UNIVERSITAT DE BARCELONA
FACULTAT DE MATEMÀTIQUES

MONADICITY IN TOPOLOGICAL PSEUDO–BOOLEAN ALGEBRAS

by Josep M. Font

PRE–PRINT N.º 16
novembre, 1983

MONADICITY IN TOPOLOGICAL PSEUDO-BOOLEAN ALGEBRAS

Josep M. Font

ABSTRACT

We study four subclasses of topological pseudo-Boolean algebras representing increasingly strong intuitionistic counterparts of monadic Boolean algebras. The fourteen equivalent classical conditions are shown to split into six non-equivalent sets of equivalent conditions, whose inter-connections are all determined. We also deal with several algebraic-logic properties of our classes, such as regular,dense, Peircean elements, and others. We conclude that a closure operator derived from an interior one is not meaningless in this context of intuitionistic modal logic.

Index

0. Introduction

It is generally accepted that in intuitionistic modal logic the two modal operators cannot be completely dual in the sense of the two classical laws: $L \leftrightarrow \neg M \neg$ and $M \leftrightarrow \neg L \neg$. Most authors have chosen to work considering both operators as primitive and independent, linking them with other weaker relations; this is the case of Prior, Bull, Ono, Fischer-Servi, Sotirov, and others. In some cases, such as in part of [Bul] and [O], they avoid having M. In this paper we begin an algebraic study of the application of Gödel's proposal (that is, to have a primitive L and define $M \leftrightarrow \neg L \neg$) to an intuitionistic base. It is worth noting that the remaining alternative, having M primitive and defining $L \leftrightarrow \neg M \neg$, is not viable, as example 5.10 shows.

The work has been done and written in the algebraic side of the subject, and in order to avoid repetitions we will not refer, outside of this section, to the equivalent logical formulations of several results; some of them will be dealt with in another paper. Here we start from a system of (propositional) intuitionistic modal logic analogous to S4, whose algebraic models are topological pseudo-Boolean algebras (tpBa); these are defined as in [O] using only the interior operator I, in the Gödelian style. A deductive and implicational study of tpBa and of the logical system, whithout a mention to M, has been published in [Fo].

In section 1 we introduce the operator $\delta = \neg I \neg$ corresponding to M and we study some properties related to it. It is of special interest the analysis of several conditions (axioms or inference rules) which can be added to the S4 system to obtain S5, that is, conditions which turn a topological Boolean algebra into a monadic one. In such structures these conditions are all equivalent but it is not so in topological pseudo-Boolean algebras, due to the peculiar features of intuitionistic negation; hence the interest of its study. H. Ono has done some work concerning several conditions without the possitibility operator.

We examine fourteen different conditions involving L and/or M. Some of them come from classical modal logic, such as the laws of reduction of modalities, Becker's rule and axioms, or the M" axiom of von Wright. Some others originate in mainly algebraic works, as are those of Halmos, Davis, Monteiro, Bull, Beth and Nieland mentioned here. We determine all the equivalences and implications that hold between these conditions, and we define with some of them four subclasses of tpBa and study some of their algebraic properties. We specially deal with several concepts and results formally analogous to other classical concepts of modal or intuitionistic logic. All this is contained in sections 2,3 and 4. In section 5 we have gathered all the counterexamples we use throughout the paper.

The four logical systems that would correspond to the subclasses of tpBa here defined are all of type S5 in the sense of [Bu2], although the last one is not intuitionistically plausible. We have made no attempt to single out one of them as a "true" analogue of S5, but we rather study the properties that establish differences between them, thus finding they have an increasing "degree of monadicity". On the other hand, H. Ono has proven in [O] that there is an infinite number of systems of intuitionistic modal logics analogous to S5, and G. Fischer-Servi has worked on this subject with remarkable deepness in [Fi1], [Fi2] and [Fi3], among other papers.

We recall some of the definitions and notational conventions introduced in [Fo] and that will be used here. A <u>topological pseudo-Boolean algebra (tpBa)</u> $\langle A, I, \neg, \wedge, \vee, \rightarrow \rangle$ is a pseudo-Boolean algebra

$\langle A,\neg,\wedge,\vee,\rightarrow\rangle$ where 1 denotes the maximum and 0 the minimum, with an interior operator I on A. The <u>open</u> elements are those of $B = \{a \in A\colon Ia=a\}$, and the <u>deductive systems</u> of A are the sets in $\mathcal{D} = \{D \subseteq A\colon 1 \in D,$ if $a, a \rightarrow b \in D$ then $Ia, b \in D\}$. If for each $D \in \mathcal{D}$ we define $\equiv_D$ by $a \equiv_D b$ iff $a \rightarrow b \in D$ and $b \rightarrow a \in D$ then the correspondence $D \mapsto \equiv_D$ is a lattice isomorphism between $\mathcal{D}$ and the congruence lattice of A. We denote by $\underline{D}$ the consequence operator associated with $\mathcal{D}$ and $\underline{L} = \langle A, \underline{D}\rangle$ is the associated <u>abstract logic</u>. This abstract logic has the following properties: The <u>Adjunction Principle</u> $\underline{D}(a,b) = \underline{D}(a \wedge b)$, the <u>Strong Disjunction Principle</u> $\underline{D}(X,a) \cap \underline{D}(X,b) = \underline{D}(X, a \overset{*}{\vee} b)$ where $a \overset{*}{\vee} b = Ia \vee Ib$, the <u>Deduction Principle</u> $b \in \underline{D}(X,a)$ iff $a * b \in \underline{D}(X)$ and the <u>Pseudo-Reductio ad Absurdum Principle</u> $\neg^* a \in \underline{D}(X)$ iff $\underline{D}(X,a) = A$, where $\neg^* a = a * 0$, and $*$ is any of the following implication operations: We weak implication $a \nrightarrow b = Ia \rightarrow b$, the intuitionist implication $a \Rightarrow b = I(Ia \rightarrow Ib)$, and the strange implication $a \rightsquigarrow b = Ia \rightarrow Ib$. Hence the logic $\underline{L}$ has an intuitionistic character when we take into account the preceding connectives, and this is reinforced by the following construction: with each $D \in \mathcal{D}$ we associate the relation $\sim_D$ defined as $a \sim_D b$ iff $\underline{D}(D,a) = \underline{D}(D,b)$ iff $a * b \in D$ and $b * a \in D$. Then $\sim_D$ is a logical congruence of $\underline{L}$ in the sense of [B-S] and the quotient $A/\sim_D$ is a pseudo-Boolean algebra with respect to the operations there induced by $\neg^*, \wedge, \overset{*}{\vee}$ and $*$.

The purely intuitionistic character of these structures is shown in results as the following: for all $D \subseteq A$, $D \in \mathcal{D}$ iff $1 \in D$ and D is closed by Modus Ponens with respect to $*$ (that is, if $a, a * b \in D$ then $b \in D$). We can extend the analogy with intuitionistic structures by introducing two kinds of elements related to implication and negation as in pseudo-Boolean algebras: the <u>$*$-dense</u> elements $D_* = \{a \in A\colon \neg^* a = 0\}$ and the <u>$*$-Peircean</u> elements $P_* = \{a \in A\colon a = ((b * c) * b) * b$ for some $b, c \in A\}$. We have the usual characterization of the (maximal) radical of A: $R(A) = \underline{D}(P_*)$ and the following relations: $P_{\Rightarrow} \subseteq P_{\rightsquigarrow} \subseteq P_{\nrightarrow} = R(A) = D_{\Rightarrow} \supseteq D_{\rightsquigarrow} = D_{\nrightarrow}$, and $D_{\nrightarrow} \cap B \subseteq P_{\Rightarrow} = I(P_{\nrightarrow}) = P_{\rightsquigarrow} \cap B = R(A) \cap B$. Moreover for each $a \in A$ we have that $a \in P_{\Rightarrow}$ iff $a = ((a \Rightarrow 0) \Rightarrow a) \Rightarrow a$. In the present paper these results will be strengthened for some of the subclasses of tpBa studied and more properties of these special elements and of other related concepts will be obtained.

1. Negation and possibility in tpBas.

In all pseudo-Boolean algebras we have the closure operator $x \mapsto \neg\neg x$ whose properties are well-known; see for instance [M-T] or [R-S]. In tpBas there is another closure operator which is defined as follows:

1.1 Definition. In every tpBa A the closure operator associated with the interior operator I is $\delta a = \neg I \neg a$ for all $a \in A$. The closed elements of A are those of $T = \{a \in A: a = \delta a\}$.

1.2. Proposition. In all tpBa A the following hold:

(1) $Ia \leq a \leq \neg\neg a \leq \delta a$ for all $a \in A$,

(2) $\delta 0 = 0$, $a \leq \delta a = \delta^2 a$ for all $a \in A$, and if $a \leq b$ then $\delta a \leq \delta b$ for all $a,b \in A$, and

(3) T is $\wedge$-closed, contains 0 and 1, and for all $a \in A$, $\delta a = = \min\{t \in T: a \leq t\}$.

Proofs: All trivial.■

We observe that (2) states that δ is an order closure satisfying $\delta 0=0$ and, from (3), $\delta(\delta a \wedge \delta b) = \delta a \wedge \delta b$ for all $a,b \in A$. While I is a lattice-interior operator, δ is not a lattice-closure operator because it does not necessarily satisfy $\delta(a \vee b) = \delta a \vee \delta b$ nor even $\delta(\delta a \vee \delta b) = \delta a \vee \delta b$, as example 5.7 shows. We now give some relations and properties satisfied by I,δ and $\neg$ which we shall use from now on without mentioning them.

1.3. Proposition. In every tpBa A and for all $a \in A$:

(1) $\delta a = \delta \neg\neg a = \neg\neg \delta a$,

(2) if $a \in T$ then $a = \neg\neg a$,

(3) if $\neg a = 0$ then $\delta a = 1$,

(4) $\delta I \neg a = I\delta a$,

(5) $I\neg a \leq \neg\delta a \leq \neg a \leq \delta\neg a \leq \neg Ia$, and

(6) $\neg Ia \in T$.

Proofs: All reduce to easy computations dealing only with the definitions of δ and T and with elementary properties of negation in pseudo-Boolean algebras.■

Now we ask whether the negation induces any relation between open and closed elements. As it is well-known, in topological Boolean algebras, as in ordinary topological spaces, there is a perfect duality between them, and there are four valid implications: (1) if a is open then $\neg a$ is closed; (2) if $\neg a$ is closed then a is open; (3) if a is closed then $\neg a$ is open; and (4) if $\neg a$ is open then a is closed. In our case the results are more limited:

1.4. Proposition: If a is an open element of a tpBa then $\neg a$ is closed.

Proof: From $a \leq \neg\neg a$ we deduce $a = Ia \leq I\neg\neg a$ and then $\delta\neg a = \neg I\neg\neg a \leq$ $\leq \neg Ia = \neg a$, so $\delta\neg a = \neg a$.■

We have proven that implication (1) is always true. The remaining three are not true in general, as we see in example 5.3, although (2) and (3) can hold in some cases, as in 5.2. This is not the case of (4), which turns out to be characteristic for topological Boolean algebras:

1.5. Proposition. A tpBa A is a topological Boolean algebra iff for all $a \in A$, if $\neg a$ is open then a is closed.

Proof: If A is Boolean there is nothing to prove. If it is not, then there is an $a \in A$, $a \neq 1$, such that $\neg a = 0$. This implies that $\neg a$ is open and that a is not closed since by 1.3(3), $\delta a = 1 \neq a$.■

The most direct relations between open and closed elements, namely $B \subseteq T$ and $T \subseteq B$, will be shown to be equivalent to the definitions of some subclases of tpBa in sections 3 and 4. Another kind of relations between B and T are given by the so-called "connecting conditions" of [Fi3]; here we have one of them:

1.6. Proposition: In every tpBa A the following hold:

(1) $\delta(a\to b)\leq Ia\to\delta b$ for all $a,b\in A$, and

(2) $B\to T\subseteq T$.

Proof: We have $a\to b\leq\neg b\to\neg a = (b\to 0)\to(a\to 0)$ and therefore $b\to 0\leq$ $\leq (a\to b)\to(a\to 0) = a\to((a\to b)\to 0)$ and $a\leq(b\to 0)\to((a\to b)\to 0)$, that is, $a\leq\neg b\to\neg(a\to b)$; now $Ia\leq I(\neg b\to\neg(a\to b))\leq I\neg b\to I\neg(a\to b)\leq$ $\leq\neg I\neg(a\to b)\to\neg I\neg b = \delta(a\to b)\to\delta b$ which is equivalent to (1). Now (2) is a trivial consequence of (1).■

The remaining condition $\delta a\to Ib\leq I(a\to b)$, which is equivalent to $T\to B\subseteq B$, will be dealt with at the end of section 3.

The rest of this section is devoted to the introduction of the concept of regularity in tpBas. The concept of regular element comes from topology and was introduced in [M-T] by using the negation of pseudo-Boolean algebras. With respect to the pseudo-Boolean algebra underlying a tpBa A, we denote by $Reg_H(A) = \{a\in A: a = \neg\neg a\}$ the set of H-regular elements, that is, the set of regular elements of the pseudo-Boolean algebra (or Heyting algebra). Now 1.3(2) is read $T\subseteq Reg_H(A)$; it is not possible to strengthen this relation: in 5.7 the inclusion is proper, and even in monadic Boolean algebras it can be so.

In topological pseudo-Boolean algebras there are there operations $\neg^*$ which have the behaviour of logical negations of the intuitionistic type; but we have a topological interior operator and an associated closure operator, and so it can have some interest trying to write down the original topological ideas. We think that finding some coincidences between the two formulations is not merely casual.

1.7 Proposition. In every tpBa A and for all $a\in A$ the following conditions are equivalent:

(1) $a = \delta Ia$,

(2) $a = \neg_{\nrightarrow}\neg_{\nrightarrow} a$, and

(3) $a = \neg_{\leadsto}\neg_{\leadsto} a$.

Proof: From the definition of $\neg_{\nrightarrow}$ and $\neg_{\leadsto}$ we see that $\neg_{\nrightarrow} a = \neg_{\leadsto} a = \neg Ia$ and then $\neg_{\nrightarrow}\neg_{\nrightarrow} a = \neg_{\leadsto}\neg_{\leadsto} a = \neg I \neg Ia = \delta Ia$.■

1.8. Proposition. In every tpBa A and for all a A the following conditions are equivalent:

(1) $a = I\delta a$, and

(2) $a = \neg_{\Rightarrow}\neg_{\Rightarrow} a$.

Proof: From the definition of $\neg_{\Rightarrow}$ we see that $\neg_{\Rightarrow} a = I\neg Ia$ and then $\neg_{\Rightarrow}\neg_{\Rightarrow} a = I\neg I\neg Ia = I\delta Ia$, but $a = I\delta a$ implies a is open and so $a = I\delta Ia = \neg_{\Rightarrow}\neg_{\Rightarrow} a$. Conversely $a = \neg_{\Rightarrow}\neg_{\Rightarrow} a = I\delta Ia$ also implies a is open and therefore $a = I\delta a$.■

In this situation we are nearly forced to give the two following definitions.

1.9. Definitions. In a tpBa A the elements $a \in A$ such that $a = \delta Ia$ will be called regular, and those satisfying $a = I\delta a$ will be called I-regular. We denote by $Reg(A)$ and $Reg_I(A)$ the sets of regular and I-regular elements, respectively.

The most immediate properties of regular elements are the following:

1.10. Proposition. In every tpBa A we have:

(1) $\{0,1\} \subseteq Reg(A) \subseteq T \subseteq Reg_H(A)$,

(2) $\{0,1\} \subseteq T \cap B = Reg(A) \cap B \subseteq Reg_I(A) \subseteq B$, and

(3) $Reg(A) \cap R(A) = Reg(A) \cap D_{\nrightarrow} = \{1\}$.

Proof. (1) and (2) are direct consequences of the definitions. For (3), we have $Reg(A) \cap D_{\nrightarrow} \subseteq Reg(A) \cap R(A)$ and if $a \in Reg(A) \cap R(A)$ then, taking into account that $R(A) = D_{\Rightarrow}$, we have $a = \neg_{\nrightarrow}\neg_{\nrightarrow} a = (a \nrightarrow 0) \nrightarrow 0 = (a \Rightarrow 0) \nrightarrow 0 = 0 \nrightarrow 0 = 1$.■

Examples 5.2 and 5.3 show us that the inclusions of 1.10 are not equalities in general. Later we will complete the analysis of the concept(s) of regularity and some of this properti es will be partially improved.

2. Weakly monadic tpBas

The concept of monadic Boolean algebras was invented by P.R. Halmos to set out an algebraic description of the monadic predicate calculus. He defined them in [H] with the specific axiom $\delta(a \wedge \delta b) = \delta a \wedge \delta b$, and showed that this was equivalent to the addition of the condition $\delta \neg \delta = \neg \delta$ to a topological Boolean algebra. Independently and at nearly the same time C. Davis defined in [D] the class of " S5 operators" on a Boolean algebra with the condition "if $a \wedge b = 0$ then $\delta a \wedge \delta b = 0$", and showed that it was also equivalent to $\delta \neg \delta = \neg \delta$. The motivation fcr Davis' work was modal logic, and in fact this last condition corresponds to the specific axiom of von Wright's system M" presented in [vW]; as it is well-known, B. Sobociński showed the equivalence between M" and S5. Now we see the equivalence of these conditions on a topological pseudo-Boolean algebra.

2.1. Theorem. In every tpBa A the following conditions are equivalent:

(1) $\delta \neg \delta a = \neg \delta a$ for all $a \in A$ (that is, T is closed by negation),

(2) if $a \wedge \delta b = 0$ then $\delta a \wedge \delta b = 0$ for all $a, b \in A$, and

(3) $\delta(a \wedge \delta b) = \delta a \wedge \delta b$ for all $a, b \in A$.

Proof: (1)$\Rightarrow$(2): $a \wedge \delta b = 0$ is equivalent to $a \leq \neg \delta b$ and from this and (1) we have $\delta a \leq \delta \neg \delta b = \neg \delta b$, which is equivalent to $\delta a \wedge \delta b = 0$.

(2)$\Rightarrow$(1) because we always have $\neg \delta a \leq \delta \neg \delta a$, and since $\neg \delta a \wedge \delta a = 0$, by (2) we have $\delta \neg \delta a \wedge \delta a = 0$, thus establishing $\delta \neg \delta a \leq \neg \delta a$.

(2)$\Rightarrow$(3): $a \leq \delta a$ and so $a \wedge \delta b \leq \delta a \wedge \delta b \in T$ as T is $\wedge$-closed. By 1.2(3), to show (3) it suffices to show that for any $t \in T$, if $a \wedge \delta b \leq t$ then $\delta a \wedge \delta b \leq t$. But $a \wedge \delta b \leq t$ is equivalent to $\delta b \leq a \rightarrow t \leq$

$\leq \neg t \to \neg a$ which in turn is equivalent to $\delta b \wedge \neg t \leq \neg a$ and hence to $\delta b \wedge \neg t \wedge a = 0$. Now $\delta b \wedge \neg t \in T$ and so we have $\delta(\delta b \wedge \neg t) \wedge a = 0$, and applying (2) we obtain $\delta b \wedge \neg t \wedge \delta a = \delta(\delta b \wedge \neg t) \wedge \delta a = 0$ from where we can infer $\delta a \wedge \delta b \leq \neg\neg t = t$ by 1.3(2). Thus we have proved that $\delta(a \wedge \delta b) = \delta a \wedge \delta b$.

(3) $\Rightarrow$ (2) is trivial. ■

The preceding result justifies the following

2.2 Definition. A tpBa is called weakly monadic iff it satisfies any of the conditions in Theorem 2.1.

We give a list of several useful rules for weakly monadic tpBas and some elementary properties.

2.3. Proposition. In every weakly monadic tpBa A the following hold:

(1) $\delta I \neg a = \neg \delta a = \neg I \delta a$ for all $a \in A$,

(2) $\neg Ia = \delta Ia$ for all $a \in A$,

(3) $\neg\neg Ia = \delta Ia$ for all $a \in A$,

(4) if $a \in B$ then $\delta a = \neg\neg a$, and

(5) if $a \in T$ then $\neg a = \neg Ia$ and $a = \neg\neg Ia$.

Proofs: They all are straightforward computations making use of 2.1(1); for (5) recall that $T \subseteq Reg_H(A)$. ■

2.4. Proposition. In every tpBa the following conditions are equivalent:

(1) A is weakly monadic,

(2) $\neg\neg Ia \in T$ for all $a \in A$, and

(3) $a = \neg\neg Ia$ for all $a \in T$.

Proof. In 2.3(3) we have seen that (1) implies (2), and in 2.3(5) we show that (1) implies (3). If we assume (2) and apply it to $\neg a$ we have $\neg \delta a = \neg\neg I \neg a \in T$ and we see that T is closed by negation, which is equivalent to (1). Similarly if we assume (3) and apply it to δa we find $\delta a = \neg\neg I \delta a = \neg\neg I \neg I \neg a = \neg\neg I \neg\neg\neg I \neg a = \neg \delta \neg \delta a$ and by negation $\neg \delta a = \neg\neg \delta \neg \delta a = \delta \neg \delta a$ because $T \subseteq Reg_H(A)$. Therefore we have (1) again. ■

2.5. <u>Proposition</u>. In every weakly monadic tpBa A and for every $a \in A$ we have:

(1) $\neg^{\Rightarrow} a=0$ iff $\neg^{\leftrightarrow} a=0$, that is, $I\neg Ia = 0$ iff $\neg Ia = 0$, and

(2) if $a \in T$ and $Ia = 0$ then $a = 0$.

<u>Proof</u>. For (1) there is nothing to prove in one direction; and if $I\neg Ia = 0$ then using 2.3(5) for $\neg Ia \in T$ we obtain $\neg Ia = \neg\neg I\neg Ia = \neg\neg 0 = 0$. (2) is also a direct consequence of 2.3(4).■

2.6. <u>Corollary</u>. In every weakly monadic tpBa A, $R(A) = D_{\Rightarrow} = D_{\leftrightarrow} = D_{\leadsto}$ and if we put $R_H(A) = \{a \in A: \neg a=0\}$ (the maximal radical of pseudo-Boolean algebra under A) then $R(A) \subseteq R_H(A)$.■

It is worth noting that, according to 2.6, in weakly monadic tpBas there is only one kind of <u>dense</u> elements, $D_* = \{a \in A: \neg^* a = 0\}$ because of the coincidence of $D_{\Rightarrow}$, $D_{\leftrightarrow}$ and $D_{\leadsto}$ (example 5.1 shows that this is not general). Therefore, there is a unified characterization of the radical in terms of dense elements: $R(A) = D_*$. This characterization can be logically interpreted as follows: if we think of dense elements as representing "almost true" sentences in the sense that their logical negation is false, and if we think of the elements in the radical as representing "almost true" sentences in the sense that they belong to every complete consistent theory, then the equality $R(A) = D_*$ can be read as stating the equivalence of these two kinds of "almost true" sentences.

There is no coincidence among the three types of Peircean elements, as exemple 5.3, where the inclusions $P_{\Rightarrow} \subseteq P_{\leadsto} \subseteq P_{\leftrightarrow}$ are proper, shows. In the same example we see that it is not possible to improve, for weakly monadic tpBas, the results obtained in section 1 concerning the relations between open and closed elements via negation. Concerning 2.5(1), we announce that it is not true in every weakly monadic tpBa A that $\neg^{\Rightarrow} a = \neg^{\leftrightarrow} a = \neg^{\leadsto} a$ for all $a \in A$: in 3.5 we will show that this fact characterizes an effective subclass of those algebras.

It is well known that the set of H-regular elements in a pseudo-Boolean algebra A is a Boolean algebra with respect to $\neg, \wedge, \dot{\vee}$ and $\rightarrow$, where the join is $a \dot{\vee} b = \neg\neg(a \vee b)$, and also that it is isomorphic to the ordinary quotient of A by its radical. In tpBas we have two kinds of regular elements and a radical linked with the dense elements. We shall obtain for these concepts, in weakly monadic tpBas, several results partially similar to the classical ones.

2.7. Proposition. In every weakly monadic tpBa A, we have that T = = Reg(A) and this is a Boolean algebra with respect to $\neg, \wedge, \dot{\vee}, \rightarrow$.

Proof: According to 1.10(1) we always have Reg(A) $\subseteq$ T; and if $a \in T$ then using 2.3(1) twice we obtain $\delta Ia = \delta I \delta a = \delta I \neg I \neg a = \neg \delta I \neg a = \neg\neg \delta a = \neg\neg a = a$, and so $a \in \mathrm{Reg}(A)$. On the other hand $T \subseteq \mathrm{Reg}_H(A)$ and we know that $\mathrm{Reg}_H(A)$ is a Boolean algebra with respect to the desired operations; hence we only need to prove that T is closed with respect to them. T is always closed by $\wedge$, and if A is weakly monadic then T is closed by $\neg$. Moreover we have $a \dot{\vee} b = \neg\neg(a \vee b) = \neg(\neg a \wedge \neg b)$, therefore T is closed by $\dot{\vee}$, and since in $\mathrm{Reg}_H(A)$ $a \rightarrow b = \neg a \dot{\vee} b$ then T is also closed by $\rightarrow$. As a result T = Reg(A) is a Boolean algebra with $\neg, \wedge, \dot{\vee}$ and $\rightarrow$ ∎

Note that in weakly monadic tpBa the set T is closed by $\neg$, and $\rightarrow$. However it is not a subalgebra of A, since it does not need to be closed by $\vee$ (see 5.7), and this condition will later on play a role. The point whether B is or is not closed by $\neg$ and $\rightarrow$ will also play an important role in the next section, but we can say nothing about this now. On the other hand, in 2.7 we have turned one of the inclusions of 1.10 into an equality; we cannot do the same for the remaining ones, for all weakly monadic tpBas, as the examples 5.3 (for the first inclussion of 1.10(1) and the ones of (2)) and 5.2 (for the rest) show. Therefore T = Reg(A) $\subseteq \mathrm{Reg}_I(A)$ and the inclusion can be proper; thus we still have two different kinds of regular elements. This is not an impediment to show that one of them has all the properties cited above. We first need a general result:

2.8. Lemma. In every tpBa A we have that $a \sim_{R(A)} \neg\neg Ia$ for all $a \in A$.

Proof: On one side we have $Ia \leq I\neg\neg a$ and from this $a \leftrightarrow \neg\neg Ia = Ia \rightarrow \neg\neg Ia = 1 \in R(A)$. On the other side, from $0 \leq a$ we deduce $(a \leftrightarrow 0) \rightarrow 0 \leq (a \leftrightarrow 0) \leftrightarrow 0 \leq (a \leftrightarrow 0) \leftrightarrow a$ and so $\neg\neg Ia \leftrightarrow a = I\neg\neg Ia \rightarrow a = ((a \leftrightarrow 0) \rightarrow 0) \leftrightarrow a \geq ((a \leftrightarrow 0) \leftrightarrow a) \leftrightarrow a \in P_{\leftrightarrow} = R(A)$ and, as $R(A)$ is a deductive system and hence an order filter, we find that $\neg\neg Ia \rightarrow a \in R(A)$. Then $a \sim_{R(A)} \neg\neg Ia$.■

It is easy to show (see [Fo]) that in every tpBa A and for every $D \in \mathcal{D}$, the quotient $A/\sim_D$ is a semisimple pseudo-Boolean algebra (i.e., a Boolean algebra) if and only if $D \supseteq R(A)$; this is the case of $R(A)$ itself, therefore $A/\sim_{R(A)}$ is a Boolean algebra with respect to the operations $\bar{\neg}, \bar{\wedge}, \bar{\vee}, \bar{\rightarrow}$ induced in the quotient by $\neg^*$, $\wedge, \overset{*}{\vee}, {}^*$ respectively. Then we have:

2.9. Theorem. In every weakly monadic tpBa A the Boolean algebras T and $A/\sim_{R(A)}$ are isomorphic.

Proof: We define the following mapping from A to T: $h(a) = \delta Ia$ for all $a \in A$. Recall that here $\delta Ia = \neg^{\leftrightarrow} \neg^{\leftrightarrow} a = \neg\neg Ia$. This mapping is onto, because $T = Reg(A)$, and Shell $h = h^{-1}(\{1\}) = \{a \in A: \delta Ia = 1\} = \{a \in A: \neg^{\leftrightarrow}\neg^{\leftrightarrow} a = 1\} = \{a \in A: \neg^{\leftrightarrow} a = 0\} = D_{\leftrightarrow} = R(A)$.

We first show that for all $a, b \in A$, $h(a \leftrightarrow b) \leq h(a) \leftrightarrow h(b)$: using 1.6(1) we have $h(a \leftrightarrow b) = \delta I(Ia \rightarrow b) \leq \delta(Ia \rightarrow Ib) \leq Ia \rightarrow \delta Ib$, and so $\delta I(a \leftrightarrow b) \wedge Ia \leq \delta Ib$ from where, applying 2.1(3), we obtain $\delta I(a \leftrightarrow b) \wedge I\delta Ia \leq \delta I(a \leftrightarrow b) \wedge \delta Ia = \delta(\delta I(a \leftrightarrow b) \wedge Ia) \leq \delta Ib$ and therefore $\delta I(a \leftrightarrow b) \leq I\delta Ia \rightarrow \delta Ib = \delta Ia \leftrightarrow Ib$, that is, $h(a \leftrightarrow b) \leq h(a) \leftrightarrow h(b)$ as we desired.

We can now show that for all $a, b \in A$, $h(a) = h(b)$ iff $a \sim_{R(A)} b$: if $h(a) = h(b)$ we have $\delta Ia = \delta Ib$ and by 2.8 we have $a \sim_{R(A)} \delta Ia$ and $b \sim_{R(A)} \delta Ib$, so $a \sim_{R(A)} b$. Conversely if $a \sim_{R(A)} b$ we have that $a \leftrightarrow b \in R(A) =$ Shell h and $b \leftrightarrow a \in R(A) =$ Shell h, and then $1 = h(a \leftrightarrow b) \leq h(a) \leftrightarrow h(b)$ and $1 \leq h(b) \leftrightarrow h(a)$; therefore $h(a) \leftrightarrow h(b) = h(b) \leftrightarrow h(a) = 1$ which implies $Ih(a) = Ih(b)$. But $h(a), h(b) \in T = Reg(A)$, so $h(a) = \delta Ih(a)$ and $h(b) = \delta Ih(b)$, and finally $h(a) = h(b)$.

It follows from what has been done until now that the induced mapping $\bar{h}$ maps $A/\sim_{R(A)}$ onto T and is indeed a bijection between them. We must only prove that h is a morphism, and then it will be the required isomorphism. It suffices to show it for the negation $\neg$ and the meet $\wedge$: $\bar{h}(\neg\bar{a}) = \bar{h}(\overline{\neg\mapsto a}) = \bar{h}(\overline{\neg Ia}) = \delta I\neg Ia = \neg\delta Ia = \neg\bar{h}(\bar{a})$ using 2.3(1); if we recall that $\delta Ia \in T$ for all $a \in A$ and that $T = Reg(A)$ is $\wedge$-closed, then we have that $\bar{h}(\bar{a}\mathbin{\bar{\wedge}}\bar{b}) = \bar{h}(\overline{\delta Ia}\mathbin{\bar{\wedge}}\overline{\delta Ib}) = \bar{h}(\overline{\delta Ia \wedge \delta Ib}) = \delta I(\delta Ia \wedge \delta Ib) = \delta Ia \wedge \delta Ib = \bar{h}(\bar{a}) \wedge \bar{h}(\bar{b})$. ■

3. Monadic and strongly monadic algebras

In this section we present two subclasses of the topological pseudo-Boolean algebras related to five classical conditions belonging to S5, that is, to monadic Boolean algebras. First, the condition $I\neg I = \neg I$, dual to another one studied in the previous section, has been used by A. Monteiro in [M], and in its logical form it appears in the 1933 axiomatics given by M. Wajsberg in [W]. We must say that A.N. Prior calls it insistently "the Gödelian axiom for S5" (see for instance [P1] page 20 and [P2] page 312) although the reference given [G] does not seem to provide reasons for this. Second, the law of reduction of modalities $I\delta = \delta$ already used in its logical and strict form by C.I. Lewis to define S5 over S1. Third, the axiom $a \leq I\delta a$ which can be used to produce S5 from S4 and characterizes a system called "Bronwerian" because of a comment of O. Becker in [B] about the intuitionistic character of a property of the strict negation or impossibility in some modal systems. Fourth, the rule "if $\delta a \leq b$ then $a \leq Ib$" which has been classically regarded as equivalent to the last axiom. And finally an interesting axiom involving the most elementary operators, namely implication and interior: $I(Ia \to b) = Ia \to Ib$; it was used by E.W. Beth and J.F.F. Nieland to give an axiomatization of S5 out from S4 in [B-N]. We begin our study by showing that the first four preceding conditions are equivalent in our case.

3.1. Theorem. In every tpBa A the following conditions are equivalent:

(1) $I\neg Ia = \neg Ia$ for all $a \in A$, that is, B is closed by negation,

(2) $I\delta a = \delta a$ for all $a \in A$, that is, $T \subseteq B$,

(3) $a \leq I\delta a$ for all $a \in A$, and

(4) if $\delta a \leq b$ then $a \leq Ib$ for all $a,b \in A$.

Proof: $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial. If $\delta a \leq b$ then $I\delta a \leq Ib$ and by (3) $a \leq Ib$ and so we have (4). Finally, as for every $a \in A$ we have $\delta\, Ia = \neg Ia$ by 1.3(6), in particular $\delta\neg Ia \leq Ia$, and if we apply (4) then we have $\neg Ia \leq I\neg Ia$ and we obtain (1).■

3.2. Definition. A tpBa A is called monadic iff it satisfies any of the conditions in Theorem 3.1.

3.3. Proposition. Every monadic tpBa is weakly monadic.

Proof: From 3.1(1) we have $\delta\neg\delta a = \neg I\neg\neg\neg I\neg a = \neg I\neg I\neg a = \neg\neg I\neg a = \neg\delta a$. So we obtain 2.1(1).■

The converse of 3.3 is not true, as example 5.3 shows. Thus we have a proper subclass of all weakly monadic tpBa. We now give some properties of the several operations of negation we have at hand; we begin by improving the relations between open and closed elements via negation.

3.4. Proposition. In every monadic tpBa if $a \in A$ is closed then $\neg a$ is open.

Proof. $a \in T$ implies $\neg a \in T$ by 3.3, and $T \subseteq B$ by 3.1(2).■

We already know that the converse does not hold unless A is Boolean (Prop. 1.5). Only one implication remains (if $\neg a$ is closed then a is open) and 5.6 shows that it is not true for all monadic tpBas, although it can be true for particular cases, as in 5.7.

3.5. Proposition. A tpBa is monadic iff the three logical negations $\neg_{\Rightarrow}$, $\neg_{\leftrightarrow}$ and $\neg_{\rightsquigarrow}$ coincide.

Proof: It is enough to observe that $\neg^{\rightsquigarrow} a = \neg^{\leftrightarrow} a = \neg Ia$ and $\neg^{\Rightarrow} a =$ $= I\neg Ia$ and look at 3.1(1).■

3.6. Corollary. In every monadic tpBa A, we have $T = \mathrm{Reg}(A) =$
$= \mathrm{Reg}_I(A) = B \cap \mathrm{Reg}_H(A)$.

Proof: In weakly monadic tpBas $T = \mathrm{Reg}(A)$ and by 3.5 $\mathrm{Reg}(A) =$ $= \mathrm{Reg}_I(A)$. On the other hand by 2.3(4) we have $B \cap \mathrm{Reg}_H(A) \subseteq T$ and always $T \subseteq \mathrm{Reg}_H(A)$; but if A is monadic we have in addition that $T \subseteq B$ and so the last equality is proved.■

Therefore we see that in monadic tpBa there is only one kind of regular elements, thus emphasizing the result in 2.9. We see in 5.4 and 5.5 that we still have three types of Peircean elements.

3.7. Proposition. In every monadic tpBa A, if $a \in A$ is open then
$a \vee \neg a$ and $\neg\neg a \to a$ are dense.

Proof: If a is open so will be $\neg a$ and $a \vee \neg a$, and then $\neg^{\leftrightarrow}(a \vee \neg a) =$ $= \neg I(a \vee \neg a) = \neg(a \vee \neg a) = \neg a \wedge \neg\neg a = 0$, that is, $a \vee \neg a$ is dense. In every pseudo-Boolean algebra $\neg a \vee b \leq a \to b$ for all $a, b \in A$, so $\neg a \vee a = \neg\neg\neg a \vee a \leq \neg\neg a \to a$, and then $\neg\neg a \to a$ is dense as $R(A)$ is an order filter.■

The logical interpretation of this result is as follows: If we read dense as "almost true" and open as "necessary" then 3.7 describes a partially classical behaviour of necessary sentences in the sense that two strictly classical laws concerning negation are almost true when referred to necessary sentences. This classical character of the set of open elements will be total in the semisimple tpBas. But before let us say something about an intermediate class of tpBa given by the condition of Beth and Nieland.

3.8. Definition: A tpBa A will be called strongly monadic iff it
satisfies $I(Ia \to b) = Ia \to Ib$ for all $a, b \in A$.

3.9. Proposition. In every tpBa the following conditions are equivalent:

(1) A is strongly monadic,

(2) $I(Ia \to Ib) = Ia \to Ib$ for all $a, b \in A$, that is, B is closed by $\to$,

(3) B is a subalgebra of A, and

(4) $a \Rightarrow b = a \rightsquigarrow b$ for all $a, b \in A$.

Proof: (2) and (4) say exactly the same thing, and they are equivalent to (3) because B is always closed by $\wedge, \vee$ and $0 \in B$; to be a subalgebra of A it only needs to be closed by $\neg$ and $\to$, but $\neg a = a \to 0$, so we see that (2) and (3) are equivalent. Putting Ib for b in the definition we see that (1) implies (2), and if we assume (2) then $Ia \to Ib = I(Ia \to Ib) \leq I(Ia \to b \leq Ia \to Ib$, so we have (1).■

3.10. Corollary. Every strongly monadic tpBa is monadic.■

Example 5.4 shows that the converse is not true. Now we shall see two kinds of topological pseudo-Boolean algebras which are always strongly monadic. One of them is any tpBa defined on a linearly ordered set:

3.11. Proposition. If A is a tpBa whose underlying ordering relation is linear, then A is strongly monadic.

Proof: In a linearly ordered set there is only one binary operation $\to$ which can give it the structure of a Hilbert algebra (and hence of a pseudo-Boolean algebra), namely $a \to b = 1$ iff $a \leq b$ and $a \to b = b$ otherwise. It is then trivial that any interior operator on this set will produce an open set B closed by $\to$.■

Another kind of strongly monadic tpBa are functional algebras as defined in [H] for the Boolean case. We call functional every tpBa of the form $A = H^X$ where H is a complete (or at least sup-complete) pseudo-Boolean algebra and $X \neq \emptyset$ is any set, with the pointwise defined pseudo-Boolean structure and the interior operator is $(If)(y) = \sup\{f(x): x \in X\}$ for all $f \in A$, $y \in X$. We then have:

3.12. Proposition. Every functional tpBa is strongly monadic.

Proof: From the definition it follows that the open elements of a functional tpBa are the constant functions, and these have the structure of H, thus forming a subalgebra of A. ■

In the following result we see that 3.9(4) is the only general coincidence among the implication operations that can hold in any kind of (non trivial) tpBa.

3.13 Proposition. In every tpBa A the following conditions are equivalent:

(1) $a \Rightarrow b = a \nrightarrow b$ for all $a,b \in A$,

(2) $a \rightsquigarrow b = a \nrightarrow b$ for all $a,b \in A$, and

(3) $Ia = a$ for all $a \in A$.

Proof: (3) trivially implies (1) and (2); and putting a=1 in (1) or in (2) we obtain (3). ■

A trivial although worth mentionning consequence of 3.9(4) is:

3.14 Proposition. In every strongly monadic tpBa, $P_{\Rightarrow} = P_{\rightsquigarrow}$. ■

However, this set of Peircean elements must not be equal to $P_{\nrightarrow}$, as example 5.6 shows. We also observe that examples 5.6 and 5.7 are strongly monadic algebras and so we cannot improve the relations between open and closed elements we have found for monadic tpBas. Finally we find here the second "connecting condition" of Fischer-Servi:

3.15 Proposition: In every strongly monadic tpBa A we have

(1) $T \rightarrow B \subseteq B$, and

(2) $\delta a \rightarrow Ib \leq I(a \rightarrow b)$ for all $a,b \in A$.

Proof: By 3.10, 3.1(2) and 3.9(2) we have $T \rightarrow B \subseteq B \rightarrow B \subseteq B$ and so (1) holds. In every tpBa $a \leq \delta a$ and $Ib \leq b$, so $\delta a \rightarrow Ib \leq a \rightarrow b$; by (1) $\delta a \rightarrow Ib \in B$ and therefore $\delta a \rightarrow Ib \leq I(a \rightarrow b)$. ■

Example 5.5 shows us that condition (2) does not hold in every monadic tpBa. Moreover, it cannot be characteristic of strongly monadic tpBas because it can hold even in non-weakly-monadic tpBa, as 5.2.

4. Semisimplicity in tpBas

Semisimplicity is a property of every monadic Boolean algebra, as Halmos showed, and it is a sufficient condition for a topological Boolean algebra to be monadic, as Monteiro (easily) showed. Actually, semisimple topological Boolean algebras are exactly monadic Boolean algebras. We shall examine in our case the logical significance of this algebraic concept and some interesting consequences. The concepts of simple and semisimple tpBa are of course the usual ones of universal algebra: an algebra is *simple* iff its only congruences are the trivial ones, and they are different; in tpBa this is equivalent to say that $B = \{0,1\}$ and $0 \neq 1$. An algebra is *semisimple* iff it is (isomorphic to) a subdirect product of simple algebras. We then have:

4.1. Proposition. Every semisimple tpBa is strongly monadic.
Proof: $\{0,1\}$ is always a subalgebra, so every simple tpBa is strongly monadic. The condition in 3.8 is an equation, so it is preserved under the formation of direct products and subalgebras; thus every semisimple tpBa will satisfy it.■

The converse is obviously false, as for instance in 5.6, 5.7 and 5.8. Although we have not shown any example, it is easy to see that there are semisimple tpBa which are not simple. We now give several characterizations of semisimplicity, the first ones with more algebraic contents and the others having a more logical significance.

4.2. Theorem. In every tpBa A the following conditions are equivalent:

(1) A is semisimple,

(2) $R(A) = \{1\}$,

(3) $((a * b) * a) * a = 1$ for all $a, b \in A$,

(4) $((a * 0) * a) * a = 1$ for all $a \in A$, and

(5) $Ia = (a \Rightarrow 0) \Rightarrow a$ for all $a \in A$,

where $*$ stands for any of $\Rightarrow$, $\rightsquigarrow$ and $\leftrightarrow$.

Proof: It is easy to see that the simple quotients of any tpBa are in correspondence with its maximal deductive systems. Then by known results of universal algebra (1) becomes equivalent to (2). The equivalence between (2) and (3) results from $R(A) = \underline{D}(P_*)$. (3) trivially implies (4), and from $0 \leqslant b$ follows that $((a * 0) * a) * a \leq$ $\leq ((a * b) * a) * a$, so (3) follows from (4). Finally, $Ia \leq (a \Rightarrow 0) \rightarrow Ia$, so $Ia \leq (a \Rightarrow 0) \Rightarrow a$ in general; replacing $*$ by $\Rightarrow$, (4) says that $(a \Rightarrow 0) \Rightarrow a \leq Ia$, so we see that (4) for $\Rightarrow$ is equivalent to (5). But the three versions of (4) are mutually equivalent, as it is clear from (1) and (2), so (4) and (5) are completely equivalent.∎

4.3. Proposition. A strongly monadic tpBa A is semisimple iff $Ia =$ $= (a \rightsquigarrow 0) \rightsquigarrow a$ for all $a \in A$.

Proof: From 3.9(4) and 4.2(5).∎

4.4. Proposition. A weakly monadic tpBa is semisimple iff 1 is the only dense element of the algebra.

Proof: Trivial by 4.2(2) and 2.6.∎

Note that the preceding result could have been stated for general tpBa by refering only to $\Rightarrow$-dense elements. This result reminds us that a pseudo-Boolean algebra is Boolean (semisimple) iff 1 is its only (H-)dense element.

4.5. Theorem. A tpBa A is semisimple iff its set of open elements B is a Boolean subalgebra of A.

Proof: We already know that if A is semisimple then B is a subalgebra of A and thus a pseudo-Boolean algebra where $\rightarrow$ and the three coincide. Now 3.7 and 4.4 tell us that $a \vee \neg a = 1$ for all $a \in B$, so B is Boolean. Conversely, if B is a Boolean subalgebra of A, we must have $((Ia \to 0) \to Ia) \to Ia = 1$ for all $a \in A$, and this implies $Ia = (Ia \to 0) \to Ia = (a \rightsquigarrow 0) \rightsquigarrow a$ because B is closed by $\to$. But we also have that A is strongly monadic, so 4.3 completes the proof.■

We observe that 4.5 says that a tpBa is a semisimple algebra iff its open set is a semisimple subalgebra of it. In connection with this setting we note that a tpBa is a simple algebra iff its open set is a simple subalgebra of it.

The third important characterization of semisimplicity involves three conditions already known, namely the law of reduction of modalities $\delta I = I$, the law $\delta Ia \leq a$ which is dual to Becker's 3.1(3), and the axiom $I\neg Ia \vee Ia = 1$ which appears in [Bu1].

4.6. Theorem. In every tpBa A the following conditions are equivalent:

(1) A is semisimple,

(2) $\delta a = \min\{b \in B: b \geq a\}$ for all $a \in A$,

(3) $\delta Ia = Ia$ for all $a \in A$, that is, $B \subseteq T$,

(4) $\delta Ia \leq a$ for all $a \in A$, and

(5) $I\neg Ia \vee Ia = 1$ for all $a \in A$.

Proof: (1) ⇒ (2): If A is semisimple then it is also monadic, and so $T \subseteq B$; then $\delta a \in B$ and $a \leq \delta a$, and if $b \in B$ is such that $a \leq b$, it follows that $\neg b \in B$ and $\neg b \leq \neg a$, so $\neg b \leq I\neg a$ and then $\delta a \leq \neg\neg b = b$ as B is Boolean. Thus we have (2).

(2) ⇒ (3) ⇒ (4) are trivial.

(4) ⇒ (5): If we assume (4) we have $\neg I\neg Ia \to Ia = 1$, but $\neg a \to b \leq \neg\neg(a \vee b)$ for all $a, b \in A$, so we also have $1 = \neg\neg(I\neg Ia \vee Ia) = \neg\neg I(I\neg Ia \vee Ia) \leq \delta I(I\neg Ia \vee Ia) \leq I\neg Ia \vee Ia$ applying (4) once more. Now we have obtained (5).

(5) ⇒ (1): If $a \neq 1$ then $Ia \neq 1$ and the assumption of (5) forces us to accept that $I\neg Ia \neq 0$. From this it follows that there is a maximal deductive system $D \in \mathcal{D}$ such that $\neg Ia \in D$ and therefore $a \notin D$ (see theorem 3.7 of [Fo]); then $a \notin R(A)$ and this establishes (1) via 4.2(2).■

As a consequence of the two preceding theorems we observe that in a semisimple tpBa the open elements and the closed elements are the same and form a subalgebra which is Boolean, that is, these elements have a completely classical behaviour. Hence we have, among other properties, that for all $a \in A$, $\neg\neg Ia = Ia$. The validity of such formula is considered by Bull as an "intuitionistically implausible thesis" in [Bu1], and consequently all systems containing it are rejected as genuine intuitionistic analogues of S5 according to the criteria of [Bu2]. We must say that the logical system that would correspond to semisimple tpBa is weaker than the one initially considered by Bull, because this one has the interdefinability of the two modal operators, which is not true in every semisimple tpBa as 5.8 shows.

We next examine the semisimplicity of the two special kinds of tpBas dealt with in 3.11 and 3.12; we find that there is no proper semisimple tpBa among them:

4.7. Proposition. A linearly ordered tpBa is semisimple iff it is simple.

Proof: If A is a semisimple tpBa, then B will be a linearly ordered Boolean algebra, and this implies $B = \{0,1\}$, so A is simple. The converse is general.■

4.8. Proposition. A functional tpBa $A = H^X$ is semisimple iff H is a semisimple pseudo-Boolean algebra, that is, iff A is a monadic Boolean algebra.

Proof: We only need to consider that the set of open elements of A has the same structure of H, as we already said in 3.12.■

We now come to the last two conditions of the fourteen ones mentioned at the beginning of the paper.

4.9. Proposition. In every semisimple tpBa A:

(1) T is a subalgebra of A,

(2) $I(a \vee Ib) = Ia \vee Ib$ for all $a,b \in A$, and

(3) $\delta(a \vee b) = \delta a \vee \delta b$ for all $a,b \in A$.

Proof: (1) is a consequence of 4.5 and 4.6, as we have already observed. (2) and (3) are true in every simple tpBa, as it is easily checked, and therefore they are true in every semisimple tpBa.■

Condition (1) is equivalent to saying that T is a Boolean subalgebra of A, and in this form it appears in Halmos' definition of monadic Boolean algebras as equivalent to 2.1(1). Condition (2) is dual to 2.1(3) and it has been explicitly used by Monteiro. Condition (3) is of a different character: it is a property of all topological Boolean algebras. This three conditions have in common that they are not equivalent to the semisimplicity of tpBas, as 5.6 shows. We are going to find all the relations between these conditions and between the four classes of tpBas we have introduced in sections 2 and 3. In the first place, taking 2.1(1) into account we see that

4.10. Proposition. If in a tpBa A the set T is a subalgebra of A then A is weakly monadic.■

Example 5.7 shows us that the converse is not true even in monadic or strongly monadic tpBas. However in 5.3 we see that condition (1) can hold in any kind of weakly monadic tpBa. Condition (2) is even more independent from monadicity than (1), because it can hold in very general tpBa (as the one of 5.1) and it can fail in strongly monadic tpBa (such as in 5.8). The same is true for condition (3), as it is shown in examples 5.1 and 5.7. The only relation among the three conditions is the following:

4.11. Proposition. If in a tpBa A the set T is a subalgebra of A then $\delta(a \vee b) = \delta a \vee \delta b$ for all $a,b \in A$.

Proof: We always have $\delta(a \vee b) \leq \delta a \vee \delta b \leq t$ for all $t \in T$ such that $a \leq t$, $b \leq t$. If T is a subalgebra then $\delta a \vee \delta b \in T$ and by 1.2(3) this gives us $\delta a \vee \delta b = \delta(a \vee b)$.■

There are no other implications, because the converse of 4.11 is not true as we see in 5.1. Condition (1) does not imply (2) as 5.4 shows, so (3) does not either, and conversely (2) does not imply (3) because of 5.7, and then it does not imply (1) either. The following scheme summarizes all implications we have found between the classically equivalent conditions; note that we have shown that there are no other implications between them.

$$\begin{tikzcd}
 & \text{semisimple} \arrow[dddl] \arrow[d] \arrow[ddr] & \\
 & \text{strongly monadic} \arrow[dd] & \\
 & & \text{T subalgebra} \arrow[d] \arrow[ddl] \\
I(a \vee Ib) = Ia \vee Ib & \text{monadic} \arrow[d] & \delta(a \vee b) = \delta a \vee \delta b \\
 & \text{weakly monadic} &
\end{tikzcd}$$

We will close this section by giving several characterizations of semisimplicity in terms of the lattice $\mathcal{D}$ of all deductive systems of A and in terms of the abstract logic $\underline{L}$. Particularizing what we have summarized in section 0, we can associate an equivalence relation to the least deductive system $\{1\}$: $a \sim b$ iff $\underline{D}(a) = \underline{D}(b)$ iff $Ia = Ib$, which is a logical congruence with respect to $\neg^*$, $\wedge$, $\overset{*}{\vee}$ and $*$, and by Theorem IX-6 of [B-S] it is the maximum of $\Theta_{\underline{L}}$. Then we have:

4.12. Theorem. In every tpBa A the following conditions are equivalent:

(1) A is semisimple,

(2) $A/\sim$ is semisimple (that is, a Boolean algebra), and

(3) $\underline{L}$ satisfies the Reductio ad Absurdum Principle with respect to $\neg^*$: $\underline{D}(X, \neg^* a) = A$ iff $a \in \underline{D}(X)$ for all $a \in A$, $X \subseteq A$.

Proof: By 4.2(3) A is semisimple iff $((a * b) * a) * a = 1$ for all $a, b \in A$, and this is equivalent to $((\bar{a} \dot{\to} \bar{b}) \dot{\to} \bar{a}) \dot{\to} \bar{a} = \bar{1}$ because $\bar{1} = \{1\}$, and this is equivalent to the semisimplicity of $A/\sim$ as a pseudo-Boolean algebra. Thus (1) and (2) are equivalent. To see that (2) and (3) are equivalent we consider the canonical projection from A onto $A/\sim$: it is a bilogical morphism between $\underline{L}$ and the logic $\underline{\tilde{L}}$ asociated with all filters of $A/\sim$. Then $\underline{\tilde{L}}$ satisfies the four Principles mentioned in section 0, and by Theorem 13 of [V], in such situation a necessary and sufficient condition for $A/\sim$ to be Boolean (that is, semisimple) is that $\underline{\tilde{L}}$ satisfies the Reductio ad Absurdum Principle with respect to $\overline{\neg}$. Again by the already mentioned bilogical morphism this is equivalent to the fact that $\underline{\tilde{L}}$ satisfies it.■

The Strong Disjunction Principle $\underline{D}(X,a)\cap\underline{D}(X,b) = \underline{D}(X,a\overset{*}{\vee}b)$ gives the operation $\overset{*}{\vee}$ a certain character of a "logical disjunction". This can be extended if we call prime every deductive system $D\in\mathcal{D}$ such that for all $a,b\in A$, if $a\overset{*}{\vee}b\in D$ then $a\in D$ or $b\in D$. The prime deductive systems turn out to be exactly the (finitely) irreducible deductive systems, and we also find the following characterization of semisimplicity, of a clear Boolean flavour:

4.13. Theorem. A tpBa is semisimple iff the prime deductive systems and the maximal deductive systems of $\mathcal{D}$ coincide.

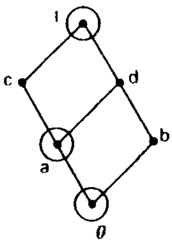
Proof: The semisimplicity of A is equivalent by 4.12 to the semsimplicity of $A/\sim$, and this one is equivalent to the coincidence of its irreducible and maximal filters. But the bilogical morphism between A and $A/\sim$ induces a lattice isomorphism between $\mathcal{D}$ and the lattice of all filters of $A/\sim$. Therefore the last coincidence is equivalent to the one stated in the Theorem, which is proved.■

More details about the most basic concepts that are involved in the last part of this section can be seen in [Fo]. Some of them were introduced in [R] for topological Boolean algebras.

5. Examples and counterexamples

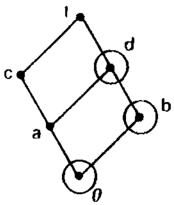
We gather here all distinct examples of tpBa that have been used in several places throughout the paper. They are in all cases finite algebras. For brevity's sake we do not give the table of any operation; they all can be produced from the Hasse diagram of the algebra (recall that there is one and only one operation $\to$ on a finite distributive lattice, namely $a\to b = \max\{c\in A\colon a\wedge c\leq b\}$, which gives it the structure of a pseudo-Boolean algebra). The open elements are indicated by a circle in the diagrams; then the interior operator is $Ia = \max\{b\in B\colon b\leq a\}$. We do not show any explicit computation but simply state the properties that each algebra has or has not and which have been mentioned in the paper.

5.1. Example



Here $B = \{0,a,1\}$ and $T = \{0,b,1\}$. This tpBa is not weakly monadic ($c\wedge\delta b = 0$ but $\delta c\wedge\delta b = b\neq 0$) and therefore T is not a subalgebra ($b\in T$ but $\neg b = c\notin T$). It satisfies $I(x\vee Iy) = Ix\vee Iy$ and $\delta(x\vee y) = \delta x\vee\delta y$ for all $x,y\in A$. Here we have $D_{\leftrightarrow} = \{1\}$ and $D_{\Rightarrow} = \{1,a,c,d\}$, so $D_{\leftrightarrow} \subsetneq D_{\Rightarrow}$.

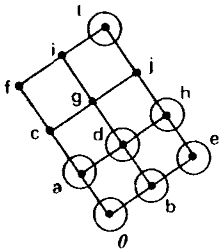
5.2. Example



Here $B = \{0,b,d,1\}$ and $T = \{0,b,1\}$. This is also a non weakly monadic tpBa ($\delta\neg\delta c = 1\neq b = \neg\delta c$). It satisfies: if $x\in T$ then $\neg x\in B$ and if $\neg x\in T$ then $x\in B$ for all $x\in A$. It satisfies $T\to B\subseteq B$. Here $Reg(A) = \{0,1\}$ and $Reg_H(A) = \{0,b,c,1\}$, therefore $Reg(A)\subsetneq T\subsetneq Reg_H(A)$.

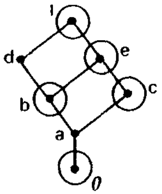
5.3. Example

Here $B = \{0,a,b,d,e,h,1\}$ and $T = \{0,e,f,1\}$. This is a weakly monadic tpBa which is not monadic



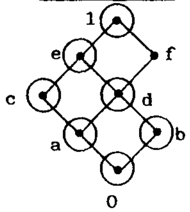
($e \in B$ but $\neg e = f \notin B$). Here $e \in T$ but $\neg e = f \notin B$, $\neg c = e \notin T$ but $c \in B$ and $\neg c = e \notin B$ but $c \notin T$. Here $\mathrm{Reg}(A) = T = \{0,e,f,1\}$ is a subalgebra of A (indeed, a Boolean one), $\mathrm{Reg}_I(A) = \{0,a,e,1\}$, $P_{\Rightarrow} = \{1,d,h\}$, $P_{\rightsquigarrow} = \{1,d,h,i\}$ and $P_{\leftrightarrow} = \{1,d,g,h,i,j\}$. We see that $P_{\Rightarrow} \subsetneq P_{\rightsquigarrow} \subsetneq P_{\leftrightarrow}$, that $\{0,1\} \subsetneq T \cap B \subsetneq \mathrm{Reg}_I(A) \subsetneq B$ and that $\{0,1\} \subsetneq \mathrm{Reg}(A)$.

5.4. Example



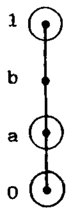
Here $B = \{0,b,e,1\}$ and $T = \{0,1\}$. This makes a monadic tpBa which is not strongly monadic ($e,b \in B$ but $e \rightarrow b = d \notin B$). T is a subalgebra of A. $I(c \vee Ib) = e \neq b = Ic \vee Ib$. $P_{\rightsquigarrow} = \{1,b,e\}$ and $P_{\leftrightarrow} = \{1,b,d,e\}$, so $P_{\rightsquigarrow} \subsetneq P_{\leftrightarrow}$.

5.5. Example



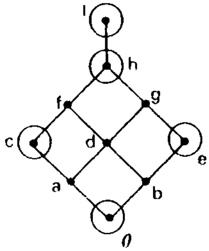
Here $B = \{0,a,b,c,d,e,1\}$ and $T = \{0,b,c,1\}$. This is monadic tpBa not strongly monadic ($c,d \in B$ but $c \rightarrow d = f \notin$). It does not satisfy the condition $T \rightarrow B \subseteq B$ (because $c \in T$, $d \in B$ but $c \rightarrow d = f \notin B$). $P_{\Rightarrow} = \{1,d,e\}$ and $P_{\rightsquigarrow} = P_{\leftrightarrow} = \{1,d,e,f\}$, so $P_{\Rightarrow} \subsetneq P_{\rightsquigarrow}$.

5.6. Example



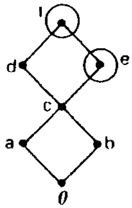
Here $B = \{0,a,1\}$ and $T = \{0,1\}$. This tpBa is strongly monadic (see 3.11) but is not semisimple, as it is not simple (see also 4.7). $\neg b = 0 \in T$ but $b \notin B$. This algebra satisfies that T is a subalgebra of A, $\delta(x \vee y) = \delta x \vee \delta y$, and $I(x \vee Iy) = Ix \vee Iy$ for all $x,y \in A$. $P_{\Rightarrow} = P_{\rightsquigarrow} = \{a,1\}$ and $P_{\leftrightarrow} = \{a,b,1\}$, so $P_{\rightsquigarrow} \subsetneq P_{\leftrightarrow}$.

5.7. Example



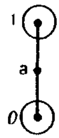
$B = \{0,c,e,h,1\}$ and $T = \{0,c,e,1\}$. This is a strongly monadic tpBa which is not semisimple: $D = \{1,h\} \neq \{1\}$. T is not a subalgebra of A, as it is not closed by $\vee$. We have that $c,e \in T$ and $\delta(c \vee e) = 1 \neq h = \delta c \vee \delta e$. This algebra satisfies that if $\neg x \in T$ then $x \in B$ for all $x \in A$, and $I(x \vee Iy) = Ix \vee Iy$ for all $x,y \in A$.

5.8. Example



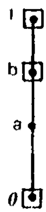
$B = \{0,e,1\}$ and $T = \{0,1\}$, which is a subalgebra of A. The tpBa is strongly monadic and not semi-simple. $I(d \vee Ie) = 1 \neq e = Id \vee Ie$. $a = \neg\neg a$, $b = \neg\neg b$ although $a,b \notin T$, and also $\delta a \neq \neg\neg a$ for $a \notin B$.

5.9. Example



$B = T = \{0,1\}$, that is, we have a semisimple tpBa (actually a simple one!) where $I \neq \neg\delta\neg$ because $Ia = 0 \neq 1 = \neg\delta\neg a$.

5.10. Example



This is an example of a totally different kind from the preceding ones. Take the pseudo-Boolean algebra A of four linearly ordered elements of the diagram. Define the following operator: $\delta 0 = 0$, $\delta a = \delta b = b$, $\delta 1 = 1$. This is a closure operator on A, and indeed a lattice closure, as it satisfies $\delta(x \vee y) = \delta x \vee \delta y$ for all $x,y \in A$. Moreover it satisfies $\delta(x \wedge \delta y) = \delta x \wedge \delta y$ and $\delta\neg\delta x = \neg\delta x$ for all $x,y \in A$. The operator I defined on A as $Ix = \neg\delta\neg x$ for all $x \in A$ is $I0 = 0$, $Ia = Ib = I1 = 1$. It satifies $I1 = 1$, $I^2 x = Ix$, and $I(x \wedge y) = Ix \wedge Iy$ for all $x,y \in A$. Moreover it satisfies $x \leq I\delta x$ and if $\delta x \leq y$ then $x \leq Iy$ for all $x,y \in A$. But it is not an interior operator of any kind, because $b < Ib$, $a < Ia$.

This example shows that it is not a good choice to have a primitive operator of possibility M and then define necessity as $L \leftrightarrow \neg M \neg$, even if we put stronger conditions on M. It is easy to prove that $\delta = \neg I \neg$ always produces a closure operator from an interior operator, while the dual formula $I = \neg\delta\neg$ does not.

References

[B] O. Becker, Zur Logik der Modalitäten, Jahrb. für Phil. und Phän. Forschung 11 (1930) 497-548.

[B-N] E. Beth and J.F.F. Nieland, Semantic Construction of Lewis Systems S4 and S5, in: Addison, Henkin, Tarski, eds., The Theory of Models (North-Holland, Amsterdam, 1965) 17-24.

[B-S] D.J. Brown and R. Suszko, Abstract Logics, Dissert. Math. CII (1973) 9-40.

[Bu1] R.A. Bull, Some modal calculi based on IC, in: Crossley and Dummett, eds., Formal Systems and Recursive Functions (North-Holland, Amsterdam, 1965) 3-7.

[Bu2] ----------, A modal extension of intuitionist logic, Notre Dame Jour. Formal Logic 6 (1965) 142-145.

[D] C. Davis, Modal operators, equivalence relations and projective algebras, Am. Jour. Math. 76 (1954) 747-762.

[Fi1] G. Fischer-Servi, On modal logic with an intuitionistic base, Studia Logica 36 (1977) 141-149.

[Fi2] ----------, Semantics for a class of intuitionistic modal calculi, in : Dalla Chiara, ed., Italian Studies in the Philosophy of Science (Reidel, Dordrecht, 1980) 59-72.

[Fi3] ----------, An intuitionistic analogue of S4 as a logical modeling for science, J. Phil. Logic, to appear.

[Fo] J.M. Font, Implication and deduction in some intuitionistic modal logics, Rep. Math. Logic. 17, (1983) to appear.

[G] K. Gödel, Eine Interpretation des Intuitionistischen Aussagenkalküls, Ergebnisse 4 (1933) 39-40.

[H] P.R. Halmos, Algebraic Logic I: Monadic Boolean algebras, Comp. Math. 12 (1955) 217-249.

[M] A. Monteiro, La semisimplicité des algèbres de Boole topologiques et les systêmes déductifs, Revista de la U.M.A. 25 (1971) 417-448.

[M-T] J.C.C. McKinsey and A. Tarski, On closed elements in closure algebras, Ann. of Math. 47 (1946) 122-162.

[O] H. Ono, On some intuitionistic modal logics, Publ. R.I.M.S. Kyoto Univ. 13 (1977) 687-722.

[P1] A.N. Prior, Time and Modality (Oxford Univ. Press, 1957).

[P2] ----------, Formal Logic (Oxford Univ. Press, 1962).

[R] H. Rasiowa, An algebraic approach to non-classical logics (North-Holland, Amsterdam, 1974).

[R-S] H. Rasiowa and R. Sikorski, The mathematics of the matemathematics (P.W.N., Warszawa, 1970).

[V] V. Verdú, Distributive and Boolean logics, Stochastica 3 (1979) 97-108.

[vW] G.H. von Wright, An essay in modal logic (North-Holland, Amsterdam, 1951).

[W] M. Wajsberg, Ein erweiterter Klassenkalkül, Monat. Math. Phys. 40 (1933) 113-126.

Josep M. Font

Dept. of Algebra and Foundations

Fac. of Mathematics

University of Barcelona

Gran Vía 585, Barcelona 7

Spain

publicacions
edicions
universitat
de barcelona

Dipòsit Legal B.: 37.493-1983
BARCELONA – 1983