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ANALYSIS OF SOME DEGENERATE QUADRUPLE COLLISIONS

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Analysis of some degenerate quadruple collisions

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Abstract.- We consider the trapezoidal problem of four bodies. This is a special problem where only three degrees of freedom are involved. The blow up method of McGehee can be used to deal with the quadruple collision. Two degenerate cases are studied in this paper: the rectangular and the collinear problems. They have only two degrees of freedom and the analysis of total collapse can be done in a way similar to the one used for the collinear and isosceles problems of three bodies. We fully analyze the flow on the total collision manifold, reducing the problem of finding heteroclinic connections to the study of a single ordinary differential equation. For the collinear case from which arises a one parameter family of equations the analysis for extreme values of the parameter is done and numerical computations fill up the gap for the intermediate values. Dynamical consequences for possible motions near total collision as well as for regularization are obtained.

§1. Introduction. The trapezoidal problem of four bodies consists in the description of the motion of four particles of mases $m_1, m_2 = m_1, m_3, m_4 = m_3$ with initial coordinates (a,b), (-a,b), (c,d) and (-c,d), respectively and velocities such that the symmetry of coordinates is keeped for all time. We can suppose that the center of masses remains at the origin, i.e., $m_1 b + m_3 d = 0$. New variables $x=2a$, $y=2c$, $z=b-d$ can be introduced (see fig. 1). The motions near quadruple collision for that problem have been partially described

fig. 1 here

in [5]. In order to give a complete picture of the flow on the total collision manifold we restrict ourselves to two degenerate cases: the rectangular and the collinear. In the first the four masses are equal and a=c, b=d. In the second one has b=d=0 but we still have one parameter: the mass ratio $\alpha = m_2/m_1$. Then the total colli-- sion manifold is two dimensional (see [6] and [1]) and the invariant manifolds associated to the critical points are one dimensional. The study can be done on the same lines as the one found in [6] and [7] for the collinear three body problem, or in [1], [8] and [2] for the isosceles problem. However the analysis of the behavior of the invariant manifolds is done using a single ordinary differential equation. A similar method was formulated in [4] and [3].

§2. The rectangular case. First we set the masses equal to one for the bodies. We write down the Lagrangian

$$L = \tfrac{1}{2}(\dot x^2 + \dot y^2) - \frac{2}{x} - \frac{2}{y} - \frac{2}{(x^2+y^2)^{1/2}}$$

and the corresponding Hamiltonian

$$H = \tfrac{1}{2}(p_x^2+p_y^2) - \frac{2}{x} - \frac{2}{y} - \frac{2}{(x^2+y^2)^{1/2}}$$

where the coordinates are described in fig. 2

fig. 2 here

The resulting Hamilton equations are

$$\dot x = p_x, \qquad \dot p_x = -\frac{2}{x^2} - \frac{2x}{\zeta^3},$$

$$\dot y = p_y, \qquad \dot p_y = -\frac{3}{y^2} - \frac{2y}{\zeta^3},$$

where $\zeta = (x^2+y^2)^{1/2}$.

Let us introduce the change of variables (see [6]):

$$X = x\zeta^{-1}, \qquad Y = y\zeta^{-1}, \qquad P_X = p_x\zeta^{1/2}, \qquad P_Y = p_y\zeta^{1/2}, \qquad ' = \frac{d}{d\tau} = \zeta^{3/2}\frac{d}{dt}$$

Then we have $X^2+Y^2 = 1$ and $\dot\zeta = \zeta^{-1}(xp_x+yp_y)$. Introducing $V=XP_X + +YP_Y$ we get the blown up equations:

$$X' = P_X - XV \qquad P_X' = -\frac{2}{X^2} - 2X + \tfrac{1}{2} VP_X ,$$
$$Y' = P_Y - YV \qquad P_Y' = -\frac{2}{Y^2} - 2Y + \tfrac{1}{2} VP_Y ,$$

On $\zeta=0$ (total collision manifold C) the equations are still regular and we shall use the description of the flow on C to get information about passage near total collision. As we know the change of variables is a diffeomorphism for $\zeta>0$. The change has blown up the point x=y=o to the manifold C. This has no physical meaning nor the fact that the new time τ on C is obtained by and infinite slowing down of the physical time. However, the regularity of the equations on C gives informations for small positive values of ζ and this has a clear physical importance.

On C the equations of the energy is $\tfrac{1}{2}(P_X^2+P_Y^2)-U = 0$ where $U = \frac{2}{X} + \frac{2}{Y} + 2$ and we get $V' = U - \frac{V^2}{2}$.

The equilibrium points are $U_c = U_{min} = (X=Y=1/\sqrt{2}) = 2+4\sqrt{2}$ and $V_c = \pm\sqrt{8\sqrt{2}+4}$.

We introduce a new change of coordinates: $X = \cos\theta$, $Y = \sin\theta$ and therefore

$$X' = -\sin\theta\cdot\theta' \quad , \quad \theta' = -\frac{P_X - XV}{\sin\theta} .$$

But $P^2 = P_X^2 + P_Y^2 = 2U$ and $\operatorname{Arg}\vec{P} = \gamma$ allow us to write $P\cos(\gamma-\theta)=$ $= V$. Therefore

$$P_X = P\cos\gamma = V\cos\theta - \sqrt{P^2-V^2}\,\sin\theta .$$

After substitution we have

$$\theta' = \pm\sqrt{2V'} \qquad V' = U(\theta) - \frac{V^2}{2} , \quad U(\theta) = \frac{2}{\cos\theta} + \frac{2}{\sin\theta} + 2$$

that we integrate from $\theta = \pi/4$, $V = -\sqrt{8\sqrt{2}+4}$ on to obtain the unstable manifold W_1^u of the lower equilibrium point A (see fig. 3). Now we have several possibilities for studying the equations of the manifold. We can obtain $d\theta/dV$ (see §3) or we can use the arc para meter σ along W_1^u as independent variable. The new equations are

$$\frac{dV}{d\sigma} = (1 + 2/V')^{-1/2}\ , \qquad \frac{d\theta}{d\sigma} = \pm(1 + V'/2)^{-1/2}\ ,$$

avoiding all the singularities. The change of sign in $\frac{d\theta}{d\sigma}$ is produced when $\theta=0$ or $\pi/2$.

fig. 3 here

§3. Numerical computations and analytical estimations for the rectangular case. The last equations have been integrated starting at A up to arriving to $V=0$ (point B). The values obtained are $\theta(B)= 0.5877$, $\sigma(B) = 4.459$. It is clear, using the symmetry with respect to $\theta = \pi/4$ and $V=0$, that to have a connection between lower, A, and upper, D, equilibrium points requires $\theta(B)$ be a multiple of $\pi/4$. The value 0.5877 is quite different from 0 and $\pi/4$. However for people who deslikes results obtained through numerical integration we offer a proof of the fact that $W^s_D \not\subset W^u_A$ that involves only inequalities and a few evaluations of trigonometric and hyperbolic functions.

Dividing θ' by V' we have

$$\left|\frac{d\theta}{dV}\right| = \sqrt{2/V'} = \sqrt{1/\sec\theta+\operatorname{cosec}\theta+1-\frac{V^2}{4}}\ .$$

we intend to show that starting at B_1 and going backwards we reach the curve $V=\sqrt{2U(\theta)}$ to the right of the point A and starting at B_2 we reach $\theta=\pi/4$ above the point A (see fig. 4)

fig. 4 here

To prove the first part we show that this is true for a vectorfield F such that $\left|\frac{d\theta}{dV}\right| \leqslant F$ and for the second, that it is true for a G such that $\left|\frac{d\theta}{dV}\right| \geqslant G$.

Let $k_i = \min_{\theta\in[\theta_i,\theta_{i+1}]} (\sec\theta + \operatorname{cosec}\theta)$. In this range of θ we take $F = 1/\sqrt{k_i+1-\frac{V^2}{4}}$. If we set $\frac{d\theta}{dV} = F$ one has $\Delta\theta = \int_{V_i}^{V_{i+1}} dV/\sqrt{k_i+1-\frac{V^2}{4}}$. Letting $V = 2\sqrt{k_i+1}\sin\alpha$ we obtain $\Delta\theta = 2\Delta\alpha$. Now we split the range of θ in the following set of intervals (in degrees) [0°,5°] , [5°,15°],[15°,30°],[30°,60°],[60°,75°],[75°,85°],[85°,95°],[95°,105°] [105°,120°],[120°, 135°]. For angles greater then 90° we take the symmetrical with respect to 90°. The points separing intervals are $\theta_0=0°$, $\theta_1=5°$, ..., $\theta_9=120°$, $\theta_{10}=135°$. At each one of such points we shall compute V_i. Note that for each V_i we have two values of α, α_{2i-1}, α_{2i}, depending on the value of k_i used, the one related to the left or right interval. Using symmetry and convexity $k_0 = \sec 5° + \operatorname{cosec} 5° = k_6$, $k_1 = 2\sqrt{6} = k_5 = k_7$, $k_2 = 2+2/\sqrt{3} = = k_4 = k_8$, $k_3 = 2\sqrt{2} = k_9$.

We set up te recurrence $\alpha_{2i+1} = \alpha_{2i} + (\theta_{i+1}-\theta_i)/2$, $\sqrt{k_i+1}\sin\alpha_{2i+1} = \sqrt{k_{i+1}+1}\sin\alpha_{2i+2}$, $i = 0 \div 8$, starting with $\alpha_0=0$. A few computations of trigonometric functions give the values $\alpha_1 = \pi/72$, $\alpha_3 = 0.153246335$, $\alpha_5 = 0.313807297$, $\alpha_7 = 0.589183175$, $\alpha_9 = 0.693534686$, $\alpha_{11} = 0.653534211$, $\alpha_{13} = 0.501227454$, $\alpha_{15} = .900181094$ $\alpha_{17} = 1.335010689$ and then we obtain $\sin\alpha_{18} > 1$ showing that under F we reach the value $V=V_c$ to the right of point A.

Now we proceed to study the solutions of $\frac{d\theta}{dV} = G$ starting at $V = 0$, $\theta = \pi/4$. Consider the interval $[a,b] \subset [0,\pi/4]$. Suppose that $V(a) < V(b)$. Then we take as $1/G$ the function $\sqrt{\frac{d}{\theta} + \sec(a) + 1 - \frac{V(a)^2}{4}}$ where $d = b/\sin b$. We have $\Delta V = \int_a^b 1/G(\theta)\, d\theta$. Let $\varphi(m,\theta) = \sqrt{\theta+m\theta^2} + \frac{1}{\sqrt{m}} \operatorname{argtanh} \sqrt{\frac{m\theta}{1+m\theta}}$ if $m>0$ and $\sqrt{\theta+m\theta^2} + \frac{1}{\sqrt{-m}} \operatorname{arctg} \sqrt{\frac{-m\theta}{1+m\theta}}$ if $m < 0$. Define $g = \sec(a)+1-\frac{V(a)^2}{4}$. Then $\Delta V = \sqrt{d}(\varphi(\frac{g}{d},b) - \varphi(\frac{g}{d},a))$. When θ goes from $\pi/4$ to $\pi/2$ and again to $\pi/4$ and V decreases, the variation ΔV is equal to the variation obtained going from $\pi/4$ to 0 and again to $\pi/4$. Using the partition $[\pi/6,\pi/4]$, $[\pi/12, \pi/6]$, $[\pi/36, \pi/12]$, $[0,\pi/36]$ twice (the same partition used for F) we have

$$\Delta V = \sum_{i=1}^{8} \sqrt{di}\,\left(\varphi\left(\frac{g_i}{di}, \theta_{i+1}\right) - \varphi\left(\frac{g_i}{di}, \theta_i\right)\right),$$

where in g_i the value V is taken as $\sum_{j=1}^{i-1}$. The values θ_i are $\pi/4$, $\pi/6$, $\pi/12$, etc. The evaluation of the required inverse trigonometric and hyperbolic functions gives $\Delta V = 3.856090805 < \sqrt{4+8\sqrt{2}} = |V_c|$ proving that under G we reach $\theta = \pi/4$ on a point above A, as desired. As a conclusion we have proved the following result.

Theorem 2.1. The right part of the invariant unstable manifold of the lower equilibrium point A reaches the value $V=0$ for $\theta \in (0, \pi/4)$.

The conclusions about the remaining part of W_1^u can be obtained by symmetry. After a sequence of binary collisions (of course, couples of simultaneous double collisions) of types 1 and 2 (see fig. 5) slightly below or above the quadruple collision point A, the bodies escape as shown in fig. 5. A similar behavior is obtained for left hand side collisions.

fig. 5 here

§4. The collinear case. Let $m_1=m_2=1$, $m_3=m_4=\alpha$ be the masses of the four bodies and $x, -x, y/\sqrt{\alpha}, -y/\sqrt{\alpha}$ the coordinates (see fig. 6). We write down again the Lagrangian

$$L = \dot{x}^2 + \dot{y}^2 - \frac{1}{2x} - \frac{\alpha^{5/2}}{2y} - \frac{2\alpha^{3/2}}{y-x\sqrt{\alpha}} - \frac{2\alpha^{3/2}}{y+x\sqrt{\alpha}}$$

and the corresponding Hamiltonian, setting, $p_x=2\dot{x}$, $p_y=2\dot{y}$,

$$H = \frac{p_x^2}{y} + \frac{p_y^2}{y} - \frac{1}{2x} - \frac{\alpha^{5/2}}{2y} - \frac{2\alpha^{3/2}}{y-x\sqrt{\alpha}} - \frac{2\alpha^{3/2}}{y+x\sqrt{\alpha}} = \frac{p_x^2}{y} + \frac{p_y^2}{y} - U(x,y).$$

fig. 6 here

Introducing $\zeta = (2x^2+2y^2)^{½}$ and the same change of coordinates of §2 we get again

$$X' = \frac{P_X}{2} - XV\,, \quad P_X' = -\frac{1}{2X^2} - \frac{2\alpha}{(\frac{Y}{\sqrt{\alpha}}+X)^2} + \frac{2\alpha}{(\frac{Y}{\sqrt{\alpha}}-X)^2} + ½\, V\, P_X\,,$$

$$Y' = \frac{P_Y}{2} - YV\,, \quad P_Y' = -\frac{\alpha^{5/2}}{2Y^2} - \frac{2\,\alpha^{1/2}}{(\frac{Y}{\sqrt{\alpha}}-X)^2} - \frac{2\alpha^{1/2}}{(\frac{Y}{\sqrt{\alpha}}+X)^2} + ½\, V\, P_Y\,,$$

where $V = XP_X + YP_Y$ as before. We have again $V' = U - ½\, V^2$ on $\zeta=0$.

Introducing $X = \frac{1}{\sqrt{2}} \cos\theta$, $Y = \frac{1}{\sqrt{2}} \sin\theta$, the equation $\theta' = \pm\sqrt{2V'}$ is obtained.

The equilibrium points are obtained in the following way: let $z = y/\sqrt{\alpha}$. From $\frac{\ddot{x}}{x} = \frac{\ddot{z}}{z}$ and letting $z = \mu x$ we have

$$\alpha = \frac{\mu^3(\mu^2-1)^2-8\mu^2(\mu^2+1)}{17\ \mu^4-2\mu^2+1}.$$

When α ranges from 0 to ∞ the parameter μ does between μ_o and ∞, where μ_o is the zero of $\mu(\mu^2-1)^2 = 8(\mu^2+1)$ (approximately μ_o=2.396812289). The minimum value of θ is given by θ_o=arctg$\sqrt{\alpha}$ and the critical one by θ_c = arctg $(\mu\sqrt{\alpha})$.

In order to study the connection of the invariant manifolds starting at points $(\theta_c, \pm\sqrt{2U(\theta_c)})$ we introduce a new change of coordinates (only useful for this purpose). Let $a = \frac{\pi/2-\theta_o}{2}$, $b = \frac{\pi/2+\theta_o}{2}$ and $\theta = b+a \sin\gamma$. Then we get

$$\frac{dV}{d\gamma} = \pm\frac{a}{\sqrt{2}} \sqrt{V'\cos^2\gamma}$$

and

$$V'\cos^2\gamma = \left(\frac{\alpha^{5/2}}{\sqrt{2}\sin\theta} + \frac{(2\alpha)^{3/2}\cos\theta_o}{\sin(\theta+\theta_o)} - \frac{V^2}{2}\right)\cos^2\gamma + \frac{1/\sqrt{2}\cos^2\gamma}{\sin(a(1-\sin\gamma))} + \frac{(2\alpha)^{3/2}\cos\theta_o\cos^2\gamma}{\sin(a(1+\sin\gamma))} .$$

The term $\frac{\cos^2\gamma}{\sin(a(1-\sin\gamma))}$ (and the one with the + sign in a similar way) has an avoidable singularity. If $\gamma = \frac{\pi}{2} + \varepsilon$, for instance, we merely write

$$\frac{\cos^2\gamma}{\sin(a(1-\sin\gamma))} = \frac{4\cos^2\varepsilon/2}{\frac{\sin(2a\psi)}{\psi}} ,$$

where $\psi = \sin^2\varepsilon/2$ and compute $\frac{\sin(2a\psi)}{\psi}$ as $2a-\frac{4}{3}a^3\psi^2+\frac{4}{15}a^5\psi^4 - \frac{8}{315}a^7\psi^6 + \ldots$ The computation must be started with $\gamma = \gamma_c = \arcsin\left(\frac{\theta_c-b}{a}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $V_c = -\sqrt{2U(\theta_c)}$. In $\frac{dV}{d\gamma}$ the + sign is used for the unstable manifold (right branch) and the - sign (with γ decreasing) for the left branch.

§ 5. Numerical computations for the collinear case. Using the equation numerically regularized as described in § 4 we have computed the point $\gamma_+ (\gamma_-)$ where the right (left) branch of the unstable manifold of the lower equilibrium point reaches the value V=0.

The independent parameter has been the parameter μ . Table 1 shows some results. Figure 7 offers a rough representation of $\gamma_\pm$ as a function of α including the region of small values of α. The computations have been done using a RK routine of fourth order with step equal to 0.02. Some errors can be introduced for this value of the step for large values of α.

fig. 7 here

In order to study the possible motions on the total collision manifold as a function of α we need the connections between the equilibrium points. For $\gamma_+ = \frac{2k-1}{2}\pi$, $k \in \mathbb{N}$ or $-\gamma_- = \frac{2k+1}{2}\pi$, one of the branches of W_1^u coincides with one of the branches of W_u^s. For $\gamma_+ - \gamma_- = 2k\pi$, $k \in \mathbb{N}$, both branches meet due to the symmetry. Table 2 offers some values of α for which such connections are established.

6. Analytical study of the limiting cases. We study the behavior of γ_+, γ_- and, incidentally, γ_c, V_c for $\alpha \downarrow 0$, $\alpha \uparrow \infty$.

For $\alpha=0$ we have $\theta_o=\theta_c=0$, $V_c=-2^{1/4}$, $a=b=\pi/4$, $\gamma_c=-\pi/2$. The differential equation is $\frac{dV}{d\theta} = \frac{1}{2}\sqrt{\sqrt{2}\sec\theta - V^2}$, and scaling $V=2^{1/4}\bar{V}$ we get $\bar{V}^2+(\frac{d\bar{V}}{d\theta})^2 4 = \sec\theta$ with $\bar{V} = -1$ for $\theta=0$.

Lemma. The solution of $\bar{V}^2 + 4(\frac{d\bar{V}}{d\theta})^2 = \sec\theta$ such that $\bar{V}(0) = -1$ reaches $\bar{V}=0$ for $\theta = \pi/2$.

Proof: It is enough to check that the solution is given by $\bar{V}(\theta) = -\sqrt{\cos\theta}$.

Corollary. For $\alpha=0$ we have $\gamma_+ = \pi/2$, $-\gamma_- = 3\pi/2$.

μ	α	γ_c	V_c	γ_+	γ_-	μ	α	γ_c	V_c	γ_+	γ_-
2.39682	.000008	-1.4718	-1.1892	1.5712	-4.7120	3.3	1.24254	.2775	-5.0787	7.4640	-6.5645
2.39685	.000036	-1.4244	-1.1894	1.5729	-4.7119	3.4	1.43195	.3126	-5.5993	8.1957	-6.8339
2.3969	.000086	-1.3886	-1.1896	1.5752	-4.7118	3.5	1.63314	.3442	-6.1455	8.7929	-7.1057
2.397	.000183	-1.3501	-1.1900	1.5789	-4.7116	3.6	1.84650	.3729	-6.7191	9.2465	-7.3841
2.398	.001155	-1.2179	-1.1940	1.6041	-4.7107	3.7	2.07239	.3992	-7.3219	9.6424	-7.6717
2.400	.003103	-1.1153	-1.2019	1.6416	-4.7093	3.8	2.31119	.4233	-7.9556	10.016	-7.9690
2.402	.005055	-1.0534	-1.2099	1.6733	-4.7082	3.9	2.56326	.4457	-8.6220	10.385	-8.2741
2.406	.008966	- .9692	-1.2259	1.7297	-4.7064	4.0	2.82898	.4666	-9.3229	10.761	-8.5845
2.412	.014856	- .8833	-1.2498	1.8054	-4.7044	4.5	4.37492	.5531	-13.410	12.715	-10.578
2.420	.022752	- .8011	-1.2815	1.8977	-4.7026	5.0	6.31619	.6195	-18.648	15.986	-13.400
2.430	.032692	- .7237	-1.3212	2.0055	-4.7013	5.5	8.69752	.6730	-25.294	18.383	-15.688
2.438	.040698	- .6736	-1.3523	2.0882	-4.7011	6.0	11.5634	.7176	-33.628	22.057	-19.281
2.45	.052802	- .6106	-1.4002	2.2087	-4.7018	6.5	14.9582	.7556	-43.946	24.966	-22.134
2.46	.062974	- .5660	-1.4396	2.3070	-4.7034	7.0	18.9261	.7887	-56.564	28.993	-26.150
2.47	.073226	- .5266	-1.4789	2.4043	-4.7057	7.5	23.5114	.8178	-71.813	32.716	-30.459
2.48	.083555	- .4913	-1.5182	2.5008	-4.7089	8.0	28.7583	.8438	-90.042	37.139	-34.200
2.49	.093965	- .4592	-1.5575	2.5969	-4.7128	8.5	34.7109	.8672	-111.616	41.624	-38.698
2.5	.104456	- .4297	-1.5967	2.6930	-4.7174	9.0	41.4134	.8883	-136.917	46.709	-43.730
2.6	.213869	- .2215	-1.9896	3.6463	-4.8028	9.5	48.9100	.9077	-166.343	50.925	-47.973
2.7	.331810	- .0904	-2.3877	4.5299	-4.9564	10.0	57.2446	.9254	-200.308	56.344	-53.323
2.8	.458723	.0051	-2.7955	5.1956	-5.1746	11.99	99.5948	.9838	-389.219	78.643	-75.591
2.9	.595040	.0800	-3.2163	5.6878	-5.4415	12.00	99.8476	.9841	-390.416	78.726	-75.671
3.0	.741177	.1414	-3.6528	6.1072	-5.7289	12.01	100.1008	.9843	-391.616	78.808	-75.755
3.1	.897541	.1933	-4.1074	6.5080	-6.0150	20.0	467.9	1.1196	-2610.1	191.0	-187.9
3.2	1.06453	.2381	-4.5821	6.9345	-6.2932	30.0	1584.4	1.2034	-11917.9	366.9	-363.8

Table 1

k	α_k	type
1	.09297	$-\gamma_- = 3\pi/2$
2	.36153	$\gamma_+ = 3\pi/2$
3	.90788	$\gamma_+ - \gamma_- = 4\pi$
4	1.3452	$\gamma_+ = 5\pi/2$
5	2.2181	$-\gamma_- = 5\pi/2$
6	2.6362	$\gamma_+ - \gamma_- = 6\pi$
7	2.9986	$\gamma_+ = 7\pi/2$
8	4.4984	$-\gamma_- = 7\pi/2$
9	4.8210	$\gamma_+ - \gamma_- = 8\pi$
10	5.1229	$\gamma_+ = 9\pi/2$
11	7.0515	$-\gamma_- = 9\pi/2$
12	7.3237	$\gamma_+ - \gamma_- = 10\pi$
13	7.5859	$\gamma_+ = 11\pi/2$
14	9.8469	$-\gamma_- = 11\pi/2$
15	10.0878	$\gamma_+ - \gamma_- = 12\pi$
16	10.3230	$\gamma_+ = 13\pi/2$
17	12.8688	$-\gamma_- = 13\pi/2$
18	13.0880	$\gamma_+ - \gamma_- = 14\pi$
19	13.3035	$\gamma_+ = 15\pi/2$
20	16.1072	$-\gamma_- = 15\pi/2$
21	16.3105	$\gamma_+ - \gamma_- = 16\pi$
22	16.5115	$\gamma_+ = 17\pi/2$
23	19.5572	$-\gamma_- = 17\pi/2$
24	19.7484	$\gamma_+ - \gamma_- = 18\pi$
25	19.9379	$\gamma_+ = 19\pi/2$
26	23.215	$-\gamma_- = 19\pi/2$
27	23.397	$\gamma_+ - \gamma_- = 20\pi$
28	23.578	$\gamma_+ = 21\pi/2$
29	27.080	$-\gamma_- = 21\pi/2$
30	27.254	$\gamma_+ - \gamma_- = 22\pi$
31	27.427	$\gamma_+ = 23\pi/2$

Table 2

fig. 8 here

Now we study what happens for $\alpha > 0$ sufficiently small. First of all we have, approximately $\theta_o = \sqrt{\alpha}$, $\theta_c = \mu_o\sqrt{\alpha}$. Therefore $U(\theta_c) = \frac{1}{\sqrt{2}} + \alpha(\frac{1}{\sqrt{2}} \;\; \frac{\mu_o^2}{2} + \frac{2^{3/2}}{\mu_o+1} + \frac{2^{3/2}}{\mu_o-1})\alpha + O(\alpha^2)$ and, $V_c = -\sqrt[4]{2} \; -\sqrt[4]{2}\,(\frac{\mu_o^2}{4} + \frac{2}{\mu_o+1} + \frac{2}{\mu_o-1})\alpha + O(\alpha^2)$. In order to check the behavior observed in §5 we have to prove two things: $\gamma_+ > \pi/2$, $-\gamma_- < 3\pi/2$.

We start at $P(V=0,\ \theta=\pi/2)$ and follow the differential equation $\frac{dV}{d\theta} = \sqrt{\frac{U(\theta)}{2} - \frac{V^2}{4}}$ backwards.

Writting down $V = -2^{1/4}\sqrt{\cos\theta} + w$, $w(\pi/2) = 0$ and retaining first order terms we get:

$$\frac{dw}{d\theta} = \frac{w}{2^{5/4}}\;\frac{\cos\theta}{\sin\theta} + \frac{2\alpha^{3/2}\cos^{1/2}\theta}{\sin\theta}\left(\frac{1}{\sin(\theta+\theta_o)} + \frac{1}{\sin(\theta-\theta_o)}\right).$$

The solution of the homogeneous equation is $w = C(\sin\theta)^{-1/2^{5/4}}$ and the method of variation of the constants gives us

$$\frac{dc}{d\theta} = \frac{2\alpha^{3/2}\cos^{1/2}\theta}{(\sin\theta)^{(1-1/2^{5/4})}}\left(\frac{1}{\sin(\theta+\theta_o)} + \frac{1}{\sin(\theta-\theta_o)}\right).$$

Therefore $w(\theta_c) = -(\mu_o\sqrt{\alpha})^{-1/2^{5/4}}\,\Delta c$, where $\Delta c = \int_{\theta c}^{\pi/2}\frac{dc}{d\theta}$. The value Δc can be estimated in the following way $\int_{\theta}^{\pi/2} = \int_{\theta c}^{z} + \int_{z}^{\pi/2}$, where z is a small but finite quantity and so

$$\int_{z}^{\pi/2} = O(\alpha^{3/2}).$$

It remains to compute the main contribution $\int_{\theta c}^{z}$. We bound $\cos^{1/2}\theta$ by 1, put $\frac{1}{\sin(\theta+\theta_o)} + \frac{1}{\sin(\theta-\theta_o)} < \frac{2\sin\theta}{\sin(\theta+\theta_o)\sin(\theta-\theta_o)}$ and approximate the sinus by the angles. We get

$$\Delta c \simeq \int_{\theta c}^{z}\frac{4\alpha^{3/2}\theta^{1/2^{5/4}}}{\theta^2-\alpha} < 4\alpha^{3/2}\;\frac{\mu_o^2}{\mu_o^2-1}\int_{\theta c}^{z}\theta^{-2+1/2^{5/4}} \simeq$$

$$\simeq \frac{4\alpha^{3/2}\,\mu_o^2}{(1-1/2^{5/4})(\mu_o^2-1)}\;(\mu_o\sqrt{\alpha})^{-1+1/2^{5/4}} .$$

Then $w(\theta_c) \approx -\dfrac{4\ \alpha\mu_o}{(1-1/2^{5/4})(\mu_o^2-1)}$. As $\dfrac{4\mu_o}{(1-1/2^{5/4})(\mu_o^2-1)} <$ $< 2^{1/4}\left(\dfrac{\mu_o^2}{4} + \dfrac{2}{\mu_o+1} + \dfrac{2}{\mu_o-1}\right)$ the point Q (fig. 8) is above the equilibrium point $(\theta_c,\bar{V}_c)$, showing that $\gamma_+ > \pi/2$.

Now let us look for the point R (see fig. 8). The first order terms for γ_c give us $\gamma_c = -\frac{\pi}{2} + \sqrt{\frac{8(\mu_o-1)}{\pi}}\ \alpha^{1/4}$. On the other side the main term in $\frac{dV}{d\gamma}$ is $-\sqrt{\pi/4\sqrt{2}}$ near the left hand side collision. Therefore the value of ΔV from the point (θ_c, V_c) to R is $2\sqrt{\pi/4\sqrt{2}}\ (\pi/2+\gamma_c) = \sqrt{8(\mu_o-1)/\sqrt{2}}\ \alpha^{1/4}$, showing that $-\gamma_- < 3\pi/2$. We have proved the following result.

Proposition 6.1. For α small enough $\gamma_+ > \pi/2$, $-\gamma_- < 3\pi/2$.

For α large we have $\theta_o \approx \frac{\pi}{2} - \frac{1}{\sqrt{\alpha}}$, $\theta_c \approx \frac{\pi}{2} - \frac{1}{\sqrt{\alpha}\sqrt{17\alpha}}$. Introducing $\bar{V} = V/\alpha^{5/4}$ and retaining the dominant term in the differential equation we get $V_c \approx -\alpha\sqrt{2\alpha}$, $\dfrac{d\bar{V}}{d\gamma} = \dfrac{a}{\sqrt{2}}\sqrt{1-\dfrac{\bar{V}^2}{\sqrt{2}}}\ |\cos\gamma|$ where $a = \dfrac{1}{2\sqrt{\alpha}}$. Then

$$\int_{-\sqrt{2}}^{o} \frac{d\bar{V}}{\sqrt{1-\bar{V}^2/2}} = \frac{a}{\sqrt{2}}\ T\ \frac{2}{\pi} ,$$

where T is the γ interval and $\frac{2}{\pi}$ is the average value of $|\cos\gamma|$. We get immediately $T = 2^{-1/4}\pi^2\alpha^{1/2}$. We state the result, showing good agrement with table 1.

Proposition 6.2. For α sufficiently large $\gamma_+ \approx \pi^2\ 2^{-1/4}\alpha^{1/2}$ and $\gamma_+ + \gamma_- \to \pi$.

Corollary 6.3. There are infinite values for which the left hand branch of W_1^u coincides with the left hand branch of W_u^s and for which the right hand one coincides with the right hand one and for which $W_1^u \equiv W_u^s$. In the last case the left hand branch of W_1^u coincides with the right hand one of W_u^s and viceversa.

The first one values for which those coincidences are obtained were given in table 2.

§7. Some dynamical consequences. Let α_1 be the unique value $\alpha > 0$ such that $-\gamma_-(\alpha_1) = 3\pi/2$, α_2 such that $\gamma_+(\alpha_2) = 3\pi/2$, α_3 such that $\gamma_+(\alpha_3) - \gamma_-(\alpha_3) = 4$, etc. Figure 9 shows a qualitative picture of the invariant manifolds of the lower equilibrium point for a initial range of values of α containing those values $(0<\alpha_1< \alpha_2< \alpha_3)$.

Fig. 9 here

The consequences with respect to orbits passing near quadruple collision are now obtained easily in the same way as they where obtained for the rectangular case (see orbits type 1,2 in fig. 3). We recall that other necessary conditions for regularization can be obtained (for the good values of α , i.e., such that $W_1^u \equiv W_u^s$)in the way introduced in [7]. Sufficient conditions will be given in a forthcoming paper [9].

From fig. 9 the way of escaping after approaching a quadruple collision and the number of collisions taking place between central bodies or simulateous double collisions between external bodies can be predicted.

Pictures similar to fig. 9 can be given for the full range of values of α. (Note that according to table 2 there is α_4 similar to α_2, α_5 similar to α_1 and α_k similar to α_{k-3} for all $k \geq 6$).

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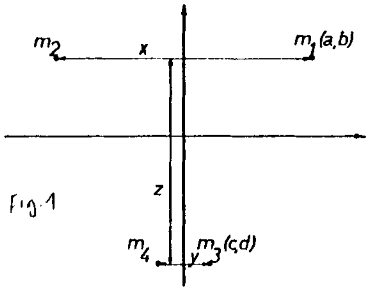


Fig.1

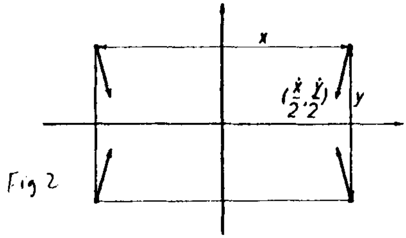


Fig 2

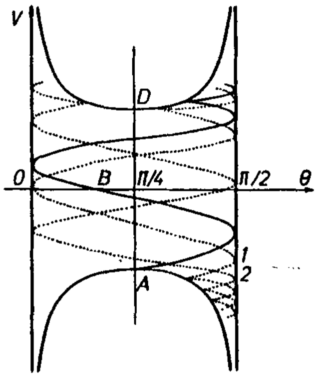
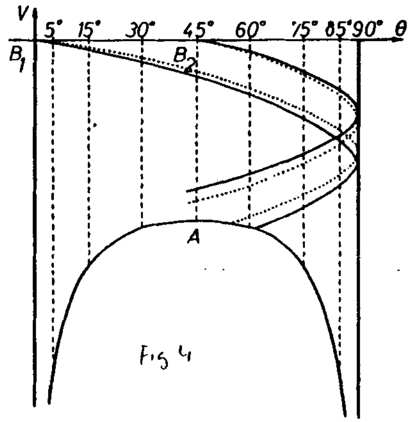


Fig.3



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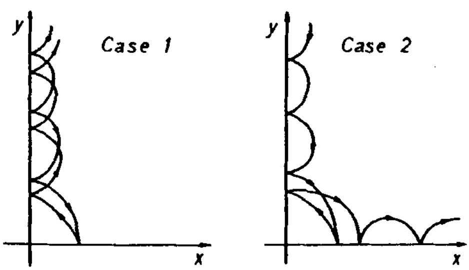
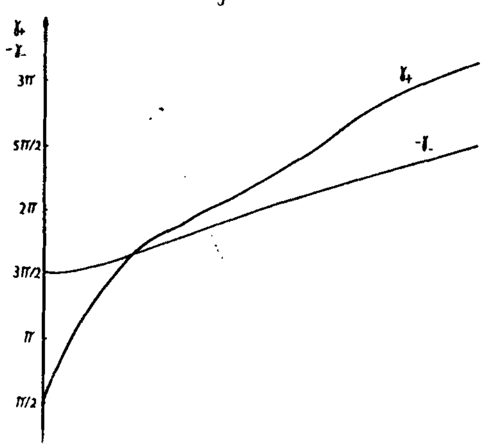
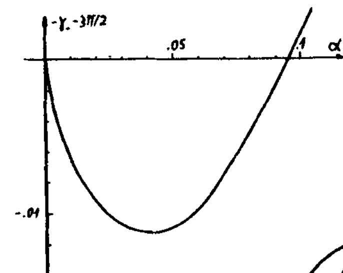
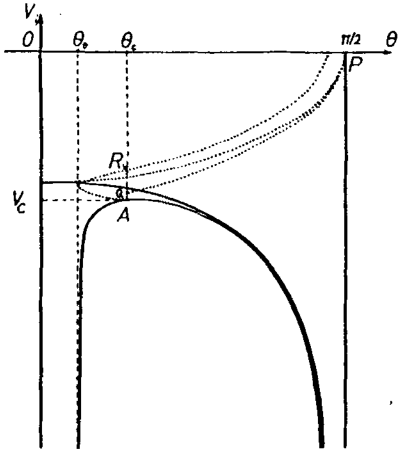


Fig 5

Fig. 6

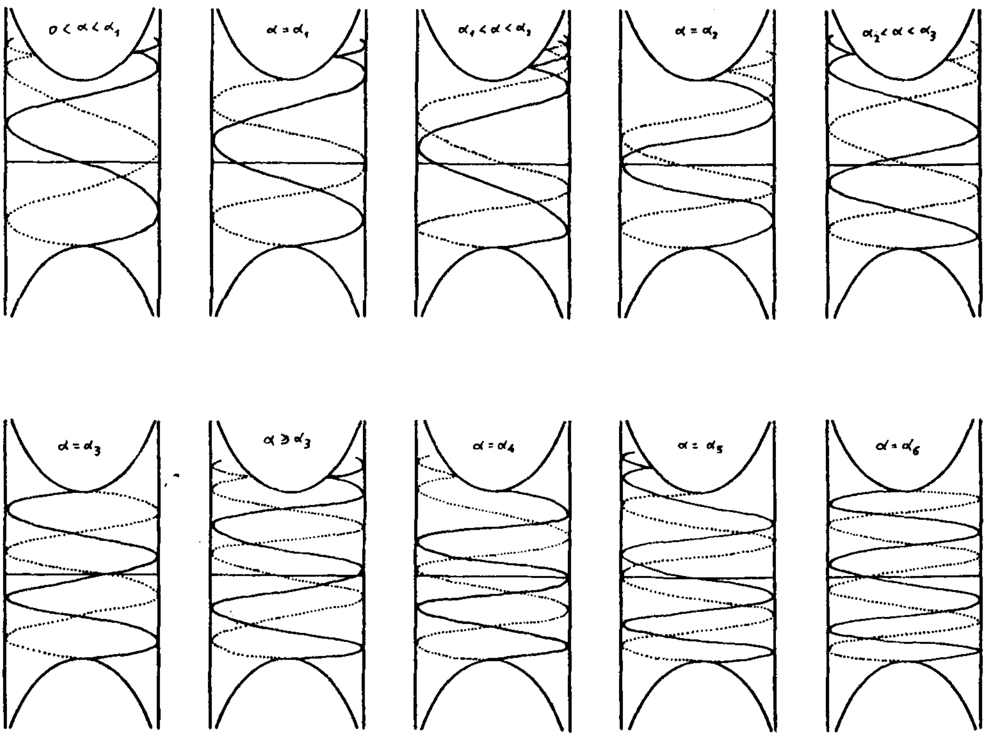






V
0
θ_0
θ_c
$\pi/2$
θ
P
R
V_c
A

Fig 8



$0 < \alpha < \alpha_1$
$\alpha = \alpha_1$
$\alpha_1 < \alpha < \alpha_2$
$\alpha = \alpha_2$
$\alpha_2 < \alpha < \alpha_3$
$\alpha = \alpha_3$
$\alpha \geq \alpha_3$
$\alpha = \alpha_4$
$\alpha = \alpha_5$
$\alpha = \alpha_6$

Fig 9

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