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ABSTRACT. The author uses a method of J. Diestel to show that an Orlicz space $L_\Phi(\mu,X)$ is WCG when X is, assuming a standard hypothesis on Φ. The proof depends upon the Davis-Figiel-Johnson-Pełczyński factorization lemma and a criterion for weak compactness in $L_\Phi(\mu,X)$ stated recently by F. Bombal. Similar results are obtained for spaces of Bochner and Pettis integrable functions, and for the space of continuous functions on an Eberlein compactum, using tensor product representations of these spaces.

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Our purpose is to show that certain Banach spaces of X-valued functions are weakly compactly generated (WCG), for a WCG Banach space X. We start with Orlicz spaces. If Φ , Ψ are a pair of complementary Young functions, (Ω, Σ, μ) is a finite measure space and X is a Banach space, $L_\Phi(\mu, X)$ stands for the space of measurable functions $f\colon \Omega \longrightarrow X$ such that , for some $k>0$:

$$M_\Phi(kf) < +\infty,$$

where:

$$M_\Phi(f) = \int_\Omega \Phi(\|f\|)\, d\mu .$$

$L_\Phi(\mu, X)$ is a Banach space (via the obvious identifications) endowed with the norm:

$$\||f|\|_\Phi = \inf \{ {}^1/_k : k>0 \text{ and } M_\Phi(kf)<1\}.$$

This norm is known to be equivalent to:

$$\|f\|_\Phi = \sup \{ \int_\Omega \|f\|\, h\, d\mu : h\in L_\Psi(\mu) \text{ and } M_\Psi(h)\le 1\}$$

(see [1] for more details, and [6] for references).

We say that Φ satisfies the Δ_2-condition when:

$$\lim_{u\to\infty} \frac{\Phi(2u)}{\Phi(u)} <+ \infty.$$

If X has Radon-Nikodým property (RNP) and Φ satisfies Δ_2-condition, $L_\Phi(\mu, X)^*$ is topologically isomorphic to $L_\Psi(\mu, X^*)$ (see [1] 4.b for comments). In the following, we suppose that Φ satisfies Δ_2-condition. First, we state:

LEMMA: *If X, Y are Banach spaces and T: $X \longrightarrow Y$ is a bounded linear operator with dense range, the operator:*

$$S: L_\Phi(\mu,Y) \longrightarrow L_\Phi(\mu,X)$$
$$f \longrightarrow T\circ f$$

is bounded and has dense range.

Proof. The boundedness of S follows from that of T and from the solidity of Luxemburg norm:

$$\||f|\|_\Phi \leq \||g|\|_\Phi \text{ holds for } f,g \in L_\Phi(\mu,X) \text{ with } \|f\| \leq \|g\|.$$

Simple functions are dense in $L_\Phi(\mu,X)$ (Diestel, [3]), and it suffices for us to see that $T(Y)$-valued simple functions are dense in the subspace of X-valued ones. Hence, it is sufficient to approximate a function of type $x\chi_A$, with $x \in X$, $A \in \Sigma$, by some $Ty\chi_A$, with $y \in Y$. We only point out that, choosing $t>0$ with $\Phi(t)<\mu(A)^{-1}$, and putting $k = t\|x\|^{-1}$, we have:

$$\Phi(\|kx\|) < \frac{1}{\mu(A)},$$

and thus:

$$M_\Phi(kx\chi_A) \leq 1,$$

and finally:

$$\frac{\|x\|}{t} = \frac{1}{k} \geq \||x\chi_A|\|_\Phi.$$

F. Bombal has stated in [1] a criterion for $\sigma(L_\Phi(\mu,X),L_\Psi(\mu,X^*))$-relative sequential compactness of a subset $K \subset L_\Phi(\mu,X)$ in case X has RNP. The conditions are:

i) K is norm bounded.

ii) For each $E\in\Sigma$, the set:

$$K(E) = \{ \int_E f \, d\mu : f\in K\}$$

is weakly relatively compact in X.

iii) For each $g\in L_\Phi(\mu,X^*)$:

$$\lim_{\mu(E)\to 0} \sup_{f\in K} \int_E \langle f,g\rangle \, d\mu = 0.$$

Using this criterion and the Lemma, we obtain, following the method of [4]:

PROPOSITION 1: <u>If</u> X <u>is WCG</u>, $L_\Phi(\mu,X)$ <u>is WCG</u>.

Proof. There exists a reflexive Banach space Y and a bounded linear operator $T: Y \longrightarrow X$ with dense range. This is a consequence of the Davis-Figiel-Johnson-Pełczyński factorization lemma ([2], Corollary 3). By the Lemma, we can restrict ourselves to the reflexive case.

Consider now the set:

$$K = \{x\chi_A : A\in\Sigma, \ \|x\|\leq 1\},$$

which is total in $L_\Phi(\mu,X)$. We apply the above criterion to check weak relative compactness (here, $L_\Psi(\mu,X^*)$ is the dual of $L_\Phi(\mu,X)$, and Eberlein-Smulian theorem gives the equivalence of the different notions of compactness). K is norm bounded by the remark we have made to finish the proof of the Lemma. If $E\in\Sigma$:

$$K(E) = \{x\mu(E\cap A) : \|x\|\leq 1, \ A\in\Sigma\}$$

is bounded, and hence weakly relatively compact. Finally, if g belongs to $L_\Psi(\mu,X^*)$:

$$\sup_{f\in K} \int_E \langle f,g\rangle \, d\mu = \sup \left\{ \int_{E\cap A} \langle x,g(\omega)\rangle \, d\mu(\omega) : \|x\|\leq 1, \ A\in\Sigma \right\}.$$

Using Young's inequality:

$$\left| \int_{E\cap A} \langle x,g(\omega)\rangle \, d\mu(\omega) \right| \leq \int_{E\cap A} \|x\| \, \|g(\omega)\| \, d\mu(\omega) \leq$$

$$\leq \int_E (\Phi(\|x\|)+\Psi(\|g(\omega)\|)) \, d\mu(\omega) = \Phi(\|x\|)\mu(E) + \int_E \Psi(\|g\|) \, d\mu,$$

and we have finished.

The proof of Proposition 1 gives as a particular case Diestel's proof (see [4]) for $L_1(\mu,X)$. Nevertheless, $L_1(\mu,X)$ can be shown to be WCG for a WCG Banach space X without using the factorization lemma nor any criterion of compactness, as we see next. The key of the proof will be the identification of $L_1(\mu,X)$ and the projective tensor product $L_1(\mu) \hat{\otimes}_\pi X$. We also remark that if Ψ satisfies the Δ_2-condition, $L_\Phi(\mu,X)$ is reflexive when X is reflexive, and the argument is simpler, as in Diestel'proof for $L_p(\mu,X)$, $1<p<\infty$ (see [5], VIII, 4.11).

PROPOSITION 2: Let X be a WCG Banach space. Then, $L_1(\mu,X)$ is a WCG Banach space.

Proof. Consider the canonical bilinear mapping:

$$\begin{aligned} U:\ L_1(\mu)\times X &\longrightarrow L_1(\mu,X) \\ (f,x) &\longrightarrow x\otimes f \ (= xf) \end{aligned}$$

If C denotes the subset of $L_1(\mu)$ formed by the characteristic functions of sets of Σ and $K\subset X$ is a weakly compact total subset of X, $U(C\times K)$ is total in $L_1(\mu,X)$, and we have finished if we can show that it is relatively compact. $C\times K$ is weakly compact in $L_1(\mu)\times X$, and, using the Eberlein-Smulian theorem, it suffices to show that U is sequentially continuous with respect to the weak topologies of $L_1(\mu)$, X, $L_1(\mu,X)$.

The dual of $L_1(\mu,X)$ can de identified to the space of continuous bilinear functionals on $L_1(\mu)\times X$. If λ is such a functional, $\{x_n\}_{n=1}^{\infty}$ converges weakly to x in X, and $\{f_n\}_{n=1}^{\infty}$ to f weakly in $L_1(\mu)$, we can write:

$$\lambda(f_n,x_n) - \lambda(f,x) = \lambda(x_n-x,f_n-f) + \lambda(x_n-x,f)+\lambda(x,f_n-f) \quad , \; n\geqslant 1.$$

The two last terms converge to zero for $n\to\infty$. On the other hand, $\lambda((x_n-x,.)$ is a continuous linear functional on $L_1(\mu)$ for each n, and the first term on the right hand side converges to zero because of the Dunford-Pettis property of $L_1(\mu)$.

A similar argument can be used for the tensor product $L_1(\mu)\hat{\otimes}_\varepsilon X$ which can be identified to the completion of the space $P_1(\mu,X)$ of measurable functions $f\colon \Omega \longrightarrow X$ which are Pettis integrable, endowed with the norm:

$$\|f\|_{P_1} = \sup_{\|x^*\|\leqslant 1} \int_\Omega |x^*f| \, d\mu.$$

Using the density of simple functions in $P_1(\mu,X)$, we only need to show that the canonical bilinear map is weak-weak sequentially continuous, and for this, it suffices to use the representation of the dual $(L_1(\mu)\hat{\otimes}_\varepsilon X)^*$ by integral bilinear functionals and to apply the Lebesgue dominated convergence theorem. Thus:

PROPOSITION 3: *If* X *is a WCG Banach space,* $\hat{P}_1(\mu,X)$ *is a WCG Banach space.*

In the same way the case $C(K)\hat{\otimes}_\varepsilon X$ can be treated, and the following is obtained:

PROPOSITION 4: *If* X *is a WCG Banach space and* K *is an Eberlein compactum,* C(K,X) *is a WCG Banach space.*

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