I. INTRODUCTION

Populations of motile entities, from bacteria to synthetic microswimmers, can spontaneously self-organize into phases which are unattainable in equilibrium passive matter. Examples range from the spectacular murmuration of starlings and bacterial swarming [1–4], to collective behavior of synthetic self-propelled grains [5,6], emulsion droplets [7], and assemblies of robots [8]. Active colloids [9], in particular, spontaneously form living clusters in low-density suspensions which, unlike equilibrium clusters, continuously break up and reform [9–15]. Remarkably, when active particles align with their neighbors, they can self-organize into a polarly ordered phase, featuring long-range order in two dimensions (2D) [16–21]. Thus, motile entities, like wildebeests or sheep, can order at long distances and all run in the same direction, whereas immotile entities like spins in a 2D magnet cannot display (true) long-range order when locally coupled [22,23]. Likewise, oscillators which are localized in space, like metronomes or neurons [24,25], can synchronize only locally in 2D or three dimensions (3D) when coupled to their neighbors [26–28].

Here, we consider ensembles of chiral active particles, such as certain microorganisms [29–33], cell components [34–36], and synthetic microswimmers [37–44]. We interpret them as active oscillators and ask for their large-scale and long-time synchronization behavior: (i) In active matter, previous studies on chiral active particles have focused on pattern formation [44–49] but leave the synchronization behavior (frequency locking) of large ensembles essentially open, irrespective of previous work focusing on phase locking in monofrequent ensembles [50] and on similarities between the Kuramoto model and Vicsek models [51]. Notice that the understanding of the large-scale synchronization in generic chiral active particles with a frequency distribution is a fundamentally different problem than the emergence of long-range order in a system of identical active particles as studied, e.g., in the Vicsek and the Toner-Tu models. To see this, consider that for “passive oscillators” (Kuramoto model with local coupling), the lower critical dimension $D_c = 2$ for the emergence of synchronization but $D_c = 4$ for oscillators with a frequency distribution $D_c = 4$ [26–28]. Therefore, while we know that identical chiral active particles can synchronize over long distances in 2D [16,17,44], it is fully unclear if generic ensembles can synchronize in fewer than four dimensions. (ii) In the field of synchronization, in turn, much is known on the large-scale synchronization of oscillators which are localized in space [24,25,52] or move in a way that is unaffected by their phases [53–56]—but not for active oscillators showing a two-way coupling between phase and displacement through space.

Here, we show that generic active oscillators featuring a frequency distribution can synchronize over large distances.
linearly with system size. This chiral phase [see Fig. 1(e), orange domain, and the snapshots Figs. 1(c) and 1(d)]. For ensembles with a comparatively large frequency dispersion, the active oscillators spatially segregate according to their chirality and form rotating macroscopic clusters [see snapshots Figs. 1(f) and 1(g)] and growing linearly with system size. This chiral phase constitutes a second example where activity leads to synchronization on the macroscale. Notice that the chiral self-sorting underlying this phase allows active particles to segregate by chirality, without requiring chemical reactions [58], external flows, or environmental chirality [40].

To demonstrate the impact of the feedback between phase and displacement on the large-scale synchronization of coupled oscillators, we consider $N$ overdamped particles with 2D positions $r_\alpha$ and alignment interactions of strength $K$ in a square box of length $L$, which self-propel with a constant speed $v$ along direction $n_\alpha = (\cos \theta_\alpha, \sin \theta_\alpha)$, as described by

$$\dot{r}_\alpha = vn_\alpha, \quad (1)$$

$$\dot{\theta}_\alpha = \omega_\alpha + \frac{K}{\pi R^2} \sum_{\nu \neq \alpha} \sin(\theta_\nu - \theta_\alpha) + \sqrt{2D} \eta_\alpha. \quad (2)$$

Here, $\eta$ represents Gaussian white noise with unit variance and zero mean. The intrinsic frequencies $\omega_\alpha$ are randomly drawn from a distribution $\Delta(\omega)$. Note that, unlike more complicated models, involving, e.g., hydrodynamic interactions, excluded volume effects, or sophisticated couplings, the present model focuses on the essential ingredients required to demonstrate the generic synchronization scenario shown by active oscillators. Moreover, direct experimental realizations of the present model are also possible, e.g., using 3D-printed granular particles on vibrated plates [6,59,60]. For example, disk-shaped granulates featuring asymmetric legs [6] both swim in circles and locally align with each other [6,59]. References [35,61,62] provide other very recent setups containing the key ingredients of the present model. Since we want to understand the synchronization behavior arising from the competition between the distribution of natural frequencies and the coupling for chiral active particles, we consider only distributions with zero mean. Indeed, here we establish the role played by the frequency dispersion [through its variance $\text{Var}(\omega)$] on the collective behavior of active oscillators. The sum of Eq. (2) runs over all the neighbors of particle $\alpha$, defined by the cutoff distance $R$. For $v = 0$, our model reduces
attached to self-propelled particles for which direction of motion—i.e., a model of nonidentical oscillators—has not been explored in detail. Models considering agents whose phase, or a different internal state, affect the way they move in space appeared recently [70–74] but did not consider particles self-propelling in the direction of their phase (with a distribution of natural frequencies and local coupling), which is the key ingredient of the present model. To understand the key control parameters of the present system, we express times and lengths in units of $1/D$ and $R$, respectively, leading to the dimensionless quantities: (i) $g = K/(\pi R^2 D_r)$, the reduced coupling; (ii) $Pe = v/(D_r R)$, the rotational Péclet number; (iii) $\Omega_i = \omega_i/D_r$, the reduced rotation frequency; and (iv) $\rho_0 = NR^2/L^2$, the average number density. We denote by $\rho^{(i)} = N^{(i)}R^2/L^2$ the density of particles with natural frequency $\Omega_i$ (species $i$), where $N^{(i)}$ counts particles sharing the same natural frequency. For simplicity, we exemplify our key results for two species with equal overall density and opposite chirality. As we will show, the synchronization behavior is controlled by $\text{Var}(\Omega)$, such that this distribution is largely representative of cases of several species and continuous frequency distributions.

### II. HYDRODYNAMIC THEORY

We first derive a set of field equations, describing the collective dynamics of chiral active particles, mainly following Refs. [64,75–77]. Here, we assume that the system contains $M$ different species with frequencies $\Omega_i$ ($i = 1 \ldots M$) and describe the dynamics of species $i$ particles using the density field $\rho \equiv \rho^{(i)}(x,t)$ and the polarization density $\mathbf{w} \equiv w^{(i)}(x,t)$, where $\mathbf{w}/\rho$ is the average self-propulsion direction of particles of species $i$. The resulting equations read (see the SM for details [57]):

\begin{equation}
\dot{\rho} = -Pe \nabla \cdot \mathbf{w} \quad (3)
\end{equation}

\begin{align*}
\dot{\mathbf{w}} = -\mathbf{w} + \sum_{i=1}^{M} \frac{g \rho^{(i)}}{2} w^{(i)} + \Omega w_{\perp} - \frac{Pe}{2} \nabla \rho \\
+ \frac{Pe^2}{2b} \nabla^2 \mathbf{w} + \frac{Pe^2 \Omega}{4b} \nabla^2 w_{\perp} \\
- \frac{g^2}{b} \left( \mathbf{w} + \frac{\Omega w_{\perp}}{2} \right) \left( \sum_{i=1}^{M} w^{(i)} \right)^2 + O(\nabla \mathbf{w}^2) 
\end{align*}

\begin{equation}
(4)
\end{equation}

Here, $b = 2(4 + \Omega^2)$, $w_{\perp} = (-w_x, w_y)$, and $O(\nabla \mathbf{w}^2)$ represent terms which are both nonlinear in $\mathbf{w}$ and involve gradients, and these are not of interest for our purposes. Equation (3) simply reflects that particles on average self-propel in the direction of $\mathbf{w}$. The first term in Eq. (4) represents a decay of the polarization due to rotational diffusion of the particles, which happens in competition with the second term, creating alignment among all species; the third term represents a rotation of the average (local) polarization direction with a species-specific frequency while the fourth term expresses a statistical tendency for self-propulsion away from high-density regions; remaining terms “smear out” regions of high and low polarization and lead to saturation. For weak interactions, the system is in a disordered uniform phase, given by...
($\rho, w) = (\rho_0, 0)$, which is a solution of Eqs. (3) and (4) and represented by the blue domain in the phase diagram Fig. 1(e).

III. MUTUAL FLOCKING PHASE

To understand the onset of synchronization, we perform a linear stability analysis for a bimodal frequency distribution $\Omega_1 = -\Omega_2 := \Omega$ of the uniform disordered phase $\rho^{(1,2)} = \rho_0/2, w_1 = w_2 = 0$ [57]. It is instructive to first focus on the zero wave-number limit ($q = 0$). Here, we find that the uniform phase loses stability if $g\rho_0 > 2(1 + \Omega^2)$, suggesting the existence of a nontrivial ordered uniform phase. We indeed find an exact solution of our field equations (3) and (4) representing such a phase: a state of uniform density $\rho^{(1)} = \rho^{(2)} = \rho_0/2$, but with a finite polarization, spontaneously breaking the symmetry of the disordered phase. In this state, circle swimmers of both species cooperate and form two individual and superimposed flocks moving linearly and at a relative angle $\Delta \Phi$ to each other. We thus call it the mutual flocking phase. Defining the overall polarization $P = |w_1 + w_2|/\rho_0$, where $|w_1| = |w_2| = \rho_0\sqrt{2(1 + \Omega^2)}$, the exact expressions representing the mutual flocking phase read

$$
\begin{align*}

P &= \frac{\sqrt{2}}{g\rho_0} \sqrt{(g\rho_0)^2 + 6\Omega^2(\rho_0 - 6) + g\rho_0 + 2\Omega^2 - 4}, \\

\Delta \Phi &= -i \ln \left[ \frac{2\Omega(6 - \frac{\rho_0}{2}) + 4i\sqrt{\frac{(\rho_0)^2}{4} + 3\Omega^2 \left( \frac{\rho_0}{2} - 3 \right)}}{g\rho_0(\Omega + 2i)} \right].
\end{align*}

(5)

At low frequency, $\Omega < 1/\sqrt{2}$, this solution exists (is real and positive) if $g\rho_0 > 2(1 + \Omega^2)$, i.e., precisely when the field equations show a linear instability at $q = 0$.

Brownian dynamics simulations at high coupling and high density confirm the existence of the mutual flocking phase. As shown in Fig. 1(e), we find the mutual flocking phase essentially in the whole parameter regime, where it exists according to our field theory. Figure 2 shows in turn a close quantitative agreement between theoretical predictions [Eqs. (5)] and our simulations, both for the angle between the two flocks and for the overall polarization (see movie 1 [57]). (In our simulations, we measure the partial and overall polarization, $\tilde{P} = |P_{1,2}|$, where $P_i = \frac{2}{N} \sum \delta(n_i - n_0)$, and $P = \frac{1}{N} \sum \delta(n_i)$, respectively.) In Fig. 2(d), we show the orientational self-correlation function $C(r) = \langle n_1 \cdot n_2 \rangle$, strongly suggesting the emergence of global synchronization only for systems of oscillators whose phase directly affects their motion, in contrast to immobile oscillators [$v = 0$ in Eq. (1)] or systems of mobile oscillators which move independently of their phase (see Ref. [57]).

IV. ACTIVITY-INDUCED SYNCHRONIZATION

To understand when the disordered phase loses stability, we now explore its linear stability at finite wave number $q \neq 0$. This is equivalent to accounting for the impact of motility on the onset of synchronization, as the coefficients of all gradient terms in (3) and (4) are nonvanishing only if $\text{Pe} \neq 0$. While static oscillators need a stronger coupling to synchronize as the frequency dispersion increases [25,65], our linear stability calculation shows that coupling phase and motility generally induces an instability for any $g\rho_0 > 2$ [57], independently of $\text{Var}(\Omega)$. Activity thus induces phase ordering even in parameter regimes where a corresponding nonmotile system ($\text{Pe} = 0$) would simply show asynchronous oscillations of the individual particles. As the present instability emerges for localized perturbations, we do not expect a uniform phase but rather the emergence of localized synchronous structures. Our particle-based simulations confirm this. Above the critical value $g\rho_0 \gtrsim 2$ [Fig. 1(e)], we find large rotating clusters of opposite chirality featuring internal phase synchronization, as illustrated by the snapshots in Figs. 1(f) and 1(g) (and movie 2 [57]). From $C(r)$, we extract a characteristic length $\xi$, which, as shown in Fig. 3(a), grows linearly with $\sqrt{N}$ (at fixed $\rho_0$), meaning that clusters are macroscopic in this regime and tend to phase separate. Thus, the chiral phase represents a second example where activity induces (long-range) phase synchronization among species. As shown in Fig. 3(b), above a critical coupling strength, $\tilde{P}$ starts to increase, allowing us to locate the phase boundaries [Fig. 1(e)]. (Particle segregation, quantified by the probability $\Psi$ to find an excess of particles of one chirality, offers an alternative route to identify the emergence of the chiral phase; see Ref. [57].)
V. CONTINUOUS FREQUENCY DISTRIBUTIONS

To demonstrate the generality of our results, we now simulate active oscillators with Gaussian distribution of natural frequencies [with zero mean and variance \( \text{Var}(\Omega) \)]. As for two species, we predict the stability threshold of the disordered phase at \( g_{00} > 2 \) for three and four species using linear stability analysis (see SM [57]) and confirm with simulations that the stability threshold of the disordered state does not change as compared to the two-species case, up to numerical accuracy [Fig. 1(e)]. At \( g_{00} \geq 2 \), we find the same two phases as for two species: a mutual flocking phase, further illustrated in the SM [57], and a phase composing synchronized macroclusters scaling linearly with system size (see Fig. 1(g) and movie 3 [57]). Now, the macroclusters involve a continuous range of frequencies and feature frequency synchronization across species.

VI. PHYSICAL MECHANISM

What is the physical mechanism allowing motility to qualitatively change the synchronization of oscillators? It is well known that oscillator motion itself does not affect the absence of global synchronization in \( D < D_c = 4 \) [56,57]. Hence, compared to oscillators on dynamic networks, the crucial ingredient of active oscillators (circle swimmers, Eqs. (1) and (2) and the underdamped, velocity aligning ones [50,57]) is that their phase influences their direction of motion. It follows that two circle swimmers sharing the same phase are aligned and move together, enhancing their interaction time, which in turn enhances their alignment and fosters synchronization. This can be viewed as a positive feedback between alignment (local synchronization) and interaction time, mutually supporting each other, which is absent for oscillators on dynamic networks but plays a fundamental role for synchronization of active oscillators. The impact of such feedback on the emergence of order is even more dramatic than in linear active matter: The critical dimension for the occurrence of global phase synchronization seems to change from \( D_c = 4 \) (passive oscillators) to \( D_c = 2 \) for active oscillators.

VII. CONCLUSIONS

Chiral active particles, here interpreted as 2D active oscillators, can synchronize over very long distances, even for a purely local coupling. This contrasts the synchronization behavior of the huge class of nonactive oscillators in 2D or 3D structures [25], which can only synchronize within domains. A consequence of global synchronization in active oscillators is the emergence of the mutual flocking phase as a new active matter phase, akin to the celebrated Toner-Tu phase in linear active matter. Our results transcend a knowledge boundary at the interface of active matter physics and synchronization and could be useful, e.g., to sort active enantiomers.

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