A NOTE ON PLANAR REAL CREMONA TRANSFORMATIONS
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Abstract. In this note we prove that a polynomial mapping T in two real variables such that its jacobian is constant (the so called planar real Cremona maps) is a bijection between $\mathbb{R}^2$ and $\mathbb{R}^2$. Let $(F,G)$ be the polynomial components of $T$. We give a complete global picture of the family of curves $F=\text{constant}$ and $G=\text{constant}$.

1 Introduction

The main purpose of this note is to prove the following theorem.

Theorem A. Let $F = F(x_1, x_2)$ and $G = G(x_1, x_2)$ be two real polynomials in the two real variables $x_1, x_2$ such that its jacobian $J = \det(\partial(F,G)/\partial(x_1, x_2))$ is a nonzero constant. Then the polynomial map $(F,G): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is bijective.

The key point in the proof of the injectivity of Theorem A is that the algebraic curves $F(x_1, x_2) = \text{constant}$ and $G(x_1, x_2) = \text{constant}$ are solutions of systems of ordinary differential equations of Hamiltonian type (with Hamiltonians $F$ and $G$, of course). These systems have only a singularity of index two at the infinity point which consists of two elliptic sectors (see section 2).
For the proof of the onto character we analyze the Newton polygon of $F$ and $G$. In order to make this analysis easier we note that using triangular maps of the type $(x_1, x_2) \rightarrow (x_1, x_2 + bx_1^p)$, where $b$ is a nonzero real number and $p$ an integer such that $p > 1$, we obtain another Cremona map of the type

$$(x_1, x_2) \rightarrow \left( x_1^m + F(x_1, x_2), x_1^n + G(x_1, x_2) \right), \quad (1)$$

where degree $F < m$ and degree $G < n$. When $F$ and $G$ have the form (1) the only possible diffeomorphic qualitative global pictures of the flow (for the Hamiltonians $F$ or $G$) are given in Figure 1.a and Figure 1.b (see section 2). In any case the global picture is homeomorphic to one of these two figures (see, again, section 2) where we have used the usual compactification of the plane adding one point $p$ at infinity.

As a consequence of the onto character of $T$ the flows with Hamiltonian $F$ and $G$ do not reach the infinity in finite time (see section 3).

Remark 1. The assumption that $F$ and $G$ are polynomials is necessary in Theorem A. The result is not true for analytical maps. For instance, if $F(x_1, x_2) = -\exp(x_2)\cos(x_1\exp(-2x_2))$ and $G(x_1, x_2) = \exp(x_2)\sin(x_1\exp(-2x_2))$ then $J = 1$ but the map $(F, G): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is clearly not injective.

Remark 2. There is no restriction putting $J = 1$ because, if $J = a \neq 0$, we can consider the map $(a^{-1}F, G)$ instead of $(F, G)$. 

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Remark 3. A theorem similar to Theorem A for complex polynomials in two complex variables would prove the Jacobian conjecture for two variables, that is, the inverse map is also polynomial (see [2], [6] and [7, Theorems 38 and 46]). In fact for the validity of the Jacobian conjecture it is enough to prove the injectivity.

Remark 4. To prove the injectivity for a mapping \( T = (F,G) : \mathbb{C}^2 \to \mathbb{C}^2 \), \( F \) and \( G \) complex polynomials with \( J = 1 \), we claim that it is enough to prove the injectivity for complex valued polynomials maps \( T' = (P,Q) : \mathbb{R}^2 \to \mathbb{C}^2 \) in two real variables with \( J = 1 \). Suppose \( T \) is not injective. Then there exist \( z,w \in \mathbb{C}^2 \) such that \( Tz = Tw \).

Using translation, scaling and rotation we can assume that \( z \) and \( w \) are \((0,0)\) and \((1,0)\). Let \( T'' \) be the map obtained from \( T \) by these changes of variables. The restriction of \( T'' \) to \( \mathbb{R}^2 \) is of type \( T' \) and the claim is proved. Note that \( T' \) is of type \( T' = (P_1 + iP_2, Q_1 + iQ_2) \) where \( P_1, P_2, Q_1, Q_2 \) are real valued polynomials. From Theorem A follows easily that if some of these four polynomials is identically zero \( T' \) is injective.

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2 Proof of the injectivity

Let \( F \) and \( G \) be two polynomials in the hypotheses of Theorem A. This implies that the analytical Hamiltonian system \( X_F \) given by

\[
\frac{dx_1}{dt} = \frac{\partial F}{\partial x_2} = Fx_2,
\]

\[
\frac{dx_2}{dt} = -\frac{\partial F}{\partial x_1} = -Fx_1.
\]
has no critical points. Of course $F$ is a first integral of this system.

Lemma 1. For each $y \in \mathbb{R}^2$ there is a unique solution $\phi_t(y) = (x_1(t), x_2(t))$ of $x_F$ with $\phi_0(y) = y$ defined on a maximal open interval $(\alpha, \beta) \subset \mathbb{R}$ such that $||\phi_t(y)|| = \infty$ as $t \to \alpha$ or $t \to \beta$ where $|| \cdot ||$ denotes the Euclidean norm.

It is possible $\alpha = -\infty$, $\beta = +\infty$ or both.

For a proof of this lemma see [4, p. 210].

Next we introduce the Poincaré compactification for polynomial vector fields $X(x)$ in the plane (see [5]). We consider in $\mathbb{R}^3$ the sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ and the plane $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 1\}$. For each point $x$ of $P$ of type $(x_1, x_2, 1)$ we define the map $f_+: P \to S^2$ given by $f_+(x) = (x_1, x_2, 1)/d(x) = (y_1, y_2, y_3) = y$ where $d(x) = (x_1^2 + x_2^2 + 1)^{1/2}$.

The image of $P$ under $f_+$ is the upper hemisphere $H_+$ of $S^2$. Then $f_+$ induces a field on $H_+$: $\bar{X}(y) = Df_+X(x)$.

Let $X = (P, Q)$ a polynomial vector field on $\mathbb{R}^2$ of degree $n = \max(\deg P, \deg Q)$. Let $r(y) = y_3^{n-1}$. Then we claim that the field $r\bar{X}$ can be extended analytically to a vector field on $H_+ \cup S^1$ where $S^1 = S^2 \cap \{y_3 = 0\}$, the equator of the sphere.

In order to prove the claim (see [5] for details) we use five local charts $(U_i, \phi_i)$, $i = 1, 2, 3$, $(V_i, \psi_i)$, $i = 1, 2$ where

$U_i = \{y \in S^2 : y_i > 0\}$; $V_i = \{y \in S^2 : y_i < 0\}$;

$\phi_i = \frac{1}{y_i} (y_j, y_k)$, $j < k$, $i \neq j, k$;

$\psi_i = \frac{1}{y_i} (y_j, y_k)$, $j < k$, $i \neq j, k$.

Let $y \in U_1 \cap H_+$ and $z = \phi_1(y)$. Then $(z_1, z_2) = (y_2, y_3)/y_1 = (x_2, 1)/x_1$.

The vector field $r\bar{X}$ is expressed in the $z$ coordinates as
\[ \frac{z^n}{d(z)^{n-1}} \left( -z_1 P\left( \frac{1}{z_2}, \frac{z_1}{z_2} \right) + Q\left( \frac{1}{z_2}, \frac{z_1}{z_2} \right), -z_2 P\left( \frac{1}{z_2}, \frac{z_1}{z_2} \right) \right). \]

In \( U_2 \) we get

\[ \frac{z^n}{d(z)^{n-1}} \left( P\left( \frac{z_1}{z_2}, \frac{1}{z_2} \right), -z_1 Q\left( \frac{z_1}{z_2}, \frac{1}{z_2} \right), -z_2 Q\left( \frac{z_1}{z_2}, \frac{1}{z_2} \right) \right). \]

Furthermore the expressions of \( \bar{r} \) on \( V_1, V_2 \) are the ones of \( \bar{r} \) on \( U_1, U_2 \), respectively, multiplied by \((-1)^{n-1}\). In \( U_3 \) the expression obtained is

\[ \frac{1}{d(z)^{n-1}} \left( P(z_1, z_2), Q(z_1, z_2) \right). \]

It is easy to check that the different expressions of \( r \) are analytical and compatible. Hence \( r \) is extended to \( H \cup S^1 \) and \( S^1 \) is clearly invariant under the flow.

Let \( P(x_1, x_2) = \sum_{j=0}^{n} P_j(x_1, x_2), Q(x_1, x_2) = \sum_{j=0}^{n} Q_j(x_1, x_2) \); where \( P_j, Q_j \) are homogenous polynomials of degree \( j \). The field on \( S_1 \cap U_1, S_1 \cap U_2 \) is given by

\[ \dot{z}_1 = R(z_1) = Q_n(1, z_1) - z_1 P_n(1, z_1), \]
\[ \dot{z}_1 = S(z_1) = P_n(z_1, 1) - z_1 Q_n(z_1, 1), \]
respectively. In \( S_1 \cap V_1, S_1 \cap V_2 \) we have the same expressions times \((-1)^{n-1}\).

On the other hand, let \( S^2 \) be the sphere of \( \mathbb{R}^3 \) defined by

\[ \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + (y_3 - 1/2)^2 = 1/4 \}. \]

The plane \( \mathbb{R}^2 \) may be identified with the sphere \( S^2 \) with the "north pole" \( p=(0,0,1) \) removed by means of the stereographic projection which assigns to each point \((x_1, x_2) \in \mathbb{R}^2\) the point \((y_1, y_2, y_3) \in S^2\) through the relations \( x_1 = y_1/(1-y_3), \quad x_2 = y_2/(1-y_3) \).

The differential system \( X_F \) on \( S^2 - \{p\} \) becomes
where

\[
\frac{dy_1}{dt} = F_1(y_1, y_2, y_3),
\]

\[
\frac{dy_2}{dt} = F_2(y_1, y_2, y_3),
\]

This system extends analytically the flow of \( X_F \) from \( S^2 - \{p\} \) to \( S^2 \) (at least in a neighborhood of \( p \)). Note that the point \( p \) is the unique critical point of the new flow.

**Lemma 2.** In the hypotheses of Theorem A the local phase-protrait of system (2) around the critical point \( p \) consists of two elliptic sectors and the rest fans (see [3,p.219] for definitions).

**Proof.** By the Poincaré-Hopf Index Theorem (see [3,p.366]) the index of \( p \) is equal to two. Now we use the compactification of Poincaré. As we noted in section 1 we can always suppose that \( F(x_1, x_2) = x_1^m + \bar{F}(x_1, x_2), \deg \bar{F} < m. \) Therefore \( P(x_1, x_2) = \bar{F}(x_2), \)
Q(x_1,x_2) = -mx_1^{m-1} - F_{x_1}. The vector field (P,Q) on the parts of S^1 which are in the carts U_1, V_1, U_2 and V_2 (see Figure 2) is given by R(z_1) = -m, R(z_1) = (-1)^{m-1}m, S(z_1) = mz_1^m and S(z_1) = (-1)^{m-2}mz_1^m, respectively. If we look at S^1 as \{(y_1,y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1 \}, then the only critical points are (0,1), (0,-1). We can visualize \( H_+ \cup S^1 \) as a closed disc bounded by S^1. The flow on it (recall that there are no critical points inside and use Lemma 1) has the two qualitative possibilities given in Figure 3. When we glue the S^1 into a point (and obtain S^2) the respective pictures of Figure 3 become the ones given in Figure 1.a and 1.b. This proves Lemma 2.

Remark 1. The statement of Lemma 2 is obviously true for the analytical Hamiltonian system X_G.

Remark 2. In the proof of Lemma 2 we strongly rely on the existence of the polynomial G such that the jacobian of (F,G) with respect to (x_1,x_2) is equal 1. In fact for a general analytic Hamiltonian system X_F without proper critical points the statement is false. A counterexample is displayed by F(x_1,x_2) = x_2(x_1x_2-1) which has two hyperbolic sectors and four elliptic sectors at the infinity.
Lemma 3. (i) The global flow of $X_F$ on $\mathbb{R}^2$ is diffeomorphic to the flow given in Figure 4.

(ii) If the algebraic curve $F(x_1,x_2) = \text{constant}$ on $\mathbb{R}^2$ is nonempty, then it has only one connected component. This component is diffeomorphic to $\mathbb{R}$.

(iii) For all $\bar{x} \in \mathbb{R}^2$ the algebraic curve $F(x) = F(\bar{x})$ and the trajectory $\phi_t(\bar{x})$ of $X_F$ represent the same curve in $\mathbb{R}^2$.

Proof. (i) follows from Figure 3.

(ii) The curves $F(x) = \text{constant}$ on $\mathbb{R}^2$ have only one connected component. Otherwise, between two curves with the same value of $F$ there would be a curve where $F$ is a extremum and hence it would be composed of critical points.

(iii) follows immediately from the Hamiltonian character and (ii).

Figure 4 here

Proof of the injectivity of $T$. Let $\phi_t$ and $\psi_t$ be the flows of $X_F$ and $X_G$, respectively. Since the jacobian is 1 we have that

$$\frac{d}{dt} (G \circ \phi_t) = \{G \circ \phi_t, F\} = -1,$$

$$\frac{d}{dt} (F \circ \psi_t) = \{F \circ \psi_t, G\} = 1,$$

where $\{ , \}$ denotes the Poisson bracket (for more details see [1,p.193]). Then for each point $\bar{x} \in \mathbb{R}^2$ we have that

$$G(\phi_t(\bar{x})) = -t + G(\bar{x}),$$

$$F(\psi_t(\bar{x})) = t + F(\bar{x}).$$

From (2) and Lemma 3 it follows that the curves $F(x) = \text{constant}$ and $G(\bar{x}) = \text{constant}$ have at most one point in common. So the polynomial map $(F,G) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is injective.
3 Proof of the exhaustivity

It is not excluded up to now that for \( \Phi_t \) or \( \Psi_t \) some point of \( \mathbb{R}^2 \) reaches the infinity infinite time. By Lemma 3 (ii) the image of \( \mathbb{R}^2 \) under \( F \) is an interval \( I \) of \( \mathbb{R} \). There are three possibilities: \( I \) is bounded, \( I \) is bounded from one side or \( I = \mathbb{R} \). If \( I \) is bounded from above, for each point \( \bar{x} \in \mathbb{R}^2 \) it follows from (2) that there is some time \( t_\infty(\bar{x}) < +\infty \) such that \( \Psi_t(\bar{x}) \) goes to infinity when \( t \uparrow t_\infty(\bar{x}) \). In a similar way if \( I \) is bounded from below there is some time \( t_\infty(\bar{x}) > -\infty \) such that \( \Psi_t(\bar{x}) \) goes to infinity when \( t \downarrow t_\infty(\bar{x}) \). These values \( t_\infty(\bar{x}) \), \( t_\infty(\bar{x}) \) are the values \( a \) and \( B \) of Lemma 1. Let us show that \( F \) is exhaustive, that is, \( I = \mathbb{R} \) (and therefore if follows that the orbits of \( X_G \) reach the infinity only for unbounded time).

The same will be true for \( G \).

We suppose again \( F(x_1, x_2) = x_1^m + \sum_{j<m} F_j(x_1, x_2) \). We consider the Newton polygon of \( F \) for the neighborhood of the infinity, i.e., the outer part of the boundary of the convex closure of the set \( \{(r, s) \in \mathbb{N} \times \mathbb{N} : F(x_1, x_2) = \sum a_{rs} x_1^r x_2^s \text{ with } a_{rs} \neq 0 \} \), see Figure 5. We claim that if there is some vertex \((i, j)\) in the Newton polygon with one odd coordinate then \( F \) is exhaustive. We set \( x_2 = x_1^{p/q}, q \text{ odd} \), where \(-q/p\) belongs to the interval \( J \) whose extrema are the slopes of the sides of the polygon which meet in \((i, j)\) (eventually \( p \) is negative if these two slopes are positive or one is positive and the other zero). Suppose \( i, j \) odd. Then we select \( p \) even and for \( |x_1| \) large we have

\[
F(x_1, x_2 = x_1^{p/q}) = a_{ij} x_1^{(q_1 + p_j)/q} (1 + o(1)) \text{ with } q_1 + p_j \text{ odd.}
\]

Therefore \( F \) is exhaustive. If \( i \) odd, \( j \) even, take any \( p \) such
that p/q ∈ J and again F is exhaustive. Finally, for i even, j odd it is enough to take p odd. Hence we can suppose that all the vertices have even coordinates. If the coefficients associated to the vertices change the sign we have exhaustivity. We suppose for definiteness that all these coefficients are positive.

Figure 5 here

Let us take a side of the Newton polygon. The involved terms are of the type $F_s = \sum_{k=0}^{r} a_k x_1^{i+p} x_2^{j-q}$ where $r$ is even and g.c.d. $(p, q) = 1$. These terms are dominant when $x_2 = ax_1^{p/q}$ (or when $x_1 = c = \text{constant}$, if $q = 0$). On this curve $F_s(x_1, x_2) = x_1^{q+q}/q a_j f_r(b)$ where $f_r(b) = \sum_r a_k b^k$, $b = a^{-q}$ (or $F_s(x_1, x_2) = x_2^{q+q} c^q f_r(b)$, $b = c^p$ if $q = 0$) and $F(x_1, x_2) = F_s(x_1, x_2)(1 + o(1))$ when $(x_1, x_2) \rightarrow \infty$ if $F_s(x_1, x_2)$ is unbounded on this curve.

If $f_r(b)$ has some real zero $b$ of odd multiplicity, then in any neighborhood of $b$ there are points $b_-, b_+$ such that $f_r(b_-) < 0$, $f_r(b_+) > 0$ and hence F is exhaustive.

Let us suppose that all the real zeros of $f_r(b)$ are of even (greater than zero) multiplicity. Then we consider the terms in the highest line parallel to the side through one of the points in the Newton diagram and let $f_r^{(1)}$ be the related polynomial in $b$. If for one of the zeros $b^*$ of $f_r$ we have $f_r^{(1)}(b^*) < 0$, we have done. For the case $f_r^{(1)}(b^*) > 0$ see later.

If $f_r^{(1)}(b^*) = 0$ we continue the process with other lines parallel to the selected side, obtaining $f_r^{(2)}$, $f_r^{(3)}$, etc. Let $t$ be the first index such that $f_r^{(t)}(b^*) \neq 0$. If $f_r^{(t)}(b^*) < 0$ the exhaustivity
is clear. If for all \( t, f_r^{(t)}(b^*) = 0 \), then we have that \( x_2^q - ax_1^p \) is a factor of \( F \) (or \( x_2^q x_1^{-p} = a\) if \( p = 0 \)). Therefore \( F(x_1, x_2) = (x_2^q - ax_1^p)F(x_1, x_2) + C \), where \( C \) is a constant. Then for the Hamiltonian field we get \( F = apx_1^{p-1} - (x_2^q - ax_1^p)F \), \( F = qx_2^{q-1}F + (x_2^q - ax_1^p)F \). First we suppose \( p > 0 \), \( q \neq 0 \). Then \( p > 1 \) and we have a critical point at the origin if \( q > 1 \), which is an absurdity. Therefore \( F(x_1, x_2) = (x_2 - ax_1^p)F + C \). Then the algebraic curve \( F = C \) has the real components \( x_2 - ax_1 = 0, \bar{F} = 0 \). If \( \bar{F} = 0 \) has real points in the curve \( x_2 - ax_1^p = 0 \), then these points will be critical points and this is impossible. If \( \bar{F} = 0 \) has real points outside \( x_2 - ax_1^p = 0 \), we have a contradiction with Lemma 3 (ii). If \( \bar{F} = 0 \) has no real components then \( \bar{F}(0, x_2) \) is a polynomial of even degree and therefore \( F(0, x_2) \) is of odd degree, showing exhaustivity. A similar reasoning applies to the cases \( p < 0 \) and \( q = 0 \).

If for all the sides of the Newton polygon we have no zeros of the related \( f_r(b) \) polynomials or if the zeros being even there is some \( t \) such that \( f_r^{(t)}(b^*) > 0 \) for a zero \( b^* \) of \( f_r \) then \( F \) is positive definite for large values of \( ||(x_1, x_2)|| \). Therefore the curve \( F(x_1, x_2) = K \) is closed for \( K \) large enough. This implies that this curve is a periodic orbit for \( X_F \) showing the existence of a critical point inside (see [4, p.254]) and therefore leading to an absurdity. This ends the proof that \( F \) is exhaustive. The same is true for \( G \). Figure 5 displays the dominant terms in the Newton polygon near infinity. If there are no terms with exponents \( (i, j) \), \( i < r \) but there is some point \( (r, s) \) with nonzero coefficient the Newton polygon ends on the point \( (r, s) \) with the maximum value of \( s \).
Proof of the exhaustivity of $T$. We know that the flows $\psi_t$, $\psi_t$ exist for all $t$ without going to infinity in finite time. Let $(x_0, y_0) \in \mathbb{R}^2$ be any point and $(t_0, s_0) = T(x_0, y_0)$. Then select any point $(t, s)$ in $\mathbb{R}^2$. From (2) it follows that

$$\phi_{s_0 - s} \psi_{t - t_0} (x_0, y_0) = (\bar{x}, \bar{y})$$

with $T(\bar{x}, \bar{y}) = (t, s)$. Hence $T$ is exhaustive.

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Captions for the figures

Fig. 1. Qualitative picture of the flow of $X_F$ on $S^2$.

Fig. 2. The four carts used for $S^1$.

Fig. 3. Qualitative picture of the flow of (1) in $H_1 \cup S^1 \subset S^2$. Case (a) $m$ odd; case (b) $m$ even.

Fig. 4. Qualitative picture of the flow of $X_F$.

Fig. 5. The Newton polygon of dominant terms near infinity.