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DISTRIBUTIVITY AND IRREDUCIBILITY
IN CLOSURE SYSTEMS

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DISTRIBUTIVITY AND IRREDUCIBILITY
IN CLOSURE SYSTEMS

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Abstract:

In this work we see how the property:

$$T \vee (S(x) \cap S(y)) = (T \vee S(x)) \cap (T \vee S(y))$$

where T is a closed set of the closure system associated to the closure to the operator S , is related to the distributivity of the closure system's lattice and to a characterization of the irreducible closed sets.

INTRODUCTION

The consequence operator of some logics (classic, intuitionistic, ...etc.) satisfies, relative to the connective \vee , the so called "separation of cases's rule":

$$S(X, x \vee y) = S(X, x) \cap S(X, y)$$

this property is equivalent to:

$$T \vee (S(x) \cap S(y)) = (T \vee S(x)) \cap (T \vee S(y)) \quad (*)$$

where T is a closed set of the closure system \mathcal{F} associated to the closure operator S . The later formulation is very interesting because it is expressed only by means of the lattice operations of the closure system and it allows to do an abstract study of it.

In this work we see how the property (*) is related to

the distributivity of the closure system's lattice and to a characterization of the irreducible closed sets of .

If \mathcal{F} is a closure system on the universe of an algebra of type (2) we give sufficient conditions in order to make the concept of \mathcal{F} -prime- which is a generalization of the concept of prime- and the concept of \mathcal{F} -irreducible closed set equivalent.

In almost every cases we get the above results assuming the closure system \mathcal{F} has a basis of irreducible closed sets assumption which includes the case where \mathcal{F} is algebraic. Moreover, in this last case the condition (*) implies that \cap is infinitely distributive in the lattice $(\mathcal{F}, \cap, \vee)$. Finally we obtain an application of the above results to distributive lattices.

Throughought this work we adopt the notation of Brown-Suszko [2] except for both irreducible which means to be finitely irreducible and completely irreducible which means to be arbitrarily irreducible.

Let \mathcal{F} be a closure system on the non empty set A , and S the associated closure operator.

Lemma 1

If $T, T \in \mathcal{F}$, is such that for any $a, b \in A \setminus T$, $S(a) \cap S(b) \not\subset T$, then T is \mathcal{F} -irreducible.

Proof. We suppose that T is not \mathcal{F} -irreducible, that is, there are $T_1, T_2 \in \mathcal{F}$, $T_1 \neq T \neq T_2$ and $T = T_1 \cap T_2$, then exist $a, b \in A \setminus T$, $a \in T_1$, $b \in T_2$ such that $S(a) \cap S(b) \not\subset T_1 \cap T_2 = T$, which contradicts the hipotesis.

Lemma 2

If \mathcal{F} is such that:

I.1 For any $T \in \mathcal{F}$, $a, b \in A$

$$T \vee (S(a) \cap S(b)) = (T \vee S(a)) \cap (T \vee S(b)),$$

then

I.2. I is \mathcal{F} -irreducible implies that for any $a, b \in A \setminus I$

$$S(a) \cap S(b) \not\subset I$$

Proof. We suppose that $I \in \mathcal{F}$, $a, b \in A \setminus I$ and $S(a) \cap S(b) \not\subset I$, by I.1 $I = I \vee (S(a) \cap S(b)) = (I \vee S(a)) \cap (I \vee S(b))$. Since $I \neq I \vee S(a)$ and $I \neq I \vee S(b)$, then I is not \mathcal{F} -irreducible.

Theorem 3.

If \mathcal{F} has a basis of irreducible closed sets and satisfies I.2, then the lattice $(\mathcal{F}, \cap, \vee)$ is distributive.

Proof. To prove the distributivity of $(\mathcal{F}, \cap, \vee)$ it suffices to see that: For any $T_1, T_2, T_3 \in \mathcal{F}$

$$T_1 \cap (T_2 \vee T_3) \subset (T_1 \cap T_2) \vee (T_1 \cap T_3)$$

Since the family of irreducible closed sets is a basis of \mathcal{F} - because S have a basis of irreducible closed sets- we show that for any \mathcal{F} -irreducible $I : (T_1 \cap T_2) \vee (T_1 \cap T_3) \subset I$ implies $T_1 \cap (T_2 \vee T_3) \subset I$. We suppose that is not true, then $T_1 \not\subset I$ and $T_2 \vee T_3 \not\subset I$, we can suppose that $T_1 \not\subset I$ and $T_2 \not\subset I$, hence there are $a, b \in A \setminus I$, $a \in T_1$, $b \in T_2$, thus

$$S(a) \cap S(b) \subset T_1 \cap T_2 \subset (T_1 \cap T_2) \vee (T_1 \cap T_3).$$

This contradicts I.2, and proves the theorem.

Corollary

Let \mathcal{F} be a closure system on A , $A \neq \emptyset$. If \mathcal{F} has a basis of irreducible closed sets, the following conditions are equivalents:

I.1. For any $T \in \mathcal{F}$, $a, b \in A$:

$$T \vee (S(a) \cap S(b)) = (T \vee S(a)) \cap (T \vee S(b))$$

I.2. If $I \in \mathcal{F}$ is irreducible, then for any $a, b \in A \setminus I$

$$S(a) \cap S(b) \not\subseteq I$$

I.3. The lattice $(\mathcal{F}, \cap, \vee)$ is distributive

The above results have a different formulation due to the next result whose proof is obvious.

Lemma 4

Let \mathcal{F} be a closure system on A , $A \neq \emptyset$, and $T \in \mathcal{F}$.

The following conditions are equivalents:

(i) For any $a, b \in A \setminus T$, $S(a) \cap S(b) \not\subseteq T$

(ii) For any $N, N' \subset A$, N, N' finites, $N, N' \not\subseteq T$, implies

$$S(N) \cap S(N') \not\subseteq T$$

Now we assume that A is the universe of the algebra $\mathcal{A} = (A, \cdot)$ of type (2). Let \mathcal{F} be a closure system on A

Definition

Let P be a closed set of \mathcal{F} , we call that P is f -prime, when:

if $a, b \in A$, $a \cdot b \in P$ if and only if $a \in P$ or $b \in P$.

The following result is due to B. Verdú [6].

Theorem 5

If \mathcal{F} has a basis \mathcal{B} of f -prime closed sets, then

it satisfies:

I.4 For any $T \in \mathcal{F}$, $a, b \in A$:

$$T \vee S(a.b) = (T \vee S(a)) \cap (T \vee S(b))$$

I.5 Every element of \mathcal{B} is \mathcal{F} -irreducible.

Using the theorem 5 and the above results we obtain:

Corollary 1

If \mathcal{F} has a basis \mathcal{B} of f-prime closed sets, then satisfies I.1 and: for any $a, b \in A$, $S(a.b) = S(a) \cap S(b)$

Proof. If in I.4 get $T = S(\emptyset)$ we obtain that for any $a, b \in A$ $S(a) \cap S(b) = S(a.b)$ and substituting one more this one in I.4 we obtain I.1.

Corollary 2

If \mathcal{F} has a basis \mathcal{B} of f-prime closed sets, then the lattice $(\mathcal{F}, \cap, \vee)$ is distributive

Now we give the relationship between f-prime closed sets and irreducible closed sets.

Lemma 6

If \mathcal{F} satisfies that for any $a, b \in A$, $S(a.b) = S(a) \cap S(b)$ then every f-prime closed set is \mathcal{F} -irreducible.

Proof. Let T be f-prime closed set, we suppose that $a, b \in T$, thus $a.b \in T$ and since $S(a.b) \in T$, so we have $S(a) \cap S(b) \subset T$.

By lemma 1, T is \mathcal{F} -irreducible.

Theorem 7

If \mathcal{F} satisfies I.2 and for any $a, b \in A$, $S(a.b) = S(a) \cap S(b)$, then every irreducible closed

set is f -prime.

Proof. Let T be \mathcal{J} -irreducible. If $a \in T$ or $b \in T$, then by I.2 $S(a) \cap S(b) \subset T$, hence $S(a.b) \subset T$, so $a.b \in T$. Conversely if $a, b \notin T$, by I.2 and hypothesis we have $S(a.b) \not\subset T$, thus $a.b \notin T$.

Corollary 1

If \mathcal{J} satisfies the hypothesis of theorem 7, then for any $T \in \mathcal{J}$ the following conditions are equivalent:

- (i) T is f -prime
- (ii) T is \mathcal{J} -irreducible.

Corollary 2

If \mathcal{J} has a basis of f -prime closed sets, then for any $T \in \mathcal{J}$ the following conditions are equivalent:

- (i) T is f -prime
- (ii) T is \mathcal{J} -irreducible.

Proof. If \mathcal{J} has a basis of f -prime closed sets, then, by theorem 5, \mathcal{J} has a basis of irreducible closed sets, and by corollary of theorem 3 and once more by theorem 5, \mathcal{J} satisfies I.2. Hence \mathcal{J} satisfies the hypothesis of theorem 7.

Corollary 1 finishes the proof.

Corollary 3

If \mathcal{J} has a basis of irreducible closed sets, then the condition I.4 is equivalent to the following:

For any $T \in \mathcal{J}$, T is f -prime if and only if, T is \mathcal{J} -irreducible.

Proof. If I.4 holds, then by lemma 6 every f -prime closed set is \mathcal{J} -irreducible. If every irreducible closed set is f -prime, then by theorem 5 the condition I.4 hold.

Let \mathcal{J} be an algebraic closure system on A. It is already known that the smallest basis of \mathcal{J} is the family of completely irreducible closed sets, so the irreducible closed sets are a basis. Then applying the above results we have:

Lemma 8

If \mathcal{J} is an algebraic closure system, then the conditions I.1, I.2 and I.3 are equivalent.

If the closure system \mathcal{J} is algebraic we can find another condition equivalent to I.1, I.2 and I.3. This condition is II.1 in the following theorem which is an extension to algebraic closure systems of a result of A. Diego [3], Pag 28.

Theorem 9

If \mathcal{J} is an algebraic closure system and the lattice $(\mathcal{J}, \cap, \vee)$ is distributive, then:

II.1. In $(\mathcal{J}, \cap, \vee)$ \cap is infinitely distributive.

Proof. We must show that if $T \in \mathcal{J}$ and $T_i \in \mathcal{J}$, $i \in I$, then

$$T \cap (\vee (T_i / i \in I)) \subseteq \vee (T \cap T_i / i \in I).$$

If $x \in T \cap (\vee (T_i / i \in I))$, then $x \in T$ and $x \in \vee (T_i / i \in I) = S(U(T_i / i \in I))$. Since \mathcal{J} is algebraic, there exists a finite number of elements $a_{i_j} \in T_{i_j}$, $0 \leq j \leq n$, $i_0, \dots, i_n \in I$, such that $x \in T$ and

$$x \in S(\{a_{i_0}, \dots, a_{i_n}\}) = \vee (S\{a_{i_j}\} / 0 \leq j \leq n);$$

so $x \in T \cap (\vee (S\{a_{i_j}\} / 0 \leq j \leq n)) = \vee (T \cap S\{a_{i_j}\} / 0 \leq j \leq n) \subseteq \vee (T \cap T_{i_j} / i \in I)$.

Corollary 1

If \mathcal{J} is an algebraic closure system, then the



conditions I.1, I.2, I.3 and II.1 are equivalent.

We know that every closure system is isomorf to an algebraic lattice, and every algebraic lattice is isomorf to a closure system. In this direction we have the following result.

Corollary 2

If (L, \wedge, \vee) is an algebraic lattice, then the following conditions are equivalent:

(i) For any $b \in L$, $x, y \in C$, we have

$$b \vee (x \wedge y) = (b \wedge x) \vee (b \wedge y)$$

(where C is the set of all compact elements of L)

(ii) If $a \in L$ is \wedge -irreducible, then for any $x, y \in C$

$$x \triangleleft a, y \triangleleft a \text{ implies } x \wedge y \triangleleft a$$

(iii) (L, \wedge, \vee) is distributive

(iv) In (L, \wedge, \vee) \wedge is infinitely distributive.

Proof. This follows from the fact that the compact elements of an algebraic closure system are the elements which are finitely generated.

If the closure sistem \mathcal{F} is defined on A , the universe of an algebra of types (2), $\mathcal{A} = (A, \cdot)$, then

Theorem 10

The following conditions are equivalent :

(i) I.4

(ii) For any $T \in \mathcal{F}$, T is f -prime if and only if T is \mathcal{F} -irreducible.

Proof. This is a consequence of theorem 7, corollary 3.

As an application of some results of this work, in theorem 11

we give a new proof whose a result is already known (Gratzer [4] pg 99)

Theorem 11

If $\mathcal{A} = (A, \wedge, \vee)$ is a lattice, then \mathcal{A} is distributive if, and only if, the lattice of its filters is distributive.

Proof. If \mathcal{A} is ditributive, then it is already known that the prime filters are a basis of the closure system of all filters so, by theorem 5 and the corollary of theorem 3, we see that the closure system \mathcal{F} of all filters is distributive.

If \mathcal{F} is distributive, then I.4 holds. We must show that for any $x, y, z \in A$, $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. This is equivalent to show that $(x \wedge y) \vee (x \wedge z) \in S(\{x, y \vee z\})$, where S is the closure operator associated to \mathcal{F} . Since $x \wedge y \in S(\{x, y\})$ and $x \wedge z \in S(\{x, z\})$, then $(x \wedge y) \vee (x \wedge z) \in S(\{x, y\}) \cap S(\{x, z\})$.

In the next theorem we give a characterization of distributive lattices in a diferent way of Theorem 11.

Theorem 12

If $\mathcal{A} = (A, \wedge, \vee)$ is an algebra of type (2,2), then is a distributive lattice if, and only if, there exist a closure system \mathcal{F} on A such that the associated closure operator S satisfies:

- (i) For any $x, y \in A$, $S(x) = S(y)$, implies $x = y$
- (ii) For any $x, y \in A$, $X \subset A$, $\text{Card}(X) \leq 1$,

$$S(X \cup \{x \vee y\}) = S(X \cup \{x\}) \cap S(X \cup \{y\}).$$

(iii) For any $x, y \in A$, $S(\{x, y\}) = S(\{x \wedge y\})$.

Proof. It is easy to see that the conditions, (i), (iii) and $S(\{x \vee y\}) = S(\{x\}) \cap S(\{y\})$ implies that \mathcal{A} is a lattice. In order to see that \mathcal{A} is distributive, we see that the proof of distributivity of theorem 11 only uses the restricted condition (ii).

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