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SOME ALGEBRAIC STRUCTURES DETERMINED BY CLOSURE OPERATORS

by

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Some algebraic structures determined by closure operators

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In this work we obtain axiomatizations of the concepts of lattice, distributive lattice, positive implication algebra, implication algebra, relative pseudocomplemented lattice, pseudoBoolean algebra and Boolean algebra by means of the concept of closure operator. The axiomatics express the algebra as a logical quotient of a logic having properties which reflect some classical rules of inference as adjunction, excluding cases, deduction and reductio at absurdum.

If $\mathcal{S}=(S,\wedge,\vee,\rightarrow,')$ is an algebra of type $(2,2,2,1)$, C is a closure operator on S, $x,y\in S$ and T is a finite subset of S, then:

(i) $C(x,y)=C(x\wedge y)$ reflects and adjunction rule for the connective $\wedge$

(ii) $C(x,T)\cap C(y,T) = C(x\vee y,T)$ reflects an excluding cases rule for the conective $\vee$ and for finitely axiomatized theories.

(iii) $x\rightarrow y\in C(T) \Leftrightarrow y\in C(T,x)$ reflects a classical deduction rule for the connective $\rightarrow$ and for finitely axiomatized theories.

(iv) $x\in C(T) \Leftrightarrow C(T,x') = S$ reflects a reductio at absurdum rule for the connective ' and finitely axiomatized theories.

(v) $C(x) = C(y) \Rightarrow x=y$ reflects the idea of Tarski-Lindenbaum quotient.

The conditions (i) - (v) are those that are used for the axiomatizations given in this paper.

We recall that if C is a closure operator on S then the following relation on S: $x \leq y \Leftrightarrow y \in C(x)$ is a preordering and if C satisfies $C(x)=C(y) \Rightarrow x=y$, then this preording is an order and we call it the C-order.

Theorem 1. Let $\mathcal{S}=(S,\wedge,\vee)$ be an algebra of type (2.2). $\mathcal{S}$ is a lattice, where $\wedge$ is the infimum and $\vee$ is the supremum, if and only if there exists a closure operator C on S such that for all $x,y \in S$.

(i) $C(x,y) = C(x \wedge y)$,

(ii) $C(x) \cap C(y) = C(x \vee y)$,

(iii) $C(x) = C(y) \Rightarrow x=y$.

Proof:

a) It is already known that the closure operator associated with all filters of a lattice satisfies conditions (i) - (iii).

b) Condition a) implies that the following equivalence relation on S: $x \sim y \Leftrightarrow C(x) = C(y)$ is a congruence on $(S,\wedge)$ and $(S/\sim,\wedge)$ is a meet-semilattice.

Condition (ii) implies that the equivalence relation $\sim$ is a congruence on $(S,\vee)$ and $(S/\sim,\vee)$ is a join-semilattice.

The above comments and condition (iii) imply that $(S,\wedge,\vee)$ is a lattice.

It is easy to see that de C-order is the same that the order of the lattice.

Theorem 2. Let $\mathcal{S}=(S,\wedge,\vee)$ be an algebra of type (2,2). $\mathcal{S}$ is a distributive lattice if, and only if, then exists a closure operator C on S such that for all $x,y \in S$ and all finite subsets $T \subseteq S$:

(i) $C(x,y) = C(x \wedge y)$,

(ii) $C(T,x) \cap C(T,y) = C(T,x \vee y)$,

(iii) $C(x) = C(y) \Rightarrow x=y$.

Proof:

a) It is easy to see that the closure operator associated with all filters of a distributive lattice satisfies conditions (i) - (iii).

b) By Theorem 1 it suffices to prove that

$$z \wedge (x \vee y) \leq (z \wedge x) \vee (z \wedge y) \quad , \quad x,y,z \in S.$$

which is equivalent to

$$(z \wedge x) \vee (z \wedge y) \in C(z,x) \cap C(z,y),$$

which is true.

Theorem 3. Let $\mathcal{S}=(S,\rightarrow)$ be an algebra of type (2). $\mathcal{S}$ is a positive implication algebra (or a Hilbert algebra) if, and only if, there exists a closure operator C ou S such that for all $x,y \in S$ and all finite sets $T \subseteq S$.

(i) $y \in C(T,x) \Leftrightarrow x \rightarrow y \in C(T)$,

(ii) $C(x) = C(y) \Rightarrow x=y$.

Proof:

a) It is easy to see that the closure operator associated with all implicative filters of $\mathcal{S}$ satisfies conditions (i)-(iii).

b) In order to prove that $(S,\rightarrow)$ is a positive implication algebra we will prove that:

$$(x \to x) \to y = y, \quad x \to (y \to z) = (x \to y) \to (x \to z),$$

and

$$(x \to y) \to ((y \to x) \to y) = (y \to x) \to ((x \to y) \to x)$$

(see Rasiowa, p.25)

$$(x \to x) \to y = y \Leftrightarrow C((x \to x) \to y) = C(y)$$

$y \in C((x \to x) \to y)$ because $x \to x \in C(\emptyset)$ by (i)

$$(x \to x) \to y \in C(y) \Leftrightarrow y \in C(y, (x \to x)),$$

which is true, and so on. It is easy to see that the C-order is the same that the order of the positive implication algebra.

Theorem 4. Let $\mathcal{S} = (S, \to)$ an algebra of type (2). $\mathcal{S}$ is an implication algebra if, and only if, three exists a closure operator C on S such that for all $x, y \in S$ and all finite sets $T \subseteq S$:

(i) $x \to y \in C(T) \Leftrightarrow y \in C(T, x)$,

(ii) $C(x) = C(y) \Rightarrow x = y$

(iii) the closure system $\mathcal{C}$ (associated with C) has a basis of maximal closed sets.

Proof:

a) It suffices to prove (iii). It is already known that irreducible implicative filters and maximal implicative filters are the same in an implication algebra (see Rasiowa, p.33). On the other hand, the irreducible sets form a basis of $\mathcal{C}$, because $\mathcal{C}$ is algebraic, so maximal sets form a basis.

b) Assume that (i)-(iii) holds. Then $\mathcal{S}$ is a positive implication algebra. If $\mathcal{D}$ is the closure system of all implicative filters of $\mathcal{S}$, then

$\mathcal{C} \subseteq \mathcal{D}$. Then

$$C(\emptyset) = \bigcap_{\substack{T\in\mathcal{C}\\ T \text{ maximal}}} T \geq \bigcap_{\substack{T\in\mathcal{D}\\ T \text{ maximal}}} T = C_{\mathcal{D}}(\{((x\to y)\to x)\to x:\ x,y\in S\})$$

(see [3])

So $\{((x\to y)\to x)\to x:\ x,y\in S\} \subseteq C(\emptyset)$ and so the positive implication algebra is an implication algebra.

Theorem 5. Let $\mathcal{S}=(S,\to,\wedge,\vee)$ and algebra of type (2,2,2). $\mathcal{S}$ is a relatively pseudo-complemented lattice, where $x\to y$ is the pseudocomplement of x relative to y, if, and only if, there exists a closure operator C on S such that for all $x,y\in S$ and all finite sets $T\subseteq S$:

(i) $C(x,y) = C(x\wedge y)$,

(ii) $C(x)\cap C(y) = C(x\vee y)$,

(iii) $C(x) = C(y) \Rightarrow x=y$,

(iv) $y\in C(T,x) \Leftrightarrow x\to y\in C(T)$.

Proof:

a) It is easy to see that the closure operator associated with all filters of a relatively pseudo-complemented lattice satisfies conditions (i)-(iv).

b) By theorem 1 $(S,\wedge,\vee)$ is a lattice. Then it suffices to prove that

$$t\leq x\to y \Leftrightarrow t\wedge x\leq y \qquad t,x,y\in S$$

But

$$t\leq x\to y \Leftrightarrow x\to y\in C(t) \Leftrightarrow y\in C(t,x) \Leftrightarrow y\in C(t\wedge x) \Leftrightarrow t\wedge x\leq y.$$

Corollary: Let $\mathcal{S}=(S,\to,\wedge,\vee,0)$ an algebra of type (2,2,2,0). $\mathcal{S}$ is a pseudo-boolean algebra if, and only if, there exists a closure operator C on S such that (i)-(iv) of theorem 5 hold and there exists an $x \in S$ such that $C(x)=S$.

Theorem 6. Let $\mathcal{S}=(S,\wedge,')$ and algebra of type (2,1). $\mathcal{S}$ is a Boolean algebra, where $\wedge$ is the infimum and $'$ is the complement, if, and only if, there exists a closure operator C on S such that for all $x,y \in S$:

(i) $C(x,y) = C(x \wedge y)$,

(ii) $y \in C(x) \Leftrightarrow C(x,y') = S$,

(iii) $C(x) = C(y) \Rightarrow x=y$

Proof:

a) It is easy to see that the closure operator associated with all filters of a boolean algebra satisfies conditions (i)-(iii).

b) We begin showing that the operation $'$ satisfies:

$$x'' = x \quad \text{and} \quad x \leq y \Leftrightarrow x' \geq y'.$$

Proof: $x \in C(x'') \Leftrightarrow C(x'',x') = S \Leftrightarrow x' \in C(x')$, which is true.

$$S = C(x,x') = C(x,C(x')) \subseteq C(x,C(x'''));$$

so

$$C(x''',x) = S \text{ and then } x'' \in C(x).$$

$$x \leq y \Leftrightarrow y \in C(x) \Leftrightarrow C(x,y')=S \Leftrightarrow C(x'',y')=S \Leftrightarrow x' \in C(y')$$

So $'$ is a dual isomorphism of $(S,\leq)$.

We know that conditions (i) and (iii) imply that $(S,\wedge)$ is a meet-semilattice. As $'$ is a dual isomorphism, then $x \vee y = (x' \wedge y')'$ is the supremum of $\{x,y\}$ and so $(S,\wedge,\vee)$ is a lattice.

In order to prove that the lattice $(S,\wedge,\vee)$ is distributive, we must show that

$$C(z,x\vee y) = C(z,x)\cap C(z,y)$$

On one hand

$$x\vee y\subset C(z,x) \Leftrightarrow C(z,x,x'\wedge y')=S,$$

which is true.

A similar argument applies to show that $x\vee y\in C(z,y)$. On the other hand if $r\in C(z,x)\cap C(z,y)$, then $x'\in C(z,r')$ and $y'\in C(z,r')$: therefore $x'\wedge y'\in C(z,r')$ and so $r\in C(z,(x'\wedge y')')$.

We finally show that ' is a complement.

$C(x,x')=S$, so $x\wedge x'$ is the first element of $(S,\leq)$. On the other hand $x'\wedge x'' = 0$, so $(x'\wedge x'')'=u$ and then $x\vee x'=u$.

Theorem 7. Let $\mathcal{S}=(S,\vee,')$ and algebra of type (2,1). $\mathcal{S}$ is a Boolean algebra if, and only if, there exists a closure operator C on S such that for all $x,y\in S$ and all finite $T\subseteq S$:

(i) $C(T,x)\cap C(T,y) = C(T,x\vee y)$,

(ii) $y\in C(T) \Leftrightarrow C(T,y') = S$,

(iii) $C(x) = C(y) \Rightarrow x=y$.

Proof:

a) It is easy to see that the closure operator associated with all filters of a Boolean algebra satisfies conditions (i) - (iii).

b) In order to see that $\mathcal{S}$ is a Boolean algebra it suffices to see that there exists a binary operation $\wedge$ on S such that $C(x,y) = C(x\wedge y)$

Put $x\wedge y=(x'\vee y')'$. Then

$$x\in C((x'\vee y')') \Leftrightarrow x'\vee y'\in C(x'),$$

which is true.

$$(x' \vee y')' \in C(x,y) \Leftrightarrow C(x' \vee y', x, y) = S \Leftrightarrow C(x', x, y) \cap C(y', x, y) = S,$$

which is true.

Theorem 8. Let $\mathcal{S} = (S, \rightarrow, ')$ an algebra of type (2,1). $\mathcal{S}$ is a Boolean algebra if, and only if, there exists a closure operator C on S such that for all $x, y \in S$ and all finite $T \subseteq S$:

(i) $y \in C(x) \Leftrightarrow C(x, y') = S$,

(ii) $y \in C(T, x) \Leftrightarrow x \rightarrow y \in C(T)$,

(iii) $C(x) = C(y) \Rightarrow x = y$.

Proof:

a) It is easy to see that the closure operator associated with all implicative filters of a Boolean algebra satisfies conditions (i)-(iii).

b) We shall see that $\mathcal{S}$ is a Boolean algebra by showing that $(S, \rightarrow)$ is a positive implication algebra with first element 0 and which satisfies $(x \rightarrow 0) \rightarrow 0 \leq x$.

By Theorem 3, $(S, \rightarrow)$ is a positive implication algebra.

$$x \rightarrow x \in C(\emptyset), \quad \text{so} \quad C(\emptyset) \neq \emptyset .$$

This fact and condition (iii) imply that $C(\emptyset)$ is a singleton, and we write $C(\emptyset) = \{u\}$. By condition (i) we see that $0 = u'$ is the first element of $\mathcal{S}$. Now we show that $x' = x \rightarrow 0$.

$$x' \in C(x \rightarrow u') \Leftrightarrow C(x, x \rightarrow u') = S, \quad \text{which is true.}$$

$$x \rightarrow u' \in C(x') \Leftrightarrow u' \in C(x', x), \quad \text{which is true.}$$

Finally we have $(x \rightarrow 0) \rightarrow 0 \leq x$ because

$$(x \rightarrow 0) \rightarrow 0 = x'' = x.$$

Theorem 9. Let $\mathcal{S}=(S,\to)$ an algebra of type (2). $\mathcal{S}$ is a Boolean algebra if, and only if, there exists a closure operator C on S such that for all $x,y \in S$:

(i) $x \to y \in C(T) \Leftrightarrow y \in C(T,x)$,

(ii) the closure system $\mathcal{C}$ (associated with C) has a basis of maximal sets,

(iii) there exists an inconsistent element in C (i.e.: an $x \in S$ such that $C(x)=S$);

(iv) $C(x) = C(y) \Rightarrow x=y$

Proof:

a) It is easy to see that the closure operator associated with all implicative filters of a Boolean algebra satisfies conditions (i)-(iv)

b) From Theorem 4 we have that $(S,\to)$ is an implication algebra. Condition (iii) implies that the algebra has a first element, say 0. Finally we have that

$$(x \to 0) \to 0 = \sup(x,0) = x,$$

so as in theorem 8 we have that $(S,\to)$ is a Boolean algebra.

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