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Abstract: In this paper we stablish a bound for the number of generators of geometric determinantal ideals of codimension two. Afterwards we show that this bound is sharp for curve singularities.

1.Definitions. Throughout this article I shall use θ to denote the local ring of formal power series $k[[X_1,\dots,X_N]]$, or the local ring at the origin of $k[X_1,\dots,X_n]_{(X_1,\dots,X_N)}$, where k is an algebraically closed field of characteristic zero.

An ideal of θ will be called determinantal if there is a nxm matrix M with entries in θ , such that I is generated by the rxr subdeterminants of M and the height of I is maximum, i.e. (n-r+1)(m-r+1).

A germ (X,0) will be called determinantal if it can be defined by a determinantal ideal of θ . In the case of codimension two we have a nx(n+1) matrix M, and I is generated by the nxn subdeterminants of M. We can also assume that any minimal basis of I is obtained taking the maximal subdeterminants of a matrix M with entries in the maximal ideal of the ring θ (see [Bu]).

It is known that in codimension two, to be determinantal is equivalent to be perfect(see [Ho-E],[Bu]).

We denote by $\mu(I)$ the number of elements of a minimal basis of I, h the height of I and e the multiplicity of the local ring $\frac{\theta}{I}$.

2. The bound. Let I be a perfect ideal of Θ with height two, then

2.1 Lemma. After a suitable linear change coordinates we may assume that the cossets $\bar{X}_1,\dots,\bar{X}_N \in \frac{\Theta}{I}$ form a system of parameters of $\frac{\Theta}{I}$.

Proof: We shall proceed by induction on the dimension of Θ. From lemma5 of [Z-S] pag.287 there exists a superficial element of degree1. Now since for an element to be superficial depends only on the initial form, we can assume that there exists a cosset $\bar{X}_{\bullet}$ such that is a superficial element of degree 1. Set $\bar{X}_{\bullet}=\bar{X}_N$.

It is easy to prove that a superficial element is a non-zero divisor, therefore it is suficient to consider the quotient

$$\frac{\Theta}{I + X_N\Theta}$$

which is isomorphic to

$$\frac{\Theta / X_N\Theta}{I + X_N\Theta / X_N\Theta}$$

and this ring is Cohen-Macaulay.

By induction on $\frac{\Theta}{X_N\Theta}$ we have a regular sequence $\bar{X}_3,\dots,\bar{X}_N$ in the ring $\frac{\Theta}{I}$, and from theorem 16.B of [Mat] it follows that the above sequence form a system of parameters of $\frac{\Theta}{I}$.

2.2 Theorem. Let I be a perfect ideal of Θ of height two then

$$\nu(I)(\nu(I)-1) \leq 2e .$$

Proof: From corollary two to theorem five of [Bu] we know that $I \subset m^{\nu-1}$

where m is the maximal ideal of Θ. By the above lemma we may assume that $\bar{X}_3,\ldots,\bar{X}_N$ are a system of parameters of $\frac{\Theta}{I}$.

Therefore

$$\dim_k\left(\frac{\Theta}{I+(X_3,\ldots,X_N)}\right) = e$$

Moreover the cossets $\bar{X}_1^a\,\bar{X}_2^b$ with $a+b\leq \nu-2$ form a k-independent set, because if we had a linear relation

$$\sum \lambda_{a,b}\,\bar{X}_1^a\,\bar{X}_2^b = \bar{0}$$

with $\lambda_{a,b}\in k$, then

$$\sum \lambda_{a,b}\,X_1^a\,X_2^b = f + g \quad , \; f\in I \;,\; g\in (X_1,\ldots,X_N)\Theta$$

and hence we would get in the ring $\frac{\Theta}{m^{\nu-1}}$ that

$$\sum \lambda_{a,b}\,\tilde{X}_1{}^a\,\tilde{X}_2{}^b = \tilde{g} \in (\tilde{X}_3,\ldots,\tilde{X}_N)\left(\frac{\Theta}{m^{\nu-1}}\right)$$

which yields $\lambda_{a,b} = 0$. The number of cossets $X_1^a\,X_2^b$ with $a+b\leq\nu-2$ is $\frac{\nu(\nu-1)}{2}$, and consequentaly $2.e \geq \nu(\nu-1)$.

<u>3. The examples of Macaulay and Moh.</u> It is a classical problem to find prime ideals of Θ which need at least n generators, for each $n\geq 1$.

Macaulay constructed prime ideals that requires exactly n generators([Mac] pp. 36-37). Abhyankar found that this construction was " rather mysterious today and ... in need of proof" ([Ab]). However Abhyankar substitutes the original claim of " exactly n generators" by " at least n generators". Now I will show how theorem 2.2 allows us to recover Macaulay's claim.

3.1 Theorem. Let M_n be Macaulay's prime ideals for the ring

$$k\left[X_1,X_2,X_3\right]_{(X_1,X_2,X_3)} \quad \text{then} \qquad \nu(M_n) = n .$$

Proof: It is known that the multiplicity e of the ideal M_n is $\frac{1}{2}n(n-1)$. From the paper of Abhyankar we get $\nu \geq n$, therefore by theorem 2.2

$$e = \frac{n(n-1)}{2} \leq \frac{\nu(\nu-1)}{2} \leq e$$

hence $n = \nu$.

More recently T.T.Moh showed ([Mo]) that in the case of the ring $\Theta = k\left[\left[X_1,X_2,X_3\right]\right]$ there exist prime ideals $P_n \in \mathrm{Spec}(\Theta)$ which have n+1 as their minimal number of generators. Like the case above we have

$$\nu(P_n)(\nu(P_n)-1) = 2e.$$

Thus we see that our bound is sharp for germs of algebraic and algebroid space curves.

Notice that the examples of Macaulay and Moh have minimal multiplicity for ν fixed, or maximal ν for a fixed multiplicity.

4. Other bounds. G. Valla in his paper ([Va]) gives a general bound that in our hypothesis is

$$\nu(I) \leq \frac{e+3}{2}$$

Now it is easy to show that the bound of theorem 2.2 is

$$\nu(I) \leq \frac{1+\sqrt{1+8e}}{2}$$

which for any $e \geq 3$ is better than Valla's.

Remark. I refer to Valla's paper for a comparison of this bound with previously obtained bounds by Becker, Sally, Boratynski-Eisenbud-Rees.

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