A SHARPED BOUND FOR THE NUMBER OF GENERATORS OF IDEALS DEFINING SPACE CURVE SINGULARITIES
by Joan Elias
Abstract: In this paper we establish a bound for the number of generators of geometric determinantal ideals of codimension two. Afterwards we show that this bound is sharp for curve singularities.

1. Definitions. Throughout this article I shall use \(\Theta\) to denote the local ring of formal power series \(k[[x_1,\ldots,x_n]]\), or the local ring at the origin of \(k[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}\), where \(k\) is an algebraically closed field of characteristic zero.

An ideal of \(\Theta\) will be called determinantal if there is a \(nxn\) matrix \(M\) with entries in \(\Theta\), such that \(I\) is generated by the \(nxn\) subdeterminants of \(M\) and the height of \(I\) is maximum, i.e. \((n-r+1)(m-r+1)\).

A germ \((X,0)\) will be called determinantal if it can be defined by a determinantal ideal of \(\Theta\). In the case of codimension two we have a \(nx(n+1)\) matrix \(M\), and \(I\) is generated by the \(nxn\) subdeterminants of \(M\). We can also assume that any minimal basis of \(I\) is obtained taking the maximal subdeterminants of a matrix \(M\) with entries in the maximal ideal of the ring \(\Theta\) (see [Bu]).

It is known that in codimension two, to be determinantal is equivalent to be perfect (see [Ho-E], [Bu]).

We denote by \(\gamma(I)\) the number of elements of a minimal basis of \(I\), \(h\) the height of \(I\) and \(e\) the multiplicity of the local ring \(\Theta/I\).
2. The bound. Let $I$ be a perfect ideal of $\Theta$ with height two, then

2.1 Lemma. After a suitable linear change coordinates we may assume

that the cossets $\bar{x}_1, \ldots, \bar{x}_N \in \frac{\Theta}{I}$ form a system of parameters of $\frac{\Theta}{I}$.

Proof: We shall proceed by induction on the dimension of $\Theta$. From lemma 5 of [2-S] pag. 237 there exists a superficial element of degree 1. Now since for an element to be superficial depends only on the initial form, we can assume that there exists a cosset $\bar{x}$ such that is a superficial element of degree 1. Set $\bar{x} = x_1 \ldots x_N$.

It is easy to prove that a superficial element is a non-zero divisor, therefore it is sufficient to consider the quotient

$$\frac{\Theta}{I + x_N \Theta}$$

which is isomorphic to

$$\frac{\Theta}{x_N \Theta} / \frac{I + x_N \Theta}{x_N \Theta}$$

and this ring is Cohen-Macaulay.

By induction on $\frac{\Theta}{x_N \Theta}$ we have a regular sequence $\bar{x}_3, \ldots, \bar{x}_N$ in the ring $\frac{\Theta}{I}$, and from theorem 16.8 of [Mat] it follows that the above sequence form a system of parameters of $\frac{\Theta}{I}$.

2.2 Theorem. Let $I$ be a perfect ideal of $\Theta$ of height two then

$$\nu(I)(\nu(I)-1) \leq 2e.$$ 

Proof: From corollary two to theorem five of [Bu] we know that $I \subset m^{\nu-1}$.
where \( m \) is the maximal ideal of \( \Theta \). By the above lemma we may assume that \( x_3, \ldots, x_N \) are a system of parameters of \( \frac{\Theta}{I} \).

Therefore

\[
\dim_k \left( \frac{\Theta}{I + (x_3, \ldots, x_N)} \right) = e
\]

Moreover the cossets \( x_1^a x_2^b \) with \( a+b \leq r-2 \) form a \( k \)-independent set, because if we had a linear relation

\[
\sum \lambda_{a,b} x_1^a x_2^b = 0
\]

with \( \lambda_{a,b} \in k \), then

\[
\sum \lambda_{a,b} x_1^a x_2^b = f + g, f \in I, g \in (x_1, \ldots, x_N)\Theta
\]

and hence we would get in the ring \( \frac{\Theta}{m^{r-1}} \) that

\[
\sum \lambda_{a,b} x_1^a x_2^b = \gamma \in (x_1, \ldots, x_N)\left( \frac{\Theta}{m^{r-1}} \right)
\]

which yields \( \lambda_{a,b} = 0 \). The number of cossets \( x_1^a x_2^b \) with \( a+b \leq r-2 \) is \( \frac{r(r-1)}{2} \), and consequently \( 2.e > r(r-1) \).

3. The examples of Macaulay and Moh. It is a classical problem to find prime ideals of \( \Theta \) which need at least \( n \) generators, for each \( n \geq 1 \).

Macaulay constructed prime ideals that require exactly \( n \) generators ([Mac] pp. 36-37). Abhyankar found that this construction was "rather mysterious today and ... in need of proof" ([Ab]). However Abhyankar substitutes the original claim of "exactly \( n \) generators" by "at least \( n \) generators". Now I will show how theorem 2.2 allows us to
recovery of Macaulay's claim.

3.1 Theorem. Let $\mathfrak{M}_n$ be Macaulay's prime ideals for the ring

$$k\left[\frac{X_1,X_2,X_3}{(X_1,X_2,X_3)}\right]$$

then

$$r(M_n) = n.$$ 

Proof: It is known that the multiplicity $e$ of the ideal $\mathfrak{M}_n$ is $\frac{1}{2} n(n-1)$. From the paper of Abhyankar we get $r \geq n$, therefore by theorem 2.2

$$e = \frac{n(n-1)}{2} \leq \frac{r(r-1)}{2} \leq e$$

hence $n = r$.

More recently T.T. Moh showed ([Mo]) that in the case of the ring $\Theta = k[\frac{X_1,X_2,X_3}{(X_1,X_2,X_3)}]$ there exist prime ideals $\mathfrak{P}_n \in \text{Spec}(\Theta)$ which have $n+1$ as their minimal number of generators. Like the case above we have

$$r(P_n)(r(P_n)-1) = 2e.$$ 

Thus we see that our bound is sharp for germs of algebraic and algebroid space curves.

Notice that the examples of Macaulay and Moh have minimal multiplicity for $r$ fixed, or maximal $r$ for a fixed multiplicity.

4. Other bounds. G. Valla in his paper ([Va]) gives a general bound that in our hypothesis is

$$r(I) \leq \frac{e+3}{2}.$$ 

Now it is easy to show that the bound of theorem 2.2 is

$$r(I) \leq \frac{1 + \sqrt{1 + 8e}}{2}$$

which for any $e \geq 3$ is better than Valla's.
Remark. I refer to Valla's paper for a comparison of this bound with previously obtained bounds by Becker, Sally, Boratynski-Eisenebod-Rees.

REFERENCES


DEPARTAMENT DE GEOMETRIA I TOPOLOGIA. FACULTAT DE MATEMÀTIQUES.
UNIVERSITAT DE BARCELONA.