

GLOBAL DYNAMICS OF THE REAL SECANT METHOD

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ABSTRACT. We investigate the root finding algorithm given by the secant method applied to a real polynomial p as a discrete dynamical system defined on \mathbb{R}^2 . We study the shape and distribution of the basins of attraction associated to the roots of p , and we also show the existence of other stable dynamics that might affect the efficiency of the algorithm. Finally we extend the secant map to the punctured torus \mathbb{T}_∞^2 which allow us to better understand the dynamics of the secant method near ∞ and facilitate the use of the secant map as a method to find all roots of a polynomial.

Keywords: Root finding algorithms, dynamical systems, secant method

1. INTRODUCTION

Root finding algorithms

$$(1) \quad x_{n+1} = R(x_{n-\ell}, \dots, x_n), \quad \ell \geq 0, n \geq \ell, x \in X$$

are iterative systems so that for most initial *seeds* (x_0, \dots, x_ℓ) the sequence $\{x_n\}_{n \geq 0}$ converges to a solution of a given nonlinear equation, namely $F(x) = 0$, $x \in X$. Since many real problems can be modelled in terms of nonlinear equations which do not admit explicit solutions, the applicability of those algorithms is wide over all areas like engineering, economics, sociology or biology. Through the whole paper we will assume that F is a polynomial p of degree at least 2 and hence our goal is to solve the equation $p(x) = 0$, $x \in X$ with $X = \{\mathbb{R}, \mathbb{C}\}$.

There are many natural questions about the efficiency of those algorithms. If the equation has more than one solution, as it happens in most cases, how to find different seeds converging to each of them? Is it possible to have regions of positive measure where seeds do not converge to any solution of $p(x) = 0$? What can be said about the speed (number of steps) to have a reasonable approximation of the solution (order of convergence)? To answer these questions, or more ambitious, to have a deeper understanding of those algorithms, we treat them as discrete dynamical systems (see, for instance, [HSS01]).

Roughly speaking a discrete dynamical system over Y is a map $f : Y \rightarrow Y$ and the orbits induced by this map starting at $y_0 \in Y$, that is, $\{y_n := f^n(y_0)\}_{n \geq 1}$ where

$$f^n = \underbrace{f \circ \dots \circ f}_{n \text{ times}}.$$

A main goal when studying dynamical systems is to describe the asymptotic behaviour of those orbits when y_0 runs over all Y . In particular the study of *fixed points*, i.e. y_0 in Y such that $f(y_0) = y_0$. Those points are classified according to the behaviour of the nearby seeds and play a key role on the global dynamics. In particular, a fixed point y_0 in Y is called *attracting* if there exists $\varepsilon > 0$ such that $f^n(y) \rightarrow y_0$, as $n \rightarrow \infty$, for all y in

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$D_\varepsilon(y_0) := \{y \in Y \mid d(y, y_0) < \varepsilon\}$. Accordingly, given an attracting fixed point $y_0 \in Y$, its *basin of attraction* is denoted by $A(y_0)$ and given by

$$A(y_0) = \{y \in Y \mid f^n(y) \rightarrow y_0, \text{ as } n \rightarrow \infty\},$$

which is open and nonempty by definition. The connected component of $A(y_0)$ which contains y_0 is called the *immediate basin of attraction* of y_0 and it is denoted by $A^*(y_0)$. We also consider *periodic points of (minimal) period q* , or *q -periodic points*, i.e. y_0 in Y such that there exist $q \geq 2$ satisfying $f^q(y_0) = y_0$ and $f^\ell(y_0) \neq y_0$ for all $\ell < q$. Similarly, we can define attracting q -periodic points and their basins of attraction.

Therefore a root finding algorithm (1) for the equation $p(x) = 0$, $X = \{\mathbb{R}, \mathbb{C}\}$ can be seen as a discrete dynamical system generated by the map

$$f_p : Y \rightarrow Y, \quad Y = X \times \overset{(\ell)}{\dots} \times X$$

for which the orbits $\{y_n := f_p^n(y_0)\}_{n \in \mathbb{N}}$ converge, for *most* initial conditions, to fixed points of f_p which are in correspondence to the roots of p . The most well-known and universal root finding algorithm applied to $p(z) = 0$, $z \in \mathbb{C}$ is the so called *Newton's method*

$$N_p : \mathbb{C} \mapsto \mathbb{C}, \quad N_p(z) = z - \frac{p(z)}{p'(z)}.$$

Easy computations show that if $\zeta \in \mathbb{C}$ is a simple root of p then $N_p(\zeta) = \zeta$ and $N_p'(\zeta) = 0$, so ζ is an attracting fixed point. It turns out that for *most* initial conditions $z_0 \in \mathbb{C}$ the sequence $\{z_n := N_p^n(z_0)\}_{n \geq 1}$ converges to some root of p . Certainly the dynamical system is not well defined at the critical points of p since the denominator of N_p vanishes but we go over this problem by extending the phase space from \mathbb{C} to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere. One can show by the use of the charts defined on $\hat{\mathbb{C}}$ that the Newton's map is well defined at the whole Riemann sphere, and in particular ∞ is always a repelling fixed point of N_p .

The literature on Newton's method as a root finding algorithm as well as a dynamical system is extremely large and in fact it was the starting point of *holomorphic dynamics* (see [Cay79a, Cay79b, Cay80]). For instance we refer to M. Shishikura [Shi09] who proved a remarkable theorem implying the simple connectivity of the immediate basins of attraction (see also [Prz89]), and we refer to J. Hubbard, D. Schleicher and S. Sutherland [HSS01], who provided a universal set (only depending on the degree of the polynomial) of initial conditions to find all roots of a polynomial of a given degree. As a counterpart it is known that for certain polynomials of degree larger than two there are open sets of initial conditions for which N_p does not converge to any root of p (see [McM87]) and so Newton's method for those polynomials might fail as a root finding algorithm. Finally we refer to [BFJK14] and [BFJK18] for Newton's method applied to transcendental maps.

Alternative to Newton's method, another well known root finding algorithm is the secant method, although few references can be found in the literature. The main goal of this paper is twofold. On the one hand we explore the secant method as a dynamical system (iterates of the secant map) and on the other hand we provide arguments for a better implementation of the secant method as a root finding algorithm. The *secant method* is given by the iterates of the *secant map*

$$S := S_p : \mathbb{C}^2 \mapsto \mathbb{C}^2, \quad S : \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} w \\ w - p(w) \frac{w-z}{p(w)-p(z)} \end{pmatrix}.$$

We notice that the Newton's map N_p defines a dynamical system on \mathbb{C} (that is, we have $\ell = 0$ in (1)) while the secant map S_p defines a dynamical system on \mathbb{C}^2 (that is, we have $\ell = 1$ in (1)). Of course this fact makes the general study more difficult. A first step towards the understanding of the complexity of this dynamical system is to restrict the attention to the *real version* of the secant method.

More precisely, let

$$(2) \quad p(x) = \sum_{j=0}^k a_j x^j, \quad a_k = 1, \quad a_j \in \mathbb{R},$$

with $k > 1$, be a monic polynomial having exactly n real roots denoted by $\alpha_1 < \alpha_2 < \dots < \alpha_n$ with $1 \leq n \leq k$, all simple. The (real) secant map is given by

$$(3) \quad S := S_p : \mathbb{R}^2 \mapsto \mathbb{R}^2, \quad S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ y - p(y) \frac{y-x}{p(y)-p(x)} \end{pmatrix},$$

and the orbit of the seed $(x_0, y_0) \in \mathbb{R}^2$ is given by the iterates of the map; that is, the sequence $\{(x_n, y_n) = S^n(x_0, y_0) \in \mathbb{R}^2\}_{n \in \mathbb{N}}$.

We notice that the real version of Newton's method and secant method is based on a similar idea. In Newton's method, for a given seed $x_0 \in \mathbb{R}$, the point $x_1 = N_p(x_0)$ is the intersection between the x -axis and the tangent line through the point $(x_0, p(x_0))$ while in the secant case for a given seed $(x_0, y_0) \in \mathbb{R}^2$, the point $(x_1, y_1) = S(x_0, y_0)$ is given by $x_1 = y_0$ and y_1 is the intersection between the x -axis and the secant line through the points $(x_0, p(x_0))$ and $(y_0, p(y_0))$.

Certainly, the secant method has disadvantages with respect to Newton's method, like for instance that locally, near simple roots, Newton's method has quadratic convergence while the secant method has superlinear convergence $(1 + \sqrt{5})/2$. However, if we are only interested on the real roots of p , which is very plausible in multiple applications, the secant method studied in this paper might be a more powerful tool than Newton's method. If, for instance, most of the roots of the polynomial are complex and we use the algorithm developed in [HSS01] we are making a tremendous numerical effort to compute roots which are not of our interest.

For a review on numerical analysis and root finding algorithms (local order of convergence, computing implementation, etc) we refer to [Tra64]. While this paper has been written we learned about [BF19] where the authors have studied independently the secant map as a dynamical system on \mathbb{C}^2 (see Theorem C for a further discussion). Finally in [CCTV15] several root finding algorithms, including the secant method, are applied to degree two polynomials.

Rational planar maps. There are several papers studying discrete dynamical systems on \mathbb{C}^2 or $\mathbb{C}\mathbb{P}^2$, induced by injective maps. For example there are many papers on polynomial automorphisms of \mathbb{C}^2 (see for instance [BS91a, BS91b, BS92, Duj04, DL15]), on the complex version of the (polynomial) Hénon map (see for instance [HOV94, HOV95, HPV00, BS06, FsS92, ABFPne]), or birational maps (see [CMn11, BD05, Bed03, CZ14]). In contrast the secant map defined on \mathbb{R}^2 or \mathbb{C}^2 is not an injective map, and so most of the tools used on the above papers fail in this case. Accordingly, the natural framework for studying S as a planar dynamical system is the iteration of rational-like maps on \mathbb{R}^2 (see [BGM99, BGM03, BGM05], and references therein, for a more complete discussion). We introduce here the notation we need to state our main results. Consider the family of maps (which include the secant map S)

$$(4) \quad T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} F(x, y) \\ N(x, y)/D(x, y) \end{pmatrix},$$

where F , N and D are differentiable functions. Set

$$\delta_T = \{(x, y) \in \mathbb{R}^2 \mid D(x, y) = 0\} \quad \text{and} \quad E_T = \mathbb{R}^2 \setminus \bigcup_{n \geq 0} T^{-n}(\delta_T).$$

Points in E_T define a natural subset of \mathbb{R}^2 where all iterates of T are well-defined, and so $T : E_T \rightarrow E_T$ defines a smooth dynamical system. In contrast, roughly speaking, T sends points of δ_T to infinity since the denominator D is zero, except at those points of δ_T where also the numerator N is zero and hence the value of T is uncertain.

Denote by T_2 the second component of T . We say that a point Q in δ_T is a **focal point** if T_2 evaluated at Q takes the form $0/0$ (i.e. $N(Q) = D(Q) = 0$), and there exists a smooth simple arc $\gamma := \gamma(t)$, $t \in (-\varepsilon, \varepsilon)$, with $\gamma(0) = Q$, such that $\lim_{t \rightarrow 0} T_2(\gamma)$ exists and it is finite. Moreover, the line $L_Q = \{(x, y) \in \mathbb{R}^2 \mid x = F(Q)\}$ is called the **prefocal line**. If we assume that γ passes through Q , not tangent to δ_T , with slope m (that is $\gamma'(0) = m$), then $T(\gamma)$ will be a curve passing through some point $(F(Q), y(m)) \in L_Q$ (at $t = 0$). Precisely

$$(5) \quad y(m) = \lim_{t \rightarrow 0} \frac{N(\gamma(t))}{D(\gamma(t))}.$$

In particular T , as a map from \mathbb{R}^2 to \mathbb{R}^2 , is not continuous at the focal points. See Figure 1. However the next result shows that the relation between the slope $m = \gamma'(0)$ and the point $y(m) \in L_Q$ is a continuous and one-to-one map.

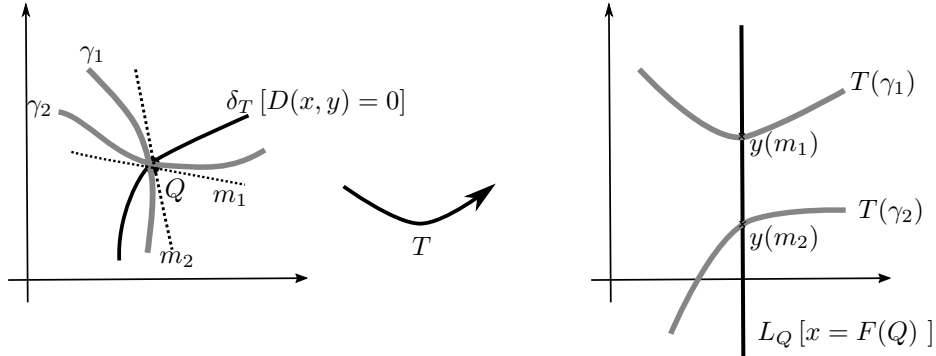


FIGURE 1. Discontinuity of T at a focal point Q .

Theorem 1.1 ([BGM99]). *Let T be the rational map described in (4). Let Q be one of its focal points and assume $N_x(Q)D_y(Q) - N_y(Q)D_x(Q) \neq 0$. Then there is a one-to-one correspondence between the slopes m of an arc γ through Q not tangent to δ_T , and the points $(F(Q), y) \in L_Q$. The correspondence writes as*

$$(6) \quad \begin{aligned} m &\mapsto (F(Q), y(m)) \quad \text{with} \quad y(m) = \frac{N_x(Q) + mN_y(Q)}{D_x(Q) + mD_y(Q)} \\ (F(Q), y) &\mapsto m(y) \quad \text{with} \quad m(y) = \frac{D_x(Q)y - N_x(Q)}{N_y(Q) - D_y(Q)y} \end{aligned}$$

We denote by \mathcal{Q}_T the set of all focal points of T given by (4). Notice that all focal points \mathcal{Q}_T belong to δ_T but they play a key role on the understanding of the global dynamics of the dynamical system generated by T on E_T . We are ready to state our main results.

Statement of the main results. Our first result, Theorem A, is about the shape and distribution of the basins of attraction of the fixed points of S . Precisely it states that any focal point belongs to the boundary of the basin of attraction of all the roots of the polynomial p (see statement (d)). An interesting corollary is that near the focal points there are initial seeds corresponding to all the basins of attraction (say, *good* initial seed). We do not ignore (see statement (c)) that those focal points are defined using the the roots of p , which we do not know! Nonetheless we believe that a deeper study (in process) in this direction might allow us to determine a (sort of) universal set of initial conditions to numerically find *all* zeroes of a polynomial of a given degree.

Theorem A. *The secant map S induces a smooth dynamical system on E_S . Moreover the following statements hold.*

- (a) *The only fixed points of S are the points $(\alpha_\ell, \alpha_\ell)$, $\ell = 1, \dots, n$, and they are all attracting.*
- (b) *Each basin of attraction $A(\alpha_\ell)$, $\ell = 1, \dots, n$, is unbounded. If $n = 1$ or $n = 2$, then $A^*(\alpha_1)$ (and $A^*(\alpha_2)$, if $n = 2$) are unbounded. If $n \geq 3$ then $A^*(\alpha_1)$ and $A^*(\alpha_n)$ are unbounded while $A^*(\alpha_\ell)$ with $2 \leq \ell \leq n - 1$ are bounded.*
- (c) *Let $Q_{i,j} := (\alpha_i, \alpha_j)$ for $i, j = 1 \dots n$, $i \neq j$. The set of focal points of the secant map is given by*

$$\mathcal{Q} := \mathcal{Q}_S = \bigcup_{\substack{i,j=1,\dots,n \\ i \neq j}} Q_{i,j}.$$

- (d) *Each focal point belongs to the common boundary of all basins of attraction, that is,*

$$Q_{i,j} \in \bigcap_{\ell=1,\dots,n} \partial A(\alpha_\ell).$$

The second and third result of this paper (Theorem B and Theorem C) deal with the existence of (unwanted) stable dynamics; that is the existence of open regions on the dynamical plane where seeds do not converge to the fixed points of S associated to the roots of the polynomial p . In other words both results bound the efficiency of the secant map as a root finding algorithm (see also [McM87]), again a dynamical result with a relevant impact on numerical computations.

Theorem B. *Let $S : E_S \rightarrow E_S$ be the dynamical system induced by the secant map defined on (3). The following statements hold.*

- (a) *S has no periodic orbits of minimal period either two or three.*
- (b) *There exists a polynomial p^* such that S_{p^*} exhibits an attracting periodic orbit of minimal period four. In particular the dynamical plane has open regions of initial conditions for which S_{p^*} does not converge to any root of the polynomial p^* .*

In contrast to Theorem B where it is shown that S (defined on E_S) has no periodic orbits of period two and three, Theorem C shows that *stable* period three cycles exist, for all polynomials p , if we extend the map to *infinity*. As we said seeds converging to this three cycle should be discarded when using S as a root finding algorithm. Here \mathbb{T}_∞^2 is a torus minus one point (see Section 3 for details).

Theorem C. *The map $\hat{S} : \mathbb{T}_\infty^2 \setminus \mathcal{Q} \mapsto \mathbb{T}_\infty^2 \setminus \mathcal{Q}$ defines a smooth extension of S . Moreover if x_0 satisfies $p'(x_0) = 0$ then the point (x_0, x_0) is periodic of minimal period three, namely $(x_0, x_0) \mapsto (x_0, \infty) \mapsto (\infty, x_0) \mapsto (x_0, x_0)$. The eigenvalues of $D\hat{S}^3(x_0, x_0)$ are 0 and 1 if the degree of the polynomial p is greater or equal than 3.*

While this paper has been written we learned that Theorem C has been proved independently in [BF19] where the authors deal with the secant map on \mathbb{C}^2 . In fact the authors determine an open region of \mathbb{C}^2 belonging to the attracting basin of the three cycle (see our Figure 8 for a numerical evidence).

We organize the paper as follows. In Section 2 we prove Theorems A and B. In Section 3 we present the extension of S over the torus \mathbb{T}_∞^2 and prove Theorem C.

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2. THE SECANT MAP ON THE REAL PLANE: PROOF OF THEOREMS A AND B

The aim of this section is to prove Theorems A and B. The proof of Theorem A splits in several technical lemmata. Remember that S is the secant map defined on (3) applied to a real polynomial of degree $k \geq 2$

$$p(x) = a_k x^k + a_{k-1} x^{k-1} + \dots + a_1 x + a_0, \quad a_k = 1,$$

having $1 \leq n \leq k$ real roots $\alpha_1 < \dots < \alpha_n$, all simple. The $n - k \geq 0$ roots are, if any, complex conjugate. Set

$$(7) \quad \begin{aligned} \delta_S^1 &= \{(x, y) \in \mathbb{R}^2 \mid p(x) = p(y) \text{ with } x \neq y\}, \quad \delta_S^2 = \{(x, x) \in \mathbb{R}^2 \mid p'(x) = 0\}, \\ \delta_S &= \delta_S^1 \cup \delta_S^2 \quad \text{and} \quad E_S = \mathbb{R}^2 \setminus \bigcup_{n \geq 1} S^{-n}(\delta_S). \end{aligned}$$

We remark that δ_S is a symmetric plane real algebraic curve intersecting the line $y = x$ precisely at points in δ_S^2 . According to previous discussion there is an implicit uncertainty on how to define the image at points where the denominator of the second component of S is zero, i.e., where $p(x) - p(y) = 0$. The following lemmas show that indeed S is well defined and smooth on $\mathbb{R}^2 \setminus \delta_S$. We define the following auxiliary polynomials

$$(8) \quad \begin{aligned} q_j(x, y) &:= x^{j-1} + x^{j-2}y + \dots + xy^{j-2} + y^{j-1}, \quad j = 1, \dots, n \\ q(x, y) &:= \sum_{j=1}^k a_j q_j(x, y). \end{aligned}$$

Lemma 2.1. *The symmetric polynomial $q(x, y)$ defined above satisfies*

$$p(x) - p(y) = (x - y)q(x, y).$$

Moreover $q(x, x) = p'(x)$ and $\frac{\partial q}{\partial x}(x, x) = \frac{\partial q}{\partial y}(x, x) = \frac{1}{2}p''(x)$.

Proof. Fix a natural $j \geq 1$. Simple computations show that

$$x^j - y^j = (x - y)(x^{j-1} + x^{j-2}y + \dots + xy^{j-2} + y^{j-1}) = (x - y) q_j(x, y).$$

Thus,

$$(9) \quad p(x) - p(y) = \sum_{j=1}^k a_j (x^j - y^j) = (x - y) \sum_{j=1}^k a_j q_j(x, y) = (x - y) q(x, y).$$

In other words the factor $(x - y)$ divides $p(x) - p(y)$ and the resultant quotient is a symmetric polynomial. Moreover, since $q_j(x, x) = jx^{j-1}$, we get

$$q(x, x) = \sum_{i=1}^k a_i q_i(x, x) = \sum_{j=1}^k a_j j x^{j-1} = p'(x).$$

Since

$$\frac{\partial q_j}{\partial x}(x, y) = (j - 1)x^{j-2} + (j - 2)x^{j-3}y + \dots + y^{j-2}$$

we have

$$\frac{\partial q_j}{\partial x}(x, x) = \frac{1}{2}j(j - 1)x^{j-2} \quad \text{and} \quad \frac{\partial q}{\partial x}(x, x) = \frac{1}{2} \sum_{j=1}^k a_j j(j - 1)x^{j-2} = \frac{1}{2}p''(x).$$

The result for $\frac{\partial q}{\partial y}(x, x)$ follows similarly. □

Lemma 2.2. *The secant map defined on (3) writes as*

$$(10) \quad S(x, y) = \left(y, \frac{yq(x, y) - p(y)}{q(x, y)} \right)$$

for all $(x, y) \in \mathbb{R}^2 \setminus \delta_S$. In particular $S : E_S \rightarrow E_S$ defines a smooth dynamical system. Moreover

$$(11) \quad DS(x, y) = \begin{pmatrix} 0 & 1 \\ A(x, y) & B(x, y) \end{pmatrix},$$

where

$$A(x, y) = \frac{p(y)}{q^2(x, y)} \frac{\partial q}{\partial x}(x, y) \quad \text{and} \quad B(x, y) = \frac{p(x)}{q^2(x, y)} \frac{\partial q}{\partial y}(x, y).$$

Proof. From the previous lemma, we have that

$$q(x, y) = \frac{p(x) - p(y)}{x - y},$$

is a symmetric polynomial not vanishing outside δ_S . So S is well-defined and smooth map on $\mathbb{R}^2 \setminus \delta_S$ defining a smooth dynamical system on E_S . The differential matrix $DS(x, y)$ is a direct computation. □

Remark 1. *Observe that for a given $(x, y) \in \mathbb{R}^2$ with $x \neq y$ the value of $q(x, y)$ is the slope of the secant line through the points $(x, p(x))$ and $(y, p(y))$. Moreover $S(x, x) = (x, N_p(x))$ where N_p is the Newton method applied to p . In particular*

$$(12) \quad DS(x, x) = \begin{pmatrix} 0 & 1 \\ \frac{p(x)p''(x)}{2[p'(x)]^2} & \frac{p(x)p''(x)}{2[p'(x)]^2} \end{pmatrix}.$$

Next lemma determines the focal points and prefocal lines for the secant map S .

Lemma 2.3. *Let S be the secant map. The following statements hold.*

- (a) Let $Q_{i,j} := (\alpha_i, \alpha_j)$ for $i, j = 1 \dots n$, $i \neq j$. The set of focal points of the secant map is given by

$$\mathcal{Q} := \mathcal{Q}_S = \bigcup_{\substack{i,j=1,\dots,n \\ i \neq j}} Q_{i,j}.$$

- (b) For a given focal point $Q_{i,j} := (\alpha_i, \alpha_j)$, $i \neq j$, the prefocal line is given by $L_{Q_{i,j}} = \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_j\}$. Hence, for a given j , the focal points $Q_{i,j}$, $i = 1, \dots, n$, $i \neq j$ share the same prefocal line.

Proof. By definition, if $Q = (x_0, y_0) \in \mathbb{R}^2$ is a focal point of S then the evaluation of $S_2(Q)$ (where S_2 denotes the second component of S) takes the form $0/0$. According to (10) there are no focal points on the line $x = y$, since otherwise $p(Q) = p'(Q) = 0$, a contradiction with the assumption that p has no multiple real roots. Therefore, again from (10), focal points should be solutions of

$$\begin{cases} yq(x, y) - p(y) = 0, \\ q(x, y) = 0, \\ x \neq y. \end{cases}$$

If $x_0 \neq y_0$ and $q(x_0, y_0) = 0$, we conclude that $p(y_0) = 0$, which in turns implies (see Lemma 2.1) $p(x_0) = 0$. Therefore we conclude that S has precisely $n(n-1)$ focal points located at $Q_{i,j} = (\alpha_i, \alpha_j)$, $i, j = 1, \dots, n$, $i \neq j$. This proves (a).

The prefocal line related to the focal point $Q_{i,j}$, is given by $x = S_1(Q_{i,j})$ being S_1 the first component of S . Thus,

$$L_{Q_{i,j}} = \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_j\},$$

and the lemma follows. In Figure 2 we sketch the distribution of focal points and prefocal lines of the secant map applied to a polynomial with three simple real roots. \square

Next lemma shows the unboundedness of the basins of attraction of the fixed points of S .

Lemma 2.4. *Let S be the secant map. The following statements hold.*

- (a) *The only fixed points of S are the points $(\alpha_\ell, \alpha_\ell)$, $\ell = 1, \dots, n$, and they are all attracting.*
 (b) *Each basin of attraction $A(\alpha_\ell)$, $\ell = 1, \dots, n$, is unbounded. If $n = 1$ or $n = 2$, then $A^*(\alpha_1)$ (and $A^*(\alpha_2)$, if $n = 2$) are unbounded. If $n \geq 3$ then $A^*(\alpha_1)$ and $A^*(\alpha_n)$ are unbounded while $A^*(\alpha_\ell)$ with $2 \leq \ell \leq n-1$ are bounded.*

Proof. From (10) we know that (x_0, y_0) is a fixed point of S if and only if $y_0 = x_0$ and $p(x_0) = 0$. Thus the fixed points of S are of the form $(\alpha_\ell, \alpha_\ell)$, where $p(\alpha_\ell) = 0$ and $p'(\alpha_\ell) \neq 0$, with $\ell = 1, \dots, n$. Moreover, from (12) we have

$$DS(\alpha_\ell, \alpha_\ell) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \ell = 1, \dots, n.$$

So the two eigenvalues of $DS(\alpha_\ell, \alpha_\ell)$ are equal to 0, proving thus that fixed point $(\alpha_\ell, \alpha_\ell)$ are all attracting. This proves statement (a).

Fix now $\ell = 1, \dots, n$. Set

$$r_H := \{(x, y) \in \mathbb{R}^2 \mid y = \alpha_\ell\} \text{ and } r_V := \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_\ell\},$$

the horizontal and vertical lines passing through the point $(\alpha_\ell, \alpha_\ell)$, respectively. It is easy to see from (10) that, on the one hand, if $x \neq \alpha_j$ then $S(x, \alpha_\ell) = (\alpha_\ell, \alpha_\ell)$; and, on the other

hand, if $y \neq \alpha_j$ then $S(\alpha_\ell, y) = (y, \alpha_\ell)$, $j = 1, \dots, n$, $j \neq \ell$. This implies that $A(\alpha_\ell)$ is unbounded since

$$R_H := r_H \setminus \bigcup_{j \neq \ell} Q_{j,\ell} \quad \text{and} \quad R_V := r_V \setminus \bigcup_{j \neq \ell} Q_{\ell,j}$$

belong to $A(\alpha_\ell)$. This proves the first assertion of statement (b).

The cases when $n = 1$ and $n = 2$ are straightforward. So we assume $n \geq 3$. Let $\ell = 2, \dots, n-1$. We claim that $A^*(\alpha_\ell) \subset R_\ell$ where R_ℓ is the rectangle with vertices at the points $\{(\alpha_{\ell-1}, \alpha_{\ell-1}), (\alpha_{\ell+1}, \alpha_{\ell-1}), (\alpha_{\ell+1}, \alpha_{\ell+1}), (\alpha_{\ell-1}, \alpha_{\ell+1})\}$. The claim follows since, according to the arguments above, $\partial R_\ell \subset A(\alpha_{\ell-1}) \cup A(\alpha_{\ell+1})$. If $\ell = 1$ or $\ell = n$, then an unbounded piece of the line r_H belongs to $A^*(\alpha_\ell)$ proving that it is unbounded. See Figure 2 where different colours illustrate points on the basins of attraction of the three fixed points. This finishes the proof of the lemma \square

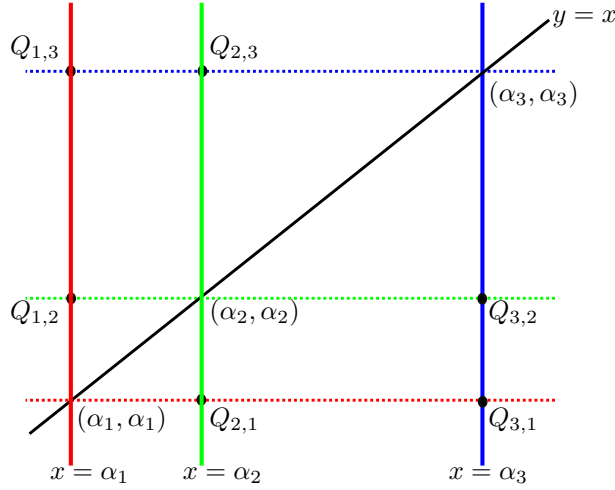


FIGURE 2. Sketch of the dynamical plane of S_p where p is a polynomial with three simple real roots $\alpha_1 < \alpha_2 < \alpha_3$. The focal points $Q_{2,1}$ and $Q_{3,1}$ share the prefocal line $x = \alpha_1$. The focal points $Q_{1,2}$ and $Q_{3,2}$ share the prefocal line $x = \alpha_2$, and finally, the focal points $Q_{1,3}$ and $Q_{2,3}$ share the prefocal line $x = \alpha_3$. Red points are seeds converging to (α_1, α_1) , green points converge to (α_2, α_2) and blue points converge to (α_3, α_3) , as shown in Lemma 2.4.

The following consequence of L'Hôpital's rule will be needed in the proof of statement (d) of Theorem A.

Lemma 2.5. *Let $f, g : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ be two smooth functions such that for some $a \in \mathbb{R}$ and $b \neq 0$ we have*

$$\lim_{t \rightarrow 0} f(t) = \lim_{t \rightarrow 0} g(t) = 0, \quad \lim_{t \rightarrow 0} f'(t) = a \quad \text{and} \quad \lim_{t \rightarrow 0} g'(t) = b.$$

Then

$$\lim_{t \rightarrow 0} \left(\frac{f(t)}{g(t)} \right)' = \frac{1}{2b} \left(f''(0) - g''(0) \frac{a}{b} \right).$$

Proof. Since

$$\lim_{t \rightarrow 0} \left(\frac{f(t)}{g(t)} \right)' = \lim_{t \rightarrow 0} \frac{f'(t)g(t) - g'(t)f(t)}{g^2(t)} = \frac{0}{0},$$

we apply Hôpital's rule to obtain

$$\lim_{t \rightarrow 0} \left(\frac{f(t)}{g(t)} \right)' = \lim_{t \rightarrow 0} \left(\frac{f''(t)}{2g'(t)} - \frac{g''(t)f(t)}{2g'(t)g(t)} \right) = \frac{1}{2b} \left(f''(0) - g''(0) \frac{a}{b} \right).$$

□

Proof of Theorem A. A major part of the proof follows from previous lemmas. From Lemma 2.2 we know that the dynamical system $S : E_S \rightarrow E_S$ is smooth. Statements (a) and (b) follows from Lemma 2.4. Statement (c) follows from Lemma 2.3. So, to finish the proof of Theorem A we deal with statement (d).

Fix $Q := Q_{i,j} = (\alpha_i, \alpha_j)$, a focal point (so $i \neq j$) and let $U := U_{i,j}$ be a sufficiently small neighborhood of Q , in particular U does not intersect other focal points. See Figure 3. We know from previous arguments (see proof of Lemma 2.4) that the segment of $x = \alpha_i$ in $U \setminus Q$ belongs to $A(\alpha_i)$ and that the segment of $y = \alpha_j$ in $U \setminus Q$ belongs to $A(\alpha_j)$, thus it follows that $Q \in \partial A(\alpha_j) \cap \partial A(\alpha_i)$. To finish the proof we need to show that $Q \in \partial A(\alpha_\ell)$ for all $\ell \neq i, j$.

Putting together (4) and (10) we have

$$(13) \quad F(x, y) = y, \quad N(x, y) = yq(x, y) - p(y), \quad D(x, y) = q(x, y).$$

Moreover from Lemma 2.1 we also have

$$(14) \quad N_x(Q) = \frac{\alpha_j p'(\alpha_i)}{\alpha_i - \alpha_j}, \quad N_y(Q) = \frac{-\alpha_i p'(\alpha_j)}{\alpha_i - \alpha_j}, \quad D_x(Q) = \frac{p'(\alpha_i)}{\alpha_i - \alpha_j} \text{ and } D_y(Q) = -\frac{p'(\alpha_j)}{\alpha_i - \alpha_j}.$$

So, we conclude

$$(15) \quad N_x(Q)D_y(Q) - N_y(Q)D_x(Q) = \frac{p'(\alpha_i)p'(\alpha_j)}{\alpha_i - \alpha_j} \neq 0.$$

Let $\gamma_m = \gamma_m(t)$, $t \in (-\varepsilon, \varepsilon)$ be a curve passing through Q (at $t = 0$) with slope $m := \gamma_m'(0)$ not tangent to δ_S . From (15) and Theorem 1.1 we know there is a one-to-one correspondence between $m \in \mathbb{R} \cup \{\infty\}$ and the points of the prefocal line $L_Q = \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_j\}$. Moreover, from (6) and (14) we conclude that the bijection is given by

$$(16) \quad y(m) = \frac{\alpha_j p'(\alpha_i) - \alpha_i p'(\alpha_j) m}{p'(\alpha_i) - p'(\alpha_j) m} \quad \text{or} \quad m(y) = \frac{p'(\alpha_i)(\alpha_j - y)}{p'(\alpha_j)(\alpha_i - y)},$$

and so if $m \neq m_\ell$, $\ell = 1, \dots, n$, with

$$(17) \quad m_\ell = \frac{p'(\alpha_i)(\alpha_j - \alpha_\ell)}{p'(\alpha_j)(\alpha_i - \alpha_\ell)},$$

the curve $S(\gamma_m)$ crosses the prefocal line L_Q through a point (α_j, y) (blue, in Figure 3) not being a focal point. Hence, shrinking U is necessary, $\gamma_m \cap U \subset A(\alpha_j)$.

According to the previous paragraph we fix in what follows $\ell \neq i, j$, and consider the family of curves depending on the parameter $\kappa \in \mathbb{R}$ given by

$$\gamma_{m_\ell, \kappa}(t) =: (x(t), y(t)) = (\alpha_i, \alpha_j) + (1, m_\ell)t + \frac{1}{2}(1, \kappa)t^2 = \left(\alpha_i + t + \frac{1}{2}t^2, \alpha_j + m_\ell t + \frac{1}{2}\kappa t^2 \right).$$

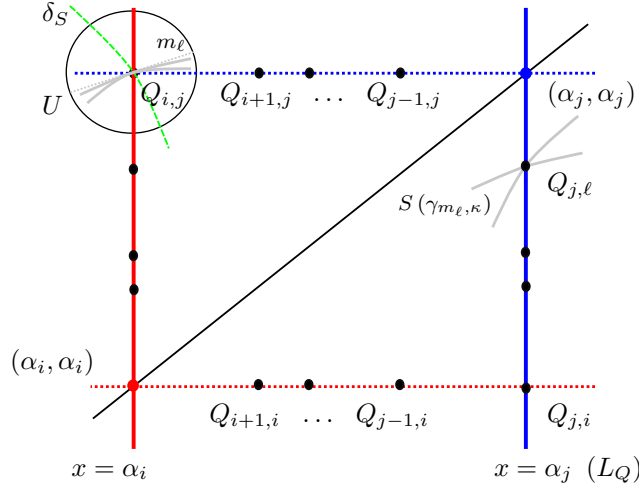


FIGURE 3. Sketch of the proof of Theorem A.

We notice that all curves in the family pass through Q with *singular slope* m_ℓ , $\ell \neq i, j$. Easily we can check that γ_{m_ℓ} , $\ell = 1, \dots, n$, is not tangent to δ_S . That is, $S(\gamma_{m_\ell, \kappa})$ cross the prefocal line L_Q at a focal point $Q_{j, \ell}$. The parameter κ is related with the curvature of $\gamma_{m_\ell, \kappa}$ at Q .

We claim that there exists a one-to-one correspondence between the parameter $\kappa \in \mathbb{R}$ and $(S \circ \gamma_{m_\ell, \kappa})'(0)$, or equivalently, we claim that choosing different values of $\kappa \in \mathbb{R}$ the curves $S(\gamma_{m_\ell, \kappa})$ pass through $Q_{j, \ell}$ with all possible slopes. See Figure 3. Assuming the claim is true this would imply that applying S once more, i.e. $S^2(\gamma_{m_\ell, \kappa})$, we will get curves passing through all points on the prefocal line $L_{Q_{j, \ell}}$, in particular passing through the point $(\alpha_\ell, \alpha_\ell) \in A(\alpha_\ell)$, and so we would conclude that $Q_{i, j} \in \partial A(\alpha_\ell)$. Since this argument works for all $\ell \neq i, j$ we would have the desired result.

Now we prove the claim. Observe that

$$(S \circ \gamma_{m_\ell, \kappa})'(0) = \frac{1}{m_\ell} \left(\frac{f(t)}{g(t)} \right)'_{|t=0}$$

where $f(t) := N(x(t), y(t))$ and $g(t) := D(x(t), y(t))$, and N and D (numerator and denominator of the second component of S) are written explicitly on (13). Since Q is a focal point we have

$$\lim_{t \rightarrow 0} \frac{f(t)}{g(t)} = \frac{0}{0}.$$

Some computations show that

$$\lim_{t \rightarrow 0} f'(t) = \frac{\alpha_j p'(\alpha_i) - \alpha_i p'(\alpha_j) m_\ell}{\alpha_i - \alpha_j} := a \quad \text{and} \quad \lim_{t \rightarrow 0} g'(t) = \frac{p'(\alpha_i) - p'(\alpha_j) m_\ell}{\alpha_i - \alpha_j} := b.$$

We claim that $b \neq 0$. Indeed, otherwise, $y(m_\ell) = \infty$ while the slope m_ℓ corresponds to focal points of the form (α_j, α_ℓ) , $\ell \neq j$. Hence, we are on the hypothesis of Lemma 2.5 to get

$$(18) \quad \lim_{\tau \rightarrow 0} \left(\frac{f}{g} \right)'(\tau) = \frac{1}{2b} \left(f''(0) - g''(0) \frac{a}{b} \right).$$

Finally some computations show that

$$\begin{aligned} f''(t) &= (x'(t), y'(t)) H(N)(x(t), y(t)) (x'(t), y'(t))^T + \nabla(N)(x(t), y(t)) (x''(t), y''(t))^T \\ g''(t) &= (x'(t), y'(t)) H(D)(x(t), y(t)) (x'(t), y'(t))^T + \nabla(D)(x(t), y(t)) (x''(t), y''(t))^T \end{aligned}$$

where H denotes the Hessian matrix.

Thus,

$$(19) \quad \begin{aligned} f''(0) &= N_{xx}(Q) + 2N_{xy}(Q)m_\ell + N_{yy}(Q)m_\ell^2 + N_x(Q) + N_y(Q)\kappa \\ g''(0) &= D_{xx}(Q) + 2D_{xy}(Q)m_\ell + D_{yy}(Q)m_\ell^2 + D_x(Q) + D_y(Q)\kappa \end{aligned}$$

Substituting on the right hand side expression of (18) we see that the κ -coefficient is given by

$$\frac{1}{2bm_\ell} \left(N_y(Q) - D_y(Q) \frac{a}{b} \right) = \frac{(\alpha_\ell - \alpha_i) p'(\alpha_j)}{2m_\ell (p'(\alpha_i) - p'(\alpha_j)m_\ell)} \neq 0,$$

as desired. Hence statement (d) follows. \square

In Figure 4(a)-(b) we illustrate Theorem A for a concrete polynomial of degree three with three real roots. Each root (α_j, α_j) , $j = 1, 2, 3$ has its own basin of attraction drawn in colour red, green and blue, respectively. Theorem A asserts that each focal point belongs to the boundary of the three basins of attraction. Fix the attention to the focal point $Q_{3,2}$ (see Figure 4(b)) and its prefocal line given by the vertical line $x = \alpha_2$ (see Figure 4(a)). The lines $y = \alpha_2$ and $x = \alpha_3$ are coloured in green and blue since they belong to $A(\alpha_2)$ and $A(\alpha_3)$, respectively. Hence $Q_{3,2} \in \partial A(\alpha_2) \cap \partial A(\alpha_3)$. We also know that the image of a curve $\gamma := \gamma_m(t)$, $t \in (-\varepsilon, \varepsilon)$ passing through $Q_{3,2}$ with slope m will be a curve through the prefocal line $x = \alpha_2$, which it is contained in $A(\alpha_2)$ except for the (focal) points $Q_{2,j}$, $j = 1, 3$. Accordingly *everything* near $Q_{3,2}$ is green except the vertical direction (basically blue and little red) and the *singular* direction $m = m_1 \approx (2\pi)/3$. The images of curves crossing $Q_{3,2}$ with slope m_1 are curves crossing the prefocal line at the focal point $Q_{2,1}$ and so we see all colours (depending on the curvature κ).

2.1. Proof of Theorem B. We show statement (a) by contradiction. We first assume the existence of a periodic orbit of minimal period 2 in E , that is $S(a, b) = (c, d)$ and $S(c, d) = (a, b)$ for some $a, b, c, d \in \mathbb{R}$ such that (a, b) and (c, d) are in E . From (10) we conclude that $c = b$ and $d = a$. So, $S(a, b) = (b, a)$ and $S(b, a) = (a, b)$ with $a \neq b$ (otherwise we would have a fixed point). From (10), if $S(a, b) = (b, a)$ we conclude that

$$a = b - p(b) \frac{b - a}{p(b) - p(a)}.$$

Notice that $p(a) \neq p(b)$ since $(a, b) \in E$. The above equation writes as

$$0 = (b - a) \left[1 - \frac{p(b)}{p(b) - p(a)} \right] = p(a) \frac{(b - a)}{p(b) - p(a)}.$$

Since $a \neq b$ the above equation concludes $p(a) = 0$. Interchanging the role of a and b we also conclude $p(b) = 0$. All together this implies $(a, b) \notin E$.

We secondly assume the existence of a periodic orbit of minimal period 3 in E , that is $S(a, b) = (c, d)$, $S(c, d) = (e, f)$ and $S(e, f) = (a, b)$ for some $a, b, c, d, e, f \in \mathbb{R}$ such that (a, b) , (c, d) and (e, f) are in E . Arguing in a similar way as before we have that $c = b$, $e = d$ and $f = a$, so we have that $S(a, b) = (b, d)$, $S(b, d) = (d, a)$ and $S(d, a) = (a, b)$. Since the minimal period of the orbit is three we conclude that the three real numbers a, b and d are different.

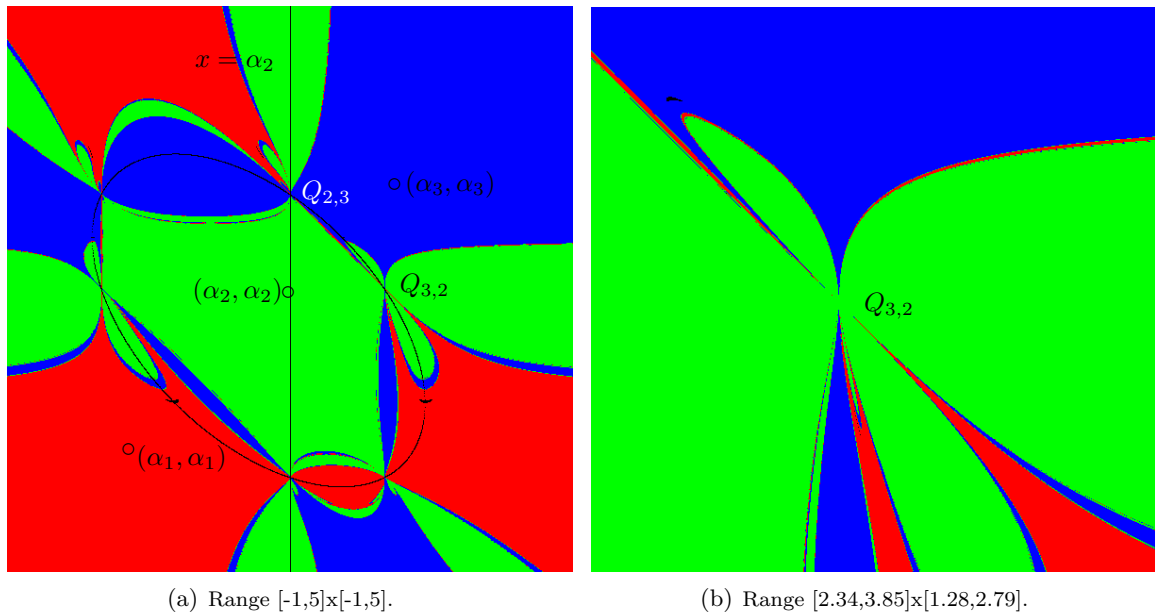


FIGURE 4. (a) The dynamical plane of S applied to the polynomial $p(x) = x(x - 2)(x - 3)$. We show in red the basin of attraction of $(\alpha_1, \alpha_1) = (0, 0)$, in green the basin of attraction of $(\alpha_2, \alpha_2) = (2, 2)$ and in blue the basin of attraction of $(\alpha_3, \alpha_3) = (3, 3)$. We show the set δ_S where the map S is not well defined. We also plot the prefocal line $x = \alpha_2$ associated to the focal points $Q_{3,2}$ and $Q_{1,2}$. (b) Zoom in the dynamical plane around the focal point $Q_{2,3}$. It is also shown some small moon-shaped black regions. We will explain their meaning in next section.

Without loss of generality we assume $a < b < d$ (otherwise we rename the points). Since the secant line through $(a, p(a))$ and $(d, p(d))$ should cut the line $y = 0$ at the point $x = b$ (observe that $S(d, a) = (a, b)$) we know that $p(a)p(d) < 0$. Assume $p(a) > 0$ and $p(d) < 0$ (the other case is similar). This force $p(b) > 0$, since the secant line passing through $(a, p(a))$ and $(b, p(b))$ should intersect the line $y = 0$ at $x = d$ (observe that $S(a, b) = (b, d)$). Accordingly the secant line through $(b, p(b))$ and $(d, p(d))$ will intersect the line $y = 0$ at a point $\eta \in (b, d)$, a contradiction with $S(b, d) = (d, a)$ and $a < b$. This finish the proof of statement (a).

Now we deal with statement (b) by showing of the existence of (attracting) periodic S -orbits of minimal period 4. We denote by a, b, c and d four real numbers such that $a < b < c < d$.

Arguing in a similar way as we did above, and after relabelling the real numbers involved in the construction of the four periodic orbit, the configuration should be as follows (see Figure 5)

$$(20) \quad \begin{aligned} & a < b < c < d \quad \text{and} \\ & S(a, b) = (b, d) \quad S(b, d) = (d, c) \quad S(d, c) = (c, a) \quad S(c, a) = (a, b). \end{aligned}$$

To simplify the construction we assume that the secant lines passing through $(a, p(a))$ and $(b, p(b))$, and through $(c, p(c))$ and $(d, p(d))$ have slope equal to -1 (observe that this is equivalent to assume $q(a, b) = q(d, c) = -1$). Under this assumption and the fact that

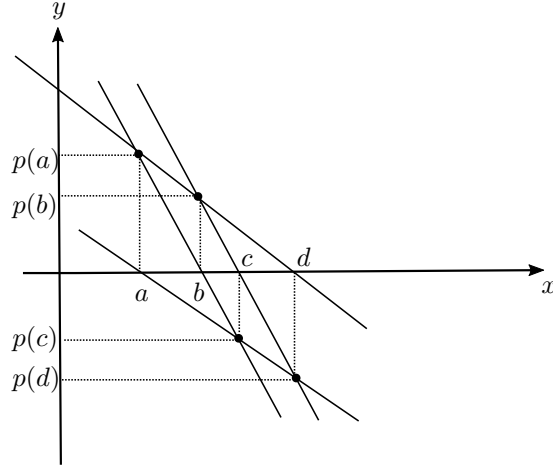


FIGURE 5. Configuration of the period 4 cycle.

$S(a, b) = (b, d)$ and $S(d, c) = (c, a)$ we get

$$(21) \quad p(a) = d - a > 0, \quad p(b) = d - b > 0, \quad p(c) = a - c < 0 \quad \text{and} \quad p(d) = a - d < 0.$$

Of course the inequalities in (21) are not enough to fulfil (20). We should further impose that the secant line passing through $(b, p(b))$ and $(d, p(d))$ crosses the line $y = 0$ at the point $x = c$, and that the line passing through $(a, p(a))$ and $(c, p(c))$ crosses the line $y = 0$ at the point $x = b$. Easy computations show that these two conditions write as

$$(22) \quad c = d - \frac{(a-d)(d-b)}{a+b-2d} \quad \text{and} \quad b = a - \frac{(d-a)(a-c)}{c+d-2a}.$$

Again doing some straightforward computations one can see that the following parameters

$$\bar{a} = 1 < \bar{b} = 2 < \bar{c} = \frac{1}{2} (3 + \sqrt{5}) \approx 2.618 < \bar{d} = \frac{1}{2} (5 + \sqrt{5}) \approx 3.618$$

determine a unique interpolating polynomial

$$p_{\bar{a}, \bar{b}, \bar{c}, \bar{d}}(x) = 2.61803 - (x - 1) - 2.61803(x - 1)(x - 2) + 2(x - 1)(x - 2)(x - 2.61803)$$

satisfying (21). Moreover, by construction, $S := S_{\bar{p}}$ has a four periodic orbit at the points

$$\{(\bar{a}, \bar{b}), (\bar{b}, \bar{d}), (\bar{d}, \bar{c}), (\bar{c}, \bar{a})\}.$$

Observe that the arguments used above implicitly provide a huge family of polynomials for which the secant method has a four periodic orbit. Our aim is to find one for which the four periodic orbit is attracting. The strategy will be to keep the parameters $\bar{a} < \bar{b} < \bar{c} < \bar{d}$ satisfying (21), but modifying the value of the derivatives of p at those points.

Since $S_{\bar{p}} : E \rightarrow E$ is smooth (see Theorem A) the local character of the four cycle is governed by the differential matrix

$$(23) \quad \Lambda = DS(\bar{a}, \bar{b}) DS(\bar{b}, \bar{d}) DS(\bar{d}, \bar{c}) DS(\bar{c}, \bar{a}),$$

where DS is the differential matrix of S given in (11). Substituting the parameter values and remembering that $q(x, y)$ is precisely the slope of the secant line through the point $(x, p(x))$

and $(y, p(y))$ we have

$$\begin{aligned} A(\bar{a}, \bar{b}) &= -\frac{1}{2} (1 + \sqrt{5}) (1 + p'(\bar{a})) & B(\bar{a}, \bar{b}) &= \frac{1}{2} (3 + \sqrt{5}) (1 + p'(\bar{b})) \\ A(\bar{b}, \bar{d}) &= \frac{1}{2} (-2 + \sqrt{5}) (3 + \sqrt{5} + 2p'(\bar{b})) & B(\bar{d}, \bar{d}) &= \frac{1}{4} (7 - 3\sqrt{5}) (3 + \sqrt{5} + 2p'(\bar{d})) \\ A(\bar{d}, \bar{c}) &= -\frac{1}{2} (1 + \sqrt{5}) (1 + p'(\bar{d})) & B(\bar{d}, \bar{c}) &= \frac{1}{2} (3 + \sqrt{5}) (1 + p'(\bar{c})) \\ A(\bar{c}, \bar{a}) &= \frac{1}{2} (-2 + \sqrt{5}) (3 + \sqrt{5} + 2p'(\bar{c})) & B(\bar{c}, \bar{a}) &= \frac{1}{4} (7 - 3\sqrt{5}) (3 + \sqrt{5} + 2p'(\bar{a})). \end{aligned}$$

At this point we are free to choose the values of $\{p'(\bar{a}), p'(\bar{b}), p'(\bar{c}), p'(\bar{d})\}$. In particular if we take $p'(\bar{a}) = p'(\bar{b}) = p'(\bar{c}) = p'(\bar{d}) = -1$ we get

$$DS(\bar{a}, \bar{b}) = DS(\bar{d}, \bar{c}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad DS(\bar{b}, \bar{d}) = DS(\bar{c}, \bar{a}) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} (3 - \sqrt{5}) & \sqrt{5} - 2 \end{pmatrix},$$

and hence the matrix

$$\Lambda = \begin{pmatrix} \frac{1}{4} (3 - \sqrt{5})^2 & \frac{1}{2} (5\sqrt{5} - 11) \\ 0 & 0 \end{pmatrix}$$

has eigenvalues given by 0 and $\frac{1}{4} (3 - \sqrt{5})^2 \approx 0.14589803 < 1$; both of modulus less than 1. Therefore using Hermite's interpolation with data

$$\begin{aligned} p(\bar{a}) &= \bar{d} - \bar{a}, & p(\bar{b}) &= \bar{d} - \bar{b}, & p(\bar{c}) &= \bar{a} - \bar{c}, & p(\bar{d}) &= \bar{a} - \bar{d}, \\ p'(\bar{a}) &= p'(\bar{b}) = p'(\bar{c}) = p'(\bar{d}) &= -1, \end{aligned}$$

we obtain the degree seven polynomial

$$(24) \quad \begin{aligned} p^*(x) &= 2.61803 - (x - 1) - 2.61803(x - 1)^2(x - 2)^2 \\ &\quad + 11.70820(x - 1)^2(x - 2)^2(x - 2.61803) \\ &\quad - 9.23607(x - 1)^2(x - 2)^2(x - 2.61803)^2 \\ &\quad + 7.05573(x - 1)^2(x - 2)^2(x - 2.61803)^2(x - 3.61803). \end{aligned}$$

According to the previous arguments S_{p^*} exhibits an attracting four periodic orbit. In Figure 6 we show the dynamical plane of the secant map applied to this interpolating polynomial p^* . \square

3. THE SECANT MAP ON A TORUS: PROOF OF THEOREM C

The aim of this section is to extend the secant map S to points in $\delta_S \setminus \mathcal{Q}$ by means of extending the map at infinity.

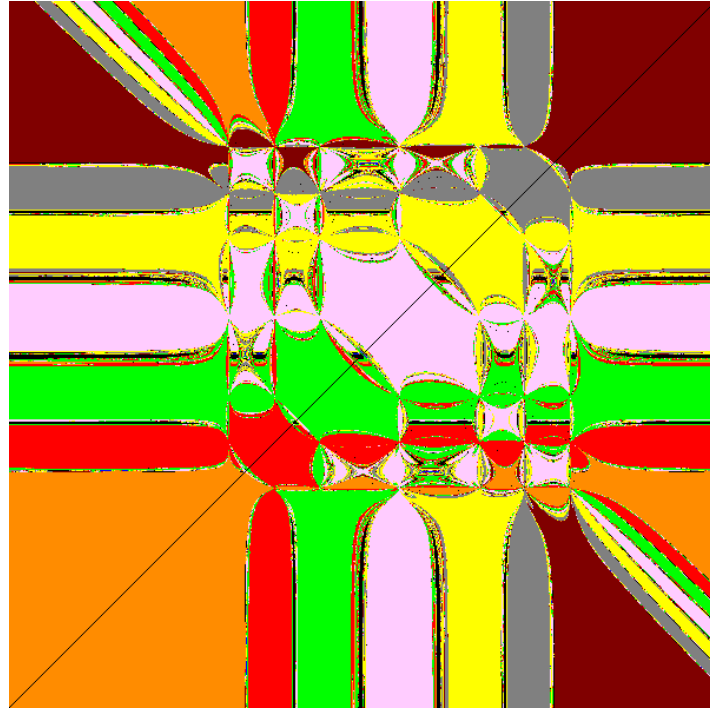
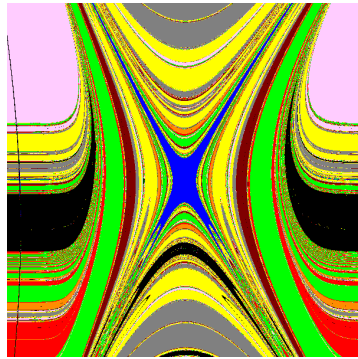
Lemma 3.1. *The following statements hold.*

- (a) $\lim_{y \rightarrow \pm\infty} S(x_0, y) = (\pm\infty, x_0)$,
- (b) $\lim_{x \rightarrow \pm\infty} S(x, y_0) = (y_0, y_0)$.

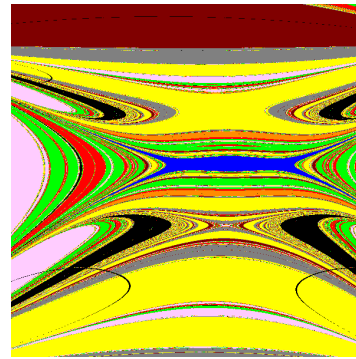
Proof. We prove (a). From (3) we have

$$\lim_{y \rightarrow \pm\infty} S(x_0, y) = \lim_{y \rightarrow \pm\infty} \left(y, y - p(y) \frac{y - x_0}{p(y) - p(x_0)} \right) = \lim_{y \rightarrow \pm\infty} \left(y, \frac{p(y)x_0 - yp(x_0)}{p(y) - p(x_0)} \right) = (\pm\infty, x_0),$$

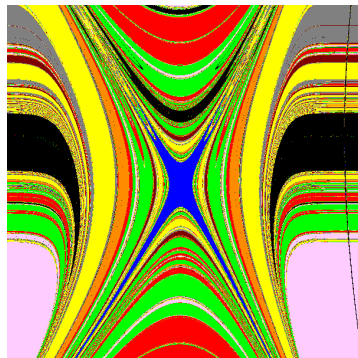
where the last equality uses that the degree of p is at least 2. Statement (b) follows similarly. \square

(a) Range $[-1, 5] \times [-1, 5]$ 

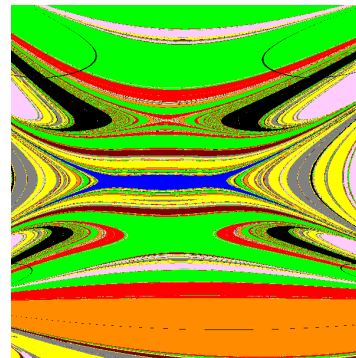
(b)



(c)



(d)



(e)

FIGURE 6. (a) The dynamical plane of S_{p^*} with p^* given in (24). Each basin of attraction is drawn in a different colour (yellow, orange, pink, red, brown, green, grey). In (b)-(e) images we show, in blue, the four attracting periodic orbit not corresponding to any root of p^* as proved in Theorem B. Black regions will be explained in the next section. Figures ((b)-(e)) correspond to zooms of (a) centered at the points $(1, 2)$, $(2, (5 + \sqrt{5})/2)$, $((5 + \sqrt{5})/2, (3 + \sqrt{5})/2)$ and $((3 + \sqrt{5})/2, 1)$, respectively.

This lemma shows that S can be extended to points of the form $(x, \pm\infty)$ and $(\pm\infty, y)$. To formalize this extension we would need to identify the symbols $+\infty$ and $-\infty$ so that the final domain of the extended map will be a torus minus one point.

If we set $R = \{(x, y) \in \mathbb{R}^2; 0 < x < 1, 0 < y < 1\}$, then a natural topological model for the torus \mathbb{T}^2 is given by $\mathbb{T}^2 := \bar{R}/\sim$ with the identifications $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$. See Figure 7(a). On the other hand we might also consider $\mathbb{T}_0^2 := R/\sim$ which turns to be a torus minus one point $(0, 0) \sim (1, 0) \sim (0, 1) \sim (1, 1)$. See Figure 7(b).

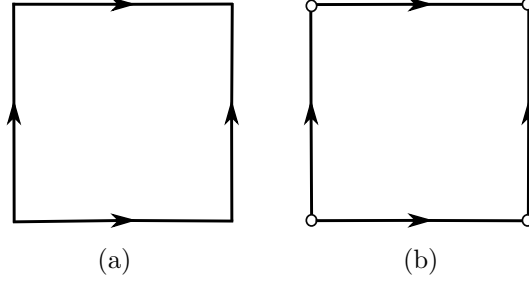


FIGURE 7. The topological model of the Torus \mathbb{T}^2 and the torus minus one point (erasing the four corners of the unit square).

This topological model together with Lemma 3.1 adapts precisely to our goal to extend S to the set

$$R_\infty = \{(x, y) \in \mathbb{R}^2\} \cup \{(x, \pm\infty), x \in \mathbb{R}\} \cup \{(\pm\infty, y), y \in \mathbb{R}\}$$

with the identifications $(x, +\infty) \sim (x, -\infty)$ and $(+\infty, y) \sim (-\infty, y)$. Similarly we define $\mathbb{T}_\infty^2 := R_\infty/\sim$ which corresponds precisely to the torus minus one point

$$(-\infty, -\infty) \sim (-\infty, \infty) \sim (\infty, -\infty) \sim (\infty, \infty).$$

Equivalently,

$$\mathbb{T}_\infty^2 = \{(x, y) \in \mathbb{R}^2\} \cup \{(x, \infty), x \in \mathbb{R}\} \cup \{(\infty, y), y \in \mathbb{R}\}.$$

The following three charts

$$(25) \quad \begin{aligned} \varphi_1(x, y) &:= \text{Id}(x, y) \text{ if } (x, y) \in \mathbb{R}^2, & \varphi_2(x, y) &= \begin{cases} (x, 0) & \text{if } y = \infty \\ (x, \frac{1}{y}) & \text{if } y \neq \{0, \infty\} \end{cases} \text{ and} \\ \varphi_3(x, y) &= \begin{cases} (0, y) & \text{if } x = \infty \\ (\frac{1}{x}, y) & \text{if } x \neq \{0, \infty\} \end{cases}, \end{aligned}$$

define a smooth atlas for the surface which allow us to do the needed computations. Given a point $P \in \mathbb{T}_\infty^2$ we denote by \mathcal{U}_P a small enough open neighbourhood of P in \mathbb{T}_∞^2 . Set

$$(26) \quad G(x, y) = \left(y, \frac{yq(x, y) - p(y)}{q(x, y)} \right), \quad (x, y) \in \mathbb{R}^2.$$

Then we might use the map G and the atlas $\{\varphi_j, j = 1, 2, 3\}$ to extend the secant map to \mathbb{T}_∞^2 . We denote the resultant map by \hat{S} and its expression is given by

$$(27) \quad \hat{S}(x, y) := \begin{cases} (\varphi_1^{-1} \circ G \circ \varphi_1)(x, y) & \text{if } (x, y) \in \mathcal{U}_P \text{ where } P \in \mathbb{R}^2 \setminus \delta_S \\ (\varphi_2^{-1} \circ G \circ \varphi_1)(x, y) & \text{if } (x, y) \in \mathcal{U}_P \text{ where } P \in \delta_S \setminus \mathcal{Q} \\ (\varphi_3^{-1} \circ G \circ \varphi_2)(x, y) & \text{if } (x, y) \in \mathcal{U}_P \text{ where } P = (x_0, \infty) \\ (\varphi_1^{-1} \circ G \circ \varphi_3)(x, y) & \text{if } (x, y) \in \mathcal{U}_P \text{ where } P = (\infty, y_0) \end{cases}$$

3.1. Proof of Theorem C. For all points in $\mathbb{R}^2 \setminus \delta_S$ the map \hat{S} (in \mathbb{R}^2 -coordinates) is G (see (26)). Hence Theorem A concludes that \hat{S} is smooth on this domain. To check the differentiability of \hat{S} at the points $(x, y) \in \delta_S \setminus \mathcal{Q}$ we split the arguments in three cases. All computations below follow from (25), (26) and (27).

Case (i). Let $(x_0, y_0) \in \delta_S \setminus \mathcal{Q} \subset \mathbb{T}_\infty^2$ and let \mathcal{U}_P be a neighbourhood of (x_0, y_0) . Then the map \hat{S} in \mathbb{R}^2 -coordinates near a point $\varphi_1(x_0, y_0) = (x_0, y_0)$ is given by

$$(28) \quad \hat{S}(x, y) = \left(y, \frac{q(x, y)}{yq(x, y) - p(y)} \right).$$

Since $(x_0, y_0) \in \delta_S \setminus \mathcal{Q}$, it is immediate to see that on the one hand $\hat{S}(x_0, y_0) = (y_0, 0)$ and on the other hand the map is locally a rational map with non zero denominator. So \hat{S} is smooth.

Case (ii). Let $(x_0, \infty) \in \mathbb{T}_\infty^2$. Doing some computations the expression of \hat{S} in \mathbb{R}^2 -coordinates near a point $\varphi_2(x_0, \infty) = (x_0, 0)$ is given by

$$(29) \quad \hat{S}(x, y) = \left(y, \frac{r(y)x - p(x)y^{k-1}}{r(y) - p(x)y^k} \right),$$

where $r(y) = 1 + a_{k-1}y + \dots + a_1y^{k-1} + a_0y^k$. We notice that on the one hand $\hat{S}(x_0, 0) = (0, x_0)$ and on the other hand the map is locally a rational map with non zero denominator. So \hat{S} is smooth.

Case (iii). Let $(\infty, y_0) \in \mathbb{T}_\infty^2$. Doing some computations the expression of \hat{S} in \mathbb{R}^2 -coordinates near a point $\varphi_3(\infty, y_0) = (0, y_0)$ is given by

$$(30) \quad \hat{S}(x, y) = \left(y, \frac{p(y)x^{k-1} - yr(x)}{p(y)x^k - r(x)} \right).$$

We notice that on the one hand $\hat{S}(0, y_0) = (y_0, y_0)$ and on the other hand the map is locally a rational map with non zero denominator. So \hat{S} is smooth.

The existence of a periodic orbit of minimal period three on \mathbb{T}_∞^2 (compare with Theorem B) is a direct application of the definition of \hat{S} . Indeed, if x_0 is such that $p'(x_0) = 0$ then $(x_0, x_0) \in \delta_S \setminus \mathcal{Q}$ and its \hat{S} -orbit is given by,

$$\hat{S}(x_0, x_0) = (x_0, \infty), \quad \hat{S}(x_0, \infty) = (\infty, x_0) \quad \text{and} \quad \hat{S}(\infty, x_0) = (x_0, x_0).$$

Fix $k \geq 3$. We need to see that the eigenvalues of $D\hat{S}^3(x_0, x_0)$ are 0 and 1. To do that we use the corresponding \mathbb{R}^2 -coordinates, equations (28), (29) and (30), obtaining

$$\begin{aligned} D\hat{S}^3(x_0, x_0) &= \hat{S} \left(\hat{S}^2(x_0, x_0) \right) D\hat{S} \left(\hat{S}(x_0, x_0) \right) D\hat{S}(x_0, x_0) = \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{p''(x_0)}{p(x_0)} & -\frac{p''(x_0)}{p(x_0)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Remark 2. As a consequence of Theorem C every point (x_0, x_0) with $p'(x_0) = 0$ generates a periodic orbit $(x_0, x_0) \mapsto (x_0, \infty) \mapsto (\infty, x_0) \mapsto (x_0, x_0)$ which is not a priori attracting, since one of the eigenvalues is 1. However in Figure 8 we show in one particular example the existence of an open region, having (x_0, x_0) on its boundary, of initial seeds converging to the three periodic orbit (similar regions can also be observed on the dynamical plane of previous examples; see Figures 4 and 5). In [BF19] (Theorem 3.2) the authors give a explicit domain of seeds converging to the periodic three cycle.

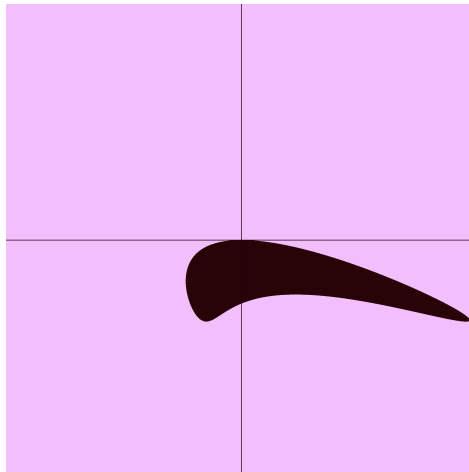


FIGURE 8. Dynamical plane of the secant map applied to the polynomial $p(x) = \frac{1}{3}x^3 - 4x + 3$ near the critical point $(2,2)$. Range of the picture $[1.92, 2.08] \times [1.92, 2.08]$. Points in colour black denote points converging to $(2,2)$ under \hat{S}^3 , while points in colour pink denote points attracted by the root $x = 3$ of p . We also show the lines $x = 2$ and $y = 2$, thus the point $(2,2)$ is located at the center of the picture.

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