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WAJSBERG ALGEBRAS

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ABSTRACT

We present the basic theory of the most natural algebraic counterpart of the $\aleph_0$-valued Łukasiewicz calculus, strictly logically formulated. After showing its lattice structure and its relation to C.C. Chang's MV-algebras we study the implicative filters and prove its equivalence to congruence relations. We present some properties of the variety of all Wajsberg algebras, among which there is a representation theorem. Finally we give several characterizations of linear, simple and semisimple algebras.

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0. INTRODUCTION

It is well known that Mordchaj Wajsberg was the first (1935) to show that the $\aleph_0$-valued Łukasiewicz propositional calculus was complete with respect to the axiomatics conjectured by Łukasiewicz itself. Unfortunately his proof, announced in [17], was never published. Only in 1958 A. Rose and J.B. Rosser did publish a sophisticated proof; in the same year C.C. Chang introduced the so-called MV-algebras, a kind of algebraic counterpart of the calculus, which allowed him to give in 1959 another completeness proof using model theory devices. Chang has no implication operator, and his presentation is very complicated and not natural, and the logical contents of the subject is not apparent.

In this paper we present a different approach to the study of the algebraic models of Lukasiewicz calculus, which we call "Wajsberg algebras" or "W-algebras"; they are defined with the same primitive operations and axioms of Łukasiewicz and Wajsberg. Our study follows the lines of modern algebraic logic (schools of Monteiro, Rasiowa, Suszko) and universal algebra. The resulting presentation seems to us more natural from both the algebraic and specially the logical point of view. In the present paper we gather and re-arrange several results obtained individually or jointly by the authors and mostly unpublished.

In sections 1 and 2 we define the W-algebras and study its elementary properties and its lattice structure. In section 3 we prove its equivalence with the MV-algebras and show several properties related to the characteristic operations of the latter. This allows us to give in section 4 Chang's examples of W-algebras, among others. In section 5 we begin the study of implicative filters and in section 6 we give some special properties related to the variety of W-algebras, the representation theory and the study of linear W-algebras. Finally in section 7 we study and classify the simple W-algebras and the finite linear ones, after some properties of maximal implicative filters; we also give an axiomatization of the class of semisimple W-algebras by means of generalized implications. These three last sections form the main part of the paper.

The material we present here forms a basis for this theory and will allow us to present in forthcoming papers several subclasses of W-algebras, such as the archimedean ones, the n-bounded ones or the n-valued ones.

1. DEFINITION AND FIRST PROPERTIES

Along all the paper and unless there is another indication, $\mathfrak{A} = (A, \to, \neg, 1)$ will represent an algebra of type (2,1,0) with universe A.

Definition: We say that $\mathfrak{A}$ is a Wajsberg algebra (W-algebra for short) if and only if it satisfies the following equations:

(1.1) $1 \to x = x$

(1.2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$

(1.3) $(x \to y) \to y = (y \to x) \to x$

(1.4) $(\neg x \to \neg y) \to (y \to x) = 1$

We can give another definition of W-algebras using Sales algebras (algebras of type (2,0) satisfying (1.1) to (1.3)); they are order-dual to commutative BCK algebras (see [8]) and it is easy to prove that they are ordered (see (2.1)) and that if they have a lower bound 0, then there is only one 1-ary operation, namely $\neg x = x \to 0$, which satisfies (1.4). Conversely any W-algebra is a Sales algebra with a lower bound $0 = \neg 1$ (see [12] and [14]).

The variety of all W-algebras will be denoted by $\mathbb{W}$.

Proposition: Any $\mathfrak{A} \in \mathbb{W}$ satisfies the following equations and implications:

(1.5) $x \to x = 1$

(1.6) If $x \to y = y \to x = 1$ then $x = y$

(1.7) $x \to 1 = 1$

(1.8) $x \to (y \to x) = 1$

(1.9) If $x \to y = y \to z = 1$ then $x \to z = 1$

(1.10) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$

(1.11) $x \to (y \to z) = y \to (x \to z)$

(1.12) $x \to 0 = x \to \neg 1 = \neg x$

(1.13) $\neg(\neg x) = x$

(1.14) $\neg x \to \neg y = y \to x$

Proofs: (1.5): By (1.1) and (1.2) $1 = (1 \to 1) \to ((1 \to x) \to (1 \to x)) = x \to x$.

(1.6): By (1.1) and (1.3) if $x \to y = y \to x = 1$ then $x = 1 \to x = (y \to x) \to x = (x \to y) \to y = 1 \to y = y$.

(1.7): Using (1.3), (1.1) and (1.5) we have $(x \to 1) \to 1 = (1 \to x) \to x = x \to x = 1$ and by (1.2), $(1 \to x) \to ((x \to 1) \to (1 \to 1)) = 1$ and so $1 = (1 \to x) \to ((x \to 1) \to 1) = (1 \to x) \to 1 = x \to 1$.

(1.8): By (1.1), (1.2) and (1.7) we have $1 = (x \to 1) \to ((1 \to y) \to (x \to y)) = 1 \to (y \to (x \to y))$ and by (1.6) and (1.7) we obtain (1.8).

(1.9): Is immediate from (1.1) and (1.2).

(1.10): To show this we shall see a weak form of (1.11):

(1.11') If $x \to (y \to z) = 1$ then $y \to (x \to z) = 1$: if $x \to (y \to z) = 1$ then by (1.1), (1.2) and (1.9) we have $((y \to z) \to z) \to (x \to z) = 1$ and by (1.3) $((z \to y) \to y) \to (x \to z) = 1$. Now by (1.8) $1 = y \to ((z \to y) \to y)$ and finally by (1.9) $y \to (x \to z)=1$. Now (1.10) follows from (1.11') and (1.2).

(1.11): Using (1.8), (1.10), (1.9) and (1.11') we successively obtain $y \to ((z \to y) \to y)=1$, $((y \to z) \to z) \to ((x \to (y \to z)) \to (x \to z))=1$, $y \to ((x \to (y \to z)) \to (x \to z) = 1$ and $(x \to (y \to z)) \to (y \to (x \to z)) = 1$. By symmetry we also have $(y \to (x \to z)) \to (x \to (y \to z)) = 1$ and by (1.6) we obtain (1.11).

(1.12): First of all, by (1.8) $\neg x \to (\neg(\neg 1) \to \neg x) = 1$; by (1.4) we have $(\neg(\neg 1) \to \neg x) \to (x \to \neg 1) = 1$ and then by (1.9) $\neg x \to (x \to \neg 1) = 1$ $(^{o})$.

On the other hand by (1.4) $1 = (\neg x \to \neg 1) \to (1 \to x) = (\neg x \to \neg 1) \to x$, by (1.1) and (1.10) $1 = (x \to \neg 1) \to ((\neg x \to \neg 1) \to \neg 1)$, by (1.3) $1 = (x \to \neg 1) \to ((\neg 1 \to \neg x) \to \neg x)$ and then by (1.11) $1 = (\neg 1 \to \neg x) \to ((x \to \neg 1) \to \neg x)$.

We now show $\neg 1 \to \neg x = 1$ $(^{oo})$: By (1.4) and (1.1) $(\neg(\neg x) \to \neg 1) \to \neg x = 1$, by (1.1) and (1.10) $(\neg 1 \to (\neg(\neg x) \to \neg 1)) \to (\neg 1 \to \neg x) = 1$ and by (1.8) and (1.1) finally $\neg 1 \to \neg x = 1$. Now we have $(x \to \neg 1) \to \neg x = 1$, and this with $(^{o})$ and (1.6) gives (1.12).

(1.13): $\neg(\neg x) = (x \to \neg 1) \to \neg 1 = (\neg 1 \to x) \to x = x$ by $(^{oo})$ and (1.1).

(1.14): is straigthforward from (1.13) and (1.4).□

It is worth noting that (1.1) to (1.11) have been proved by the only means of (1.1) to (1.3) and so they are properties of all Sales algebras, too.

2. THE LATTICE STRUCTURE OF WAJSBERG ALGEBRAS

It is easy to see, from (1.5), (1.6) and (1.9), that in every W-algebra $\mathfrak{A} = (A, \to, \neg, 1)$ the relation defined by

(2.1) $\quad a \leq b$ if and only if $a \to b = 1$ for all $a, b \in A$

is a partial order. Moreover in (1.7) and (^{oo}) in the proof of (1.12) we have shown that 1 is the upper bound and $0 = \neg 1$ the lower bound. We shall see in this section that this order relation gives a De Morgan lattice structure with 0, that is, a De Morgan algebra or quasi-Boolean algebra in the sense of Rasiowa [10].

We define in A the following operations:

(2.2) $\quad a \vee b = (a \to b) \to b$

(2.3) $\quad a \wedge b = \neg(\neg a \vee \neg b)$

and we have:

Theorem 1: If $\mathfrak{A} \in \mathbb{W}$ then with the above notations $(A, \wedge, \vee, \neg, 0, 1)$ is a De Morgan algebra.

We will give the proof of this Theorem in two parts.

Proposition 2: If $\mathfrak{A} \in \mathbb{W}$ then the operation (2.2) is a supremum with respect to the order of (2.1), (2.3) is an infimum and $\neg$ is a quasi-complementation.

Proof: We must show that $x \vee y = \sup\{x, y\}$ with respect to (2.1). By (1.3) and (1.6) we have that $x \to ((x \to y) \to y) = x \to ((y \to x) \to x) = 1$ and so $x \leq x \vee y$ and by symmetry $y \leq x \vee y$. Now let z be such that $x \leq z$, $y \leq z$. Then $x \to z = 1$ and by (1.1) $(x \to z) \to z = z$, and by (1.10) $1 = y \to z \leq (z \to x) \to (y \to x)$ so $z \to x \leq$ $\leq y \to x$. If we repeat the process we get $(y \to x) \to x \leq (z \to x) \to x = (x \to z) \to z =$ $= z$. From (1.13) and (1.14) we obtain:

(2.4) $\quad$ if $x \leq y$ then $\neg y \leq \neg x$

which with (1.13) tells us that $\neg$ is a quasi-complementation. And from this it follows that $x \wedge y = \inf\{x, y\}$. □

We also need some properties of the operations we have just introduced.

Proposition: Any W-algebra satisfies the following implications and equations

(2.5) If $x \leq y$ then $x \to z \geq y \to z$

(2.6) If $x \leq y$ then $z \to x \leq z \to y$

(2.7) $x \leq y \to z$ if and only if $y \leq x \to z$

(2.8) $\neg(x \vee y) = \neg x \wedge \neg y$

(2.9) $\neg(x \wedge y) = \neg x \vee \neg y$

(2.10) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(2.11) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

(2.12) $(x \to y) \vee (y \to x) = 1$

(2.13) $x \to (y \vee z) = (x \to y) \vee (x \to z)$

(2.14) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

(2.15) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

<u>Proofs</u>: (2.5), (2.6) and (2.7) are trivial.

(2.8) and (2.9) follow from (1.13) and (2.3).

(2.10): By Proposition 2 and (2.5) we have that $(x \vee y) \to z \leq x \to z$ and $(x \vee y) \to z \leq y \to z$, so $(x \vee y) \to z \leq (x \to z) \wedge (y \to z) = t$. Then $t \leq x \to z$ and $t \leq y \to z$, so by (2.7) $x \leq t \to z$ and $y \leq t \to z$ from where $x \vee y \leq t \to z$ and again by (2.7) $t \leq (x \vee y) \to z$.

(2.11): By (1.14), (2.10) and (2.9) we can write $x \to (y \wedge z) = \neg(y \wedge z) \to \neg x =$ $= (\neg y \vee \neg z) \to \neg x = (\neg y \to \neg x) \wedge (\neg z \to \neg x) = (x \to y) \wedge (x \to z)$.

(2.12): After (2.10) it is clear that $x \to y = (x \vee y) \to y$ and that $y \to x =$ $= (x \vee y) \to x$, so by (1.14), (1.11), (2.2) and (2.3) we have: $(x \to y) \to (y \to x) =$ $= ((x \vee y) \to y) \to ((y \vee x) \to x) = (\neg y \to \neg(x \vee y)) \to (\neg x \to \neg(y \vee x)) = \neg x \to ((\neg y \to$ $\to \neg(x \vee y)) \to \neg(x \vee y)) = \neg x \to (\neg y \vee \neg(x \vee y)) = \neg(\neg y \vee \neg(x \vee y)) \to x =$ $= (y \wedge (x \vee y)) \to x = y \to x$ and so $((x \to y) \to (y \to x)) \to (y \to x) = 1$ that is, we obtain (2.12).

(2.13): By (2.6) it is clear that (*) $x \to (y \vee z) \geq (x \to y) \vee (x \to z)$. Moreover from (2.12), (2.10), (2.1), (1.10) and (*), we have $1 = (z \to y) \vee (y \to z) =$ $= ((y \vee z) \to y) \vee ((y \vee z) \to z) \leq ((x \to (y \vee z)) \to (x \to y)) \vee ((x \to (y \vee z)) \to (x \to z)) \leq$ $\leq (x \to (y \vee z)) \to ((x \to y) \vee (x \to z))$ and then $1 = (x \to (y \vee z)) \to ((x \to y) \vee (x \to z))$, that is, $x \to (y \vee z) \leq (x \to y) \vee (x \to z)$ which with (*) shows (2.13).

(2.14) is proved in the same way as (2.11) but using (2.13) instead of (2.10).

(2.15): By (2.2), (2.13) and (2.10) we have that $(x \wedge y) \vee z = ((x \wedge y) \to z) \to z =$ $= ((x \to z) \vee (y \to z)) \to z = ((x \to z) \to z) \wedge ((y \to z) \to z) = (x \vee z) \wedge (y \vee z)$. □

Proof of Theorem 1: It follows immediately from (1.3), (2.4), (2.8), (2.9), (2.15) and Proposition 2. □

The property (2.10) can be generalized by the introduction of a really useful notation in the following way.

Proposition 3: If in any $\mathfrak{A} \in W$ we put $x^0 \to y = y$ and $x^{n+1} \to y = x \to (x^n \to y)$ for all $n \in \omega$, then $\mathfrak{A}$ satisfies the equations

(2.16) $(x \vee y)^n \to z = \bigwedge_{r+s=n} (x^r \to (y^s \to z))$ for every $n \in \omega$, $n > 1$.

Proof: It is easy to show (by induction) that for all $n \in \omega$, $x^{n+1} \to y = x^n \to (x \to y)$. Now we show (2.16) by induction. For n=1 it becomes (2.10). If we assume $n > 1$ and (2.16) for all $t < n$, then by (2.10), (2.11) and (1.11) we have that $(x \vee y)^n \to z = (x \vee y)^{n-1} \to ((x \vee y) \to z) = (x \vee y)^{n-1} \to ((x \to z) \wedge (y \to z)) = \bigwedge_{r+s=n-1} (x^r \to (y^s \to ((x \to z) \wedge (y \to z)))) = \bigwedge_{r+s=n-1} ((x^{r+1} \to (y^s \to z)) \wedge (x^r \to (y^{s+1} \to z))) = \bigwedge_{r+s=n} (x^r \to (y^s \to z))$. □

3. WAJSBERG ALGEBRAS AND MV-ALGEBRAS

The first aim of this section is to see that W-algebras are another presentation of MV-algebras, as we announced. We recall the definition and some properties of these algebras.

Definition (C.C. Chang [4]): Let $\mathfrak{B} = (B,+,\cdot,\neg,0,1)$ be an algebra of type (2,2,1,0,0) and universe B with the two following abbreviations:

$$x \wedge y = (x + \neg y) \cdot y$$
$$x \vee y = (x \cdot \neg y) + y$$

We say that $\mathfrak{B}$ is an MV-algebra if and only if it satisfies the following equations:

(3.1)	$x+y = y+x$	(3.1')	$x \cdot y = y \cdot x$
(3.2)	$x+(y+z) = (x+y)+z$	(3.2')	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
(3.3)	$x+\neg x = 1$	(3.3')	$x \cdot \neg x = 0$
(3.4)	$x+1 = 1$	(3.4')	$x \cdot 0 = 0$
(3.5)	$x+0 = x$	(3.5')	$x \cdot 1 = x$

(3.6)	$\neg(x+y) = \neg x \cdot \neg y$	(3.6')	$\neg(x \cdot y) = \neg x + \neg y$
(3.7)	$\neg(\neg x) = x$	(3.8)	$\neg 0 = 1$
(3.9)	$x \vee y = y \vee x$	(3.9')	$x \wedge y = y \wedge x$
(3.10)	$x \vee (y \vee z) = (x \vee y) \vee z$	(3.10')	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(3.11)	$x+(y \wedge z) = (x+y) \wedge (x+z)$	(3.11')	$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$.

The proofs of the following properties can be found in [4], pp. 468–470.

Proposition: If $\mathfrak{B}$ is an MV-algebra then it satisfies:

(3.12) $\neg 1 = 0$

(3.13) $(B, \wedge, \vee, 0, 1)$ is a bounded lattice where the order relation is given by $x \leq y$ if and only if $x \wedge y = x$ iff $\neg x + y = 1$ iff $x \cdot \neg y = 0$.

(3.14) $\neg(x \vee y) = \neg x \wedge \neg y$ and $\neg(x \wedge y) = \neg x \vee \neg y$.

(3.15) if $x \leq y$ then $x+z \leq y+z$

(3.15') $x \cdot y \leq x \wedge y$.

Theorem 4: Let $\mathfrak{B}$ be an MV-algebra. If we define

(3.16) $$x \dotplus y = \neg x + y \quad (\text{or } x \dotplus y = \neg(x \cdot \neg y) \text{ equivalently})$$

then $(B, \dotplus, \neg, 1)$ is a W-algebra such that $\neg x \dotplus y = x+y$ and $\neg(x \dotplus \neg y) = x \cdot y$.

Proof: We must show that $(B, \dotplus, \neg, 1)$ satisfies (1.1) to (1.4).

(1.1) follows immediately from (3.3)

(1.2): By the above definitions, and using (3.6), (3.7), (3.1), (3.2), (3.13), (3.15) and (3.3) we have: $(x \dotplus y) \dotplus ((y \dotplus z) \dotplus (x \dotplus z)) = \neg(\neg x+y) + (\neg(\neg y+z)+(\neg x+z)) = (x \cdot \neg y)+((y \cdot \neg z)+(\neg x+z)) = ((x \cdot \neg y)+\neg x)+((y \cdot \neg z)+z) = (\neg x \vee \neg y)+(y \vee z) \geq \neg y + y = 1$ and so we obtain (1.2).

(1.3) is straightforward from (3.9) because $x \vee y = (x \cdot \neg y)+y = (x \dotplus y) \dotplus y$

(1.4) follows from (3.1) and (3.13).□

Theorem 5: Let $\mathfrak{U} = (A, \dotplus, \neg, 1)$ a W-algebra. If we define

(3.17) $$x+y = \neg x \dotplus y \quad \text{and}$$

(3.17') $$x \cdot y = \neg(x \dotplus \neg y)$$

then $(A, +, \cdot, \neg, 0, 1)$ is an MV-algebra such that $\neg x+y = x \dotplus y$.

Proof: We must show that $(A, +, \cdot, \neg, 0, 1)$ satisfies (3.1) to (3.11'); recall that we can use (1.1) to (1.14) and (2.1) to (2.15).

(3.1) and (3.1') follow easily from (1.14).

(3.2) By (1.14), (1.13) and (1.11) we have that $x \oplus (y \oplus z) = \neg x \to (\neg y \to z) =$

$= \neg x \to (\neg z \to y) = \neg z \to (\neg x \to y) = z \oplus (x \oplus y) = (x \oplus y) \oplus z.$

(3.2') analogous to (3.2).

(3.3) and (3.3') are trivial from (1.5).

(3.4) and (3.4') from (1.7) and (1.14).

(3.5) and (3.5') from (1.1), (1.12) and (1.13).

(3.6) and (3.6') from (1.13) and (1.14).

(3.7) is (1.13) and (3.8) also follows from it.

It is clear that from (1.13) and (3.17) and (3.17') we have

$$x \vee y = (x \to y) \to y = (x \cdot \neg y) \oplus y \quad \text{and}$$
$$x \wedge y = (x \oplus \neg y) \cdot y$$

Then by Theorem 1 we obtain (3.9), (3.9'), (3.10) and (3.10').

(3.11) By (2.11) we have $x \oplus (y \wedge z) = \neg x \to (y \wedge z) = (\neg x \to y) \wedge (\neg x \to z) =$

$= (x \oplus y) \wedge (x \oplus z).$

(3.11') is analogous to (3.11) using (2.14) and (1.14). □

From the two preceding theorems it is clear that we can speak equally of W-algebras or MV-algebras and we can use the operations $\to, \neg, \oplus, \cdot, \wedge, \vee$ with the relations given by (2.2), (2.3), (3.16), (3.17) and (3.17'). Moreover we can use all the results in [4] and [5]. In spite of the equivalence between these two kinds of algebras, after having compared their respective presentations we prefer the one of W-algebras, because they stress the logical significance of every property, and specially that of the primitive operations $\neg$ and $\to$. In the rest of this section we relate the W-algebras to other kinds of structures, and we begin with a residuation result:

<u>Theorem 6</u>: If $\mathfrak{A} \in \mathbf{W}$ then the operation $\to$ is the residual of the operation defined by (3.17'), that is, $\mathfrak{A}$ satisfies

(3.18) $\qquad x \cdot y \leq z$ if and only if $x \leq y \to z$.

<u>Proof</u>: By (2.1) $x \cdot y \leq z$ iff $(x \cdot y) \to z = 1$ and using (1.14) and (1.11) we have $1 = (x \cdot y) \to z = \neg(x \to \neg y) \to z = \neg z \to (x \to \neg y) = x \to (\neg z \to \neg y) = x \to (y \to z)$ and this proves the Theorem. □

This last result has a stronger form which characterizes the class of W-algebras:

Theorem 7: If $(A,\to,\neg,1)$ is a W-algebra then $(A,\cdot,\leq,1)$ is a residuated ordered abelian monoid with a lower bound 0, the upper bound 1 and where the residual $\to$ satisfies (1.3). Conversely if $(A,\cdot,\leq,1)$ is a residuated ordered abelian monoid where the residual $\to$ satisfies (1.3), 1 is the upper bound and there is a lower bound 0, then, putting $\neg x = x \to 0$, $(A,\to,\neg,1)$ is a W-algebra.

Proof: The first part follows from several preceding results. To prove the second part we begin by showing:

(A) $x \leq y$ iff $x \to y = 1$: As $x = 1 \cdot x$, $x \leq y$ iff $1 \leq x \to y$ iff $1 = x \to y$. Secondly we prove:

(B) if $x \leq y$ then $z \to x \leq z \to y$: this is because $\overline{z \to x} \leq z \to x$ implies $z \cdot (z \to x) \leq x \leq y$ and so $z \to x \leq z \to y$. Third,

(C) $x \cdot (y \to z) \leq y \to (x \cdot z)$: this follows from (3.2') because then $(x \cdot (y \to z)) \cdot y = x \cdot ((y \to z) \cdot y) \leq x \cdot z$.

Now we can prove that $(A,\to,\neg,1)$ is a W-algebra:

(1.1): From $1 \cdot x \leq x$ we have $x \leq 1 \to x$ and moreover $1 = (x \to 1) \to 1 = (1 \to x) \to x$, so by (A) $1 \to x \leq x$ and we obtain $1 \to x = x$.

(1.2): By (C) we put $(y \to z) \cdot (x \to y) \leq x \to (y \cdot (y \to z)) \leq x \to z$ and so $x \to y \leq (y \to z) \to (x \to z)$ and now (1.2) follows by (A).

(1.3) is in the hypotheses, and finally

(1.4): By (A) we have that $0 \to x = 1$ and so $(x \to 0) \to 0 = (0 \to x) \to x = 1 \to x = x$. But recall that (1.11) follows from (1.1) to (1.3) and we can use it to obtain $\neg x \to \neg y = (x \to 0) \to (y \to 0) = y \to ((x \to 0) \to 0) = y \to x$ which with (A) gives (1.4). □

Another interesting result which gives us the differences and/or similarities between W-algebras and Boolean algebras, and makes a little more clear the significance of $\to$ and $\cdot$ is the following:

Theorem 8: A W-algebra is a Boolean algebra if and only if it satisfies any of the following conditions (which are equivalent):

1. $x+x = x$	4. $x \cdot y = x \wedge y$	7. $x \wedge (x \to y) = x \wedge y$
2. $x \cdot x = x$	5. $x \vee \neg x = 1$	8. $x^2 \to y = x \to y$
3. $x+y = x \vee y$	6. $x \wedge \neg x = 0$	9. $x^2 \to 0 = x \to 0$

Proof: In the case of conditions 1 to 6 the Theorem was proved by C.C. Chang in [4]. It is easy to see that 7 implies 6 (take $y=0$) and from $0 \leq y$ and (2.6) it follows that 6 implies 7. The equivalence of 9 and 5 follows from $x \to 0 \leq$ $\leq x^2 \to 0$ and $(x^2 \to 0) \to (x \to 0) = x \vee \neg x$. On the other hand $x \to y \leq x^2 \to y$ is always true, and $x \vee \neg x = x \vee (x \to 0) \leq x \vee (x \to y) = (x^2 \to y) \to (x \to y)$, so 8 and 5 are equivalent. □

We close this section with two notations of an arithmetical flavour:

(3.19) $a^{\lfloor 0} = 1$ and $a^{\lfloor n+1} = a^{\lfloor n} . a$ for all $n \in \omega$

(3.20) $0 \cdot a = 0$ and $(n+1) \cdot a = (n \cdot a) + a$ for all $n \in \omega$.

The following properties are easily checked:

Proposition: Any W-algebra satisfies, for all $n, m \in \omega$:

(3.21) $x^{\lfloor n+m} = x^{\lfloor n} \cdot x^{\lfloor m}$

(3.22) $(x^{\lfloor n})^{\lfloor m} = x^{\lfloor n \cdot m}$

(3.23) if $n \leq m$ then $x^{\lfloor n} \leq x^{\lfloor m}$

(3.24) $x^n \to y = (x^{\lfloor n}) \to y$

(3.25) $n \cdot (\neg x) = \neg (x^{\lfloor n})$. □

Observe that after this proposition we can write x^n in the place of $x^{\lfloor n}$ because (3.24) makes shure that there can be no confusion.

4. EXAMPLES OF WAJSBERG ALGEBRAS

The main example is, of course, the Tarski-Lindenbaum algebra of Łukasiewicz's $\aleph_0$-valued propositional calculus in the syntactical setting conjectured by him (and presumably proved by Wajsberg) after deleting the dependent fifth axiom.

(4.1) Let P(X) be the usual propositional language, that is, the free algebra of type (2,1) with X as free generating family; we represent the operations by $\to$ and $\neg$. We take as our set of axioms all formulae of P(X) which agree with any of the following schemes:

Ax.1 $p \to (q \to p)$

Ax.2 $(p \to q) \to ((q \to r) \to (p \to r))$

Ax.3 $((p \to q) \to q) \to ((q \to p) \to p)$

Ax.4 $(\neg p \to \neg q) \to (q \to p)$

and we take as our rule of inference Modus Ponens:

$$\frac{\begin{array}{l} p \longrightarrow q \\ p \end{array}}{q}$$

Now we can obtain in the usual way a finitary syntactical consequence relation $\vdash$ and in our case the relation $p \sim q$ iff $\vdash p \to q$ and $\vdash q \to p$ is a congruence relation over $P(X)$. The set of all theorems is an equivalence class denoted by 1. Then it is easy to show that the quotient algebra $(P(X)/\sim, \to, \neg, 1)$ is the free W-algebra with free generating family $\{[x]: x \in X\}$. See [12] for more details, and [2] for generalities.

Among the many examples that could be shown we will emphasize a construction which plays a central role in this theory.

(4.2) Let $(G,+,-,0,\leq)$ a totally ordered abelian group, and $e \in G$ with $e > 0$. In the positive segment $[0,e] = \{x \in G: 0 \leq x \leq e\}$ we define: $x \to y = \min\{e, y+e-x\}$ and $\neg x = e-x$.

Then $\underline{G[e]} = ([0,e], \to, \neg, e)$ is a W-algebra.

Two specially interesting particular cases of $G[e]$ are $\underline{\mathbb{R}[1]}$, where $\mathbb{R}$ is the additive real line, and $\underline{\mathbb{Q}[1]}$ where $\mathbb{Q}$ is the ordered additive group of rational numbers. Obviously $\mathbb{Q}[1]$ is a subalgebra of $\mathbb{R}[1]$.

(4.3) Let $n \in \omega$, $n \neq 0$. We represent by $\underline{S}_n$ the subalgebra of $\mathbb{R}[1]$ whose universe is $\{r/n: 0 \leq r \leq n,\ r \in \omega\}$. If we consider the additive subgroup of $\mathbb{Q}$ with universe $\{m/n: m \in \omega\}$, then S_n is its "unit segment".

As it is well known, the W-algebras of (4.3) are the original Łukasiewicz matrices for his n+1-valued propositional calculus, while it is clear that the construction of (4.2) is a generalization of Lukasiewicz matrix for the $\aleph_0$-valued calculus.

(4.4) Another generalization of the infinite-valued logic of Łukasiewicz is found in the basic structure of the theory of fuzzy sets. Given any set $A \neq \emptyset$, the set of all fuzzy subsets of A is I^A where $I = [0,1]$ is the universe of $\mathbb{R}[1]$. It has obviously the structure of a W-algebra with the operations defined pointwise from those of $\mathbb{R}[1]$. Thus $(\neg f)(x) = 1-f(x)$ and $(f \to g)(x) = \min\{1, 1+g(x)-f(x)\}$ for all $x \in A$, $f, g \in I^A$. The set I^A is also represented by $\underset{\sim}{P}(A)$.

5. IMPLICATIVE FILTERS AND CONGRUENCE RELATIONS

The concept of an implicative filter is the most basic one for the study of Wajsberg algebras, both from an algebraic point of view (Theorem 10) and from a logical one; this is clear from the definition itself and from the fact that several authors (specially Monteiro and his followers) have called them "deductive systems".

Definition: Let $\mathfrak{A} \in \mathbb{W}$ and $F \subseteq A$. We say that F is an implicative filter of if and only if it satisfies:

(5.1) $1 \in F$, and $x \in F$, $x \to y \in F$ imply $y \in F$.

We will represent the family of implicative filters of $\mathfrak{A}$ by $\mathcal{F}(\mathfrak{A})$. It is easy to check that $\mathcal{F}(\mathfrak{A})$ is an algebraic closure system; its associated closure operator will be represented by $\mathbb{F}^{\mathfrak{A}}$ (that is, $\mathbb{F}^{\mathfrak{A}}(X)$ is the least implicative filter containing X). Then, $(\mathcal{F}(\mathfrak{A}), \cap, \vee)$ is an algebraic lattice, where the meet $\cap$ is the set-theoretical intersection and the join $\vee$ is given by $F_1 \vee F_2 = \mathbb{F}^{\mathfrak{A}}(F_1 \cup F_2)$. Moreover its compact elements are $\mathbb{F}^{\mathfrak{A}}(N)$ for all $N \in S_\omega(A)$. The minimum is $\{1\}$ and A is the maximum; however, we usually consider only proper filters, i.e. different from A. If there is no possible confusion we will write $\mathcal{F}$ and $\mathbb{F}$ instead of $\mathcal{F}(\mathfrak{A})$ and $\mathbb{F}^{\mathfrak{A}}$. We next present two alternative forms of the above definition.

Proposition 9: Let $\mathfrak{A} \in \mathbb{W}$ and $F \subseteq A$. Then the following conditions are equivalent:

(5.1) $F \in \mathcal{F}$.

(5.2) F is a nonvoid order filter closed by $\cdot$, that is, $1 \in F$, if $x \in F$ and $x \leq y$ then $y \in F$, and if $x, y \in F$ then $x \cdot y \in F$.

(5.3) The set $\overline{F} = \{\neg x:\ x \in F\}$ is an ideal of $(A,+,\cdot,\neg,0,1)$ in the sense of Chang, that is, $0 \in \overline{F}$, if $x \in F$ and $y \leq x$ then $y \in \overline{F}$, and it $x,y \in F$ then $x+y \in F$.

Proof: (5.1) $\Rightarrow$ (5.2) $1 \in F$ by (5.1), and $x \leq y$ iff $x \to y = 1$, so if $x \in F$ then $y \in F$ by (5.1). If $x,y \in F$ then by (1.14), $x \to (x \cdot y) = \neg x + (x \cdot y) = \neg x \vee y \in F$ because $y \leq \neg x \vee y$ and by (5.1) also $x \cdot y \in F$.

(5.2) $\Rightarrow$ (5.3) follows easily from the "duality" properties $0 = \neg 1$, (2.4) and (3.6).

(5.3) $\Rightarrow$ (5.1) Clearly $1 = \neg 0 \in F$. If $x, x \to y \in F$ then $\neg x \vee \neg y = (\neg y \to \neg x) \to \neg x = x \to (x \to y) = \neg x + \neg(x \vee y)$ $\overline{F}$ and so $\neg y \in \overline{F}$ because $\neg y \leq \neg x \vee \neg y$, and then $y \in F$. □

Corollary: For every $\mathcal{U} \in \mathbb{W}$, $\mathcal{F}(\mathcal{U})$ is a subfamily of the family of all lattice filters of $(A,\wedge,\vee)$, that is, every implicative filter is closed by $\wedge$.

Proof: We have $x \wedge y = x \cdot (y + \neg x) = x \cdot (x \to y)$ and $y \leq x \to y$, so if $x,y \in F$ then by (5.2) it follows that $x \wedge y \in F$. □

The following result tells us that implicative filters can be identified with congruence relations in a natural way, as is usual in other kinds of algebraic structures.

Theorem 10: If $\mathcal{U} \in \mathbb{W}$ and we denote by $(\mathcal{C}(\mathcal{U}), \cap, \vee)$ the algebraic lattice of all congruence relations of $\mathcal{U}$, then this lattice is isomorphic to $(\mathcal{F}(\mathcal{U}), \cap, \vee)$ and the isomorphism is given by the mapping

(5.4) $\varphi : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{C}(\mathcal{U}) \quad \varphi(F) = \{(x,y) \in A \times A:\ (x \to y) \wedge (y \to x) \in F\}$

and its inverse

(5.5) $\varphi^{-1}:\ \mathcal{C}(\mathcal{U}) \longrightarrow \mathcal{F}(\mathcal{U}) \quad \varphi^{-1}(\theta) = \{x \in A:\ (x,1) \in \theta\}$.

Proof: A proof could be given by using (5.3) and a result of Chang ([4] page 484) but we prefer to give a direct proof of an implicative character based on a result of H. Rasiowa. By (2.1) we know that $(A,\to,1)$ is an implicative algebra satisfying (1.1), (1.2) and (1.10). Moreover $\neg x = x \to 0$, so $\mathcal{C}(\mathcal{U})$ is the family of all congruence relations of $(A,\to,1)$. By 1.6 of [10] (page 19), we know that φ is a one-to-one morphism, that φ^{-1} is an onto morphism, and that $\varphi^{-1} \circ \varphi = I_{\mathcal{F}(\mathcal{U})}$. It remains only to show that $\varphi \circ \varphi^{-1} = I_{\mathcal{C}(\mathcal{U})}$. If $(x,y) \in \theta$ then $(x \to y,\ y \to y) \in \theta$ and $(x \to x,\ y \to x) \in \theta$, so by (1.5) $x \to y \in \varphi^{-1}(\theta)$

and $y \to x \in \varphi^{-1}(\theta)$. By the Corollary of Theorem 9, $(x \to y) \wedge (y \to x) \in \varphi^{-1}(\theta)$, that is, $(x,y) \in \varphi(\varphi^{-1}(\theta))$. Conversely, s. $(x,y) \in \varphi(\varphi^{-1}(\theta))$ then $(x \to y) \wedge (y \to x) \in \varphi^{-1}(\theta)$ and by (5.2) $x \to y$, $y \to x \in \varphi^{-1}(\theta)$, that is, $(x \to y, 1) \in \theta$ and $(y \to x, 1) \in \theta$ Now by (1.1) $((x \to y) \to y, y) \in \theta$ and $((y \to x) \to x, x) \in \theta$, so by (1.3) we get $(x,y) \in \theta$. □

Corollary: If $\mathfrak{A}, \mathfrak{B} \in \mathbb{W}$ and f is an homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then Ker f = $= \{x \in A: f(x) = 1^{\mathfrak{B}}\} \in \mathcal{F}(\mathfrak{A})$, and moreover f is one-to-one if and only if Ker f $= \{1^{\mathfrak{A}}\}$. □

For the sake of simplicity we write $\mathbb{F}(X,a)$ in the place of $\mathbb{F}(X \cup \{a\})$ when $a \in A$ and $X \subseteq A$, and we write $\mathfrak{A}/F$ and A/F in the place of $\mathfrak{A}/\varphi(F)$ and $A/\varphi(F)$, for every $F \in \mathcal{F}$.

Theorem 11: If $\mathfrak{A} \in \mathbb{W}$ then for every $a \in X$ and $X \subseteq A$:

(5.6) $$\mathbb{F}(X,a) = \{y \in A: a^{r} \to y \in \mathbb{F}(X) \text{ for some } n \in \omega\}.$$

Proof: We denote by D the right member of (5.6). It is easy to see from (5.1) that $D \subseteq \mathbb{F}(X,a)$ and also that $X \cup \{a\} \subseteq D$, because $\mathbb{F}(X)$ is an order filter. It only remains to show that D is an implicative filter. Trivially $1 \in D$ as $a \to 1 = 1$. Now if $x, x \to y \in D$ then $a^n \to x$, $a^m \to (x \to y) \in \mathbb{F}(X)$ for some $n,m \in \omega$. By (2.1) and (1.10) we have that $(a^n \to x) \to (a^{n+m} \to y) \geq (a^{n-1} \to x) \to (a^{n+m-1} \to y)$ and by iteration and (1.11) be obtain $(a^n \to x) \to (a^{n+m} \to y) \geq x \to (a^m \to y) = a^m \to (x \to y) \in \mathbb{F}(X)$, so we can conclude that $a^{n+m} \to y \in \mathbb{F}(X)$ and so $y \in D$. This proves that $D \supseteq \mathbb{F}(X,a)$ and also (5.6). □

Note that (5.6) is exactly a Deduction Theorem, which can be considered an extension or generalization of the ones for the classical case (n=1) or the finite-valued case (a fixed n).

Corollary 1: If $\mathfrak{A} \in \mathbb{W}$ then for all $a,b \in A$ the following conditions are equivalent:

1) $b \in \mathbb{F}(a)$
2) $a^n \to b = 1$ for some $n \in \omega$.
3) $a^n \leq b$ for some $n \in \omega$. □

And taking into account the algebraic character of $\mathbb{F}$:

Corollary 2: If $\mathfrak{A} \in \mathbb{W}$ then for all $b \in A$ and all $X \subseteq A$, $b \in \mathbb{F}(X)$ if and only if there are $n \in \omega$, $k_1, \ldots, k_n \in \omega$, and $a_1, \ldots, a_n \in X$ such that $a_1^{k_1} \to (a_2^{k_2} \to \ldots$ $\ldots (a_n^{k_n} \to b) \ldots) = 1$, that is, $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n} \leq b$.□

Corollary 3: If $\mathfrak{A} \in \mathbb{W}$ then $\mathcal{F}(\mathfrak{A})$ is the family of all lattice filters of $(A, \wedge, \vee)$ if and only if $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra.

Proof: It is well known that in Boolean algebras implicative filters and lattice filters are the same. And conversely we can prove the condition 7 of Theorem 8: it is clear that x, $x \to y \in \mathbb{F}(x \wedge (x \to y))$ so $y \in \mathbb{F}(x \wedge (x \to y))$; but our assumption implies that $\mathbb{F}(x \wedge (x \to y))$ is the lattice filter generated by $x \wedge (x \to y)$, so $x \wedge (x \to y) \leq y$. Taking into account that $y \leq x \to y$ we conclude $x \wedge y = x \wedge (x \to y)$, which, by Theorem 8, implies that $\mathfrak{A}$ is Boolean.□

6. THE VARIETY OF WAJSBERG ALGEBRAS

The same definition of Wajsberg algebras tells us that $\mathbb{W}$ is a variety, and so $\mathbb{W} = \mathbb{H}\,\mathbb{S}\,\mathbb{P}(\mathbb{W})$, where $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$ are the operators homomorphic images, subalgebras and direct products respectively. But $\mathbb{W}$ is something more:

Theorem 12: $\mathbb{W}$ is an arithmetical variety. More precisely the 2/3-minority term is $p(x,y,z) = ((x \to y) \to z) \wedge ((z \to y) \to x) \wedge (x \vee z)$.

Proof: According to a result of Pixley (see [3], for instance) it suffices to show that $\mathbb{W} \models p(x,y,x) = p(x,y,y) = p(y,y,x) = x$, and this is easily checked.□

Corollary: If $\mathfrak{A} \in \mathbb{W}$ then the algebraic lattice $(\mathcal{F}(\mathfrak{A}), \cap, \vee)$ is distributive.□

From Theorem 10 and the fact that $\mathbb{W}$ is a variety the following results are immediately shown:

Proposition 13: a) If $\mathfrak{A} \in \mathbb{W}$ and $\mathfrak{B} \in \mathbb{S}(\mathfrak{A})$ then $\mathcal{F}(\mathfrak{B}) = \{B \cap F : F \in \mathcal{F}(\mathfrak{A})\}$.

b) If $\mathfrak{A}, \mathfrak{B} \in \mathbb{W}$ and $f \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ then $f^{-1}(F) \in \mathcal{F}(\mathfrak{A})$ for every $F \in \mathcal{F}(\mathfrak{B})$ and $f(F') \in \mathcal{F}(f(\mathfrak{A}))$ for all $F' \in \mathcal{F}(\mathfrak{A})$.

c) If $(\mathfrak{A}_i \mid i \in I) \subseteq \mathbb{W}$ then we represent by $\bullet\,\mathfrak{A}_i$ the direct product of the family and by p_i the canonical projection from $\bullet\,\mathfrak{A}_i$ onto $\mathfrak{A}_i$. Then if

$F \in \mathcal{F}(\otimes \mathfrak{A}_i)$, $F \subseteq \otimes p_i(F) \in \mathcal{F}(\otimes \mathfrak{A}_i)$ and if $F_i \in \mathcal{F}(\mathfrak{A}_i)$ for all $i \in I$, then $\otimes F_i \in$ $\in \mathcal{F}(\otimes \mathfrak{A}_i)$.

As in all varieties, every W-algebra can be represented as a subdirect product of subdirectly irreducible algebras of $\mathbb{W}$. It is well known that these last algebras are those whose minimum congruence is meet-irreducible, and this, by Theorem 10, is equivalent to $\{1\}$ being meet-irreducible in $C(\mathfrak{A})$. On the other hand every meet-irreducible implicative filter is finitely meet-irreducible. From this comments and the next results the interest of studying the finitely meet-irreducible implicative filters will be clear Recall that they form a basis of the closure system $\mathcal{F}(\mathfrak{A})$.

If $\mathfrak{A} \in \mathbb{W}$ then $(A, \wedge, \vee)$ is a lattice and then we have that all the implicative filters of $\mathfrak{A}$ are lattice filters of $(A, \wedge, \vee)$. We can consider the concept of prime implicative filters (those $F \in \mathcal{F}(\mathfrak{A})$ such that if $x \vee y \in F$ then $x \in F$ or $y \in F$) and we find that this is not a new concept:

Theorem 14: If $\mathfrak{A} \in \mathbb{W}$ and $F \in \mathcal{F}(\mathfrak{A})$ then F is finitely meet-irreducible if and only if F is prime.

Proof: As $\mathcal{F}(\mathfrak{A})$ is an algebraic distributive closure system, it is enough to show that for all $a, b \in A$, $\mathbb{F}(a \vee b) = \mathbb{F}(a) \cap \mathbb{F}(b)$ (see [6] and [16]). By (5.2) it is clear that $\mathbb{F}(a \vee b) \subseteq \mathbb{F}(a) \cap \mathbb{F}(b)$. Now let $x \in \mathbb{F}(a) \cap \mathbb{F}(b)$. By Corollary 1 of Theorem 11 there are $n, m \in \omega$ such that $a^n \to x = 1$ and $b^m \to x = 1$; we then put $t = \max\{n, m\}$ and show that $(a \vee b)^{2t} \to x = 1$: If $r, s \in \omega$ with $r + s = 2t$ and $r \geq t$, then $r \geq n$ and by repeated use of (1.8) and (2.6) we have $a^r \to (b^s \to x) =$ $= b^s \to (a^r \to x) \geq b^s \to (a^n \to x) = b^s \to 1 = 1$ while if $r < t$ then $s \geq t \geq m$ and by a similar argument we also find $a^r \to (b^s \to x) = 1$. Now by (2.16) $(a \vee b)^{2t} \to x =$ $= \bigwedge_{r+s=2t} (a^r \to (b^s \to x)) = 1$. This proves that $x \in \mathbb{F}(a \vee b)$. □

The above result has important consequences:

Definition: We say that a Wajsberg algebra is linear if and only if the order relation given by (2.1) is total. The class of all linear W-algebras will be denoted by $\mathbb{WL}$.

Theorem 15: If $\mathfrak{A} \in \mathbb{W}$ and $F \in \mathcal{F}(\mathfrak{A})$ then F is finitely meet-irreducible if and only if $\mathfrak{A}/F \in \mathbb{WL}$.

Proof: By (2.12) we always have that for all $a,b \in A$, $(a \to b) \vee (b \to a) \in F$. If F is finitely meet-irreducible then by Theorem 14 $a \to b \in F$ or $b \to a \in F$, that is, $[a] \to [b] = [1]$ or $[b] \to [a] = [1]$ in $\mathfrak{A}/F$: the quotient is linear. Conversely assume that $\mathfrak{A}/F$ is linear: $[a] \leq [b]$ or $[b] \leq [a]$ for all $a,b \in A$. If $a \vee b \in F = [1]$ then $[a] \; [b] = [1]$ and we must conclude that $[a] = [1]$ or $[b] = [1]$, that is, $a \in F$ or $b \in F$. Again by Theorem 14 we find that F is finitely meet-irreducible.□

Corollary: a) If $\mathfrak{A} \in \mathbb{W}$ then $\mathfrak{A} \in \mathbb{WL}$ if and only if $\{1\}$ is finitely meet-irreducible.

b) Every W-algebra is a subdirect product of linear W-algebras.

c) $\mathbb{W} = \mathbb{S}\,\mathbb{P}(\mathbb{WL})$.

Proofs: a) is only a particular case of the Theorem. b) follows from a) and the comment we have made before Theorem 14. For c), it is clear that $\mathbb{S}\,\mathbb{P}(\mathbb{WL}) \subseteq \mathbb{W}$ as $\mathbb{WL} \subseteq \mathbb{W}$. On the other hand by a) and the same comment we see that all subdirectly irreducible W-algebras are linear, and so $\mathbb{W} \subseteq \mathbb{S}\mathbb{P}(\mathbb{WL})$. □

From the results we have just given C.C. Chang proves in [5] the following fundamental Theorem, which implicitly states a completeness result for the $\aleph_0$-valued Łukasiewicz calculus:

Theorem 16: If $\mathbb{Q}[1]$ is the W-algebra described in (4.2) then $\mathbb{W} = \mathbb{H}\,\mathbb{S}\,\mathbb{P}(\mathbb{Q}[1])$, that is, the variety $\mathbb{W}$ is generated by $\mathbb{Q}[1]$: an equation is true in every W-algebra if and only if it is true in $\mathbb{Q}[1]$.

We will give only a sketch of the proof. It is worth mentioning the basic construction, consisting in the association of a totally ordered abelian group to each linear W-algebra and conversely. This is done in the following way:

If $\mathfrak{A} \in \mathbb{WL}$ we define $\mathcal{G}(\mathfrak{A}) = (A^*, +, -, 0, \leq)$ as follows:

A^* is the set resulting from $\mathbb{Z} \times A$ after the identification $(n,1) = (n+1,0)$;

$(m,x)+(n,y) = (m+n, \neg x \to y) = (m+n, x+y)$ when $\neg x \to y < 1$, and $(m,x)+(n,y) = (m+n+1, \neg(x \to \neg y)) = (m+n+1, x \cdot y)$ when $\neg x \to y = 1$;

$-(n,x) = (-n-1, \neg x)$;

$0 = (0,0)$;

and the order $\leq$ results from the lexicographical ordering of $\mathbb{Z} \times A$.

Then one can show that $\mathcal{G}(\mathfrak{A})$ is a totally ordered abelian group with strong unit (0,1) and such that $\mathcal{U} \cong \mathcal{G}(\mathcal{U})[(0,1)]$. Conversely given a totally ordered abelian group G with strong unit e, we have seen in (4.2) that G[e] is a linear W-algebra, and moreover $G \cong \mathcal{G}(G[e])$.

The Theorem is proved through model theory. To each equation of the language of W-algebras we can associate a formula of the language of totally ordered abelian groups with one free variable (which is interpreted as a positive element of the group) and such that the equation holds in $\mathbb{W}$ if and only if the associated formula holds in every totally ordered abelian group. Then there is a jump to divisible totally ordered abelian groups. This is a complete theory and $(\mathbb{Q},+,-,0,\leq)$ is a model. Finally it results that an equation holds in $\mathbb{WL}$ if an only if it holds in $\mathbb{Q}[1]$.

7. SIMPLE AND SEMISIMPLE WAJSBERG ALGEBRAS

An algebra is simple when it has exactly two congruence relations, the maximum and the minimum ones. After Theorem 10 a W-algebra will be simple if and only if {1} is a maximal implicative filter. So we begin by studying some properties of this kind of filters. We will represent by $SpM(\mathcal{U})$ the set of all maximal (obviously proper) implicative filters.

Theorem 17: If $\mathcal{U} \in \mathbb{W}$ and $F \in \mathcal{F}$ then the following conditions are equivalent:

(7.1) $F \in SpM(\mathcal{U})$,

(7.2) $\mathcal{U}/F$ is simple,

(7.3) For every $x \in A$, $x \notin F$, there is $n \in \omega$ such that $x^n \to 0 \in F$.

Proof: (7.1) and (7.2) are equivalent since F is [1] in the quotient. If $x \notin F \in SpM(\mathcal{U})$ then $\mathbb{F}(F,x) = A$ and so $0 \in \mathbb{F}(F,x)$ and by Theorem 11 we obtain (7.3). If $F' \in \mathcal{F}(\mathcal{U})$ and $F \subsetneq F'$ then there is an $x \in F'$, $x \notin F$ and from (7.3) we have $0 \in \mathbb{F}(F,x) \subseteq F'$ so $F' = A$ and F is maximal. □

Corollary 1: If $\mathcal{U} \in \mathbb{W}$ then $\mathcal{U}$ is simple if and only if it satisfies any one of the following equivalent conditions:

(7.3') For every $x \in A$, $x \neq 1$, there is an $n \in \omega$ with $x^n \to 0 = 1$.

(7.4) For every $x \in A$, $x \neq 1$, there is an $n \in \omega$ with $x^n = 0$ (i.e, x is nilpotent)

(7.5) For every $x \in A$, $x \neq 0$, there is an $n \in \omega$ with $n \cdot x = 1$ (i.e., A is locally finite in the sense of Chang).

Proof: From Theorem 17 the simplicity of $\mathfrak{A}$ is equivalent to (7.3'). By (3.24) and (1.12) we see that (7.3') and (7.4) are equivalent, and finally the equivalence between (7.4) and (7.5) follows from (1.13), (3.6) and (3.2).□

Corollary 2: If $\mathfrak{A},\mathfrak{B} \in \mathbf{W}$ and $f \in \mathrm{Hom}(\mathfrak{A},\mathfrak{B})$ then for every $F \in \mathrm{SpM}(\mathfrak{B})$, $f^{-1}(F) \in \mathrm{SpM}(\mathfrak{A})$. If in addition f is onto then for all $G \in \mathrm{SpM}(\mathfrak{A})$ we have $f(G) = B$ or $f(G) \in \mathrm{SpM}(\mathfrak{B})$.

Proof: This is an immediate consequence of Proposition 13 and (7.3). For if $F \in \mathrm{SpM}(\mathfrak{B})$ and $x \notin f^{-1}(F)$ then $f(x) \notin F$ and so there is an $n \in \omega$ such that $f(x)^n \to 0 \in F$ and this implies $x^n \to 0 \in f^{-1}(F)$.□

Corollary 3: If $\mathfrak{A} \in \mathbf{W}$ is simple then $\mathfrak{A}$ is a linear W-algebra and every subalgebra of $\mathfrak{A}$ is also simple ($\mathfrak{A}$ is hereditarily simple).

Proof: The first part follows the theorem, because if $\mathfrak{A}$ is simple then $\{1\}$ is maximal, and thus finitely meet-irreducible, and from the Corollary of Theorem 15. The second part is trivial, since (7.3') is obviously preserved under subalgebras.□

Theorem 18: If $\mathfrak{A} \in \mathbf{W}$ then $\mathfrak{A}$ is simple iff $\mathfrak{A} \in \mathbf{S}(\mathbb{R}[1])$.

Proof: We use the construction in Theorem 16. It is easy to check that $\mathfrak{A} \in \mathbf{W}$ is simple iff $\mathfrak{A}$ is linear and $\mathcal{G}(\mathfrak{A})$ is an archimedean totally ordered abelian group. Now it follows from a known result of Hölder (see [1] page 300) that $\mathfrak{A}$ is simple iff it is linear and $\mathcal{G}(\mathfrak{A})$ is (isomorphic to) a subgroup of $(\mathbb{R},+,-,0,\leq)$. But this can be made by mapping $(0,1)$ A^* to $1 \in \mathbb{R}$, and since $\mathfrak{A} \cong \mathcal{G}(\mathfrak{A})[(0,1)]$ we find that $\mathfrak{A}$ is a subgroup of $\mathbb{R}[1]$, and these are all linear. So the Theorem is proved.□

Another important result about simple algebras is the characterization of all finite linear W-algebras. It is clear that the algebras S_n defined in (4.3) as subalgebras of $\mathbb{R}[1]$ are simple, by Theorem 18. Since there is only one order isomorphism between two finite chains of the same cardinal and every W-homomorphism is an order morphism, we find that the only simple W-algebra of cardinal number n+1 will be S_n, which is the only linear W-algebra with this number of elements. So we have shown:

Theorem 19: If $\mathfrak{A} \in \mathbf{W}$ is finite and has $n \in \omega$, $n > 1$, elements, then $\mathfrak{A}$ is linear if and only if it is simple, and if and only if $\mathfrak{A} \cong S_{n-1}$.□

Now we can study the subdirect products of simple algebras, that is, the semisimple algebras. We represent by $R(\mathfrak{A})$ the radical of $\mathfrak{A}$, that is, the intersection of all maximal (proper) implicative filters of $\mathfrak{A} \in \mathbb{W}$. Then by Theorem 10 a W-algebra is semisimple if and only if $R(\mathfrak{A})=\{1\}$. If we represent by $\mathbb{W}_S$ the class of all semisimple W-algebras then we have:

Theorem 20: The class $\mathbb{W}_S$ can be defined by generalized implications but not by implications.

Proof: $\mathbb{P}(\mathbb{W}_S) = \mathbb{W}_S$ by definition of semisimplicity, and $\mathbb{S}(\mathbb{W}_S) = \mathbb{W}_S$ by Proposition 13 and $R(\mathfrak{A}) = \{1\}$. Therefore $\mathbb{W}_S$ can be defined by generalized implications. On the other hand, the ultrapower $\bigotimes_{\mathfrak{F}} \mathbb{R}[1]$, where $\mathfrak{F}$ is a non principal ultrafilter of ω, is a W-algebra which is not semisimple (see [12]). Then $\mathbb{W}_S$ is not closed under ultraproducts and hence it cannot be defined by implications.□

Our last aim in this paper is to give the generalized implications defining $\mathbb{W}_S$.

Definitions: If $\mathfrak{A} \in \mathbb{W}$ then we say that an $a \in A$ is a quasi-maximum if and only if:

(7.6) There is a $b \in A$ such that $(a^n \to b) \to a = 1$ for all $n \in \omega$, $n \geq 1$.

The set of all quasi-maxima of $\mathfrak{A}$ will be denoted by $U(\mathfrak{A})$. Clearly $U(\mathfrak{A}) \neq \emptyset$ as $1 \in U(\mathfrak{A})$.

Corollary 1: If $\mathfrak{A} \in \mathbb{W}$ then for every $a \in A$ the following are equivalent:

(7.6) $a \in U(\mathfrak{A})$

(7.7) $(a^n \to 0) \to a = 1$ for all $n \in \omega$, $n \geq 1$

(7.8) $\neg a \leq a^n$ for all $n \in \omega$, $n \geq 1$.

Proof: It is clear from (3.24), (2.4) and (1.13) that (7.7) is equivalent to (7.8). (7.6) follows from (7.7) by taking b=0, and from (7.6) and (2.6) we see that $0 \leq b$ implies $a^n \to 0 \leq a^n \to b \leq a$ and we obtain (7.7).□

Corollary 2: (7.9) If $\mathfrak{A} \in \mathbb{W}$ and $\mathfrak{B} \in \mathbb{S}(\mathfrak{A})$ then $U(\mathfrak{B}) = U(\mathfrak{A}) \cap B$, where B is the universe of $\mathfrak{B}$.

(7.10) If $(\mathfrak{A}_i \mid i\in I)\subseteq \mathbb{W}$ then $U(\circledast \mathfrak{A}_i \mid i\in I) = \circledast(U(\mathfrak{A}_i)\mid i\in I)$.□

The above Corollary and the Corollary to Theorem 15 suggest us the study of the quasi-maxima of linear W-algebras.

<u>Proposition 21</u>: If $\mathfrak{A}\in \mathbb{WL}$ and $a\in A$, then $a\in U(\mathfrak{A})$ iff $a^n\neq 0$ for all $n\in\omega$.
<u>Proof</u>: Assume that $a\in U(\mathfrak{A})$. If $a=1$ then $a^n=1\neq 0$, and from $a\neq 1$ and (7.8) we obtain $0\leq \neg a\leq a^n$ and so $a^n\neq 0$. Conversely assuming $a^n\neq 0$ for all $n\in\omega$, if $a\notin U(\mathfrak{A})$ there would be an $m\in\omega$ with $a^m\leq \neg a$ (because is linear) and then $a^{m+1}=a^m\cdot a = \neg(a^m\to\neg a) = \neg 1 = 0$, against our assumption.□

<u>Proposition 22</u>: If $\mathfrak{A}\in\mathbb{WL}$ then $U(\mathfrak{A})\in SpM(\mathfrak{A})$.
<u>Proof</u>: We begin by proving that $U(\mathfrak{A})\in F(\mathfrak{A})$ through (5.2). It is clear that $1\in U(\mathfrak{A})$. If $x\leq y$ then (2.5) gives $x^n\to 0\geq y^n\to 0$ and so $x^n\leq y^n$. Therefore if $x\in U(\mathfrak{A})$ it is obvious that $y\in U(\mathfrak{A})$, using Proposition 21. On the other hand if $x,y\in U(\mathfrak{A})$ by the linearity of $\mathfrak{A}$ we can assume $x\leq y$ and consequently $x^n\leq y^n$ as before. Then $0\leq x^{2n} = x^n\cdot x^n\leq x^n\cdot y^n = (x\cdot y)^n$, so $x\cdot y\in U(\mathfrak{A})$. Next we show that $U(\mathfrak{A})$ is maximal. If $x\notin U(\mathfrak{A})$ then by (7.7) $x^n\to 0\not\leq x$ for some $n\in\omega$, and so $x\leq x^n\to 0$ for is linear. Then $x^{n+1}\to 0 = x\to(x^n\to 0) = 1\in U(\mathfrak{A})$. Now by (7.3) we have proved that $U(\mathfrak{A})\in SpM(\mathfrak{A})$.□

<u>Theorem 23</u>: If $\mathfrak{A}\in\mathbb{W}$ then $R(\mathfrak{A}) = U(\mathfrak{A})$.
<u>Proof</u>: Note first that every $\mathfrak{A}$ is isomorphic to a subdirect product of linear W-algebras $(\mathfrak{A}_i\mid i\in I)$, and so we have an embedding $\varphi:\mathfrak{A}\longrightarrow\circledast(\mathfrak{A}_i\mid i\in I)$ such that denoting by p_i the projections then $\varphi_i = p_i\circ\varphi$ is onto. From the above Corollary 2 we have $\varphi(U(\mathfrak{A})) = U(\varphi(\mathfrak{A})) = \varphi(\mathfrak{A})\cap U(\circledast\mathfrak{A}_i) = \varphi(\mathfrak{A})\cap\circledast(U(\mathfrak{A}_i))$ and by Proposition 13 $U(\mathfrak{A}) = \varphi^{-1}(\varphi(U(\mathfrak{A})))\in F(\mathfrak{A})$. Moreover $\mathfrak{A}_i\in\mathbb{WL}$ and so by Proposition 22 $U(\mathfrak{A}_i)\in SpM(\mathfrak{A}_i)$ and then by Corollary 2 of Theorem 17 $\varphi^{-1}(U(\mathfrak{A}_i))\in SpM(\mathfrak{A})$. Now $R(\mathfrak{A})\subseteq\cap\varphi_i^{-1}(U(\mathfrak{A}_i)) = U(\mathfrak{A})$ because $\varphi(U(\mathfrak{A})) = \circledast\varphi_i(U(\mathfrak{A}))$ for every φ_i is onto. On the other hand, suppose that $x\in U(\mathfrak{A})$ but $x\notin R(\mathfrak{A})$: there is an $F\in SpM(\mathfrak{A})$ with $x\notin F$ and so there is an $n\in\omega$, $n>1$, such that $x^n\to 0$ F. But by (7.7) $(x^n\to 0)\to x = 1\in F$ and then $x\in F$ against our assumption. So $U(\mathfrak{A})\subseteq R(\mathfrak{A})$ and the theorem is proved.□

<u>Corollary 1</u>: If $\mathfrak{A}\in\mathbb{W}$ then $\mathfrak{A}$ is semisimple iff $U(\mathfrak{A}) = \{1\}$.□

Corollary 2: The class W_S is defined by (1.1), (1.2), (1.3), (1.4) and

(7.11) If $(x^n \to 0) \to x = 1$ for all $n \in \omega$, $n \geq 1$ then $x=1$.□

References

[1] Birkhoff, G.: Lattice Theory (3rd Ed.), Ann. Math. Soc. Colloquium Publications, XXV, Providence, 1967.

[2] Brown, D.J. - Suszko, R.: Abstract Logics, Dissertationes Mathematicae CII (1973) 9-41.

[3] Burris, S. - Sankappanavar, H.P.: A course in Universal Algebra, Springer-Verlag, Berlin, 1981.

[4] Chang, C.C.: Algebraic analysis of many-valued logics, Transactions of the Am. Math. Soc. 88 (1958) 467-490.

[5] ——————: A new proof of the completeness of the Lukasiewicz axioms, Transactions of the Am. Math. Soc. 93 (1959) 74-80.

[6] Dzik, W.: On the content of the lattice of Logics, part I, Reports on Mathematical Logic 13 (1981) 17-28.

[7] Font, J.M. - Rodríguez, A.J.: Nota sobre el significado lógico de ciertas estructuras residuadas elementales, Publ. Mat. of the Univ. Autònoma de Barcelona 20 (1980) 83-86.

[8] Iseki, K. - Tanaka, S.: An introduction to the theory of BCK-algebras, Math. Japonica 23 (1978) 1-26.

[9] Komori, Y.: Super-Lukasiewicz propositional logics, Nagoya Math. J. 84 (1981) 119-133.

[10] Rasiowa, H.: An algebraic approach to non-classical logics, North-Holland, Amsterdam, 1974.

[11] Rodríguez, A.J.: El álgebra de Wajsberg de las partes difusas de un conjunto, Rev. Univ. Santander 2 (1979) part II, 927-929.

[12] ——————: Un estudio algebraico de los cálculos proposicionales de Lukasiewicz, Ph. D. Thesis, Univ. of Barcelona, 1980.

[13] ——————: Algebras de Wajsberg pseudo-complementadas, Actas de las IX Jornadas Mat. Hispano-Lusas, Salamanca, 1982 (to appear).

[14] Torrens, A.: Estudi i algebrització de certes lògiques. Algebres d-completes. Ph. D. Thesis. Univ. of Barcelona, 1980.

[15] ——————: Cadenas de Sales discretes, Publ. Mat. de la Univ. Autònoma de Barcelona 20 (1980) 101-103.

[16] ———————— - Verdú, V.: Distributivity and irreducibility in closure systems, Preprint nº8, Fac. Math., Univ. of Barcelona, 1982.

[17] Wajsberg, M.: Beiträge zum Metaaussagenkalkül I, Monat. Math. Phys. 42 (1935) page 240.

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