma(\hat{n})})$.

PROOF. This is clear if $l = 0$, so assume $l > 0$. By Lemma 2.77 we have

$$\mathfrak{E}[p, \hat{m}, \hat{n}] = \bigcup_{j=1}^{t} \tau(\mathfrak{D}^*_{l-1}[p, \mathsf{tp}(\mathfrak{m}_1), \ldots, \mathsf{tp}(\mathfrak{m}_l), \mathsf{tp}(\mathfrak{n}_j)](\vec{x}_p, \tilde{\mathfrak{m}}_1, \ldots, \tilde{\mathfrak{m}}_l, \tilde{\mathfrak{n}}_j)),$$

so all we need to show is that the variables that occur in $\tau(\mathfrak{C}_j)$ are all among $x_1, \ldots, x_{p+\Sigma(\hat{m})+\Sigma(\hat{n})}$ for every $j = 1, \ldots, t$, where

$$\mathfrak{C}_j := \mathfrak{D}^*_{l-1}[p, \mathsf{tp}(\mathfrak{m}_1), \ldots, \mathsf{tp}(\mathfrak{m}_l), \mathsf{tp}(\mathfrak{n}_j)](\vec{x}_p, \tilde{\mathfrak{m}}_1, \ldots, \tilde{\mathfrak{m}}_l, \tilde{\mathfrak{n}}_j),$$

and this is a consequence of Lemma 2.78 because the variables occurring in $\mathfrak{C}_j$ are all in $\vec{x}_{p+\Sigma(\hat{m})+\Sigma(\hat{n})}$. $\square$

To see that $\langle\{\mathfrak{E}[p,\hat{m},\hat{n}] : \hat{m},\hat{n} \in \mathsf{tr}'\} : p \in \omega\rangle$ is indeed a CDD-sequence for $\mathbf{G}'$, let $\mathfrak{P} \cup \{\mathfrak{s},\mathfrak{t}\} \subseteq \mathsf{tr}'\text{-}\mathsf{Seq}_p$ be such that $\mathsf{tp}(\mathfrak{s}) = \hat{m}$ and $\mathsf{tp}(\mathfrak{t}) = \hat{n}$.

Assume first that $l = 0$, i.e., $\rho_{\hat{m}} = \varnothing$. We have:

$$\begin{aligned}
\mathfrak{P},\mathfrak{s} \vdash_{\mathbf{G}'} \mathfrak{t} &\overset{\text{(ALG3)}}{\iff} \rho(\mathfrak{P}),\rho(\mathfrak{s}) \vdash_{\mathbf{G}} \rho(\mathfrak{t})\\
&\iff \rho(\mathfrak{P}) \vdash_{\mathbf{G}} \rho(\mathfrak{t})\\
&\overset{\text{(ALG1)}}{\iff} \tau(\rho(\mathfrak{P})) \vdash_{\mathbf{G}'} \tau(\rho(\mathfrak{t}))\\
&\overset{\text{(ALG2)}}{\iff} \mathfrak{P} \vdash_{\mathbf{G}'} \mathfrak{t}\\
&\iff \mathfrak{P} \vdash_{\mathbf{G}'} \mathfrak{E}[p,\hat{m},\hat{n}](\vec{x}_p,\mathfrak{s},\mathfrak{t}),
\end{aligned}$$

so we are done.

Assume now that $l > 0$, so that $\rho_{\hat{m}} \neq \varnothing$. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be any substitution such that:

- $\sigma(\vec{x}_p) := \vec{x}_p$.
- $\sigma(\langle x_{p+1},\ldots,x_{p+\Sigma(\hat{m})}\rangle_{\hat{m}}) := \mathfrak{s}$.
- $\sigma(\langle x_{p+\Sigma(\hat{m})+1},\ldots,x_{p+\Sigma(\hat{m})+\Sigma(\hat{n})}\rangle_{\hat{n}}) := \mathfrak{t}$.

Note that $\sigma(\tilde{\mathfrak{m}}_i) = \mathfrak{m}_i(\mathfrak{s}) \in \mathsf{tr}\text{-}\mathsf{Seq}_p$ and $\sigma(\tilde{\mathfrak{n}}_j) = \mathfrak{n}_j(\mathfrak{t}) \in \mathsf{tr}\text{-}\mathsf{Seq}_p$ for all $i = 1,\ldots,l$ and all $j = 1,\ldots,t$. Therefore, setting $\mathfrak{D}^*_j := \mathfrak{D}^*_{l-1}[p,\mathsf{tp}(\mathfrak{m}_1),\ldots,\mathsf{tp}(\mathfrak{m}_l),\mathsf{tp}(\mathfrak{n}_j)]$ we have:

$$\begin{aligned}
\mathfrak{P},\mathfrak{s} \vdash_{\mathbf{G}'} \mathfrak{t} &\overset{\text{(ALG3)}}{\iff} \rho(\mathfrak{P}),\rho(\mathfrak{s}) \vdash_{\mathbf{G}} \rho(\mathfrak{t})\\
&\iff \rho(\mathfrak{P}),\mathfrak{m}_1(\mathfrak{s}),\ldots,\mathfrak{m}_l(\mathfrak{s}) \vdash_{\mathbf{G}} \rho(\mathfrak{t})\\
&\iff \rho(\mathfrak{P}),\mathfrak{m}_1(\mathfrak{s}),\ldots,\mathfrak{m}_l(\mathfrak{s}) \vdash_{\mathbf{G}} \mathfrak{n}_j(\mathfrak{t}) \text{ for all } j = 1,\ldots,t\\
&\iff \rho(\mathfrak{P}) \vdash_{\mathbf{G}} \mathfrak{D}^*_j(\vec{x}_p,\mathfrak{m}_1(\mathfrak{s}),\ldots,\mathfrak{m}_l(\mathfrak{s}),\mathfrak{n}_j(\mathfrak{t})) \text{ for all } j = 1,\ldots,t\\
&\iff \rho(\mathfrak{P}) \vdash_{\mathbf{G}} \textstyle\bigcup_{j=1}^{t} \mathfrak{D}^*_j(\vec{x}_p,\mathfrak{m}_1(\mathfrak{s}),\ldots,\mathfrak{m}_l(\mathfrak{s}),\mathfrak{n}_j(\mathfrak{t}))\\
&\overset{\text{(ALG1)}}{\iff} \tau(\rho(\mathfrak{P})) \vdash_{\mathbf{G}'} \tau(\textstyle\bigcup_{j=1}^{t} \mathfrak{D}^*_j(\vec{x}_p,\mathfrak{m}_1(\mathfrak{s}),\ldots,\mathfrak{m}_l(\mathfrak{s}),\mathfrak{n}_j(\mathfrak{t})))\\
&\overset{\text{(ALG2)}}{\iff} \mathfrak{P} \vdash_{\mathbf{G}'} \tau(\textstyle\bigcup_{j=1}^{t} \mathfrak{D}^*_j(\vec{x}_p,\mathfrak{m}_1(\mathfrak{s}),\ldots,\mathfrak{m}_l(\mathfrak{s}),\mathfrak{n}_j(\mathfrak{t})))\\
&\iff \mathfrak{P} \vdash_{\mathbf{G}'} \tau(\textstyle\bigcup_{j=1}^{t} \sigma(\mathfrak{D}^*_j(\vec{x}_p,\tilde{\mathfrak{m}}_1,\ldots,\tilde{\mathfrak{m}}_l,\tilde{\mathfrak{n}}_j)))\\
&\iff \mathfrak{P} \vdash_{\mathbf{G}'} \textstyle\bigcup_{j=1}^{t} \tau(\sigma(\mathfrak{D}^*_j(\vec{x}_p,\tilde{\mathfrak{m}}_1,\ldots,\tilde{\mathfrak{m}}_l,\tilde{\mathfrak{n}}_j))), \text{ by Lemma 2.77}\\
&\iff \mathfrak{P} \vdash_{\mathbf{G}'} \textstyle\bigcup_{j=1}^{t} \sigma(\tau(\mathfrak{D}^*_j(\vec{x}_p,\tilde{\mathfrak{m}}_1,\ldots,\tilde{\mathfrak{m}}_l,\tilde{\mathfrak{n}}_j)))\\
&\iff \mathfrak{P} \vdash_{\mathbf{G}'} \sigma(\textstyle\bigcup_{j=1}^{t} \tau(\mathfrak{D}^*_j(\vec{x}_p,\tilde{\mathfrak{m}}_1,\ldots,\tilde{\mathfrak{m}}_l,\tilde{\mathfrak{n}}_j)))
\end{aligned}$$

$$\iff \mathfrak{P} \vdash_{\mathsf{G}'} \sigma(\mathfrak{E}[p, \hat{m}, \hat{n}](\vec{x}_{p+\Sigma(\hat{m})+\Sigma(\hat{n})}))$$
$$\iff \mathfrak{P} \vdash_{\mathsf{G}'} \mathfrak{E}[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}).$$

Conclusion: $\langle \{\mathfrak{E}[p, \hat{m}, \hat{n}] : \hat{m}, \hat{n} \in \mathsf{tr}'\} : p \in \omega \rangle$ is a CDD-sequence for G'. ■

THEOREM 3.23. *Let* K *be a quasivariety. Then,* $\mathsf{EQ}(\mathsf{K})$ *has the CDDT iff* K *has ESPRC.*

PROOF. ($\Rightarrow$) Since $\mathsf{tr}(\mathsf{EQ}(\mathsf{K})) = \{\langle 1, 1 \rangle\}$ and $\mathsf{EQ}(\mathsf{K})$ is finitary (Lemma 2.75), by Theorem 3.21(iii) there is a CDD-sequence for $\mathsf{EQ}(\mathsf{K})$ of the form $\langle \mathfrak{D}[n] : n \in \omega \rangle$, where each $\mathfrak{D}[n]$ is a finite set of $\langle 1, 1 \rangle$-sequents of Fm (equations) in the variables $\vec{x}_{n+4}$. So let us write

$$\mathfrak{D}[n] = \{\delta_i(\vec{x}_{n+4}) \approx \varepsilon_i(\vec{x}_{n+4}) : i \in I_n\},$$

with each I_n finite.

Fix any $n \in \omega$ and let $\Psi_n := \mathfrak{D}[n]$. Also, fix any $\boldsymbol{A} \in \mathsf{K}$ generated by some elements $g_1, \ldots, g_n \in A$, and let $a, b, c, d \in A$. Note that, if $n = 0$, by Theorem 1.22 we must have $Fm_0 \neq \varnothing$ because $A \neq \varnothing$. Thus, Fm_n is the universe of a subalgebra $\boldsymbol{Fm}_n$ of $\boldsymbol{Fm}$ regardless of the choice of n.

Let $h \in \mathrm{Hom}(\boldsymbol{Fm}_n, \boldsymbol{A})$ be any homomorphism such that $h(\vec{x}_n) = \vec{g}_n$. By Proposition 1.11, h is surjective, so let $\varphi_0, \varphi_1, \psi_0, \psi_1 \in Fm_n$ be such that $h(\varphi_0) = a$, $h(\varphi_1) = b$, $h(\psi_0) = c$ and $h(\psi_1) = d$.

Let $\Phi := \ker h$. Since K is a quasivariety, K is closed under isomorphisms, so Φ is a K-congruence of $\boldsymbol{Fm}_n$ by the first isomorphism theorem because $\boldsymbol{A} \in \mathsf{K}$.

By Lemma 3.18 we have

$$h^{-1}(\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b)) = \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\varphi_0, \varphi_1) \vee^{\boldsymbol{Fm}_n}_{\mathsf{K}} \Phi,$$

so

$$\langle \psi_0, \psi_1 \rangle \in \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\varphi_0, \varphi_1) \vee^{\boldsymbol{Fm}_n}_{\mathsf{K}} \Phi \iff \langle c, d \rangle \in \boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b). \tag{3.10}$$

Since $\boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\varphi_0, \varphi_1) \vee^{\boldsymbol{Fm}_n}_{\mathsf{K}} \Phi = \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\Phi, \langle \varphi_0, \varphi_1 \rangle)$, by Corollary 2.72 the left-hand side of (3.10) is equivalent to

$$\Phi, \varphi_0 \approx \varphi_1 \vDash_{\mathsf{K}} \psi_0 \approx \psi_1,$$

which by the CDDT for $\mathsf{EQ}(\mathsf{K})$ is equivalent to

$$\Phi \vDash_{\mathsf{K}} \delta_i(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1) \approx \varepsilon_i(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1) \text{ for all } i \in I_n.$$

As Φ is a congruence of $\boldsymbol{Fm}_n$, by Corollary 2.72 this is equivalent to

$$\langle \delta_i(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1), \varepsilon_i(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1) \rangle \in \Phi = \ker h \text{ for all } i \in I_n,$$

which holds iff

$$\delta^{\boldsymbol{A}}_i(\vec{g}_n, a, b, c, d) = \varepsilon^{\boldsymbol{A}}_i(\vec{g}_n, a, b, c, d) \text{ for all } i \in I_n,$$

i.e., iff

$$\boldsymbol{A} \models \eta[\![\vec{g}_n, a, b, c, d]\!] \text{ for all } \eta \in \Psi_n.$$

Conclusion: the equations in the sets Ψ_n, $n \in \omega$, witness that K has ESPRC.

($\Leftarrow$) For all $n \in \omega$, let Ψ_n be as in Definition 3.16. We prove that $\langle \Psi_n : n \in \omega \rangle$ is a CDD-sequence for $\mathsf{EQ}(\mathsf{K})$, i.e., that for every context $n \in \omega$ and every equations $\mathfrak{E} \cup \{\varphi_0 \approx \varphi_1, \psi_0 \approx \psi_1\} \subseteq \langle 1, 1\rangle\text{-}\mathsf{Seq}_n$, we have

$$\mathfrak{E}, \varphi_0 \approx \varphi_1 \models_{\mathsf{K}} \psi_0 \approx \psi_1 \iff \mathfrak{E} \models_{\mathsf{K}} \Psi_n(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1). \tag{3.11}$$

This is vacuously true for $n = 0$ if $Fm_0 = \varnothing$ because then $\langle 1, 1\rangle\text{-}\mathsf{Seq}_n = \varnothing$, and thus for the case $n = 0$ we may assume $Fm_0 \neq \varnothing$, so that Fm_n is the universe of a subalgebra $\boldsymbol{Fm}_n$ of $\boldsymbol{Fm}$ regardless of the choice of n.

Fix any $n \in \omega$ and $\mathfrak{E} \cup \{\varphi_0 \approx \varphi_1, \psi_0 \approx \psi_1\} \subseteq \langle 1, 1\rangle\text{-}\mathsf{Seq}_n$. By Corollary 2.72,

$$\begin{aligned}\mathfrak{E}, \varphi_0 \approx \varphi_1 \models_{\mathsf{K}} \psi_0 \approx \psi_1 &\iff \langle \psi_0, \psi_1\rangle \in \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\mathfrak{E}, \langle \varphi_0, \varphi_1\rangle)\\ &\iff \langle \psi_0, \psi_1\rangle \in \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\mathfrak{E}) \vee^{\boldsymbol{Fm}_n}_{\mathsf{K}} \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\varphi_0, \varphi_1).\end{aligned}$$

Let $\boldsymbol{B} := \boldsymbol{Fm}_n / \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\mathfrak{E})$, and let $\pi : \boldsymbol{Fm}_n \to \boldsymbol{B}$ be the natural projection. Then, $\boldsymbol{B} \in \mathsf{K}$, π is surjective and $\ker \pi = \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\mathfrak{E})$, so Lemma 3.18 yields:

$$\pi^{-1}(\boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(\pi(\varphi_0), \pi(\varphi_1))) = \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\varphi_0, \varphi_1) \vee^{\boldsymbol{Fm}_n}_{\mathsf{K}} \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\mathfrak{E}).$$

Therefore, $\mathfrak{E}, \varphi_0 \approx \varphi_1 \models_{\mathsf{K}} \psi_0 \approx \psi_1$ is equivalent to

$$\langle \pi(\psi_0), \pi(\psi_1)\rangle \in \boldsymbol{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(\pi(\varphi_0), \pi(\varphi_1)).$$

Since K has ESPRC and, by Proposition 1.28, $\boldsymbol{B}$ is generated by $\pi(x_1), \dots, \pi(x_n)$, this is equivalent to

$$\boldsymbol{B} \models_{\mathsf{K}} \delta_i \approx \varepsilon_i[\![\pi(x_1), \dots, \pi(x_n), \pi(\varphi_0), \pi(\varphi_1), \pi(\psi_0), \pi(\psi_1)]\!] \text{ for all } \delta_i \approx \varepsilon_i \in \Psi_n,$$

which, by Proposition 1.27, is equivalent to:

$$\langle \delta_i(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1), \varepsilon_i(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1)\rangle \in \boldsymbol{\Theta}^{\boldsymbol{Fm}_n}_{\mathsf{K}}(\mathfrak{E}) \text{ for all } \delta_i \approx \varepsilon_i \in \Psi_n.$$

By Corollary 2.72, this is equivalent to

$$\mathfrak{E} \models_{\mathsf{K}} \delta_i(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1) \approx \varepsilon_i(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1) \text{ for all } \delta_i \approx \varepsilon_i \in \Psi_n,$$

i.e., to

$$\mathfrak{E} \models_{\mathsf{K}} \Psi_n(\vec{x}_n, \varphi_0, \varphi_1, \psi_0, \psi_1).$$

Conclusion: $\langle \Psi_n : n \in \omega \rangle$ is a CDD-sequence for $\mathsf{EQ}(\mathsf{K})$. ■

The bridge theorem connecting the CDDT with having ESPRC is an immediate consequence of the two preceding theorems:

THEOREM 3.24. *Let* $\mathbf{G}$ *be an elementarily algebraizable finitary Gentzen relation, and let* K *be a quasivariety such that* $\mathbf{G} \cong \mathsf{EQ}(\mathsf{K})$*. Then,* $\mathbf{G}$ *has the CDDT iff* K *has ESPRC.*

First Proof. By Theorem 3.22, G has the CDDT iff $\mathsf{EQ}(\mathsf{K})$ has the CDDT, iff K has ESPRC by Theorem 3.23. ■

Note that the proofs of Theorem 3.22 and Theorem 3.23 are constructive, in the sense that we can easily obtain equations witnessing the property of having ESPRC from a CDD-sequence and vice versa.

3.2.3. Raftery's strategy. Now we obtain an alternative proof of Theorem 3.24 by generalizing that of Raftery's [**28**, Thm. 9.2]. This time we show that an elementarily algebraizable finitary Gentzen relation G has the CDDT iff K has ESPRC, where K is a quasivariety satisfying $\mathsf{G} \cong \mathsf{EQ}(\mathsf{K})$, by proving that both conditions are equivalent to the statement that the compact K-congruences of every finitely generated algebra form a dually Brouwerian join-semilattice.

If K is a quasivariety, then $\mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$ is a complete lattice by Corollary 1.92 for every algebra $\boldsymbol{A}$, and thus the compact K-congruences of $\boldsymbol{A}$ form a join-semilattice by Proposition 1.73.

Theorem 3.25 (cf. [**28**, Thm. 8.6]). *A quasivariety K has ESPRC iff for every finitely generated algebra $\boldsymbol{A}$, the join-semilattice of compact K-congruences of $\boldsymbol{A}$ is dually Brouwerian.*

For clarity, we restate Theorem 3.24 before its second proof:

Theorem 3.24. *Let G be an elementarily algebraizable finitary Gentzen relation, and let K be a quasivariety such that $\mathsf{G} \cong \mathsf{EQ}(\mathsf{K})$. Then, G has the CDDT iff K has ESPRC.*

Second Proof. By Theorem 2.89, G is protoalgebraic, and thus G has the CDDT iff the join-semilattice of compact G-filters of every finitely generated algebra is dually Brouwerian, by Theorem 3.21(ii).

If $\boldsymbol{A}$ is a finitely generated algebra, then $\mathcal{F}i_{\mathsf{G}}(\boldsymbol{A}) \cong \mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$ by Theorem 2.86 and Proposition 2.69, and therefore the join-semilattice of compact G-filters of $\boldsymbol{A}$ is dually Brouwerian iff the join-semilattice of compact K-congruences of $\boldsymbol{A}$ is dually Brouwerian, by Lemma 1.75.

Therefore, G has the CDDT iff the join-semilattice of compact K-congruences of every finitely generated algebra $\boldsymbol{A}$ is dually Brouwerian, and by Theorem 3.25 this is equivalent to K having ESPRC. ■

In contrast with the first proof, this one is not constructive, as it does not allow us to build a CDD-sequence from equations witnessing having ESPRC nor to obtain these equations from a CDD-sequence. For this reason, Raftery finishes [**28**] by proving a theorem that allows one to perform such translation between equations and CDD-sequences. Of course, there is no need for us to prove it because we have already given a constructive proof of Theorem 3.24.

3.3. The local CDDT

DEFINITION 3.26. Let G be a Gentzen relation with trace tr. For all contexts $n \in \omega$ and all types $\hat{m}_1, \hat{m}_2 \in \mathsf{tr}$, let $I_{n,\hat{m}_1,\hat{m}_2}$ be a non-empty set and, for all $i \in I_{n,\hat{m}_1,\hat{m}_2}$, let $\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_t) \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_t$, where $t := n + \Sigma(\hat{m}_1) + \Sigma(\hat{m}_2)$. The sequence

$$\langle\{\{\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2] : i \in I_{n,\hat{m}_1,\hat{m}_2}\} : \hat{m}_1, \hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$$

is said to be a *local CDD-sequence* for G if, for all $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_n$ with $\mathsf{tp}(\mathfrak{s}) = \hat{m}_1$ and $\mathsf{tp}(\mathfrak{r}) = \hat{m}_2$, we have:

$$\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r} \iff \mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}) \text{ for some } i \in I_{n,\hat{m}_1,\hat{m}_2}.$$

If such a sequence exists, we say that G has a *local CDDT*.

If $|I_{n,\hat{m}_1,\hat{m}_2}| = 1$ for all $n \in \omega$ and all $\hat{m}_1, \hat{m}_2 \in \mathsf{tr}$, then we identify each singleton $\{\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2] : i \in I_{n,\hat{m}_1,\hat{m}_2}\}$ with its unique element, which we denote by $\mathfrak{L}[n, \hat{m}_1, \hat{m}_2]$, so that we recover the notion of having the CDDT (Definition 3.1).

In the context of Definition 3.26, the left-to-right implication is called the *local contextual deduction theorem*, and the right-to-left is known as *local contextual detachment*.

PROPOSITION 3.27. *Let* $\langle\{\{\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2] : i \in I_{n,\hat{m}_1,\hat{m}_2}\} : \hat{m}_1, \hat{m}_2 \in \mathsf{tr}\} : n \in \omega\rangle$ *be a local CDD-sequence for a Gentzen relation* G *with trace* tr. *For every context* $n \in \omega$ *and every* $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}_n$, *with* $\mathsf{tp}(\mathfrak{s}) = \hat{m}_1$ *and* $\mathsf{tp}(\mathfrak{r}) = \hat{m}_2$, *we have:*

(i) $\vdash_{\mathsf{G}} \mathfrak{L}_i[n, \hat{m}, \hat{m}](\vec{x}_n, \mathfrak{s}, \mathfrak{s})$ *for some* $i \in I_{n,\hat{m},\hat{m}}$.
(ii) $\mathfrak{s}, \mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathsf{G}} \mathfrak{r}$ *for all* $i \in I_{n,\hat{m}_1,\hat{m}_2}$.
(iii) *For all* $i \in I_{n,\hat{m}_1,\hat{m}_2}$ *there is some* $j \in I_{n+1,\hat{m}_1,\hat{m}_2}$ *such that*

$$\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathsf{G}} \mathfrak{L}_j[n + 1, \hat{m}_1, \hat{m}_2](\vec{x}_{n+1}, \mathfrak{s}, \mathfrak{r}).$$

PROOF.

(i) Apply the local contextual deduction theorem to $\mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{s}$.
(ii) Apply local contextual detachment to

$$\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathsf{G}} \mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}).$$

(iii) As $\mathsf{tr}\text{-}\mathsf{Seq}_n \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_{n+1}$, the local contextual deduction theorem on (ii) yields (iii). ■

We cannot obtain a uniqueness result for local CDD-sequences analogous to that of Theorem 3.4 because the sets $\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2]$, with $n, \hat{m}_1, \hat{m}_2$ fixed, need not be pairwise 'comparable' with respect to $\vdash_{\mathsf{G}}$, i.e., in principle

$$\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2] \nvdash_{\mathsf{G}} \mathfrak{L}_j[n, \hat{m}_1, \hat{m}_2] \text{ and } \mathfrak{L}_j[n, \hat{m}_1, \hat{m}_2] \nvdash_{\mathsf{G}} \mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2]$$

may both be the case for some $i, j \in I_{n,\hat{m}_1,\hat{m}_2}$. The most we can prove towards uniqueness is the following proposition, which establishes a (weak) relation between any two local CDD-sequences for a given Gentzen relation:

PROPOSITION 3.28. *Let* $\langle \{ \{ \mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2] : i \in I_{n,\hat{m}_1,\hat{m}_2} \} : \hat{m}_1, \hat{m}_2 \in \mathsf{tr} \} : n \in \omega \rangle$ *and* $\langle \{ \{ \mathfrak{L}'_i[n, \hat{m}_1, \hat{m}_2] : i \in I'_{n,\hat{m}_1,\hat{m}_2} \} : \hat{m}_1, \hat{m}_2 \in \mathsf{tr} \} : n \in \omega \rangle$ *be two local CDD-sequences for a Gentzen relation* G *with trace* tr*. For every context* $n \in \omega$ *and every* $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}_n$*, with* $\mathsf{tp}(\mathfrak{s}) = \hat{m}_1$ *and* $\mathsf{tp}(\mathfrak{r}) = \hat{m}_2$*, we have that for all* $i \in I_{n,\hat{m}_1,\hat{m}_2}$ *there is some* $j \in I'_{n,\hat{m}_1,\hat{m}_2}$ *such that:*

$$\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathsf{G}} \mathfrak{L}'_j[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}).$$

PROOF. From Proposition 3.27(ii) we have $\mathfrak{s}, \mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathsf{G}} \mathfrak{r}$ for every $i \in I_{n,\hat{m}_1,\hat{m}_2}$, so $\mathfrak{L}_i[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathsf{G}} \mathfrak{L}'_j[n, \hat{m}_1, \hat{m}_2](\vec{x}_n, \mathfrak{s}, \mathfrak{r})$ holds for some $j \in I'_{n,\hat{m}_1,\hat{m}_2}$ because $\langle \{ \{ \mathfrak{L}'_i[n, \hat{m}_1, \hat{m}_2] : i \in I'_{n,\hat{m}_1,\hat{m}_2} \} : \hat{m}_1, \hat{m}_2 \in \mathsf{tr} \} : n \in \omega \rangle$ is a local CDD-sequence. ■

Items (i) and (ii) of Proposition 3.27 resemble conditions (R) and (MP) of Theorem 2.62, and in fact having a local CDDT is equivalent to being protoalgebraic. To prove it, we first need to obtain another characterization of protoalgebraicity, which first appeared for finitary sentential logics in the proof of the (only) theorem of [**12**], due to Czelakowski and Dziobiak. It appeared again in the proof of [**11**, Thm. 2.1.5] by Czelakowski, this time without the assumption of finitarity.

In order to simplify the proofs of the statements to come, we define the notion of a 'template' of a Gentzen relation, which captures the essence of Czelakowski and Dziobiak's characterization of protoalgebraicity:

DEFINITION 3.29. Let G be a Gentzen relation with trace tr. A G-*template* is a quintuple of the form $\langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle$, where $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, $\mathfrak{T} \in \mathcal{T}h(\mathsf{G})$ and $\sigma \in \mathrm{End}(\boldsymbol{Fm})$, such that:

(i) $\mathfrak{T}, \mathfrak{y}_{\mathsf{tp}(\mathfrak{s})} \vdash_{\mathsf{G}} \mathfrak{z}_{\mathsf{tp}(\mathfrak{r})}$.
(ii) $\sigma(\mathfrak{y}_{\mathsf{tp}(\mathfrak{s})}) = \mathfrak{s}$.
(iii) $\sigma(\mathfrak{z}_{\mathsf{tp}(\mathfrak{r})}) = \mathfrak{r}$.
(iv) $\mathfrak{P} \vdash_{\mathsf{G}} \sigma(\mathfrak{T})$.

REMARK 3.30. If $\langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle$ is a G-template, then $\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r}$.

THEOREM 3.31. *A Gentzen relation* G *with trace* tr *is protoalgebraic iff the following holds for every* $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$*:*

$$\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r} \text{ iff there is a theory } \mathfrak{T} \in \mathcal{T}h(\mathsf{G}) \text{ and a substitution } \sigma \in \mathrm{End}(\boldsymbol{Fm}) \text{ such that } \langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle \text{ is a } \mathsf{G}\text{-template.} \tag{3.12}$$

PROOF. ($\Rightarrow$) The right-to-left implication of (3.12) is Remark 3.30.

Assume $\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r}$. Let $\langle m, n\rangle := \mathsf{tp}(\mathfrak{s})$ and $\langle r, s\rangle := \mathsf{tp}(\mathfrak{r})$. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be any surjective substitution such that $\sigma(\mathfrak{y}_{m,n}) := \mathfrak{s}$ and $\sigma(\mathfrak{z}_{r,s}) := \mathfrak{r}$. Note that at least one such substitution exists by Proposition 1.11 because the (countably infinite) set $\mathsf{Var} \setminus \{y_1, \dots, y_{m+n}, z_1, \dots, z_{r+s}\}$ can be mapped onto Var.

Let $\mathfrak{T} := \sigma^{-1}(\mathrm{Cn}_{\mathsf{G}}(\mathfrak{P}))$. By Proposition 2.36(i) and Proposition 2.34, we have $\mathfrak{T} \in \mathcal{T}h(\mathsf{G})$. Since σ is surjective, $\sigma(\mathfrak{T}) = \mathrm{Cn}_{\mathsf{G}}(\mathfrak{P})$, i.e., $\mathfrak{P} \vdash_{\mathsf{G}} \sigma(\mathfrak{T})$. Also, σ is a strict surjective homomorphism from the G-matrix $\langle \boldsymbol{Fm}, \mathfrak{T}\rangle$ to the G-matrix $\langle \boldsymbol{Fm}, \mathrm{Cn}_{\mathsf{G}}(\mathfrak{P})\rangle$, so the correspondence theorem (Theorem 2.67(iv)) yields

$$\sigma(\mathrm{Cn}_{\mathsf{G}}(\mathfrak{T}, \mathfrak{y}_{m,n})) = \mathrm{Cn}_{\mathsf{G}}(\sigma(\mathfrak{T}), \sigma(\mathfrak{y}_{m,n})) = \mathrm{Cn}_{\mathsf{G}}(\mathrm{Cn}_{\mathsf{G}}(\mathfrak{P}), \mathfrak{s}) = \mathrm{Cn}_{\mathsf{G}}(\mathfrak{P}, \mathfrak{s})$$

and also, as a consequence,

$$\mathrm{Cn}_{\mathsf{G}}(\mathfrak{T}, \mathfrak{y}_{m,n}) = \sigma^{-1}(\mathrm{Cn}_{\mathsf{G}}(\mathfrak{P}, \mathfrak{s})).$$

Given that $\sigma(\mathfrak{z}_{r,s}) = \mathfrak{r} \in \mathrm{Cn}_{\mathsf{G}}(\mathfrak{P}, \mathfrak{s})$, we have $\mathfrak{z}_{r,s} \in \mathrm{Cn}_{\mathsf{G}}(\mathfrak{T}, \mathfrak{y}_{m,n})$, so we are done.

($\Leftarrow$) We prove that G is protoalgebraic using Theorem 2.62.

Fix any $\langle m, n\rangle \in \mathsf{tr}$. If $\langle m, n\rangle = \langle 0, 0\rangle$, then the conditions (R) and (MP) of Theorem 2.62 are satisfied taking $\mathfrak{E}_{0,0} := \varnothing$, so assume $\langle m, n\rangle \neq \langle 0, 0\rangle$. Note that, then, x_1 occurs among $\vec{x}_{m+n}$.

Since $\mathfrak{x}_{m,n} \vdash_{\mathsf{G}} \mathfrak{x}_{m,n}$, there are some $\mathfrak{T} \in \mathcal{T}h(\mathsf{G})$ and $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ such that:

(i) $\mathfrak{T}, \mathfrak{y}_{m,n} \vdash_{\mathsf{G}} \mathfrak{z}_{m,n}$.
(ii) $\sigma(\mathfrak{y}_{m,n}) = \mathfrak{x}_{m,n}$.
(iii) $\sigma(\mathfrak{z}_{m,n}) = \mathfrak{x}_{m,n}$.
(iv) $\vdash_{\mathsf{G}} \sigma(\mathfrak{T})$.

Let $\sigma^* \in \mathrm{End}(\boldsymbol{Fm})$ be given by $\sigma^*(\vec{x}_{m+n}) := \sigma^*(\vec{y}_{m+n}) := \vec{x}_{m+n}$ and $\sigma^*(u) := x_1$ for every variable u not in $\vec{x}_{m+n}, \vec{y}_{m+n}$.

Let $\sigma' \in \mathrm{End}(\boldsymbol{Fm})$ be such that $\sigma'(\vec{y}_{m+n}) := \vec{x}_{m+n}$, $\sigma'(\vec{z}_{m+n}) := \vec{y}_{m+n}$ and $\sigma'(u) := \sigma^*(\sigma(u))$ for every variable u not in $\vec{y}_{m+n}, \vec{z}_{m+n}$. By Proposition 2.9, for every formula $\varphi \in Fm$, all the variables occurring in $\sigma'(\varphi)$ are among $\vec{x}_{m+n}, \vec{y}_{m+n}$.

Define $\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n}) := \sigma'(\mathfrak{T})$. By structurality, applying σ' to both sides of (i) yields (MP).

To prove (R), recall (Definition 2.60) that $\sigma_{m,n}$ is the substitution that maps $\vec{y}_{m,n}$ to $\vec{x}_{m,n}$ and leaves all the other variables untouched.

CLAIM 3.31.1. *$\sigma_{m,n} \circ \sigma' = \sigma^* \circ \sigma$.*

PROOF. By Proposition 2.8, it suffices to prove that $(\sigma_{m,n} \circ \sigma')(u) = (\sigma^* \circ \sigma)(u)$ for every $u \in \mathsf{Var}$. For the variables in $\vec{y}_{m+n}$, we have:

$$\sigma_{m,n}(\sigma'(\vec{y}_{m+n})) = \sigma_{m,n}(\vec{x}_{m+n}) = \vec{x}_{m+n} = \sigma^*(\vec{x}_{m+n}) = \sigma^*(\sigma(\vec{y}_{m+n})).$$

For the variables in $\vec{z}_{m+n}$, we have:

$$\sigma_{m,n}(\sigma'(\vec{z}_{m+n})) = \sigma_{m,n}(\vec{y}_{m+n}) = \vec{x}_{m+n} = \sigma^*(\vec{x}_{m+n}) = \sigma^*(\sigma(\vec{z}_{m+n})).$$

Let u be a variable not occurring in $\vec{y}_{m,n}, \vec{z}_{m,n}$. Then:

$$\sigma_{m,n}(\sigma'(u)) = \sigma_{m,n}(\sigma^*(\sigma(u))) = \sigma^*(\sigma(u)).$$

$\square$

By structurality, applying σ^* to both sides of (iv) yields $\vdash_{\mathbf{G}} (\sigma^* \circ \sigma)(\mathfrak{T})$, which by the claim is equivalent to $\vdash_{\mathbf{G}} (\sigma_{m,n} \circ \sigma')(\mathfrak{T})$. And this is precisely (R) because $\sigma_{m,n}(\sigma'(\mathfrak{T})) = \sigma_{m,n}(\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n})) = \mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{x}_{m+n})$. $\blacksquare$

We can now prove that having a local CDDT is equivalent to protoalgebraicity:

THEOREM 3.32. *A Gentzen relation* $\mathbf{G}$ *has a local CDDT iff* $\mathbf{G}$ *is protoalgebraic.*

PROOF. Let $\mathsf{tr} := \mathsf{tr}(\mathbf{G})$.

($\Rightarrow$) Let $\langle\{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ be a local CDD-sequence for $\mathbf{G}$. We prove that $\mathbf{G}$ is protoalgebraic using Theorem 2.62.

Fix any $\langle m, n\rangle \in \mathsf{tr}$. By Proposition 3.27(i), there is some $i \in I_{m+n,\langle m,n\rangle,\langle m,n\rangle}$ such that $\vdash_{\mathbf{G}} \mathfrak{L}_i(\vec{x}_{m+n}, \mathfrak{x}_{m,n}, \mathfrak{x}_{m,n})$, where $\mathfrak{L}_i := \mathfrak{L}_i[m+n, \langle m, n\rangle, \langle m, n\rangle]$. For this i, we also have $\mathfrak{x}_{m,n}, \mathfrak{L}_i(\vec{x}_{m+n}, \mathfrak{x}_{m,n}, \mathfrak{y}_{m,n}) \vdash_{\mathbf{G}} \mathfrak{y}_{m,n}$ by Proposition 3.27(ii). Therefore, the set $\mathfrak{E}_{m,n}(\vec{x}_{m+n}, \vec{y}_{m+n}) := \mathfrak{L}_i(\vec{x}_{m+n}, \mathfrak{x}_{m,n}, \mathfrak{y}_{m,n})$ satisfies conditions (R) and (MP) of Theorem 2.62.

($\Leftarrow$) We build a local CDD-sequence

$$\langle\{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$$

for $\mathbf{G}$ using Theorem 3.31.

Fix any context $p \in \omega$. If $p = 0$ and $Fm_0 = \varnothing$, then $\mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r}$ holds for all $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_p$ because $\mathfrak{s} = \mathfrak{r} = \varnothing \rhd \varnothing$, so for every $\hat{m}, \hat{n} \in \mathsf{tr}$ we can let $I_{0,\hat{m},\hat{n}}$ be any singleton and $\mathfrak{L}_i[0, \hat{m}, \hat{n}] := \varnothing$ for all $i \in I_{0,\hat{m},\hat{n}}$.

Assume now that either $p > 0$ or $Fm_0 \neq \varnothing$, so that $Fm_p \neq \varnothing$ regardless of the choice of p, and fix any types $\hat{m}, \hat{n} \in \mathsf{tr}$. For every $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, by

Theorem 3.31 $\mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r}$ holds iff there are $\mathfrak{T} \in \mathcal{T}h(\mathbf{G})$ and $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ such that $\langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle$ is a $\mathbf{G}$-template. Note that, if $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr\text{-}Seq}_p$, then we may assume that $\sigma(Fm) \subseteq Fm_p$, as otherwise we can replace σ by $\sigma' \circ \sigma$, where $\sigma' \in \mathrm{End}(\boldsymbol{Fm})$ is defined by setting $\sigma'(\vec{x}_p) := \vec{x}_p$ and letting $\sigma'(z)$ be any element of $Fm_p \neq \varnothing$ for all variables $z \notin \vec{x}_p$. (By Proposition 2.9, $\sigma'(Fm) \subseteq Fm_p$).

Let $t := p + \Sigma(\hat{m}) + \Sigma(\hat{n})$ and let $I_{p,\hat{m},\hat{n}}$ be the set of all $\mathbf{G}$-templates $\langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle$ such that: $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr\text{-}Seq}_p$, $\mathsf{tp}(\mathfrak{s}) = \hat{m}$, $\mathsf{tp}(\mathfrak{r}) = \hat{n}$ and $\sigma(Fm) \subseteq Fm_p$.

Claim 3.32.1. $I_{p,\hat{m},\hat{n}} \neq \varnothing$.

Proof. Since $Fm_p \neq \varnothing$, let $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr\text{-}Seq}_p$ be such that $\mathsf{tp}(\mathfrak{s}) = \hat{m}$ and $\mathsf{tp}(\mathfrak{r}) = \hat{n}$. Let $\mathfrak{T} := \mathrm{Cn}_{\mathbf{G}}(\mathfrak{y}_{\hat{m}}, \mathfrak{z}_{\hat{n}})$, and define $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ by setting $\sigma(\mathfrak{y}_{\hat{m}}) := \mathfrak{s}$, $\sigma(\mathfrak{z}_{\hat{n}}) := \mathfrak{r}$, and letting $\sigma(u)$ be any element of Fm_p for every variable u not occurring in $\mathfrak{y}_{\hat{m}}, \mathfrak{z}_{\hat{n}}$. By Proposition 2.9, $\sigma(Fm) \subseteq Fm_p$. Finally, let $\mathfrak{P} := \sigma(\mathfrak{T}) \subseteq \mathsf{tr\text{-}Seq}_p$. Then, $\langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle \in I_{p,\hat{m},\hat{n}}$. □

For every $i \in I_{p,\hat{m},\hat{n}}$, say $i = \langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle$, let $\sigma_i \in \mathrm{End}(\boldsymbol{Fm})$ be given by:

- $\sigma_i(\mathfrak{y}_{\hat{m}}) := \langle x_{p+1}, \ldots, x_{p+\Sigma(\hat{m})} \rangle_{\hat{m}} \in \hat{m}\text{-}\mathsf{Seq}_t$.
- $\sigma_i(\mathfrak{z}_{\hat{n}}) := \langle x_{p+\Sigma(\hat{m})+1}, \ldots, x_t \rangle_{\hat{n}} \in \hat{n}\text{-}\mathsf{Seq}_t$.
- $\sigma_i(u) := \sigma(u) \in Fm_p$ for all variables u not occurring in $\mathfrak{y}_{\hat{m}}$ or $\mathfrak{z}_{\hat{n}}$.

By Proposition 2.9, $\sigma_i(Fm) \subseteq Fm_t$. Let $\mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_t) := \sigma_i(\mathfrak{T}) \subseteq \mathsf{tr\text{-}Seq}_t$.

Claim 3.32.2. *For every $\langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle \in I_{p,\hat{m},\hat{n}}$, the following hold:*

(i) $\mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) = \sigma(\mathfrak{T})$.
(ii) $\sigma_i(\mathfrak{y}_{\hat{m}}), \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_t) \vdash_{\mathbf{G}} \sigma_i(\mathfrak{z}_{\hat{n}})$.

Proof.

(i) Let $\sigma' \in \mathrm{End}(\boldsymbol{Fm})$ be such that:
- $\sigma'(\vec{x}_p) := \vec{x}_p$.
- $\sigma'(\langle x_{p+1}, \ldots, x_{p+\Sigma(\hat{m})} \rangle_{\hat{m}}) := \mathfrak{s}$.
- $\sigma'(\langle x_{p+\Sigma(\hat{m})+1}, \ldots, x_t \rangle_{\hat{n}}) := \mathfrak{r}$.

It suffices to prove that $\sigma' \circ \sigma_i = \sigma$. For the variables in $\mathfrak{y}_{\hat{m}}$, we have:

$$(\sigma' \circ \sigma_i)(\mathfrak{y}_{\hat{m}}) = \sigma'(\langle x_{p+1}, \ldots, x_{p+\Sigma(\hat{m})} \rangle_{\hat{m}}) = \mathfrak{s} = \sigma(\mathfrak{y}_{\hat{m}}).$$

For the variables in $\mathfrak{z}_{\hat{m}}$, we have:

$$(\sigma' \circ \sigma_i)(\mathfrak{z}_{\hat{m}}) = \sigma'(\langle x_{p+\Sigma(\hat{m})+1}, \ldots, x_t \rangle_{\hat{n}}) = \mathfrak{r} = \sigma(\mathfrak{z}_{\hat{m}}).$$

Let u be a variable not in $\mathfrak{y}_{\hat{m}}$ or $\mathfrak{z}_{\hat{n}}$. Then,

$$(\sigma' \circ \sigma_i)(u) = \sigma'(\sigma(u)) = \sigma(u),$$

where the last equality is due to the fact that $\sigma(u) \in Fm_p$. Therefore, by Proposition 2.8 we have $\sigma' \circ \sigma_i = \sigma$.

(ii) By structurality, applying σ_i to both sides of $\mathfrak{y}_{\hat{m}}, \mathfrak{T} \vdash_{\mathsf{G}} \mathfrak{z}_{\hat{n}}$, which holds because $\langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle$ is a G-template, yields (ii). □

Let us check that $\langle \{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega \rangle$ is indeed a local CDD-sequence for G.

Fix any context $p \in \omega$, any types $\hat{m}, \hat{n} \in \mathsf{tr}$ and any $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_p$ with $\mathsf{tp}(\mathfrak{s}) = \hat{m}$ and $\mathsf{tp}(\mathfrak{r}) = \hat{n}$. The case corresponding to $p = 0$ with $Fm_0 = \varnothing$ has already been checked, and thus, as we explained above, we may assume that each σ given by Theorem 3.31 satisfies $\sigma(Fm) \subseteq Fm_p$. We must prove:

$$\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r} \iff \mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \text{ for some } i \in I_{p,\hat{m},\hat{n}}.$$

Assume $\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r}$. By Theorem 3.31 and the construction of $I_{p,\hat{m},\hat{n}}$, there is a G-template $i \in I_{p,\hat{m},\hat{n}}$ of the form $i = \langle \mathfrak{P}, \mathfrak{s}, \mathfrak{r}, \mathfrak{T}, \sigma \rangle$. Therefore, $\mathfrak{P} \vdash_{\mathsf{G}} \sigma(\mathfrak{T})$, whence $\mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ by Claim 3.32.2(i).

Conversely, let $i \in I_{p,\hat{m},\hat{n}}$ be such that $\mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$. Choose any substitution $\sigma' \in \mathrm{End}(\boldsymbol{Fm})$ satisfying:

- $\sigma'(\vec{x}_p) := \vec{x}_p$.
- $\sigma'(\langle x_{p+1}, \ldots, x_{p+\Sigma(\hat{m})} \rangle_{\hat{m}}) := \mathfrak{s}$.
- $\sigma'(\langle x_{p+\Sigma(\hat{m})+1}, \ldots, x_t \rangle_{\hat{n}}) := \mathfrak{r}$.

Applying σ' to both sides of Claim 3.32.2(ii) yields $\mathfrak{P}, \mathfrak{s} \vdash_{\mathsf{G}} \mathfrak{r}$ by cut. ■

In [**28**, Exmpl. 3.2, 4.9], Raftery shows two examples of sentential logics having a local CDDT but lacking the CDDT. Thus, by Proposition 3.7 we can now conclude that having the CDDT is a strictly stronger condition than being protoalgebraic.

According to Raftery (cf. [**28**, p. 294]), for sentential logics the proof of Theorem 3.6(i) can be easily adapted to obtain the case $k = 0$ of [**28**, Thm. 4.5], which is a result for local CDD-sequences analogous to Theorem 3.6(i). We can actually prove much more, namely the local version of the *whole* Theorem 3.6, and for arbitrary Gentzen relations. For this, we first need to obtain the local version of Theorem 3.5:

THEOREM 3.33. *Let $\langle \{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega \rangle$ be a local CDD-sequence for a Gentzen relation G with trace tr. Then, for every $k \in \omega$, every context $p \in \omega$ and every types $\hat{m}_1, \ldots, \hat{m}_{k+1}, \hat{n} \in \mathsf{tr}$ there is a family of sets*

$$\{\mathfrak{L}^*_{k,j}[p, \hat{m}_1, \ldots, \hat{m}_{k+1}, \hat{n}](\vec{x}_t) : j \in J_{p,\hat{m}_1,\ldots,\hat{m}_{k+1},\hat{n}}\},$$

where $J_{p,\hat{m}_1,\dots,\hat{m}_{k+1},\hat{n}} \neq \varnothing$ *and* $t := p + \Sigma(\hat{m}_1) + \cdots + \Sigma(\hat{m}_{k+1}) + \Sigma(\hat{n})$*, such that, for every sequents* $\mathfrak{P} \cup \{\mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \mathfrak{r}\} \subseteq \mathsf{tr\text{-}Seq}_p$ *with* $\mathsf{tp}(\mathfrak{r}) = \hat{n}$ *and* $\mathsf{tp}(\mathfrak{s}_i) = \hat{m}_i$ *for* $i = 1, \dots, k+1$*, the following holds:*

$$\mathfrak{P}, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1} \vdash_{\mathbf{G}} \mathfrak{r} \iff \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{L}^*_{k,j}[p, \hat{m}_1, \dots, \hat{m}_{k+1}, \hat{n}](\vec{x}_p, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \mathfrak{r}) \quad \textit{for some } j \in J_{p,\hat{m}_1,\dots,\hat{m}_{k+1},\hat{n}}.$$

Proof. By induction on k. If $k = 0$ the statement of the theorem is just the defining property of any local CDD-sequence for $\mathbf{G}$, so we can take $J_{p,\hat{m}_1,\hat{n}} := I_{p,\hat{m}_1,\hat{n}}$ and $\mathfrak{L}^*_{0,j}[p, \hat{m}_1, \hat{n}] := \mathfrak{L}_j[p, \hat{m}_1, \hat{n}]$ for all $j \in J_{p,\hat{m}_1,\hat{n}}$.

Assuming that the theorem holds for k, let us consider the case of $k + 1$.

By IH, there is a family

$$\{\mathfrak{L}^*_{k,j}[p, \hat{m}_2, \dots, \hat{m}_{k+2}, \hat{n}](\vec{x}_{t-\Sigma(\hat{m}_1)}) : j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}}\},$$

with $J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}} \neq \varnothing$, such that $\mathfrak{P}, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+2} \vdash_{\mathbf{G}} \mathfrak{r}$ holds iff

$$\mathfrak{P}, \mathfrak{s}_1 \vdash_{\mathbf{G}} \mathfrak{L}^*_{k,j}[p, \hat{m}_2, \dots, \hat{m}_{k+2}, \hat{n}](\vec{x}_p, \mathfrak{s}_2, \dots, \mathfrak{s}_{k+2}, \mathfrak{r})$$

is the case for some $j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}}$.

First, let us build the index set $J_{p,\hat{m}_1,\dots,\hat{m}_{k+2},\hat{n}}$. For every $j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}}$, let us abbreviate $\mathfrak{L}^*_{k,j}[p, \hat{m}_2, \dots, \hat{m}_{k+2}, \hat{n}]$ by $\mathfrak{L}^*_{k,j}$, and let A_j be the set of all triples $\langle j, \mathfrak{t}_j, i_{\mathfrak{t}_j}\rangle$ such that $\mathfrak{t}_j \in \mathfrak{L}^*_{k,j}$ and $i_{\mathfrak{t}_j} \in I_{p,\hat{m}_1,\mathsf{tp}(\mathfrak{t}_j)}$. Let $\mathcal{C}_j$ be the collection of all subsets $B_j \subseteq A_j$ such that for every $\mathfrak{t}_j \in \mathfrak{L}^*_{k,j}$ there is at least one triple in B_j with $\mathfrak{t}_j$ as its second component. Clearly, $\varnothing \in \mathcal{C}_j$ iff $\mathfrak{L}^*_{k,j} = \varnothing$. Finally, define:

$$J_{p,\hat{m}_1,\dots,\hat{m}_{k+2},\hat{n}} := \bigcup\{\{j\} \times \mathcal{C}_j : j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}}\}.$$

Note that $J_{p,\hat{m}_1,\dots,\hat{m}_{k+2},\hat{n}} \neq \varnothing$ because by IH there is some $j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}}$, and thus $\langle j, A_j\rangle \in J_{p,\hat{m}_1,\dots,\hat{m}_{k+2},\hat{n}}$ because $A_j \in \mathcal{C}_j$.

For every $j' \in J_{p,\hat{m}_1,\dots,\hat{m}_{k+2},\hat{n}}$, say $j' = \langle j, B_j\rangle$, let

$$\mathfrak{L}^*_{k+1,j'} := \bigcup_{\langle j,\mathfrak{t}_j,i_{\mathfrak{t}_j}\rangle \in B_j} \mathfrak{L}_{i_{\mathfrak{t}_j}}[p, \hat{m}_1, \mathsf{tp}(\mathfrak{t}_j)](\vec{x}_{p+\Sigma(\hat{m}_1)}, \mathfrak{t}_j(\vec{x}_p, x_{p+\Sigma(\hat{m}_1)+1}, \dots, x_t)).$$

By the IH, $\mathfrak{P}, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+2} \vdash_{\mathbf{G}} \mathfrak{r}$ is equivalent to the statement

$$(\exists j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}})\, \mathfrak{P}, \mathfrak{s}_1 \vdash_{\mathbf{G}} \mathfrak{L}^*_{k,j}(\vec{x}_p, \mathfrak{s}_2, \dots, \mathfrak{s}_{k+2}, \mathfrak{r}),$$

which is in turn equivalent to

$$(\exists j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}})(\forall \mathfrak{t}_j \in \mathfrak{L}^*_{k,j})\, \mathfrak{P}, \mathfrak{s}_1 \vdash_{\mathbf{G}} \mathfrak{t}_j(\vec{x}_p, \mathfrak{s}_2, \dots, \mathfrak{s}_{k+2}, \mathfrak{r}). \tag{3.13}$$

And by the local CDDT, (3.13) is the case iff:

$$(\exists j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}})(\forall \mathfrak{t}_j \in \mathfrak{L}^*_{k,j})(\exists i_{\mathfrak{t}_j} \in I_{p,\hat{m}_1,\mathsf{tp}(\mathfrak{t}_j)})$$
$$\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{L}_{i_{\mathfrak{t}_j}}[p, \hat{m}_1, \mathsf{tp}(\mathfrak{t}_j)](\vec{x}_p, \mathfrak{s}_1, \mathfrak{t}_j(\vec{x}_p, \mathfrak{s}_2, \dots, \mathfrak{s}_{k+2}, \mathfrak{r})). \tag{3.14}$$

We need to prove that (3.14) is equivalent to:

$$(\exists j' \in J_{p,\hat{m}_1,\dots,\hat{m}_{k+2},\hat{n}})\, \mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{L}^*_{k+1,j'}(\vec{x}_p, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+2}, \mathfrak{r}). \tag{3.15}$$

Assume (3.14). Let C be the collection of all triples $\langle j, \mathfrak{t}_j, i_{\mathfrak{t}_j}\rangle$, with $\mathfrak{t}_j \in \mathfrak{L}^*_{k,j}$ and $i_{\mathfrak{t}_j} \in I_{p,\hat{m}_1,\mathsf{tp}(\mathfrak{t}_j)}$, such that

$$\mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{L}_{i_{\mathfrak{t}_j}}[p, \hat{m}_1, \mathsf{tp}(\mathfrak{t}_j)](\vec{x}_p, \mathfrak{s}_1, \mathfrak{t}_j(\vec{x}_p, \mathfrak{s}_2, \dots, \mathfrak{s}_{k+2}, \mathfrak{r})).$$

By (3.14), we must have $C \in \mathcal{C}_j$. So, by the way C has been defined, it is clear that we can take $j' := \langle j, C\rangle$ to make (3.15) hold.

Conversely, assume (3.15). Then, $j' = \langle j, B_j\rangle$ for some $j \in J_{p,\hat{m}_2,\dots,\hat{m}_{k+2},\hat{n}}$ and some $B_j \in \mathcal{C}_j$. If $B_j = \varnothing$, then we must have $\mathfrak{L}^*_{k,j} = \varnothing$, whence (3.14) holds trivially. Assume now that $B_j \neq \varnothing$ and fix any $\mathfrak{t}_j \in \mathfrak{L}^*_{k,j}$. Since $B_j \in \mathcal{C}_j$, there is some $i_{\mathfrak{t}_j} \in I_{p,\hat{m}_1,\mathsf{tp}(\mathfrak{t}_j)}$ such that $\langle j, \mathfrak{t}_j, i_{\mathfrak{t}_j}\rangle \in B_j$, and thus

$$\mathfrak{L}_{i_{\mathfrak{t}_j}}[p, \hat{m}_1, \mathsf{tp}(\mathfrak{t}_j)](\vec{x}_{p+\Sigma(\hat{m}_1)}, \mathfrak{t}_j(\vec{x}_p, x_{p+\Sigma(\hat{m}_1)+1}, \dots, x_t)) \subseteq \mathfrak{L}^*_{k+1,j'},$$

whence

$$\mathfrak{L}_{i_{\mathfrak{t}_j}}[p, \hat{m}_1, \mathsf{tp}(\mathfrak{t}_j)](\vec{x}_p, \mathfrak{s}_1, \mathfrak{t}_j(\vec{x}_p, \mathfrak{s}_2, \dots, \mathfrak{s}_{k+2}, \mathfrak{r})) \subseteq \mathfrak{L}^*_{k+1,j'}(\vec{x}_p, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+2}, \mathfrak{r}).$$

Therefore, (3.14) holds. ■

Now we are ready to prove the local version of Theorem 3.6:

THEOREM 3.34. *Let $\langle\{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ be a local CDD-sequence for a Gentzen relation G with trace tr, and let the family*

$$\{\mathfrak{L}^*_{k,j}[p, \hat{m}_1, \dots, \hat{m}_{k+1}, \hat{n}] : j \in J_{p,\hat{m}_1,\dots,\hat{m}_{k+1},\hat{n}}\}$$

be as in Theorem 3.33, for all $k, p \in \omega$ and all types $\hat{m}_1, \dots, \hat{m}_{k+1}, \hat{n} \in \mathsf{tr}$. Let $\boldsymbol{A}$ be an algebra finitely generated by some elements $g_1, \dots, g_p \in A$, let $\mathfrak{a}_1, \dots, \mathfrak{a}_{k+1} \in \mathsf{tr}\text{-}\mathsf{Seq}_p(A)$ and let $\hat{m}_i := \mathsf{tp}(\mathfrak{a}_i)$ for $i = 1, \dots, k+1$. Then:

(i) *For any G-filter F of $\boldsymbol{A}$ and any $\mathfrak{b} \in \mathsf{tr}\text{-}\mathsf{Seq}_p(A)$, we have:*

$$\mathfrak{b} \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathsf{G}}(F, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}) \iff \mathfrak{L}^*_{k,j}[\hat{n}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F \quad \text{for some } j \in J_{p,\hat{m}_1,\dots,\hat{m}_{k+1},\hat{n}},$$

where $\hat{n} := \mathsf{tp}(\mathfrak{b})$ and $\mathfrak{L}^*_{k,j}[\hat{n}] := \mathfrak{L}^*_{k,j}[p, \hat{m}_1, \dots, \hat{m}_{k+1}, \hat{n}]$.

(ii) *If G is finitary, then for each $\mathfrak{b} \in \mathsf{tr}\text{-}\mathsf{Seq}_p(A)$ there is a family*

$$\{\mathfrak{L}_j : j \in J_{p,\hat{m}_1,\dots,\hat{m}_{k+1},\mathsf{tp}(\mathfrak{b})}\},$$

where each $\mathfrak{L}_j$ is a finite subset of $\mathfrak{L}^*_{k,j}[p, \hat{m}_1, \dots, \hat{m}_{k+1}, \mathsf{tp}(\mathfrak{b})]$, such that, for any G-filter F of $\boldsymbol{A}$, we have:

$$\mathfrak{b} \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathsf{G}}(F, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}) \iff \mathfrak{L}^{\boldsymbol{A}}_j(\vec{g}_p, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F \quad \text{for some } j \in J_{p,\hat{m}_1,\dots,\hat{m}_{k+1},\mathsf{tp}(\mathfrak{b})}.$$

PROOF. For $i = 1, \dots, k+1$, let $\mathfrak{s}_i \in \mathsf{tr\text{-}Seq}_p$ be such that $\mathfrak{a}_i = \mathfrak{s}_i^{\boldsymbol{A}}(\vec{g}_p)$, by Corollary 2.11. Also, for every $\mathfrak{b} \in \mathsf{tr\text{-}Seq}(A)$ let $\hat{n}_{\mathfrak{b}} := \mathsf{tp}(\mathfrak{b})$.

As p and $\hat{m}_1, \dots, \hat{m}_{k+1}$ will remain fixed throughout this proof, to improve readability let us denote, for every $\mathfrak{b} \in \mathsf{tr\text{-}Seq}(A)$, $J_{p,\hat{m}_1,\dots,\hat{m}_{k+1},\hat{n}_{\mathfrak{b}}}$ by $J_{\hat{n}_{\mathfrak{b}}}$ and, for every $j \in J_{\hat{n}_{\mathfrak{b}}}$, the set $\mathfrak{L}^*_{k,j}[p, \hat{m}_1, \dots, \hat{m}_{k+1}, \hat{n}_{\mathfrak{b}}]$ by $\mathfrak{L}^*_{k,j}[\hat{n}_{\mathfrak{b}}]$.

Note that, by Theorem 3.33, for every $\mathfrak{r} \in \mathsf{tr\text{-}Seq}_p$ we have

$$\mathfrak{L}^*_{k,j}[\hat{n}_{\mathfrak{r}}](\vec{x}_p, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \mathfrak{r}), \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1} \vdash_{\mathsf{G}} \mathfrak{r} \tag{3.16}$$

for all $j \in J_{\hat{n}_{\mathfrak{r}}}$, where $\hat{n}_{\mathfrak{r}} := \mathsf{tp}(\mathfrak{r})$.

(i) Let $G := \{\mathfrak{b} \in \mathsf{tr\text{-}Seq}(A) : (\exists j \in J_{\hat{n}_{\mathfrak{r}}})\ \mathfrak{L}^*_{k,j}[\hat{n}_{\mathfrak{b}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F\}$. We shall prove that $G = \mathrm{Fg}^{\boldsymbol{A}}_{\mathsf{G}}(F, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1})$, whence (i) clearly follows.

CLAIM 3.34.1. *$\mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{F\cup\{\mathfrak{a}_1,\dots,\mathfrak{a}_{k+1}\}} \subseteq \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{G}$.*

PROOF. Let $H \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^{F\cup\{\mathfrak{a}_1,\dots,\mathfrak{a}_{k+1}\}}$. Let $\mathfrak{b} \in G$ and, by Corollary 2.11, let $\mathfrak{r} \in \mathsf{tr\text{-}Seq}_p$ be such that $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p)$. Then, by the definition of G there is some $j \in J_{\hat{n}_{\mathfrak{b}}}$ for which $\mathfrak{L}^*_{k,j}[\hat{n}_{\mathfrak{b}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F \subseteq H$, so, since $H \supseteq \{\mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}\}$ and H is a G-filter, (3.16) implies $\mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p) \in H$, i.e., $\mathfrak{b} \in H$. □

CLAIM 3.34.2. *$F \subseteq G$.*

PROOF. Let $\mathfrak{b} \in F$, and, by Corollary 2.11, let $\mathfrak{r} \in \mathsf{tr\text{-}Seq}_p$ be such that $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p)$. Since $\mathfrak{r}, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1} \vdash_{\mathsf{G}} \mathfrak{r}$, by Theorem 3.33 there is some $j \in J_{\hat{n}_{\mathfrak{r}}}$ such that

$$\mathfrak{r} \vdash_{\mathsf{G}} \mathfrak{L}^*_{k,j}[\hat{n}_{\mathfrak{r}}](\vec{x}_p, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \mathfrak{r}),$$

whence $\mathfrak{L}^*_{k,j}[\hat{n}_{\mathfrak{b}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F$ because F is a G filter of $\boldsymbol{A}$. Therefore, $\mathfrak{b} \in G$. □

CLAIM 3.34.3. *$\{\mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}\} \subseteq G$.*

PROOF. For every $i = 1, \dots, k+1$ we have $\mathfrak{s}_1, \dots, \mathfrak{s}_{k+1} \vdash_{\mathsf{G}} \mathfrak{s}_i$, so by Theorem 3.33 there is some $j_i \in J_{\hat{m}_i}$ such that

$$\vdash_{\mathsf{G}} \mathfrak{L}^*_{k,j_i}[\hat{m}_i](\vec{x}_p, \mathfrak{s}_1, \dots, \mathfrak{s}_{k+1}, \mathfrak{s}_i),$$

whence $\mathfrak{L}^*_{k,j_i}[\hat{m}_i]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \dots, \mathfrak{a}_{k+1}, \mathfrak{a}_i) \subseteq F$ because F is a G-filter of $\boldsymbol{A}$, and thus $\mathfrak{a}_i \in G$. □

CLAIM 3.34.4. *G is a G-filter of $\boldsymbol{A}$.*

PROOF. Assume $\mathfrak{P} \vdash_{\mathsf{G}} \mathfrak{q}$ holds for some $\mathfrak{P} \cup \{\mathfrak{q}\} \subseteq \mathsf{tr\text{-}Seq}$, and let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ be such that $h(\mathfrak{P}) \subseteq G$. We need to prove that $h(\mathfrak{q}) \in G$.

By Lemma 2.15, we may assume, without loss of generality, that the variables in Var_x do not occur in $\mathfrak{P} \cup \{\mathfrak{q}\}$ and, moreover, that $h(\vec{x}_p) = \vec{g}_p$. Note that this implies $h(\mathfrak{s}_i) = \mathfrak{a}_i$ for all $1 \leq i \leq k+1$.

For every variable $u \notin \mathsf{Var}_x$ we have $h(u) \in A$, so by Corollary 2.12 there is some $\eta_u \in Fm_p$ such that $h(u) = \eta_u^{\boldsymbol{A}}(\vec{g}_p) = h(\eta_u)$. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be any substitution mapping u to η_u for every variable $u \notin \mathsf{Var}_x$.

The same induction carried on in the proof of Theorem 3.6 shows that $h(\sigma(\varphi)) = h(\varphi)$ for every formula φ in which none of the variables in Var_x occurs. In particular:

(a) $h(\sigma(\mathfrak{p})) = h(\mathfrak{p}) \in G$ for all $\mathfrak{p} \in \mathfrak{P}$.
(b) $h(\sigma(\mathfrak{q})) = h(\mathfrak{q})$.

For every $\mathfrak{p} \in \mathfrak{P}$ we have $h(\mathfrak{p}) \in G$, so there is some $j_{\mathfrak{p}} \in J_{\hat{n}_{\mathfrak{p}}}$ for which

$$\mathfrak{L}^*_{k,j_{\mathfrak{p}}}[\hat{n}_{\mathfrak{p}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, h(\mathfrak{p})) \subseteq F. \tag{3.17}$$

And since $\sigma(\mathfrak{p}) \in \mathsf{tr\text{-}Seq}_p$, (3.16) implies:

$$\mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{L}^*_{k,j_{\mathfrak{p}}}[\hat{n}_{\mathfrak{p}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{p})) \vdash_{\mathsf{G}} \sigma(\mathfrak{P}). \tag{3.18}$$

Applying cut to (3.18) and $\sigma(\mathfrak{P}) \vdash_{\mathsf{G}} \sigma(\mathfrak{q})$, which holds by structurality, we obtain:

$$\mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{L}^*_{k,j_{\mathfrak{p}}}[\hat{n}_{\mathfrak{p}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{p})) \vdash_{\mathsf{G}} \sigma(\mathfrak{q}).$$

So by Theorem 3.33 there is some $j' \in J_{\hat{n}_{\mathfrak{q}}}$ for which:

$$\bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{L}^*_{k,j_{\mathfrak{p}}}[\hat{n}_{\mathfrak{p}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{p})) \vdash_{\mathsf{G}} \mathfrak{L}^*_{k,j'}[\hat{n}_{\mathfrak{q}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{q})). \tag{3.19}$$

From (a) and (3.17) we deduce that, for every $\mathfrak{p} \in \mathfrak{P}$,

$$h(\mathfrak{L}^*_{k,j_{\mathfrak{p}}}[\hat{n}_{\mathfrak{p}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{p}))) = \mathfrak{L}^*_{k,j_{\mathfrak{p}}}[\hat{n}_{\mathfrak{p}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, h(\mathfrak{p})) \subseteq F,$$

so, since $F \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})$, (3.19) and (b) imply:

$$F \supseteq h(\mathfrak{L}^*_{k,j'}[\hat{n}_{\mathfrak{q}}](\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \sigma(\mathfrak{q}))) = \mathfrak{L}^*_{k,j'}[\hat{n}_{\mathfrak{q}}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, h(\mathfrak{q})).$$

Therefore, $h(\mathfrak{q}) \in G$. □

From the previous claims it follows that $G = \mathrm{Fg}^{\boldsymbol{A}}_{\mathsf{G}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1})$, so we are done.

(ii) Let $H := \mathrm{Fg}^{\boldsymbol{A}}_{\mathbf{G}}(F, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1})$, $\hat{n} := \mathsf{tp}(\mathfrak{b})$ and, by Corollary 2.11, let $\mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$ be such that $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p)$.

If $\mathbf{G}$ is finitary, (3.16) implies that for every $j \in J_{p,\hat{m}_1,\ldots,\hat{m}_{k+1},\hat{n}}$ there is a finite $\mathfrak{L}_j \subseteq \mathfrak{L}^*_{k,j}[p, \hat{m}_1, \ldots, \hat{m}_{k+1}, \hat{n}]$ such that

$$\mathfrak{L}_j(\vec{x}_p, \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1}, \mathfrak{r}), \mathfrak{s}_1, \ldots, \mathfrak{s}_{k+1} \vdash_{\mathbf{G}} \mathfrak{r}. \tag{3.20}$$

Assume there is some $j \in J_{p,\hat{m}_1,\ldots,\hat{m}_{k+1},\hat{n}}$ for which

$$\mathfrak{L}^{\boldsymbol{A}}_j(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F.$$

Then,

$$\mathfrak{L}^{\boldsymbol{A}}_j(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \cup \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}\} \subseteq H,$$

so, since $H \in \mathcal{F}i_{\mathbf{G}}(\boldsymbol{A})$, (3.20) implies $\mathfrak{b} = \mathfrak{r}^{\boldsymbol{A}}(\vec{g}_p) \in H$.

Conversely, assume $\mathfrak{b} \in H$. By (i), there is some $j \in J_{p,\hat{m}_1,\ldots,\hat{m}_{k+1},\hat{n}}$ such that

$$\mathfrak{L}^*_{k,j}[p, \hat{m}_1, \ldots, \hat{m}_{k+1}, \hat{n}]^{\boldsymbol{A}}(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F,$$

so in particular $\mathfrak{L}^{\boldsymbol{A}}_j(\vec{g}_p, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k+1}, \mathfrak{b}) \subseteq F$. ∎

3.3.1. Axiomatic extensions. In this subsection we generalize [**28**, Thm. 4.2] to prove that [local] CDD-sequences are invariant under axiomatic extensions, i.e., that every [local] CDD-sequence for a Gentzen relation $\mathbf{G}$ is also a [local] CDD-sequence for every axiomatic extension of $\mathbf{G}$.

Definition 3.35. Let $\mathbf{G} := \langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ be a Gentzen relation with trace tr. An *extension* of $\mathbf{G}$ is a Gentzen relation $\mathbf{G}' := \langle \mathcal{L}, \vdash_{\mathbf{G}'} \rangle$ with trace tr satisfying $\vdash_{\mathbf{G}} \subseteq \vdash_{\mathbf{G}'}$. If, moreover, there is a set of sequents $\mathfrak{A} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, closed under substitutions, such that, for every $\mathfrak{P} \cup \{\mathfrak{s}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$, we have

$$\mathfrak{P} \vdash_{\mathbf{G}'} \mathfrak{s} \iff \mathfrak{A}, \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{s},$$

then $\mathbf{G}'$ is an *axiomatic extension* of $\mathbf{G}$.

Theorem 3.36. *Let $\mathbf{G}$ be a Gentzen relation with trace tr. Any local CDD-sequence for $\mathbf{G}$ is a local CDD-sequence for every axiomatic extension of $\mathbf{G}$.*

Proof. Let $\langle \{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega \rangle$ be a local CDD-sequence for $\mathbf{G}$. Let $\mathbf{G}'$ be an axiomatic extension of $\mathbf{G}$, and let $\mathfrak{A}$ be as in Definition 3.35. Fix any context $p \in \omega$, any types $\hat{m}, \hat{n} \in \mathsf{tr}$ and any sequents $\mathfrak{P} \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathsf{tr}\text{-}\mathsf{Seq}_p$ with $\mathsf{tp}(\mathfrak{s}) = \hat{m}$ and $\mathsf{tp}(\mathfrak{r}) = \hat{n}$.

Suppose there is some $i \in I_{p,\hat{m},\hat{n}}$ such that $\mathfrak{P} \vdash_{\mathbf{G}'} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$. By Proposition 3.27(ii), $\mathfrak{s}, \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{r}$, so $\mathfrak{s}, \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}'} \mathfrak{r}$. Therefore, we get $\mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}'} \mathfrak{r}$ by cut.

Conversely, suppose $\mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}'} \mathfrak{r}$. Since $\mathbf{G}'$ is an axiomatic extension of $\mathbf{G}$, we have:

$$\mathfrak{A}, \mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r}. \tag{3.21}$$

Assume first that $p = 0$ and that there are no constant symbols in the algebraic language. Then, $\mathfrak{s} = \mathfrak{r} = \varnothing \rhd \varnothing$, and thus $\mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r}$. Hence, there is some $i \in I_{p,\hat{m},\hat{n}}$ such that $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$, so $\mathfrak{P} \vdash_{\mathbf{G}'} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$.

Assume now that either $p > 0$ or there is some constant symbol c in the algebraic language. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be a substitution such that $\sigma(\vec{x}_p) := \vec{x}_p$ and that maps $\mathsf{Var} \setminus \{x_1, \ldots, x_p\}$ onto $\{x_1, \ldots, x_p\}$, if $p > 0$, and onto $\{c\}$ if $p = 0$. By structurality, applying σ to both sides of (3.21) yields $\mathfrak{A}^*, \mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r}$, where $\mathfrak{A}^* := \sigma(\mathfrak{A})$. Note that $\mathfrak{A}^* \subseteq \mathfrak{A} \cap \mathsf{tr}\text{-}\mathsf{Seq}_p$ because $\mathfrak{A}$ is closed under substitutions, so there exists some $i \in I_{p,\hat{m},\hat{n}}$ such that $\mathfrak{A}^*, \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$, whence $\mathfrak{P} \vdash_{\mathbf{G}'} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$.

Conclusion: $\langle\{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ is a local CDD-sequence for $\mathbf{G}'$. ■

Note that a CDD-sequence $\langle\{\mathfrak{D}[p, \hat{m}, \hat{n}] : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ can be seen as a local CDD-sequence $\langle\{\{\mathfrak{D}[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ in which every $I_{p,\hat{m},\hat{n}}$ is a singleton. Therefore, by Theorem 3.36 CDD-sequences also remain CDD-sequences in axiomatic extensions.

3.3.2. Directed local CDDTs. To finish the chapter we present two special cases of the local CDDT: the directed local CDDT and the M-directed local CDDT. Both notions are introduced by Raftery in [**28**, §§6-7], where he provides characterizations of both of them in terms of well-known lattice-theoretic properties of the lattices of theories. We generalize these results to Gentzen relations.

Definition 3.37. Let $\mathbf{G}$ be a Gentzen relation with trace tr. A local CDD-sequence $\langle\{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ for $\mathbf{G}$ is said to be *directed* if, for every context $p \in \omega$, all types $\hat{m}, \hat{n} \in \mathsf{tr}$, all $i, j \in I_{p,\hat{m},\hat{n}}$ and all $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}_p$ with $\mathsf{tp}(\mathfrak{s}) = \hat{m}$ and $\mathsf{tp}(\mathfrak{r}) = \hat{n}$, there is some $k \in I_{p,\hat{m},\hat{n}}$ such that

$$\mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$$

and

$$\mathfrak{L}_j[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}).$$

In this situation, $\mathbf{G}$ is said to have a *directed local CDDT*.

We are going to prove that a local CDD-sequence for a finitary Gentzen relation is directed iff its lattice of theories is distributive.

LEMMA 3.38. *Let* $\mathbf{G}$ *be a finitary Gentzen relation with trace* tr. *Then, the compact* $\mathbf{G}$*-theories form a join-semilattice with a minimum element generated, as an algebra, by* $\{\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) : \mathfrak{s} \in \mathsf{tr}\}$ *if* $\varnothing \notin \mathcal{T}h(\mathbf{G})$, *and by* $\{\varnothing\} \cup \{\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) : \mathfrak{s} \in \mathsf{tr}\}$ *if* $\varnothing \in \mathcal{T}h(\mathbf{G})$.

PROOF. Let K be the collection of all compact $\mathbf{G}$-theories. By Proposition 2.33, Proposition 1.73 and Proposition 2.34, K is the universe of a join-semilattice $\mathbf{K}$ that has a minimum element.

By Proposition 2.33, $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) \in K$ for every $\mathfrak{s} \in \mathsf{tr}$, and $\varnothing \in K$ if $\varnothing \in \mathcal{T}h(\mathbf{G})$.

Conversely, let $\mathfrak{T} \in K$ be a compact theory, with $\mathfrak{T} \neq \varnothing$. By Proposition 2.33, there are $\mathfrak{s}_1, \ldots, \mathfrak{s}_n \in \mathsf{tr}\text{-}\mathsf{Seq}$, $n \in \omega$, such that:

$$\mathfrak{T} = \mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}_1, \ldots, \mathfrak{s}_n) = \bigvee_{1 \leq i \leq n} \mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}_i) \in \mathrm{Sg}^{\mathbf{K}}(\{\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) : \mathfrak{s} \in \mathsf{tr}\}).$$

■

THEOREM 3.39. *A local CDD-sequence for a finitary Gentzen relation* $\mathbf{G}$ *is directed iff the lattice of* $\mathbf{G}$*-theories is distributive.*

PROOF. Let $\langle\{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ be a local CDD-sequence for $\mathbf{G}$, where $\mathsf{tr} := \mathsf{tr}(\mathbf{G})$, and let K be the collection of all compact $\mathbf{G}$-theories, which, by Proposition 2.33, are exactly the finitely generated ones.

($\Rightarrow$) Suppose that $\langle\{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ is directed.

By Lemma 3.38, K is the universe of a join-semilattice $\mathbf{K}$ that has a minimum element and is generated by $\{\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) : \mathfrak{s} \in \mathsf{tr}\}$, if $\varnothing \notin \mathcal{T}h(\mathbf{G})$, and by $\{\varnothing\} \cup \{\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) : \mathfrak{s} \in \mathsf{tr}\}$ if $\varnothing \in \mathcal{T}h(\mathbf{G})$.

By Lemma 1.80, Proposition 2.33 and Proposition 2.34, the lattice $\mathcal{T}h(\mathbf{G})$ is distributive iff the join-semilattice $\mathbf{K}$ is distributive. So, by Lemma 1.79, $\mathcal{T}h(\mathbf{G})$ is distributive iff the following two conditions hold:

(a) $D(\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}), \mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}))$ holds in $\mathbf{K}$ for all $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$.

(b) If $\varnothing \in \mathcal{T}h(\mathbf{G})$, then $D(\mathfrak{T}, \varnothing)$ and $D(\varnothing, \mathfrak{T})$ hold in $\mathbf{K}$ for all $\mathfrak{T} \in K$.

To prove (a), let $\mathfrak{s}, \mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}$ and $\mathfrak{T}, \mathfrak{T}' \in K$ be such that:

(i) $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}) \subseteq \mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) \vee \mathfrak{T}$.

(ii) $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}) \subseteq \mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) \vee \mathfrak{T}'$.

We need to find a $\mathfrak{T}^* \in K$ such that $\mathfrak{T}^* \subseteq \mathfrak{T} \cap \mathfrak{T}'$ and $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}) \subseteq \mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) \vee \mathfrak{T}^*$. Let $\mathfrak{P} \cup \mathfrak{P}' \subseteq \mathsf{tr}\text{-}\mathsf{Seq}$ be finite and such that $\mathfrak{T} = \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P})$ and $\mathfrak{T}' = \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P}')$. By (i) and (ii), we have $\mathfrak{P}, \mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r}$ and $\mathfrak{P}', \mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r}$, respectively.

Let $p \in \omega$ be such that $\mathfrak{P} \cup \mathfrak{P}' \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \textsf{tr-Seq}_p$, $\hat{m} := \textsf{tp}(\mathfrak{s})$ and $\hat{n} := \textsf{tp}(\mathfrak{r})$. By the local CDDT, there are $i, j \in I_{p,\hat{m},\hat{n}}$ such that $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ and $\mathfrak{P}' \vdash_{\mathbf{G}} \mathfrak{L}_j[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$. So, since the local CDD-sequence is directed, there is some $k \in I_{p,\hat{m},\hat{n}}$ such that $\mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ and $\mathfrak{P}' \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$. By Proposition 3.27(ii), we have $\mathfrak{s}, \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{r}$, so by finitarity there is a finite $\mathfrak{P}^* \subseteq \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ such that $\mathfrak{s}, \mathfrak{P}^* \vdash_{\mathbf{G}} \mathfrak{r}$. Therefore, we can take $\mathfrak{T}^* := \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P}^*) \in K$.

Finally, to prove (b) let $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4 \in K$ be such that $\varnothing \subseteq \mathfrak{T} \vee \mathfrak{T}_1$, $\varnothing \subseteq \mathfrak{T} \vee \mathfrak{T}_2$, $\mathfrak{T} \subseteq \varnothing \vee \mathfrak{T}_3 = \mathfrak{T}_3$ and $\mathfrak{T} \subseteq \varnothing \vee \mathfrak{T}_4 = \mathfrak{T}_4$. Clearly, we can take $\mathfrak{T}_1^* := \mathfrak{T}_1 \cap \mathfrak{T}_2 \in K$ and $\mathfrak{T}_2^* := \mathfrak{T}_3 \cap \mathfrak{T}_4 \in K$ to see that $D(\mathfrak{T}, \varnothing)$ and $D(\varnothing, \mathfrak{T})$ hold in K, respectively.

Conclusion: the lattice $\mathcal{T}h(\mathbf{G})$ is distributive.

($\Leftarrow$) Conversely, suppose that the lattice $\mathcal{T}h(\mathbf{G})$ is distributive. By Lemma 1.80, Proposition 2.33 and Proposition 2.34, K is the universe of a distributive join-semilattice $\mathbf{K}$.

Fix a context $p \in \omega$, any types $\hat{m}, \hat{n} \in \textsf{tr}$, any $i, j \in I_{p,\hat{m},\hat{n}}$ and any sequents $\mathfrak{s}, \mathfrak{r} \in \textsf{tr-Seq}_p$ with $\textsf{tp}(\mathfrak{s}) = \hat{m}$ and $\textsf{tp}(\mathfrak{r}) = \hat{n}$. By Proposition 3.27(ii), we have $\mathfrak{s}, \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{r}$ and $\mathfrak{s}, \mathfrak{L}_j[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{r}$, so by finitarity there are finite sets $\mathfrak{P}_i \subseteq \mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ and $\mathfrak{P}_j \subseteq \mathfrak{L}_j[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ such that $\mathfrak{s}, \mathfrak{P}_i \vdash_{\mathbf{G}} \mathfrak{r}$ and $\mathfrak{s}, \mathfrak{P}_j \vdash_{\mathbf{G}} \mathfrak{r}$. Then, $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}) \subseteq \mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) \vee \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P}_i)$ and $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}) \subseteq \mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) \vee \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P}_j)$, so, since $\mathbf{K}$ is distributive, there is a compact theory $\mathfrak{T} \in K$ such that $\mathfrak{T} \subseteq \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P}_i) \cap \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P}_j)$ and $\mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}) \subseteq \mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}) \vee \mathfrak{T}$. Let $\mathfrak{P} \subseteq \textsf{tr-Seq}$ be finite and such that $\mathfrak{T} = \mathrm{Cn}_{\mathbf{G}}(\mathfrak{P})$. Thus, we have:

$$\mathfrak{P}_i \vdash_{\mathbf{G}} \mathfrak{P} \quad \text{and} \quad \mathfrak{P}_j \vdash_{\mathbf{G}} \mathfrak{P} \quad \text{and} \quad \mathfrak{s}, \mathfrak{P} \vdash_{\mathbf{G}} \mathfrak{r}. \tag{3.22}$$

Assume first that $p = 0$ and that there are no constant symbols in the algebraic language. Then, $\mathfrak{s} = \mathfrak{r} = \varnothing \rhd \varnothing$, and thus $\mathfrak{s} \vdash_{\mathbf{G}} \mathfrak{r}$. Hence, there is some $k \in I_{p,\hat{m},\hat{n}}$ such that $\vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$, so $\mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ and $\mathfrak{L}_j[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$.

Assume now that either $p > 0$ or there is some constant symbol c in the algebraic language. Let $\sigma \in \mathrm{End}(\boldsymbol{Fm})$ be a substitution such that $\sigma(\vec{x}_p) := \vec{x}_p$ and that maps $\textsf{Var} \setminus \{x_1, \ldots, x_p\}$ onto $\{x_1, \ldots, x_p\}$, if $p > 0$, and onto $\{c\}$ if $p = 0$. By structurality, applying σ to (3.22) yields:

$$\mathfrak{P}_i \vdash_{\mathbf{G}} \sigma(\mathfrak{P}) \quad \text{and} \quad \mathfrak{P}_j \vdash_{\mathbf{G}} \sigma(\mathfrak{P}) \quad \text{and} \quad \mathfrak{s}, \sigma(\mathfrak{P}) \vdash_{\mathbf{G}} \mathfrak{r}.$$

Since $\sigma(\mathfrak{P}) \cup \{\mathfrak{s}, \mathfrak{r}\} \subseteq \textsf{tr-Seq}_p$, by the local CDDT there is some $k \in I_{p,\hat{m},\hat{n}}$ such that $\sigma(\mathfrak{P}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$, whence $\mathfrak{L}_i[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ and $\mathfrak{L}_j[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p, \hat{m}, \hat{n}](\vec{x}_p, \mathfrak{s}, \mathfrak{r})$ follow by cut and monotonicity.

Conclusion: $\langle\{\{\mathfrak{L}_i[p, \hat{m}, \hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m}, \hat{n} \in \textsf{tr}\} : p \in \omega\rangle$ is directed. ■

Finally, we turn our attention to another special case of the local CDDT, very similar to the directed version.

DEFINITION 3.40. Let $\mathbf{G}$ be a Gentzen relation with trace tr. A local CDD-sequence $\langle\{\{\mathfrak{L}_i[p,\hat{m},\hat{n}] : i \in I_{p,\hat{m},\hat{n}}\} : \hat{m},\hat{n} \in \mathsf{tr}\} : p \in \omega\rangle$ for $\mathbf{G}$ is said to be *M-directed* if, for every context $p \in \omega$, all traces $\hat{m},\hat{n} \in \mathsf{tr}$, all $i,j \in I_{p,\hat{m},\hat{n}}$ and all $\mathfrak{s},\mathfrak{r} \in \mathsf{tr}\text{-}\mathsf{Seq}_p$ with $\mathsf{tp}(\mathfrak{s}) = \hat{m}$ and $\mathsf{tp}(\mathfrak{r}) = \hat{n}$, there is some $k \in I_{p,\hat{m},\hat{n}}$ such that

$$\mathfrak{L}_i[p,\hat{m},\hat{n}](\vec{x}_p,\mathfrak{s},\mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p,\hat{m},\hat{n}](\vec{x}_p,\mathfrak{s},\mathfrak{r})$$

and

$$\mathfrak{s},\mathfrak{L}_j[p,\hat{m},\hat{n}](\vec{x}_p,\mathfrak{s},\mathfrak{r}) \vdash_{\mathbf{G}} \mathfrak{L}_k[p,\hat{m},\hat{n}](\vec{x}_p,\mathfrak{s},\mathfrak{r}).$$

In this situation, $\mathbf{G}$ is said to have an *M-directed local CDDT*.

For the M-directed local CDDT, a result analogous to Theorem 3.39 holds, with modularity in place of distributivity:

THEOREM 3.41. *A local CDD-sequence for a finitary Gentzen relation $\mathbf{G}$ is M-directed iff the lattice of $\mathbf{G}$-theories is modular.*

INDICATION FOR THE PROOF. With some minor, obvious changes, the proof is almost identical to the one of Theorem 3.39, using modularity, $M(\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}),\mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}))$ and M-directedness instead of distributivity, $D(\mathrm{Cn}_{\mathbf{G}}(\mathfrak{s}),\mathrm{Cn}_{\mathbf{G}}(\mathfrak{r}))$ and directedness, respectively. ∎

CHAPTER 4

Conclusion and Further Research

We have generalized many results from abstract algebraic logic to the context of Gentzen relations. Particularly, we have presented and discussed multiple characterizations of protoalgebraicity, given a corrected version of Raftery's [**27**, Thm. 13.4] and proved a new syntactic one that simplifies the *modus ponens* condition.

After that, we have studied the CDDT and its local variant in the framework of Gentzen relations, generalizing the bridge theorem connecting the CDDT with having ESPRC using two strategies: the first, based on Blok and Pigozzi's work in [**6**], clearly shows the connection between a CDD-sequence and the equations that semi-define the principal relative congruences, while the second generalizes the one from Raftery's [**28**] and has a much more algebraic flavour.

We now present some indications for future work that would be a natural continuation of this thesis.

4.1. Protoalgebraic multi-dimensional Gentzen relations

In [**18**, Thm. 2.17], Gil and Rebagliato prove a syntactic characterization of protoalgebraic multi-dimensional Gentzen relations similar to Theorem 2.56, which we have adapted from Pałasińska's [**23**, Thm. 5.12].

We have argued that Theorem 2.62 is more suitable when working with Gentzen relations than Theorem 2.56, mainly due to the simplification of the *modus ponens* condition, and thus a natural continuation of Subsection 2.7.2 would be to generalize Theorem 2.62 to multi-dimensional Gentzen relations. No difficulty, other than an increase in the complexity of the notation, should arise.

4.2. Filter distributivity and modularity

Recall (cf. Theorem 3.39 and Theorem 3.41) that a local CDD-sequence for a finitary Gentzen relation $\mathbf{G}$ is directed (respectively, M-directed) iff the algebraic lattice of $\mathbf{G}$-theories is distributive (respectively, modular).

In [**28**, §§6-7], Raftery goes one step further and proves that every finitary sentential logic $\mathbf{S}$ with a directed (respectively, M-directed) local CDDT is *filter* *distributive* (respectively, *filter modular*), i.e., the lattice $\mathcal{F}i_{\mathsf{S}}(\boldsymbol{A})$ is distributive (respectively, modular) for all algebras $\boldsymbol{A}$. His proofs make use of the following result, due to Czelakowski:

THEOREM 4.1 (cf. [**11**, Thm. 1.7.1]). *Let* S *be a protoalgebraic finitary sentential logic. Then, every universal sentence of first-order lattice theory that is true in the lattice* $\mathcal{T}h(\mathsf{S})$ *is also true in the lattice* $\mathcal{F}i_{\mathsf{S}}(\boldsymbol{A})$ *for all algebras* $\boldsymbol{A}$.

Czelakowski's proof of Theorem 4.1 is quite involved, but we have not found any obstacle that would make it impossible to generalize it to Gentzen relations and then use it in conjunction with Theorem 3.39 and Theorem 3.41 to prove that every finitary Gentzen relation G with a directed (respectively, M-directed) local CDDT is filter distributive (respectively, filter modular). We have not done so here because we were more focused on the bridge between the CDDT and ESPRC than on the directed variants of the local CDDT.

4.3. A family of CDDTs

The CDDT was introduced by Raftery in [**28**] as a generalization of the DDT in order to extend some of its desirable features to logics lacking a DDT.

In the literature, one can find several variants of the DDT for sentential logics; among them, the following are probably the most well known:

- The *local* DDT (LDDT).
- The *parametrised* DDT (PDDT).
- The *parametrised local* DDT (PLDDT).

The reader in need of their definitions is referred to [**14**, pp. 173-4, 334-6].

Bridge theorems for each of these versions of the DDT are known to hold in the case of sentential logics:[1]

THEOREM 4.2 (cf. [**6**, Thm. 5.5]). *Let* S *be a finitary elementarily algebraizable sentential logic, and* K *a quasivariety such that* $\mathsf{S} \cong \mathsf{EQ}(\mathsf{K})$. *Then,* S *has the DDT iff* K *has EDPRC.*

DEFINITION 4.3 (cf. [**10**, §II]). Let G be a Gentzen relation, and M a class of G-matrices. M is said to have the *filter extension property (FEP)* if, for every submatrix $\langle \boldsymbol{B}, G\rangle$ of a G-matrix $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ and every $G' \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{B})^G$, there is some $F' \in \mathcal{F}i_{\mathsf{G}}(\boldsymbol{A})^F$ such that $G' = F' \cap B$.

THEOREM 4.4 (cf. [**10**, Thm. II.1]). *Let* S *be a finitary protoalgebraic sentential logic. Then,* S *has the LDDT iff the class of all* S*-matrices has the FEP.*

[1]Note that protoalgebraicity can be viewed as an algebraic property, due to its many algebraic characterizations (cf. Subsection 2.7.3).

THEOREM 4.5 (cf. [**11**, Thm. 2.4.1]). *Let* $\mathbf{S}$ *be a protoalgebraic sentential logic. Then,* $\mathbf{S}$ *has the PDDT iff* $\mathbf{S}$ *has* 1*-FDPF (equivalently,* n*-FDPF for all* $n > 0$*).*

THEOREM 4.6 (cf. [**14**, Thm. 6.22]). *Let* $\mathbf{S}$ *be a sentential logic. Then,* $\mathbf{S}$ *has the PLDDT iff* $\mathbf{S}$ *is protoalgebraic.*

Nothing should prevent these theorems to be generalized to Gentzen relations. In fact, as we said in Section 3.2, Rebagliato and Verdú proved Theorem 4.2 for elementarily algebraizable finitary Gentzen relations in [**30**].

We have proved (Theorem 3.24) a bridge theorem analogous to Theorem 4.2 for the CDDT and having ESPRC. Also, Theorem 3.32 shows that having a local CDDT (LCDDT) is equivalent to being protoalgebraic. All these bridge theorems are summarized in Table 4.1. Note that, in the contextual case, the local version is

NON-CONTEXTUAL		CONTEXTUAL	
Logic	**Algebra**	**Logic**	**Algebra**
DDT	EDPRC	CDDT	ESPRC
LDDT	FEP	??	??
PDDT	FDPF	??	??
PLDDT	protoalgebraicity	LCDDT	protoalgebraicity

TABLE 4.1. Algebraic counterparts of the deduction-detachment theorems.

already equivalent to protoalgebraicity, whereas in the non-contextual case being protoalgebraic is equivalent to the PLDDT and there are two variants of the DDT, namely the LDDT and the PDDT, in between EDPRC and protoalgebraicity.

Therefore, a natural continuation of Chapter 3 would be to investigate the existence of variants of the CDDT equivalent to some (possibly weakened) versions of the FEP and having n-FDPF for Gentzen relations.

In Subsection 3.1.2 we presented a variant of having n-FDPF for all $n > 0$, namely having $\mathrm{FDPF}_{\mathrm{fg}}$, and we proved (Proposition 3.14) that it is implied by having the CDDT. Nevertheless, we have been unable to replicate the proofs of Theorem 4.4 and Theorem 4.5 for the contextual case and some suitable variants of the FEP and having n-FDPF, respectively, so we believe that there are no versions of the CDDT filling the gaps of Table 4.1.

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