One-seller assignment markets with multi-unit demands: core and competitive equilibrium

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Abstract

We consider an assignment market with one seller who owns several indivisible heterogeneous goods and many buyers each willing to buy up to a given capacity. In this market, the core contains the Vickrey payoff vector. Notwithstanding, core allocations may not be supported by competitive equilibrium prices, even in a finite replication of the market. We first characterize convexity of the associated coalitional game and we show that it is a sufficient condition so that the buyers-optimal core allocation is competitive. With respect to the seller-optimal core allocation, we provide a characterization of competitiveness by means of buyers' valuations. In addition, we characterize in terms of the valuation matrix the coincidence between the core and the set of competitive equilibrium payoff vectors.

JEL classification: C71, C78

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1 Introduction

We study markets with several buyers and only one seller. The seller owns many indivisible and potentially different objects on sale. Being heterogeneous, the objects are of the same type, for instance different houses or different tasks. On the other side of the market, each buyer has a non-negative valuation for each object and a desire to acquire a certain number of objects. This number is known as the capacity of the buyer. Since we are thinking of objects such as houses, cars or jobs, it makes sense to assume

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that a buyer, even if he values all of them positively, is not interested in acquiring more units than those that can be of use to him. We assume buyers have quasi-linear utility functions and value packages of objects additively up to a given capacity. Side-payments are allowed. Our aim is to determine under which conditions all core allocations can be priced by means of competitive prices.

This market is a particular case of the one considered in Jaume et al. (2012) and Massó and Neme (2014), where there are several sellers, each with a set of heterogeneous objects on sale. It also a particular case of the package auction of Ausubel and Milgrom (2002), where there is only one seller, but buyers may not value packages additively. Two-sided markets with one seller have also been considered in Camiña (2006) and Stuart (2007). Two-sided markets with transferable utility are first considered from the viewpoint of coalitional games in the assignment game (Shapley and Shubik, 1972). In this market, buyers want to buy at most one unit and the objects on sale belong to different sellers. The core is non-empty and coincides with the set of competitive equilibrium payoff vectors (Gale, 1960). It has a lattice structure with two particular core elements, one of them optimal for all buyers and the other one optimal for all sellers.

When the assumptions of the classical assignment model are relaxed, the lattice structure of the core and its coincidence with the set of competitive equilibrium payoff vectors do not hold in general. This is the case of many-to-many assignment markets where both buyers and sellers may be willing to trade more than one object and buyers value packages of objects additively up to their given capacity. The core of this game is always non-empty but has no lattice structure and it remains an open problem whether in this setting an optimal core element for each side of the market does exist. However, even under the assumptions that each seller has a set of homogeneous objects on sale, Sotomayor (2002) shows that a worst core element for each side of the market may not exist.

In the more encompassing many-to-many assignment model where each seller has several units of potentially different objects, the set of competitive equilibrium payoff vectors is non-empty and is strictly included in the core. However, let us point out that the definition of competitive equilibrium in Jaume et al. (2012) and Massó and Neme (2014) assumes that buyers demand as many copies of their preferred object as their capacities allow, being the prices given. Instead, we will follow the notion of competitive equilibrium used for more general markets in Gul and Stacchetti (1999), and also in Sotomayor (2007) and Arribillaga et al. (2014) for many-to-many assignment markets. There, competitive equilibrium is defined by means of a demand correspondence in which buyers maximize the utility of the packages they can buy given prices and their capacities. As a consequence, when buyers in a many-to-many assignment market value packages of objects additively, it may be the case that in a demanded package a buyer may obtain different utilities from the different objects that form the package.

In the present paper, where we have only one seller with heterogeneous goods and multi-unit demands, we study the relationship between the core and the set of competitive equilibria. We say that a core allocation is competitive if it is the payoff vector associated to some competitive equilibrium. Then, this core allocation is said to be supported by that vector of competitive equilibrium prices.

We first notice that the valuation functions of buyers in our model are monotone and satisfy the gross-substitute property. This implies that the characteristic function of the game is buyers-submodular (Gul and Stacchetti, 1999) and then, as an immediate consequence of Ausubel and Milgrom (2002), the core has a very simple structure: it is the non-empty set of efficient payoff vectors where each buyer gets a non-negative payoff bounded by his marginal contribution to the whole market. Hence, the core is endowed with a lattice structure by the partial order defined from the point of view of buyers and there exists one optimal core element for each side of the market. Moreover, as in the assignment game, in the buyers-optimal core allocation each buyer is paid his marginal contribution, that is, the Vickrey payoff (Vickrey, 1961). Our first aim is to analize under which conditions the two optimal core allocations are competitive.

Also for valuations that are monotone and satisfy the gross-substitute property, Gul and Stacchetti (1999) characterizes the maximum and minimum competitive prices and show that even if the Vickrey outcome is not supported by the minimum competitive price vector, it will become a competitive allocation of the enlarged market obtained by a finite replication of the original market. Compared to that, we look for sufficient conditions on the market valuations that guarantee that the buyers-optimal core allocation (the Vickrey outcome) is competitive.

In the literature, the relationship between the whole core and the set of competitive equilibria has been addressed. In particular Massó and Neme (2014) shows that for many-to-many assignment markets the core converges to the set of competitive equilibria payoff vectors in an infinite replication of the market. Although with a slightly different definition of competitive equilibria, they also show that this coincidence may not be attained with a finite replication. Also in our setting, we show that the core of the one-seller assignment game may not coincide with the set of competitive equilibrium payoff vector when the original market is replicated finitely many times. In particular, different to Gul and Stacchetti (1999) results for the Vickrey outcome, we show that the seller-optimal core allocation may not be competitive in any finite replication of the one-seller assignment market.

Further, we give necessary and sufficient conditions on the buyers' valuations so that the seller-optimal core allocation is supported by the maximum competitive prices. Finally, we also characterize those buyers' valuations under which the set of competitive equilibrium payoff vectors coincides with the core.

The paper is organized as follows. In the next section, the model is introduced and the necessary preliminaries are addressed. Section 3 is devoted to study under which conditions the buyers-optimal and the seller-optimal core allocations come from a competitive equilibrium. Finally, in Section 4 we characterize the coincidence between the set of competitive equilibrium payoff vectors and the core.

2 The model and some preliminaries

The *one-seller assignment market with multi-unit demands* is defined by a 5-tuple $(M, \{0\}, Q, A, r)$. The finite set of m buyers is denoted by M and the unique seller is denoted by 0. The seller owns a finite set Q of objects. These objects are indivisible and heterogeneous, but of a similar type, let us say different houses or different (maybe part-time) jobs.

Each buyer-object pair $(i,j) \in M \times Q$ has a potential gain $a_{ij} \in \mathbb{R}_+$, interpreted as the valuation of object j by buyer i, where $\mathbb{R}_+$ stands for the set of non-negative real numbers. Given a set S, we will denote by $|S|$ the cardinality of S and 2^S the set of all subsets of S. Without loss of generality, we normalize to zero the reservation value the seller has for each object. The valuation matrix, denoted by $A = (a_{ij})_{(i,j)\in M\times Q}$, captures each potential gain of all buyer-object pairs. Moreover, each buyer $i \in M$ can acquire $r_i \in \mathbb{N}$ objects and the vector $r = (r_i)_{i\in M} \in \mathbb{N}^M$ indicates the buyers' capacities. We assume that the seller owns some copies of a dummy object, as many as the sum of all buyers' capacities, $\sum_{i\in M} r_i$. With some abuse of notation, each copy of this dummy object is denoted by j_0 and each buyer values it at zero.

We assume that the utility of each buyer is quasi-linear in money and that buyers value packages of objects additively up to their given capacity. That is, buyer $i \in M$ values a package $R \subseteq Q$ by

$$\max_{\substack{R' \subseteq R \\ |R'| \leq r_i}} \left\{ \sum_{j \in R'} a_{ij} \right\}.$$

A *matching* μ between $S \subseteq M$ and Q in the market $(M, \{0\}, Q, A, r)$, is a subset of $S \times Q$ such that each $j \in Q$ belongs to at most one pair and each $i \in S$ belongs to exactly r_i pairs. Notice that it is possible to match any buyer with dummy objects to complete his capacity. We denote by $\mathcal{M}(S,Q)$ the set of matchings between $S \subseteq M$ and Q, $\mu(S)$ is the set of objects matched by μ to some buyer in S, and when $S = \{i\}$ we simply write $\mu(i)$. We denote by $\mu^{-1}(j)$ the buyer matched to object $j \in Q$ by μ.

Let $(M, \{0\}, Q, A, r)$ be a market. Given $S \subseteq M$, a matching $\mu \in \mathcal{M}(S,Q)$ is *optimal for* $S \cup \{0\}$ if

$$\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij} \quad \text{for all } \mu' \in \mathcal{M}(S,Q).$$

We denote by $\mathcal{M}_A(S,Q)$ the set of optimal matchings between S and Q in this market.

Let us introduce the definition of a coalitional game with transferable utility (a game). A game (N,v) is a pair formed by a finite set of players N and a characteristic function v that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. The core of a game (N,v) consists of

$$C(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \ \text{ for all } S \subseteq N \right\}.$$

Now, we consider a game associated to one-seller assignment markets. For any $(M, \{0\}, Q, A, r)$, the *one-seller assignment game* related to $(M, \{0\}, Q, A, r)$ is denoted by $(M \cup \{0\}, v_A)$. The worth of any coalition formed by only one type of agents is zero, because in these cases there is no trade. When a coalition is formed by a group of buyers $S \subseteq M$ and the seller, the worth is given by the following expression

$$v_A(S \cup \{0\}) = \max_{\mu \in \mathcal{M}(S,Q)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\}.$$

Now, we define competitive equilibrium for a one-seller assignment market with multi-unit demands $(M, \{0\}, Q, A, r)$. We define by $2^Q_{r_i} = \{R \subseteq Q;\ |R| = r_i\}$ the set of allowable packages of objects for a buyer $i \in M$. A *price vector* $p = (p_j)_{j \in Q} \in \mathbb{R}^Q_+$ consists of one price for each object, with a price of zero for each dummy object. For each $p \in \mathbb{R}^Q_+$, we denote by $D_i(p) \subseteq 2^Q_{r_i}$ the *demand set of buyer i at level prices p*, that is

$$D_i(p) = \left\{ R \in 2^Q_{r_i} \;\middle|\; \sum_{j \in R} (a_{ij} - p_j) \geq \sum_{j \in R'} (a_{ij} - p_j) \ \text{ for all } R' \in 2^Q_{r_i} \right\}.$$

The demand set of any buyer is never empty but, at sufficiently high prices, the demand set can be formed only by dummy objects.

Definition 2.1. *A* competitive equilibrium *for a one-seller assignment market* $(M, \{0\}, Q, A, r)$ *is a pair* $(p, \mu) \in \mathbb{R}^Q_+ \times \mathcal{M}(M, Q)$*, such that the following two conditions hold:*
C.1 *For all* $i \in M$, $\mu(i) \in D_i(p)$,
C.2 *For all* $j \in Q \setminus \mu(M)$, $p_j = 0$.

If a pair (p, μ) is a competitive equilibrium, we say that p is a *competitive equilibrium price vector*. The *payoff vector associated to* (p, μ) is $(U(p, \mu), V(p, \mu)) \in \mathbb{R}^M \times \mathbb{R}$, where

$$\begin{array}{rcll} U_i(p, \mu) & = & \sum_{j \in \mu(i)} (a_{ij} - p_j) & \text{for each } i \in M, \text{ and} \\ V(p, \mu) & = & \sum_{j \in Q} p_j & \text{for the seller.} \end{array} \tag{1}$$

Gul and Stacchetti (1999) shows that when all buyers value packages additively up to a given capacity, these valuations satisfy the gross-substitutes condition[1] as well as monotonicity.[2] Then, the following consequences regarding the set of competitive equilibria follow easily for one-seller assignment markets with multi-unit demands.

R.1 If (p, μ) is a competitive equilibrium, then μ is optimal and, for any optimal matching μ', (p, μ') is also a competitive equilibrium.

R.2 The set of competitive equilibrium price vectors of the market is non-empty and forms a complete lattice.

R.3 The maximum competitive equilibrium price for an object $k \in Q$ is

$$\overline{p}_k = \max_{\mu \in \mathcal{M}(M,Q)} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} - \max_{\mu \in \mathcal{M}(M,Q \setminus \{k\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\}. \tag{2}$$

In order to express the minimum competitive price of an object $k \in Q$ in a market $(M, \{0\}, Q, A, r)$, we need to consider a new type of matchings. We will allow only object k to be matched at most twice but not to the same buyer. This is equivalent to

[1] The gross-substitutes condition was introduced by Kelso and Crawford (1982) and it requires that for any two price vectors p and q such that $q \geqslant p$, and any $R \in D_i(p)$, there exists $R' \in D_i(q)$ such that $\{j \in R | p_j = q_j\} \subseteq R'$.

[2] Monotonicity simply says that if $R' \subseteq R$ then the valuation of package R' is at least as high as the valuation of R.

introducing an identical copy of object k and restrict to matchings that do not assign the two copies to the same buyer. With some abuse of notation, we will denote this set of matchings by $\mathcal{M}^k(M,Q)$. Now, we can give an expression for the minimum competitive equilibrium prices.

R.4 The minimum competitive equilibrium price of an object k does exist and it is given by

$$\underline{p}_k = \max_{\mu\in\mathcal{M}^k(M,Q)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\} - \max_{\mu\in\mathcal{M}(M,Q)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\}. \tag{3}$$

Result **R.1** and the existence of competitive equilibria can also be found in Arribillaga et al. (2014) for a more general assignment model.

In the definitions of competitive equilibria, the owners of the objects do no play any role. Hence the set of competitive equilibria of the one-seller assignment market $(M,\{0\},Q,A,r)$ equals the set of competitive equilibria of the related many-to-one market where each object belongs to a different seller. Moreover, these competitive prices are easily described by linear equalities and inequalities. A pair $(p,\mu)\in\mathbb{R}^Q_+\times\mathcal{M}(M,Q)$ is a competitive equilibrium of $(M,\{0\},Q,A,r)$ if and only if

$$\begin{aligned} &0\le p_j\le a_{ij} \text{ for all } (i,j)\in\mu, \quad p_j=0 \text{ for all } j\in Q\setminus\mu(M) \text{ and} \\ &a_{ij}-p_j\ge a_{ik}-p_k \text{ for each } i\in M, \quad \text{all } j\in\mu(i) \text{ and all } k\in Q\setminus\{\mu(i)\}. \end{aligned} \tag{4}$$

To better analyze the relationship between the core and the competitive equilibria for the one-seller assignment game, we first find a simple description of the core of the game. To this end, we use a result in Ausubel and Milgrom (2002). They introduce the notion of buyers-submodularity, which means that the marginal contribution of a buyer to a coalition containing the seller decreases as the coalition grows larger. A game $(M\cup\{0\},v)$ is *buyers-submodular* if for all $i\in M$,

$$v((T\cup\{i\})\cup\{0\})-v(T\cup\{0\})\ge v((S\cup\{i\})\cup\{0\})-v(S\cup\{0\}), \tag{5}$$

for all $T\subseteq S\subseteq M\setminus\{i\}$. The following result shows that the one-seller assignment game is buyers-submodular.

Proposition 2.2. *Let $(M,\{0\},Q,A,r)$ be a one-seller assignment market and $(M\cup\{0\},v_A)$ be its related one-seller assignment game. Then $(M\cup\{0\},v_A)$ is buyers-submodular.*

Proof. Recall that buyers' valuations satisfy the gross-substitutes condition and monotonicity (Gul and Stacchetti, 1999). Therefore, by Theorem 11 in Ausubel and Milgrom (2002), the game $(M\cup\{0\},v_A)$ is buyers-submodular. □

Together with the fact that coalitions not containing the seller have null worth, buyers-submodularity implies that the one-seller assignment game $(M\cup\{0\},v_A)$ is a big-boss game, as defined in Muto et al. (1988).[3] As a consequence of Theorem 7 in

[3]Buyers submodularity clearly implies condition B2** in page 312 of Muto et al. (1988), which implies B2 in the definition of big-boss game.

Ausubel and Milgrom (2002), or also Theorem 3.2 in Muto et al. (1988), the core of the one-seller assignment game is non-empty and can be described by

$$\left\{ (U,V) \in \mathbb{R}^M \times \mathbb{R} \;\middle|\; \sum_{i\in M} U_i + V = v_A(M \cup \{0\}),\; 0 \leq U_i \leq M_i^{v_A} \text{ for all } i \in M \right\}, \quad (6)$$

where $M_i^{v_A} = v_A(M \cup \{0\}) - v_A((M \setminus \{i\}) \cup \{0\})$ denotes the marginal contribution of buyer $i \in M$ to the grand coalition, which is also known as the Vickrey payoff for this agent. Furthermore, the core is a lattice with respect to the usual order defined on buyers' payoffs. Then, we can guarantee the existence of one optimal core allocation for each side of the market. In the buyers-optimal core allocation $(\overline{U}, \underline{V}) \in \mathbb{R}^M \times \mathbb{R}$, each buyer gets his marginal contribution, that is, $\overline{U}_i = M_i^{v_A}$ for all $i \in M$ and $\underline{V} = v_A(M \cup \{0\}) - \sum_{i\in M} M_i^{v_A}$. On the other hand, in the seller-optimal core allocation $(\underline{U}, \overline{V}) \in \mathbb{R}^M \times \mathbb{R}$, each buyer $i \in M$ gets $\underline{U}_i = 0$ and $\overline{V} = v_A(M \cup \{0\})$. Thus, the core of the one-seller assignment game has an optimal core allocation for each market sector as it happens in the classical Shapley and Shubik (1972) assignment game. This is not known to be true for other many-to-many assignment models (see e.g. Sotomayor, 2002).

A first relationship between core and competitive equilibrium for many-to-many assignment markets is well known (see for instance Theorem 36 in Arribillaga et al., 2014): the payoff vector $(U(p,\mu), V(p,\mu))$ associated to any competitive equilibrium (p,μ) of the one-seller market $(M, \{0\}, Q, A, r)$ (or of any many-to-many assignment market) belongs to the core of the associated game. However, there may be core allocations not supported by competitive equilibrium prices. In particular, we ask when the two optimal core allocations are competitive payoff vectors.

3 When optimal core allocations are competitive?

To begin the study of the relationship between optimal core allocations and the competitive equilibria, we focus on conditions on the buyers' valuations so that optimal core allocations are supported by competitive equilibria.

As it was remarked in the previous section, the set of competitive equilibrium payoff vectors is a subset of the core. Notwithstanding, the one-seller assignment market has an interesting property which does not hold in more general many-to-many assignment markets: the Vickrey outcome coincides with the buyers-optimal core allocation. The relationship between the Vickrey outcome and the set of competitive equilibria was addressed in Gul and Stacchetti (1999). They show that when the original market is replicated finitely many times, the Vickrey outcome is supported by a competitive equilibrium in the enlarged market. In fact, they show that it is sufficient to replicate the market at least as many times as the number of objects. In spite of that, we are interested in conditions to guarantee competitive optimal core allocations in the original market, with no need of replication.

The next proposition concerns the buyers-optimal core allocation and characterizes those markets in which it is possible to allocate to each buyer (one of) his most preferred package of objects. The feasibility of such an allocation characterizes the convexity of the game[4] and implies the existence of a competitive buyers-optimal core allocation.

Proposition 3.1. *Let $(M, \{0\}, Q, A, r)$ be a one-seller assignment market and $(M \cup \{0\}, v_A)$ its related one-seller assignment game. The following assertions are equivalent:*

i. $(M \cup \{0\}, v_A)$ *is convex,*

ii. There is a matching $\mu \in \mathcal{M}(M, Q)$ *such that for each* $i \in M$,

$$\sum_{j \in \mu(i)} a_{ij} \geq \sum_{j \in R} a_{ij} \text{ for all } R \in 2^Q_{r_i}. \tag{8}$$

iii. The minimum competitive equilibrium price vector is $\underline{p} = (0, ..., 0) \in \mathbb{R}^Q_+$.

Proof. *i.* ⇒ *ii.* First, it can be deduced from Proposition 3.4 in Muto et al. (1988) that $(M \cup \{0\}, v_A)$ is convex if and only if for any $S \subseteq M$,

$$v_A(S \cup \{0\}) = v_A(\{0\}) + \sum_{i \in S} M_i^{v_A}.$$

Now, assume that $(M \cup \{0\}, v_A)$ is convex. We have that for all $\mu \in \mathcal{M}_A(M, Q)$,

$$\sum_{(i,j) \in \mu} a_{ij} = v_A(M \cup \{0\}) = v_A(\{0\}) + \sum_{i \in M} M_i^{v_A} = \sum_{i \in M} M_i^{v_A}. \tag{9}$$

Since $\sum_{j \in \mu(i)} a_{ij} \geq M_i^{v_A}$ for all $i \in M$, expression (9) implies that $\sum_{j \in \mu(i)} a_{ij} = M_i^{v_A}$ for all $i \in M$. Moreover, for all $i \in M$, let $R_i \in 2^Q_{r_i}$ be such that

$$\sum_{j \in R_i} a_{ij} \geq \sum_{j \in R} a_{ij} \text{ for all } R \in 2^Q_{r_i}.$$

Since $(M \cup \{0\}, v_A)$ is convex, then

$$\sum_{j \in R_i} a_{ij} = v_A(\{i, 0\}) = M_i^{v_A} \text{ for all } i \in M,$$

and as a consequence, for each $i \in M$, we have $\sum_{j \in \mu(i)} a_{ij} = \sum_{j \in R_i} a_{ij} \geq \sum_{j \in R} a_{ij}$ for all $R \in 2^Q_{r_i}$.

ii. ⇒ *iii.* Take the matching μ of statement *ii.* and the price vector $p = (0, ..., 0) \in \mathbb{R}^Q_+$. It is immediate that by (8), $\mu(i) \in D_i(p)$ for each $i \in N$. Indeed, (p, μ) is a competitive equilibrium. The minimality of the price vector p is also immediate.

[4] A game (N, v) is convex if for all $S, T \subseteq N$

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{7}$$

iii. $\Rightarrow$ *i.* Since $\underline{p} = (0, ..., 0) \in \mathbb{R}^Q_+$ is a competitive equilibrium price vector, then there is a matching $\mu \in \mathcal{M}(M, Q)$ such that $\mu(i) \in D_i(p)$ for all $i \in N$. In fact, because of the null prices, notice that $\mu \in \mathcal{M}(M, Q)$ satisfies

$$\sum_{i \in S} \sum_{j \in \mu(i)} a_{ij} = v_A(S \cup \{0\}) \text{ for all } S \subseteq M.$$

As a consequence, it is easy to see that $M_i^{v_A} = \sum_{j \in \mu(i)} a_{ij}$ for every $i \in M$. Then, we have that for any $S \subseteq M$,

$$\sum_{i \in S} M_i^{v_A} = \sum_{i \in S} \sum_{j \in \mu(i)} a_{ij} = v_A(S \cup \{0\}),$$

which shows that the game $(M \cup \{0\}, v_A)$ is convex. □

Notice that if the above equivalent conditions hold, and μ is a matching that allocates to each buyer one of his preferred packages, then the Vickrey payoff of each $i \in M$ is $\sum_{j \in \mu(i)} a_{ij}$ and hence it is attained at the minimum competitive equilibrium. However, there are instances (see Example 3.2) in which the game is not convex but the buyers-optimal core allocation is also competitive. To see that, we only need to check whether for all $i \in M$ it holds $M_i^{v_A} = \sum_{j \in \mu(i)} (a_{ij} - \underline{p}_j)$, where μ is an optimal matching and $\underline{p} \in \mathbb{R}^Q_+$ can be computed following expression (2).

The next example also shows that the fact that the buyers-optimal core allocation is supported by competitive prices does not guarantee that all other core allocations are also competitive.

Example 3.2. *Consider a market with unitary capacities* $(M, \{0\}, Q, A, r)$ *given by* $M = \{1, 2\}$, $Q = \{1', 2'\}$ *and* $r = (1, 1)$. *For the purposes of this example, we show no dummy objects. The valuation matrix* A *is the following*

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 5 & \textcircled{4} \\ \textcircled{4} & 2 \end{pmatrix} \end{array}.$$

Consider the one-seller assignment game $(M \cup \{0\}, v_A)$. *By (6), the core can be described by the set of payoff vectors* $(U, V) \in \mathbb{R}^2_+ \times \mathbb{R}_+$ *such that* $U_1 + U_2 + V = 8$, $U_1 \leq 4$ *and* $U_2 \leq 3$.

Take the unique optimal matching $\mu = \{(1, 2'), (2, 1')\}$. *By (4), a price* $p \in \mathbb{R}^2_+$ *is a competitive price vector if and only if* $0 \leq p_{1'} \leq 4$, $0 \leq p_{2'} \leq 4$ *and* $1 \leq p_{1'} - p_{2'} \leq 2$. *Since the corresponding payoff vector* $(U_1, U_2; V)$ *satisfies* $U_1 = 4 - p_{2'}$, $U_2 = 4 - p_{1'}$ *and* $V = p_{1'} + p_{2'}$, *we have* $p_{1'} - p_{2'} = (4 - U_2) - (4 - U_1) = U_1 - U_2$. *Hence, the competitive equilibrium payoff vectors are described by* $U_1 + U_2 + V = 8$, $1 \leq U_1 - U_2 \leq 2$, $0 \leq U_1 \leq 4$ *and* $0 \leq U_2 \leq 4$. *Figure 1 depicts the core and the set of competitive equilibrium payoff vectors.*

In the above example the game is not convex and the buyers-optimal core allocation is supported by a competitive equilibrium while the seller-optimal core allocation is not. Moreover, although convexity is a strong condition, it is not sufficient to guarantee that the seller-optimal core allocation is supported by a competitive equilibrium, as the following example illustrates.

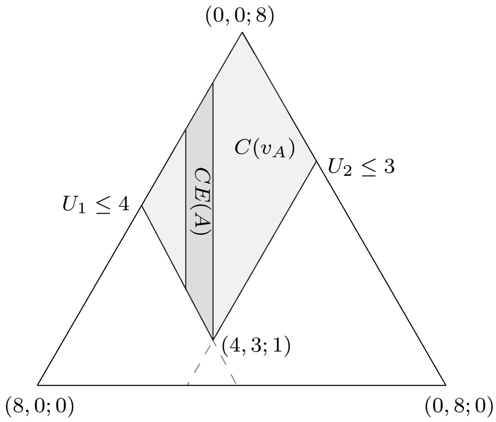

Figure 1: The set of competitive equilibria payoff vectors $CE(A)$ is strictly included in the core $C(v_A)$

Example 3.3. *Consider a market* $(M, \{0\}, Q, A, r)$ *given by* $M = \{1, 2\}$, $Q = \{1', 2', 3'\}$ *and* $r = (1, 1)$. *The valuation matrix* A *is the following*

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \left(\begin{array}{ccc} 2 & \textcircled{4} & 1 \\ \textcircled{2} & 1 & 1 \end{array} \right) \end{array}.$$

Note that the game is convex and the seller-optimal core allocation is $(\underline{U}, \overline{V}) = (0, 0; 6)$. *We can obtain the maximum competitive equilibrium prices* $\overline{p} = (1, 3, 0)$ *by means of formula (3) and we see that the corresponding payoff vector is* $(U, V) = (1, 1; 4)$ *which is not the seller-optimal core allocation.*

An interesting fact about the seller-optimal core allocation is the following. When the seller-optimal core allocation is not a competitive equilibrium payoff vector, even if the economy is replicated finitely many times as in Gul and Stacchetti (1999), it may not be supported by any competitive equilibrium in the enlarged market. To see this, consider the previous example. If we replicate the market finitely many times, the replicas of object $3'$ will be unmatched in any optimal matching. As a consequence, the price of these replicas in any competitive equilibrium will be zero. Assume there is a competitive equilibrium that supports the seller-optimal core allocation in the enlarged market. In this equilibrium each buyer will pay for each of his matched objects his own valuation of the object. But then, any buyer strictly prefers a replica of object $3'$ for free to his matched objects at the described prices, and this contradicts these prices are competitive.

In order to characterize when the seller-optimal core allocation is competitive, let us first define the set of desirable objects, Q^*_A. We say that an object is desirable if at least one buyer valuates it positively

$$Q^*_A = \{j \in Q \mid a_{ij} > 0 \text{ for some } i \in M\}.$$

The following result gives a characterization of markets so that the seller-optimal core allocation is competitive.

Proposition 3.4. *Let $(M, \{0\}, Q, A, r)$ be a one-seller assignment market. The seller-optimal core allocation is a competitive equilibrium payoff vector if and only if there is an optimal matching $\mu \in \mathcal{M}_A(M, Q)$ and the following two conditions are satisfied:*

(a) For each $j \in \mu(M)$, $a_{ij} \le a_{\mu^{-1}(j)j}$ for all $i \in M \setminus \{\mu^{-1}(j)\}$,

(b) $Q^*_A \subseteq \mu(M)$.

Proof. We first prove the 'if' part. Assume that $\mu \in \mathcal{M}_A(M, Q)$ satisfies conditions (a) and (b). Define $p_j = a_{\mu^{-1}(j)j}$ for all $j \in \mu(M)$ and $p_j = 0$ for all $j \in Q \setminus \mu(M)$. We show that $\mu(i) \in D_i(p)$ for all $i \in M$. Take any $i \in M$ and consider any $R \in 2^Q_{r_i}$. Since μ satisfies (a) and (b), and by definition of p, we have

$$\sum_{j\in R}(a_{ij} - p_j) = \sum_{j\in R\cap\mu(M)}(a_{ij} - a_{\mu^{-1}(j)j}) + \sum_{j\in R\setminus\mu(M)}(a_{ij} - 0) \le 0 = \sum_{j\in\mu(i)}(a_{ij} - p_j),$$

and thus $\mu(i) \in D_i(p)$ for all $i \in M$. Besides, by definition of p, we get $p_j = 0$ for each $j \in Q \setminus \mu(M)$. Notice that $(U(p, \mu), V(p, \mu))$ is the seller-optimal core allocation.

Now, we prove the 'only if' part. Assume that (p, μ) is a competitive equilibrium and $(U(p, \mu), V(p, \mu))$ is the seller-optimal core allocation. By property R.1 in page 5, we have that $\mu \in \mathcal{M}_A(M, Q)$. Moreover, in the seller-optimal core allocation the seller's payoff is equal to $v_A(M \cup \{0\})$.

We claim that

$$p_j = a_{\mu^{-1}(j)j} \text{ for all } j \in \mu(M). \tag{10}$$

If $p_j > a_{\mu^{-1}(j)j}$ for some $j \in \mu(M)$, then for all $R \in D_{\mu^{-1}(j)}(p)$ we have $j \notin R$, and as a consequence (p, μ) is not a competitive equilibrium. On the other hand, if $p_j < a_{\mu^{-1}(j)j}$ for some $j \in \mu(M)$ then $\sum_{j\in Q} p_j < v_A(M \cup \{0\})$ and the seller-optimal core allocation is not the payoff vector of (p, μ).

Now, taking (10) into account, we shall prove that μ satisfies condition (a) of the statement. Assume on the contrary that there is some $i \in M$ such that $a_{ij} > a_{\mu^{-1}(j)j}$ for some $j \in Q$ with $i \in M \setminus \{\mu^{-1}(j)\}$. Let $R \in 2^Q_{r_i}$ be the package formed by object j and copies of the dummy object, i.e., $R = \{j, j^1_0, j^2_0, ..., j^{r_i-1}_0\}$. Since $\sum_{j\in R}(a_{ij} - p_j) > 0 = \sum_{j\in\mu(i)}(a_{ij} - p_j)$, we obtain that $\mu(i) \notin D_i(p)$ in contradiction with (p, μ) being a competitive equilibrium. Then μ satisfies (a). In order to show (b), assume on the contrary that, there is some $j \in Q^*_A \setminus \mu(M)$. By definition of competitive equilibrium, the price of this object is $p_j = 0$. Since $j \in Q^*_A$, there is some $i \in M$ such that $a_{ij} > 0$. This implies that $\mu(i) \notin D_i(p)$ because $\sum_{j\in R}(a_{ij} - p_j) > \sum_{j\in\mu(i)}(a_{ij} - p_j)$ where $R = \{j, j^1_0, j^2_0, ..., j^{r_i-1}_0\}$. This contradicts (p, μ) being a competitive equilibrium. Hence, μ satisfies (b). □

Condition (a) above requires that each object must be allocated to the buyer who values it the most, while condition (b) simply says that each desirable object must be allocated. Notice that condition (a) is not satisfied in Example 3.2, and condition (b) is not satisfied in Example 3.3.

4 Characterization of the coincidence of the core and the competitive equilibria payoff vectors

In this section, we address the relationship between the whole core and the set of competitive equilibria. The aim is to obtain conditions so that the core coincides with the set of competitive equilibrium payoff vectors, that is, to obtain a competitive core. For more general many-to-many assignment markets but with a different definition of the demand set, Massó and Neme (2014) shows that the sequence of cores of replicated markets converges to the set of competitive equilibrium payoffs when the number of replicas tends to infinity. Moreover, for any number of replicas there is a market with a core payoff that is not a competitive equilibrium payoff. Also, for our definition of competitive equilibrium that follows Gul and Stacchetti (1999), Example 3.3 shows that the process of finite replication does not guarantee a competitive seller-optimal core allocation.

Since no core coincidence result can be achieved for arbitrary one-seller assignment markets, after a finite replication of the economy, we focus on the search of conditions on buyers' valuations that guarantee that all core allocations are competitive. Taking into account propositions 3.1 and 3.4, one might wonder if convexity together with conditions (a) and (b) of Proposition 3.4 are sufficient to obtain a competitive core. The answer is in the negative as the next example shows.

Example 4.1. *Consider a market $(M, \{0\}, Q, A, r)$ given by $M = \{1, 2\}$, $Q = \{1', 2'\}$ and $r = (1, 1)$. The valuation matrix A is the following*

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 2 & \textcircled{4} \\ \textcircled{2} & 1 \end{pmatrix} \end{array}.$$

Note that the game is convex and conditions (a) and (b) of Proposition 3.4 hold. Consider the core allocation $(4, 0; 2)$. Notice that the unique optimal matching in this market assigns object $1'$ to buyer 2 and object $2'$ to buyer 1. Because of the unitary demands, the unique price vector that supports the core allocation $(4, 0; 2)$ is $p = (2, 0)$. However, $p = (2, 0)$ is not a competitive equilibrium price vector since $\{1'\} \notin D_2(p)$.

The following theorem is the main result of this paper and characterizes the coincidence between the set of competitive equilibrium payoff vectors and the core.

Theorem 4.2. *Let $(M, \{0\}, Q, A, r)$ be a one-seller assignment market and $(M \cup \{0\}, v_A)$ be its associated one-seller assignment game. The core of $(M \cup \{0\}, v_A)$ coincides with the set of competitive equilibrium payoff vectors if and only if there is an optimal matching $\mu \in \mathcal{M}_A(M, Q)$ that satisfies the following three conditions:*

(a) For each $j \in \mu(M)$, $a_{ij} \leq a_{\mu^{-1}(j)j}$ for all $i \in M \setminus \{\mu^{-1}(j)\}$,

(b) $Q_A^* \subseteq \mu(M)$,

(c) $M_i^{v_A} \leq \sum_{j\in\mu(i)} \left(a_{ij} - \max\limits_{t \in M\setminus\{i\}} \{a_{tj}\} \right)$ for all $i \in M$.

Proof. We first prove the 'if' part. Assume that some $\mu \in \mathcal{M}_A(M,Q)$ satisfies (a), (b) and (c). We show that any $(U,V) \in C(v_A)$ is the payoff vector of some competitive equilibrium. By conditions (a) and (c), for each $i \in M$, we can find some $(\alpha_{ij})_{j\in\mu(i)} \in \mathbb{R}^{r_i}$ such that $a_{ij} \geq \alpha_{ij} \geq \max_{t\in M\setminus\{i\}}\{a_{tj}\}$ for all $j \in \mu(i)$ and

$$M_i^{v_A} = \sum_{j\in\mu(i)} (a_{ij} - \alpha_{ij}). \tag{11}$$

Take any $(U,V) \in C(v_A)$ and define $b_i = M_i^{v_A} - U_i$ for all $i \in M$. Since for all $i \in M$ we have $M_i^{v_A} \geq U_i \geq 0$, then $M_i^{v_A} \geq b_i \geq 0$.

Let us define $p \in \mathbb{R}^Q$ by

$$p_j = \begin{cases} \alpha_{\mu^{-1}(j)j} + \dfrac{a_{\mu^{-1}(j)j} - \alpha_{\mu^{-1}(j)j}}{M^{v_A}_{\mu^{-1}(j)}}\, b_{\mu^{-1}(j)} & \text{if } j \in \mu(M) \text{ and } M^{v_A}_{\mu^{-1}(j)} > 0, \\ a_{\mu^{-1}(j)j} & \text{if } j \in \mu(M) \text{ and } M^{v_A}_{\mu^{-1}(j)} = 0, \\ 0 & \text{if } j \in Q \setminus \mu(M). \end{cases} \tag{12}$$

Notice that $p \in \mathbb{R}^Q_+$. We show that $\mu(i) \in D_i(p)$ for all $i \in M$. It is sufficient to see that $a_{ij} - p_j \geq a_{ik} - p_k$ for all $j \in \mu(i)$ and all $k \in Q \setminus \mu(i)$. To this end, let us see that for all $i \in M$ and all $j \in \mu(i)$ it holds $a_{ij} - p_j \geq 0$ while $a_{ik} - p_k \leq 0$ for all $k \in Q \setminus \mu(i)$. On one hand, take $i \in M$ such that $M_i^{v_A} > 0$. Then $a_{ij} - p_j = a_{ij} - \alpha_{ij} - \frac{a_{ij}-\alpha_{ij}}{M_i^{v_A}} b_i = (a_{ij} - \alpha_{ij})(1 - \frac{b_i}{M_i^{v_A}}) \geq 0$ for all $j \in \mu(i)$. Take $i \in M$ such that $M_i^{v_A} = 0$. Then $a_{ij} - p_j = a_{ij} - a_{ij} = 0$ for all $j \in \mu(i)$. On the other hand, take $k \in \mu(M)$ such that $M^{v_A}_{\mu^{-1}(k)} > 0$. Then for any $i \in M \setminus \{\mu^{-1}(k)\}$, we have $a_{ik} - p_k = a_{ik} - \alpha_{\mu^{-1}(k)k} - \frac{a_{\mu^{-1}(k)k} - \alpha_{\mu^{-1}(k)k}}{M^{v_A}_{\mu^{-1}(k)}} b_{\mu^{-1}(k)} \leq 0$ because $a_{\mu^{-1}(k)k} \geq \alpha_{\mu^{-1}(k)k} \geq a_{ik}$. Take $k \in \mu(M)$ such that $M^{v_A}_{\mu^{-1}(k)} = 0$. Then for any $i \in M \setminus \{\mu^{-1}(k)\}$, we have $a_{ik} - p_k = a_{ik} - a_{\mu^{-1}(k)k} \leq 0$ because of assumption (a). Finally, consider $k \in Q \setminus \mu(M)$. Then for any $i \in M$, $a_{ik} - p_k = 0$ because of (b). Thus $\mu(i) \in D_i(p)$ for all $i \in M$ and $p \in \mathbb{R}^Q_+$. Hence, (p,μ) is a competitive equilibrium. Then, the payoffs in this competitive equilibrium are

$$\begin{aligned} U_i(p,\mu) &= \sum_{j\in\mu(i)} (a_{ij} - p_j) = \sum_{j\in\mu(i)} \left(a_{ij} - \alpha_{ij} - \frac{a_{ij} - \alpha_{ij}}{M_i^{v_A}} b_i \right) \\ &= \sum_{j\in\mu(i)} (a_{ij} - \alpha_{ij}) \left(1 - \frac{b_i}{M_i^{v_A}} \right) = M_i^{v_A} - b_i = U_i, \end{aligned}$$

for all $i \in M$ such that $M_i^{v_A} > 0$, where the last equality comes from expression (11). Take now any $i \in M$ such that $M_i^{v_A} = 0$. From the definition of p_j in (12), we have $U_i(p,\mu) = \sum_{j\in\mu(i)}(a_{ij} - p_j) = \sum_{j\in\mu(i)}(a_{ij} - a_{ij}) = 0 = U_i$. Since $(U(p,\mu), V(p,\mu)) \in C(v_A)$ for any competitive equilibrium (p,μ), by efficiency the seller's payoff is $V(p,\mu) = v_A(M \cup \{0\}) - \sum_{i\in M} U_i(p,\mu) = v_A(M \cup \{0\}) - \sum_{i\in M} U_i = V$. This completes the proof of the 'if' part.

Now, we prove the 'only if' part. Assume that the core and the set of payoff vectors associated with the competitive equilibria coincide. By Proposition 3.4, conditions

(a) and (b) hold for some optimal matching $\mu \in \mathcal{M}_A(M,Q)$. Then, we only have to prove (c). Assume on the contrary that for this μ, there is some buyer $i' \in M$ such that $M_{i'}^{v_A} > \sum_{j\in\mu(i')}(a_{i'j} - \max_{t\in M\setminus\{i'\}}\{a_{tj}\})$. Recall the description of the core in (6) and consider $(U,V) \in C(v_A)$ with $U_{i'} = M_{i'}^{v_A}$ for the buyer i' and $U_i = 0$ for all $i \in M \setminus \{i'\}$. By assumption, there is a competitive equilibrium (p,μ') such that (U,V) is its payoff vector. Take this competitive equilibrium price vector p and the matching $\mu \in \mathcal{M}_A(M,Q)$ such that $M_{i'}^{v_A} > \sum_{j\in\mu(i')}(a_{i'j} - \max_{t\in M\setminus\{i'\}}\{a_{tj}\})$. Then (p,μ) is a competitive equilibrium (recall R.1 in page 5). Therefore $p_j = a_{\mu^{-1}(j)j}$ for all $j \in \mu(M\setminus\{i'\})$ and $M_{i'}^{v_A} = \sum_{j\in\mu(i')}(a_{i'j} - p_j)$. We obtain $\sum_{j\in\mu(i')}(a_{i'j} - p_j) = M_{i'}^{v_A} > \sum_{j\in\mu(i')}(a_{i'j} - \max_{i\in M\setminus\{i'\}}\{a_{ij}\})$. As a consequence, $\sum_{j\in\mu(i')}\max_{i\in M\setminus\{i'\}}\{a_{ij}\} > \sum_{j\in\mu(i')} p_j$ which implies that there is some $i \in M\setminus\{i'\}$ such that $a_{ij} > p_j$ for some $j \in \mu(i')$. We have that $\mu(i) \notin D_i(p)$ because $a_{ik} - p_k = 0 < a_{ij} - p_j$ for all $k \in \mu(i)$ and the above $j \notin \mu(i)$. This contradicts that (p,μ) is a competitive equilibrium. Hence, condition (c) holds. □

The above theorem gives a characterization of the competitive core in one-seller assignment markets. Notice that the core and competitive equilibrium payoff vectors do not coincide in Example 4.1 because (c) is not satisfied.

As a consequence of Theorem 4.2, when the buyers have a sufficiently large capacity, the core coincides with the set of competitive equilibrium payoff vectors. Indeed, when there are no capacity constraints (or each buyer has a capacity greater than the number of non-dummy objects), an optimal matching assigns each object to one of the buyers who values it the most. Hence, conditions (a) and (b) are satisfied. Moreover, when a buyer leaves the market, his objects are assigned to the buyers with the second highest valuation, and this implies that for all $i \in M$, condition (c) holds with an equality.

It is also quite straightforward to check that if for some capacities $r \in \mathbb{N}^M$ the core of the market $(M,\{0\},Q,A,r)$ coincides with the set of competitive equilibrium payoff vectors, then they also coincide if capacities are increased to r' where $r'_i \geq r_i$ for all $i \in M$.

Finally, the following corollary allows us to obtain a stronger condition for the coincidence of the core and the set of competitive equilibrium payoff vectors.

Corollary 4.3. *Let $(M,\{0\},Q,A,r)$ be a one-seller assignment market and $\mu \in \mathcal{M}_A(M,Q)$ be such that: $(*)$ if $(i,j) \notin \mu$ then $a_{ij} = 0$. The core coincides with the set of competitive equilibrium payoff vectors.*

Proof. It is immediate to see that $(*)$ implies conditions (a) and (b) of Theorem 4.2. Now, notice that for any $i \in M$,

$$\sum_{j\in\mu(i)} a_{ij} = \sum_{(t,j)\in\mu} a_{tj} - \sum_{t\in M\setminus\{i\}}\sum_{j\in\mu(t)} a_{tj} \geq v_A(M\cup\{0\}) - v_A((M\setminus\{i\})\cup\{0\}) = M_i^{v_A}. \tag{13}$$

Since $(*)$ holds, expression (13) implies condition (c) of Theorem 4.2. Therefore, $(*)$ implies the coincidence between the core and the set of competitive equilibria. □

Note that property $(*)$ is stronger than convexity of the game and it represents those markets where agents only value positively objects optimally assigned to them.

The next example shows that if a buyer places a small positive valuation on some object not assigned to him, the coincidence between core and competitive equilibrium payoff vectors may be lost.

Example 4.4. *Consider a market* $(M, \{0\}, Q, A, r)$ *given by* $M = \{1, 2\}$, $Q = \{1', 2'\}$, $r = (1, 1)$ *and the following valuation matrix* A, *where* $0 < \varepsilon < 4$*:*

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} \varepsilon & \textcircled{4} \\ \textcircled{2} & 0 \end{pmatrix} \end{array}.$$

Since the marginal contributions of the two buyers are 4 *and* 2 *respectively,* $(0, 2; 4)$ *belongs to the core. This core element can only be supported by prices* $p = (0, 4)$ *but at the price of 0, buyer 1 would prefer object 1 rather than object 2. Hence, for any* $0 < \varepsilon < 4$, *the core of this market does not coincide with the set of competitive equilibrium payoff vectors.*

References

Arribillaga, R., J. Massó, and A. Neme (2014), "On the structure of cooperative and competitive solutions for a generalized assignment game." *Journal of Applied Mathematics*, URL http://dx.doi.org/10.1155/2014/190614.

Ausubel, L. and P. Milgrom (2002), "Ascending auctions with package bidding." *Frontiers of Theoretical Economics*, 1(1), Article 1, 489–509.

Camiña, E. (2006), "A generalized assignment game." *Mathematical Social Sciences*, 52, 152–161.

Gale, D. (1960), "The theory of linear economic models." *Mc Graw Hill*, New York.

Gul, F. and E. Stacchetti (1999), "Walrasian equilibrium with gross substitutes." *Journal of Economic Theory*, 87, 95–124.

Jaume, D., J. Massó, and A. Neme (2012), "The multiple-partners assignment game with heterogeneous sales and multi-unit demands: competitive equilibria." *Mathematical Methods of Operations Research*, 76, 161–187.

Kelso, A. and V. Crawford (1982), "Job matching, coalition formation and gross substitutes." *Econometrica*, 50, 1483–1504.

Massó, J. and A. Neme (2014), "On cooperative solution of a generalized assignment game: Limits theorems to the set of competitive equilibria." *Journal of Economic Theory*, 154, 187–215.

Muto, S., M. Nakayama, J. Potters, and S. Tijs (1988), "On big boss games." *The Economic Studies Quaterly*, 39, 303–321.

Shapley, L. and M. Shubik (1972), “The assignment game I: the core.” *International Journal of Game Theory*, 1, 111–130.

Sotomayor, M. (2002), “A labor market with heterogeneous firms and workers.” *International Journal of Game Theory*, 31, 269–283.

Sotomayor, M. (2007), “Connecting the cooperative and competitive structures of the multiple-partners assignment game.” *Journal of Economic Theory*, 134, 155–174.

Stuart, H. (2007), “Buyer symmetry in monopoly.” *International Journal of Industrial Organization*, 25, 615–630.

Vickrey, W. (1961), “Counterspeculation, auctions and competitive sealed tenders.” *Journal of Finance*, 16, 8–37.