ON THE DISTRIBUTION OF A DOUBLE STOCHASTIC INTEGRAL

by David Nualart
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Abstract.- Let \( \{W(z), z \in [0,1]^2\} \) be a Wiener process with a two-dimensional parameter. We evaluate the characteristic function of the stochastic integral \( \int_{[0,1]^2} W \, dW \) and obtain some properties of its moments. Also, a martingale array having this non-symmetric limit distribution is exhibited.
0. Introduction

The law of the stochastic integral \( \int_0^1 W_t dW_t \), where \( W_t \) is an ordinary Brownian motion can be obtained in an obvious way from Itô's formula:

\[
\int_0^1 W_t dW_t = \frac{1}{2}(W_1^2 - 1).
\]

For a two-parameter Wiener process \( \{W(s,t), (s,t) \in T\} \), Itô's differentiation formula (see Wong and Zakai [11]) claims that

\[
\frac{1}{2}(W_{11}^2 - 1) = \int_T W_t dW_t + \int_T \int_T 1_D dW dW,
\]

where \( D = \{(z,z') \in T \times T: z = (x,y), z' = (x',y'), x \leq x' \text{ and } y \geq y'\} \).

In this case we cannot attain from this expression the distribution of the random variables \( \int_T W_t dW_t \) and \( \int_T \int_T 1_D dW dW \). This paper is devoted to discuss the law of these variables. As we shall see they have the same law.

It is known that the law of a double Wiener stochastic integral can be computed in terms of a weighted sum of independent chi-squared random variables. See, for instance, the papers of Varberg [10] and Rosiński–Szulga [9]. In section 1, using a result of this kind for a two-parameter Wiener process, we deduce the characteristic function of \( \int_T \int_T 1_D dW dW \).

The distribution of the two-parameter Wiener process \( W_{st} \) in the space of continuous functions \( C(T) \) is the weak limit of the law of the sequence of processes \( n^{-1/2} \sum_{i=1}^n X_{st}^i Y_{st}^i \), where \( \{X^n(t), t \in [0,1], n \geq 1\} \) and \( \{Y^n(t), t \in [0,1], n \geq 1\} \) are two independent sequences of infinite dimensional Brownian motions. This result has been proved in [7]. In section 2 we will use this fact to express the indefinite integrals \( J_{st} = \int_{R_{st}} 1_D dW dW \) and \( K_{st} = \int_{R_{st}} W_t dW_t \), where \( R_{st} = [0,s] \cup [0,t] \), as the weak limit of a sequence of two-parameter continuous processes. This provides a method to compute the moments of the random variable \( J_{11} \).

The sequence of random variables converging to \( J_{11} \) can be arranged in order to exhibit an example of a martingale array \( \{X_{ni}, n \geq 1, i=1,\ldots,k_n\} \), with respect to a family of \( \sigma \)-fields \( F_{ni} \), satisfying the conditional Lindeberg condition...
The asymptotic behavior of this martingale array is similar to that of the class of degenerate U-statistics discussed by Alvo, Cabilio and Feigin in [1]. Indeed, it is proved that the sequence of conditional variances converges in distribution, as long as \( \sum_{i=1}^{n} X_{ni} \) converges in law to the non-symmetric random variable \( J_1 \).

1. Let \( W = \{ W(s, t), (s, t) \in T, T = [0, 1]^2 \} \) be a two-parameter Wiener process in a probability space \( (\Omega, F, P) \). For any function \( f \in L^2(T \times T) \) the double Itô-Wiener integral \( I(f) \) with respect to \( W \) can be defined as in Itô [6]. This stochastic integral takes into account just the values of \( f \) into the set \( \{(z, z') \in T \times T: z \neq z' \} \), and it verifies \( I(f) = 1(\tilde{f}) \), where \( \tilde{f}(z, z') = \frac{1}{2}(f(z, z') + f(z', z)) \). We are going to recall some known facts about the distribution of \( I(f) \).

Consider an orthonormal basis \( \{ \psi_k \}_{k=1}^{\infty} \) of \( L^2(T) \) and form the development \( \tilde{f}(z, z') = \sum_{j,k=1}^{\infty} a_{jk} \psi_j(z) \psi_k(z') \) of the symmetric function \( \tilde{f} \). Then, \( X_k = \int_T \psi_k dW \) is a sequence of independent standard Gaussian random variables, and we have

**Proposition 1.1.** The sequence \( \sum_{j,k=1}^{n} a_{jk} X_j X_k - \sum_{j=1}^{n} a_{jj} \) converges in quadratic mean to \( I(f) \).

**Proof:** It follows easily from the equalities \( I(\psi_j \psi_k) = (\int_T \psi_j dW) (\int_T \psi_k dW) - \delta_{jk} \).

Now consider the Hilbert-Schmidt operator \( K \) on \( L^2(T) \) given by the symmetric kernel \( \tilde{f}(z, z') \). Denote by \( \{ \mu_k \}_{k=1}^{N} \) (\( N < \infty \) or \( N = \infty \)) the sequence of non-zero eigenvalues of \( K \) (including multiplicities), and let \( \{ \phi_k \}_{k=1}^{N} \) be a sequence of orthonormal eigenfunctions of \( K \).
Proposition 1.2. I(f) has the law of the sum \( \sum_{k=1}^{N} \mu_k (k^2 - 1) \) where \( \{\xi_k\}_{k=1}^{N} \) is a sequence of independent standard Gaussian random variables. In particular, the characteristic function of I(f) can be expressed in terms of a modified Fredholm determinant (see Varberg [10]):

\[
E(e^{i\lambda I(f)}) = (\delta(2i\lambda, \tilde{\lambda}))^{-1/2} = \prod_{k=1}^{N} (1 - 2i\mu_k)^{-1/2} e^{-i\lambda \mu_k}.
\]

Proof: Apply proposition 1.1 to the development \( f(z, z') = \sum_{k=1}^{N} \mu_k \phi_k(z) \phi_k(z') \). In the sequel we will use these results to find the distribution of \( I(D) = \int_T \int_D dW \) dW, being D the set of points \( ((x, y), (x', y')) \) in \( T \times T \) such that \( x \ll x' \) and \( y \gg y' \). For any integers \( j \) and \( k \) set \( a_{jk} = (\frac{n}{2})^{(2j-1)(2k-1)} \).

Proposition 1.3. There exists a sequence \( \{X_{jk}, j, k \in \mathbb{Z}\} \) of independent standard Gaussian random variables such that

\[
I(I_D) = \sum_{j, k \in \mathbb{Z}} a_{jk} X_{jk}^2 + 8(\sum_{j \geq 1, k \in \mathbb{Z}} a_{jk} X_{jk})^2 - \frac{1}{4}.
\]

Proof: Consider the orthonormal basis of \( L^2(T) \) formed by the family of trigonometric functions \( \sqrt{2} \sin((2j-1)\pi x + (2k-1)\pi y), \sqrt{2} \cos((2j-1)\pi x + (2k-1)\pi y), j, k \) integers such that \( j \gg 1 \). Set \( G = \{(z, z') \in T \times T: (z, z') \in D \text{ or } (z', z) \in D\} \).

Then \( I(I_D) = \frac{1}{2} I(I_G) \), and the symmetric function \( \frac{1}{2} I_G \) has the following development

\[
\frac{1}{2} I_G((x, y), (x', y')) = 8 \left[ \sum_{j \geq 1, k \in \mathbb{Z}} a_{jk} \sqrt{2} \cos((2j-1)\pi x + (2k-1)\pi y) \right] + \\
\left[ \sum_{j \geq 1, k \in \mathbb{Z}} a_{jk} \sqrt{2} \cos((2j-1)\pi x + (2k-1)\pi y') \right] + \\
\sum_{j \geq 1, k \in \mathbb{Z}} a_{jk} \sqrt{2} \cos((2j-1)\pi x + (2k-1)\pi y) \cos((2j-1)\pi x + (2k-1)\pi y') + \\
2 \sin((2j-1)\pi x + (2k-1)\pi y) \sin((2j-1)\pi x + (2k-1)\pi y')).
\]

Formula (1.3) can be checked by taking the orthonormal basis \( \{e^{i(2k-1)\pi x}, k \in \mathbb{Z}\} \) in \( L^2([0,1]) \) and computing the coefficients of the Fourier expansion.
\[ \frac{1}{2} \lambda_j \lambda_{j'} \lambda_{k} \lambda_{k'} e^{-i \phi (2j-1)x + (2k-1)y + (2j'-1)x' + (2k'-1)y')} \]

The values of these coefficients are

\[ \lambda_{j,k,j',k'} = 4(n^4(2j-1)(2k-1)(2j'-1)(2k'-1))^{-1} \text{ if } j+j' \neq 1 \text{ or } k+k' \neq 1, \]

and

\[ \lambda_{j,k,1-j,1-k} = 4(n^4(2j-1)^2(2k-1)^2)^{-1} + (n^2(2j-1)(2k-1))^{-1}. \]

Define

\[ X_{j,k} = \int_T \sqrt{2} \cos((2j-1)nx + (2k-1)ny) dW_{xy} \text{ for } j \geq 1, k \in \mathbb{Z}, \text{ and} \]

\[ X_{j,k} = \int_T \sqrt{2} \sin((2j-1)nx + (2k-1)ny) dW_{xy} \text{ for } j < 0, k \in \mathbb{Z}. \]

Then, (1.2) is a consequence of (1.3), using proposition 1.1 and noting that

\[ \sum_{j \geq 1, k \in \mathbb{Z}} \alpha_{j,k}^2 = \frac{1}{4}. \]

**Proposition 1.4.** - The characteristic function of the random variable \( I(1_D) \)

has the following expression

\[ E( e^{it(1_D)}) = \]

\[ e^{-it/4} \prod_{k=1}^{\infty} \cos \left( \frac{it}{(2k-1)\pi} \right)^{-1/2} \left( 1 - 4 \sum_{k=1}^{\infty} \frac{1}{(2k-1)\pi} \tan \left( \frac{it}{(2k-1)\pi} \right) \right). \]  

(1.4)

**Proof:** From proposition 1.3 we obtain the decomposition \( I(1_D) = J_1 + J_2 \), where

\[ J_1 = \sum_{j \in \mathbb{D}, k \in \mathbb{Z}} \alpha_{j,k} X_{j,k}^2 - \frac{1}{4} \]

and

\[ J_2 = \sum_{j \geq 1, k \in \mathbb{Z}} \alpha_{j,k} X_{j,k}^2 + 8(\sum_{j \geq 1, k \in \mathbb{Z}} \alpha_{j,k} X_{j,k})^2 \]

are independent random variables. Then

\[ E(e^{itJ_1}) = e^{-it/4} \prod_{j \in \mathbb{D}, k \in \mathbb{Z}} (1 - 2it\alpha_{j,k})^{-1/2} = \]

\[ e^{-it/4} \prod_{k=1}^{\infty} \cos \left( \frac{it}{(2k-1)\pi} \right)^{-1/2} \].  

(1.5)

In order to compute the characteristic function of \( J_2 \), we put

\[ J_2 = \sum_{j,k=1}^{\infty} \alpha_{j,k} (X_{j,k}^2 - X_{j,-k}^2) + 8(\sum_{j,k=1}^{\infty} \alpha_{j,k} (X_{j,k} - X_{j,-k}))^2 = \]
\[ \sum_{j,k=1}^{\infty} M_{jk} N_{jk} + 8 \left( \sum_{j,k=1}^{\infty} N_{jk} \right)^2, \]

where \( M_{jk} \) and \( N_{jk} \) are independent random variables with distribution \( \mathcal{N}(0,2) \) and \( \mathcal{N}(0,2\alpha_{jk}^2) \), respectively. Thus,

\[
E(e^{itJ_2}) = \lim_{N \to \infty} E\left[ \exp\left(8it\sum_{j,k=1}^{N} N_{jk}\right)^2 - t^2 \sum_{j,k=1}^{N} \alpha_{jk}^2 \right] = \lim_{N \to \infty} \int_{\mathbb{R}^N} \exp\left(8it\sum_{j,k=1}^{N} x_{jk}^2 - t^2 \sum_{j,k=1}^{N} \alpha_{jk}^2 \right) \prod_{j,k=1}^{N} \frac{dx_{jk}}{\sqrt{4\pi\alpha_{jk}}} = \prod_{k=1}^{\infty} \cos \left( \frac{it}{2(k-1)\pi} \right)^{-1/2} \prod_{k=1}^{\infty} \frac{1}{\sin \left( \frac{it}{2(k-1)\pi} \right)},
\]

Finally, (1.4) follows from (1.5) and (1.6).

Note that if we write \( g(t) = \left( \prod_{k=1}^{\infty} \cos \left( \frac{it}{2(k-1)\pi} \right)^{-2} \right), \) then \( E(e^{it1(1_\mathcal{D})}) = e^{-it/4} (g(t)-2ig'(t))^{-1/2}. \) Unfortunately, as far as we know, there is not a simpler or more reduced expression for the function \( g. \)

Although we have already obtained an infinite product expansion for the characteristic function of \( I(1_\mathcal{D}) \), it may be interesting to exhibit the eigenvalues and the eigenfunctions of the integral operator \( \mathcal{K} \) on \( L^2(\mathbb{T}) \) with kernel \( \frac{1}{2} \hat{\mathcal{G}}. \) Observe that they are given by the partial differential equation

\[ \lambda \frac{\partial^2 \psi}{\partial x \partial y} = -\psi(x,y), \quad \psi(0,0) = \psi(1,1) = 0, \quad \psi(0,y) + \psi(1,y) = \psi(x,0) + \psi(x,1) = 1. \]

If a function \( \psi \in L^2(\mathbb{T}) \) has the Fourier development

\[
\psi(x,y) = \sum_{j,k \neq 0} \hat{x}_{jk} \sqrt{2} \cos((2j-1)\pi x + (2k-1)\pi y) + \hat{x}_{-j,k} \sqrt{2} \sin((2j-1)\pi x + (2k-1)\pi y),
\]

then, using the expansion (1.3), we see that the equation \( \mathcal{K}\psi = \lambda \psi \) is equivalent to the system of equations
for all $j \geq 1, k \in \mathbb{Z}$.

From these equations we can deduce the next results, which could also have been derived in a direct form from (1.4) and (1.1).

a) For any integer $h$ put $A_h = \{(j,k) \in \mathbb{Z}^2 : \alpha_{jk} = (\pi^2(2h-1))^{-1}\}$ and denote by $m_h$ the cardinal of $A_h$. Then, the numbers $\lambda = (\pi^2(2h-1))^{-1}$, $h \in \mathbb{Z}$, are eigenvalues of $K$, each one with multiplicity $m_h - 1$. The invariant subspace associated to $\lambda = (\pi^2(2h-1))^{-1}$ is $\psi \in L^2(T)$: $x_{jk} = 0$ for all $(j,k) \notin A_h$, and $\sum_{(j,k) \in A_h, j \geq 1} x_{jk} = 0$.

b) The rest of eigenvalues have multiplicity one and are the solutions of the equation $\frac{1}{8} = \sum_{j \geq 1, k \in \mathbb{Z}} \frac{\alpha_{jk}}{(\lambda - \alpha_{jk})}$, which equivalent to

$$\sum_{k=1}^\infty \frac{1}{(2k-1)\pi} \tan \frac{1}{2\lambda(2k-1)\pi} = \frac{1}{4} \quad (1.7)$$

There is exactly one solution of this equation in every one of the open intervals $(\pi^{-2}, \infty)$ and $(\pi^{-2}(2k+1)^{-1}, \pi^{-2}(2k-1)^{-1})$, $(-\pi^{-2}(2k-1)^{-1}, -\pi^{-2}(2k+1)^{-1})$, $k = 1, 2, \ldots$. Unlike the first ones, these eigenvalues are not symmetrically placed about the origin, and can only be evaluated approximately. For instance, using the development of the function $f(\lambda) = \sum_{k=1}^\infty \frac{1}{(2k-1)\pi} \tan \frac{1}{2\lambda(2k-1)\pi}$ in the interval $(\pi^{-2}, \infty)$, we can find the approximate value 0.276203 for the maximum eigenvalue.
2. Consider the two-parameter continuous processes defined by 
\[ K_{st} = \int_{R_{st}} W_{udW}, (s,t) \in T, \]
and 
\[ J_{st} = \int_{R_{st}} 1 \, dW_{udW}, (s,t) \in T, \]
where \( R_{st} = [0,s] \times [0,t] \). It is known that
\[ W^2_{st} = 2K_{st} + 2J_{st} + st. \]
The processes \( J_{st} \) and \( K_{st} \) are two-parameter martingales
with respect to the natural filtration of \( W \), and \( K_{st} \) is a strong martingale
(cf. [3]).

Let \( \{X^n(t), t \in [0,1], n \geq 1\} \) and \( \{Y^n(t), t \in [0,1], n \geq 1\} \) be two independent
infinite dimensional Brownian motions. We know (cf. [7]) that
\[ \sum_{i=1}^{n-1} X^i Y^i \]
converges weakly to \( W \).

**Lemma 2.1.** The sequences of two-parameter continuous processes

\[ S^n_{st} = n^{-1} \sum_{i,j=1}^{n} (\int_0^s X^i \, dX^j)(\int_0^t Y^i \, dY^j), \quad (2.1) \]

\[ T^n_{st} = n^{-1} \sum_{i,j=1}^{n} (\int_0^s X^i \, dX^j)(\int_0^t Y^i \, dY^j), \quad (2.2) \]

converge weakly to the processes \( K_{st} \) and \( J_{st} \), respectively.

**Proof:** We shall only prove the convergence of \( T^n_{st} \) to \( J_{st} \), and the other state-
ment has a similar demonstration. Put \( t^n_k = k2^{-n} \) for \( k = 0, 1, \ldots, 2^n \) and \( n \geq 1 \). Define

\[ J^n_{st} = \sum_{h,k=0}^{2^n-1} [W(t^n_{h+1} \wedge s, t^n_k \wedge t) - W(t^n_h \wedge s, t^n_k \wedge t)], \]

\[ W(t^n_h \wedge s, t^n_k+1 \wedge t) - W(t^n_h \wedge s, t^n_k \wedge t), \]

\[ T^{nm}_{st} = n^{-1} \sum_{i,j=1}^{n} \sum_{h,k=0}^{m-1} X^i(t^n_h \wedge s)X^j(t^n_{h+1} \wedge s) - X^i(t^n_h \wedge s)Y^j(t^n_k \wedge t) \]

\[ Y^j(t^n_{k+1} \wedge t) - Y^j(t^n_k \wedge t). \]

For any \( m \geq 1 \), the sequence \( T^{nm}_{st} \) converges weakly to \( J^n \) as \( n \) tends to in-
finity. Also, the following convergence holds

\[ \sup_n \mathbb{E}(|T^{nm}_{st} - T^n_{st}|^2) \xrightarrow{n\to\infty} 0, \quad (2.3) \]

for all \( (s,t) \in T \). In fact,
\[
E\left( \left| T_{st}^{nm} - T_{st}^{n} \right|^2 \right) = n^{-2} \sum_{i,j=1}^{n} E\left( \sum_{h=0}^{2m-1} x_i(t_h, s) x_j(t_h+1, s) - x_i(t_h, s) x_j(t_{h+1}, s) \right)
\]

\[
y_j(t_{k+1} \wedge t) y_i(t_{k+1} \wedge s) - y_i(t_k \wedge t) y_j(t_k \wedge s) \leq \int_0^s x_i(y_j(y_j(t_k \wedge s) - y_i(t_k \wedge t))) df(s)
\]

\[
2n^{-2} \sum_{i,j=1}^{n} E\left( \sum_{h=0}^{2m-1} x_i(t_h, s) x_j(t_h+1, s) - x_i(t_h, s) x_j(t_{h+1}, s) \right) - \int_0^s x_i(y_j(y_j(t_k \wedge s) - y_i(t_k \wedge t))) df(s)
\]

\[
\sum_{k=0}^{2m-1} y_j(t_k \wedge t) y_i(t_k \wedge s) \leq \int_0^s x_i(y_j(y_j(t_k \wedge s) - y_i(t_k \wedge t))) df(s) - \int_0^s x_i(y_j(y_j(t_k \wedge s) - y_i(t_k \wedge t))) df(s)
\]

\[
2Kn^{-2} \left[n^2 - n \left( E\left( \sum_{h=0}^{2m-1} B^1(t_h, s) B^2(t_{h+1}, s) - B^1(t_h, s) B^2(t_h, s) \right) \right) \right]^{1/2} + n \left( E\left( \sum_{h=0}^{2m-1} B^1(t_h, s) B^1(t_{h+1}, s) - B^1(t_h, s) B^1(t_h, s) \right) \right)^{1/2},
\]

where

\[
K^2 = \sup_{m>1} \left( E\left( \sum_{h=0}^{2m-1} B^1(t_h, s) B^1(t_{h+1}, s) B^1(t_h, s) \right) \right)^{1/4},
\]

\[
E\left( \sum_{h=0}^{2m-1} B^1(t_h, s) B^1(t_{h+1}, s) - B^1(t_h, s) B^1(t_h, s) \right) E\left( \int_0^s B^1 dB^2 \right), E\left( \int_0^s B^1 dB^1 \right), E\left( \int_0^s B^2 dB^1 \right),
\]

and \( B^1, B^2 \) are two independent standard Brownian motions. Therefore, (2.3) is proved. In addition, we have

\[
E\left( J_{st}^{nm} - J_{st}^{n} \right)^2 \xrightarrow[n \to \infty]{} 0,
\]

for all \((s,t) \in T\).

Using Cairoli-Doob's maximal inequalities for two-parameter martingales, the convergences (2.3) and (2.4) can be transformed into

\[
sup_n E\left( \sup_{s,t} \left| T_{st}^{nm} - T_{st}^{n} \right|^2 \right) \xrightarrow[m \to \infty]{} 0,
\]

and

\[
E\left( \sup_{s,t} \left| J_{st}^{nm} - J_{st}^{n} \right|^2 \right) \xrightarrow[n \to \infty]{} 0.
\]

Let \( d \) be a metric on the set of all probabilities on \( C(T) \) which induces the weak convergence, and such that \( d(L(X), L(Y)) \leq E\left( \sup_{s,t} \left| X_{st} - Y_{st} \right| \right) \) for
any $C(T)$-valued random variables $X$ and $Y$. Here $L(X)$ stands for the distribution of $X$. Then

\[
\begin{align*}
    d(L(T^n), L(J)) &\leq \sup_n E(\sup_{s,t} |T^n_{st} - T^n_{st}|) + d(L(T^n_m), L(J^m)) + \\
    &\quad + E(\sup_{s,t} |J^n_{st} - J^n_{st}|)
\end{align*}
\]

converges to zero as $n \to \infty$, and the lemma is proved. \(\Box\)

**Lemma 2.2.** For all $(s,t)$ in $T$, the random variables $K_{st}$, $K_{11}$, $J_{st}$ and $J_{11}$ are identically distributed.

**Proof:** An invariance property for the Wiener process states that for any $a > 0$ and $b > 0$, \(\sqrt{ab} W(s/a, t/b), (s,t) \in T\) has the law of a two-parameter Wiener process. Therefore, it suffices to show that $K_{11}$ and $J_{11}$ have the same distribution. To do this, set $\Delta_{ij} = (i2^{-n}, (i+1)2^{-n}] \times (j2^{-n}, (j+1)2^{-n}]$ for $0 \leq i, j \leq 2^n - 1$, $n \geq 1$, and $W(\Delta_{ij}) = W((i+1)2^{-n}, (j+1)2^{-n}) - W(i2^{-n}, (j+1)2^{-n}) - W((i+1)2^{-n}, j2^{-n}) + W(i2^{-n}, j2^{-n})$. Then, we have in the $L^2$ sense $K_{11} = \lim_n \sum_{i' < i} W(\Delta_{ij})W(\Delta_{i'j'})$ and $J_{11} = \lim_n \sum_{i' < i} W(\Delta_{ij})W(\Delta_{i'j'})$, which implies the assertion of the lemma. \(\Box\)

Next we will use lemma 2.1 to obtain some information about the moments $m_p = E(K_{11}^p) = E(J_{11}^p), p \geq 1$. For $i, j = 1, \ldots, p$ define

\[
\xi_{i,j} = (\int_0^1 x^i dx^j)(\int_0^1 y^i dy^j) + (\int_0^1 x^i dx^j)(\int_0^1 y^i dy^j),
\]

(2.7)

and

\[
\mu = E(\xi_{12} \xi_{23} \ldots \xi_{p-1,p} \xi_{p,1}).
\]

(2.8)
Proposition 2.1. - For all \( p \geq 1 \), we have

\[
m_p = \sum_{k=1}^{p} \frac{(p-1)!}{2(p-k)!(p-k)!} \mu^k \cdot m_{p-k}.
\]  

(2.9)

Proof: Set

\[
E((S_n^n)^p) = n^{-p} \sum_{(i,j) \in \{1, \ldots, n\}^p} \prod_{k=1}^{p} \int_0^1 x^i_k \, dx_k \int_0^1 x^j_k \, dx_k \].
\]  

(2.10)

In this sum all terms vanish except those corresponding to multi-indexes \((i,j)\) such that for any \( m \in \{1, \ldots, n\} \) there is an even number of indexes equal to \( m \). Denote by \( v_{ij} \) the number of different integers appearing in the multi-index \((i,j)\). Then, for all \( k=1, \ldots, p \), the sum of the terms with multi-indexes verifying \( v_{ij} = k \) is of order \( n^k \). Therefore, if \( G_p \) denotes the set of permutations of the numbers 1, 2, 2, ..., p, p, we obtain

\[
\lim_n E((S_n^n)^p) = \frac{1}{p!} \sum_{(i,j) \in G_p} E(\prod_{k=1}^{p} \int_0^1 x^i_k dx_k \cdot \int_0^1 x^j_k dx_k)^2,
\]  

(2.11)

where \((i,j)\) represents the permutation \((i_1, j_1, i_2, j_2, \ldots, i_p, j_p)\). In view of lemma 2.1, (2.11) is the value of \( m_p \). Two permutations \((i,j)\) and \((i',j')\) of \( G_p \) such that \( i_k = j_h \Leftrightarrow i'_k = j'_h \) for any \( k, h = 1, \ldots, p \), will be called equivalent and they give rise to identical terms in the sum (2.11). If \( Q_p \) stands for the quotient set, we have

\[
m_p = \sum_{(i,j) \in Q_p} E(\prod_{k=1}^{p} \int_0^1 x^i_k dx_k \cdot \int_0^1 x^j_k dx_k)^2.
\]  

(2.12)

Observe that the cardinal of \( Q_p \) is \((2p)!/p!2^p\). A permutation of \( G_p \) will be called irreductible if it cannot be a product of cycles. All permutations equivalent to an irreductible one are also irreductibles. Denote by \( I_p \subset Q_p \) the set of equivalence classes of irreductible permutations and define

\[
n_p = \sum_{(i,j) \in I_p} E(\prod_{k=1}^{p} \int_0^1 x^i_k dx_k \cdot \int_0^1 x^j_k dx_k)^2.
\]  

(2.13)
Then,
\[ n_p = \sum_{k=1}^{p} \binom{p-1}{k-1} n_k m_{p-k}, \quad \text{for all } p \geq 1, \]
(2.14)

with the convention \( n_0 = m_0 = 1 \).

Finally, if we set \( \int_0^1 x^i dx^{i+1} = \xi_i^+ \) and \( \int_0^1 x^i dx^i = \xi_i^- \), for \( i=1, \ldots, p \), with the assumption \( p+1 = 1 \), it can be shown that
\[ n_p = \frac{1}{2} (p-1)! \sum_{\varepsilon \in \{+, -\}^p} [E(\xi_1^\varepsilon \cdots \xi_p^\varepsilon)]^2 = \frac{1}{2} (p-1)! \mu_p, \]
(2.15)

which completes the proof of the proposition. \( \square \)

Remarks:

1. Suppose that for any \( \varepsilon \in \{+, -\}^p \) we define the set \( A_{\varepsilon} \) of points \( x \) in \( [0,1]^p \) such that \( x_i \leq x_{i+1} \) for all \( i=1, \ldots, p \) (with the convention \( p+1 = 1 \)), where the symbols \( + \) and \( - \) mean \( \leq \) and \( > \), respectively. Then, using the formal rules \( E(X^i x^i) = u \wedge v \), \( E(X^i dx^i) = 1_{(u>v)} dv \) and \( E(dx^i dx^i) = 1_{(u=v)} du \), it can be seen that \( \mu = \sum_{\varepsilon} |A_{\varepsilon}|^2 \), where \( |A_{\varepsilon}| \) is the Lebesgue measure of \( A_{\varepsilon} \).

2. The expectations \( E(\xi_1^\varepsilon \cdots \xi_p^\varepsilon) \) can be computed recursively by means of Itô's formula. Indeed, if we define \( J_{\varepsilon} = \{ i : \varepsilon_i = + \text{ and } \varepsilon_{i+1} = - \} \), then
\[ E(\xi_1^\varepsilon \cdots \xi_p^\varepsilon) = \frac{1}{p} \sum_{i \in J_{\varepsilon}} \left[ E(\xi_1^\varepsilon \cdots \xi_{i-1}^\varepsilon + \xi_{i+1}^\varepsilon \cdots \xi_p^\varepsilon) + E(\xi_1^\varepsilon \cdots \xi_{i-1}^\varepsilon \xi_{i+2}^\varepsilon \cdots \xi_p^\varepsilon) \right]. \]

Using this algorithm it is not hard to evaluate the first moments of \( J_{11} \). For instance, \( m_1 = m_0 = 0 \); \( \mu_2 = \frac{1}{2} \); \( m_2 = \frac{1}{4} \); \( \mu_3 = \frac{1}{6} \); \( m_3 = \frac{1}{6} \); \( \mu_4 = \frac{7}{96} \); \( m_4 = \frac{23}{48} \); \( \mu_5 = \frac{37}{720} \); \( m_5 = \frac{31}{30} \).
3. The following expression for the characteristic function of $J_{11}$ can be deduced from proposition 2.1,

$$E(e^{itJ_{11}}) = \exp\left(\sum_{k=1}^{\infty} \frac{\mu_k i^k t^k}{2k}\right).$$

Set $\phi = \frac{1}{2}(\xi^2 - 1)$, where $\xi$ is a random variable with law $N(0,1)$. An argument similar to that used in the proof of proposition 2.1 shows that $E(\phi^p) = \sum_{(i,j) \notin \mathbb{Q}} \mathbb{E}(\prod_{k=1}^{p} \int_0^1 x^i_k dx^j_k)$. Then, the following inequalities hold

$$E((\phi/2)^p) \leq \eta_p \leq E(\phi^p). \quad (2.16)$$

In fact, to verify the second inequality observe that the terms in the sum (2.12) are less or equal than one. For the first inequality note that $J_{11} + K_{11}$ has the same law as $\phi$. As a consequence of (2.16), $J_{11}$ has finite exponential moments $E(e^{t J_{11}})$ for $t<1$. The next corollary shows that really $E(e^{t J_{11}}) \leq \infty$ for $t<\sqrt{2}$.

**Corollary 2.1.**

- If $p$ is even, and $\mu_p \leq 2^{-p/2}$ if $p$ is even, and $\mu_p \leq C_2^{-p/2}$ if $p$ is odd, being $C = \sqrt{\pi}$. Moreover,

$$E(e^{t J_{11}}) \leq 2^{1/4}(\sqrt{2}+t)^{(C-1)/4}(\sqrt{2}-t)^{-1/4}(C+1)/4. \quad (2.17)$$

**Proof:** If $p$ is even, the first statement is an immediate consequence of Schwarz inequality,

$$\mu_p = \mathbb{E}[(\xi_{12} \xi_{34} \ldots)(\xi_{23} \xi_{45} \ldots)] \leq \mathbb{E}[(\xi_{12} \xi_{34} \ldots)^2]\mathbb{E}[(\xi_{23} \xi_{45} \ldots)^2]^{1/2} = 2^{-p/2}.$$

For $p=2q+1$, $q \geq 0$, we apply Itô's formula and Schwarz inequality,
Finally, (2.17) follows immediately from the preceding inequalities. □

3. Consider the sequence of random variables \( U_n = \sum_{i,j=1}^{n} \int_{0}^{1} x^i \, dx \int_{0}^{1} y^j \, dy \).

It is clear that \( U_n \) converges in distribution to \( J_{11} \). Indeed, \( U_n = S_{11}^n \) -

\[
\sum_{i=2}^{n} \int_{0}^{1} x^i \, dx \int_{0}^{1} y^j \, dy.
\]

For each \( n \geq 2 \) and \( j=2, \ldots, n \) define

\[
X_{nj} = \sum_{i=1}^{j-1} \xi_{ij},
\]

and let \( F_{nj} \), \( j=1, \ldots, n \), be the \( \sigma \)-field generated by the processes \( X_1, \ldots, X_j, Y_1, \ldots, Y_j \). Then \( X_{nj} \) is a martingale array, that means, \( X_{nj} \) are \( F_{nj} \)-measurable and \( E(X_{nj} \mid F_{nj-1}) = 0 \), for \( j=2, \ldots, n \). Furthermore, \( \sum_{j=2}^{n} X_{nj} = U_n \). Set

\[
\Sigma_{nj}^2 = \sum_{i=2}^{j} E(X_{nj}^2 \mid F_{nj-1}), \quad \text{and} \quad v_n^2 = \frac{\Sigma_{nj}^2}{n}.
\]

Lemma 3.1. - The martingale array \( X_{nj} \) satisfies the conditional Lindeberg condition (0.1).

Proof: Put \( Z_j = \sum_{i=1}^{j-1} \xi_{ij} \). Then,

\[
E(Z_j^2) \leq \text{const.} \left( E \left( \sum_{i=1}^{j-1} (\int_{0}^{1} x^i \, dx)^2 \right)^2 \right) + \]

\[
E \left( \sum_{i=1}^{j-1} (\int_{0}^{1} y^j \, dy)^2 \right)^2.
\]
+ E \left( \sum_{i=1}^{j} (f_0^i dX^j)^2 (f_0^i dY)^2 \right)^2 = \text{const.} j^2 + o(j).

Therefore,

\[ n^{-2} \sum_{j=2}^{n} E(Z_j^2 1_{\{|Z_j| > n\varepsilon\}}) \leq n^{-4} \varepsilon^{-2} \sum_{j=2}^{n} E(Z_j^4) \to 0, \]

as \( n \to \infty \), for all \( \varepsilon > 0 \).

If this Lindeberg condition holds, Hall [5] and Rootzen [8] have shown that \( \frac{V_n^2}{n} \xrightarrow{P} \frac{\eta}{\sigma} \) with \( P(\eta > 0) = 1 \) implies that \( U_n \) converges in distribution to a mixture of normal distributions with characteristic function \( E(\exp(-\frac{1}{2} t^2 \eta)) \). If there is only convergence in distribution of the sequence \( V_n^2 \), this result may fail as it has been proved by a counterexample of Dvoretzky [4]. Also, Alvo, Cabilio and Feigin [1] exhibit a class of martingales, which are degenerate U-statistics, and such that the sequence \( U_n \) of row sums converge in distribution to a weighted sum of chi-squared independent random variables as long as the sequence of conditional variances converges in law. The next result shows that the martingale array (3.1) satisfies these same properties.

**Proposition 3.1.** The sequence \( \frac{V_n^2}{n} \) converges in law to the random variable

\[ f_{[0,1]^3} (W_{stu} + \tilde{W}_{stu})^2 ds du, \]

where \( \tilde{W}_{stu} = W_{st-u} - W_{st} + W_{su} + W_{su} \), and \( \{W_{stu}, (s,t,u) \in [0,1]^3\} \) is a zero mean continuous Gaussian process with covariance function \( E(W(s_1,t_1,u_1)) \).

\[ W(s_2,t_2,u_2) = (s_1 \wedge s_2)(t_1 \wedge t_2)(u_1 \wedge u_2). \]
Proof: Compute

$$E(X_{nj}^2 / F_{n,j-1}) =$$

$$\Sigma_{i=1}^{j-1} \left( \int_0^1 (X_{i}^2 - X_{i}^1)^2 du \right) + \Sigma_{i=1}^{j-1} \left( \int_0^1 (Y_{i}^2 - Y_{i}^1)^2 du \right) +$$

$$2 \Sigma_{i,i'=1}^{j-1} \left( \int_0^1 (X_{i}^2 - X_{i}^1)(X_{i'^2} - X_{i'^1}) du \right) + \Sigma_{i,i'=1}^{j-1} \left( \int_0^1 (Y_{i}^2 - Y_{i}^1)(Y_{i'^2} - Y_{i'^1}) du \right) +$$

$$2 \Sigma_{i,i'=1}^{j-1} \left( \int_0^1 (X_{i}^2 - X_{i}^1)(X_{i'}^2 - X_{i'}^1) du \right) + \Sigma_{i,i'=1}^{j-1} \left( \int_0^1 (Y_{i}^2 - Y_{i}^1)(Y_{i'}^2 - Y_{i'}^1) du \right) =$$

$$\int_0^1 \left( \Sigma_{i=1}^{j-1} \left( (X_{i}^2 - X_{i}^1)(Y_{i}^2 - Y_{i}^1) + (X_{i}^2 - X_{i}^1)^2 dsdt \right) \right).$$

Then,

$$V_n^2 = n^{-2} \int_0^1 \Sigma_{j=1}^{n-1} \left( \Sigma_{i=1}^{j} \left( (X_{i}^2 - X_{i}^1)(Y_{i}^2 - Y_{i}^1) + (X_{i}^2 - X_{i}^1)^2 dsdt \right) \right).$$

(3.3)

Denote by $D_3$ the set of functions from $[0,1]^3$ to $\mathbb{R}$ which are continuous from above, with limits from below, and define the $D_3$-valued processes

$$Z_n(s,t,u) = n^{-1/2} \int_0^1 [\nu] (X_{i}(s)Y_{i}(t)).$$

(3.4)

Using theorem 6 of Bickel and Wichura [2], we obtain the weak convergence of the sequence $Z_n(s,t,u)$ to $W(s,t,u)$. Further, the mapping $x(s,t,u) \rightarrow \int_{[0,1]^3} [x(1,l,u) - x(1,t,u) - x(s,l,u) + x(s,t,u)]^2 dudsdt$ from $D_3$ to $\mathbb{R}$ is continuous. Therefore, noting that

$$V_n^2 = \int_{[0,1]^3} [Z_n(1,l,u) - Z_n(1,t,u) - Z_n(s,l,u) + 2Z_n(s,t,u)] dsdtdu,$$

the proof of the proposition is complete. $\square$
References:


David Nualart
Departament d'Estadística
Facultat de Matematiques
Universitat de Barcelona
Gran Via 585, Barcelona-7
SPAIN