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Let (A,m) be a (conmutative, noetherian) local ring. An A-module M is to be said a balanced big Cohen-Macaulay module (b.b.C.M. A-module) if $mM \neq M$ and any system of parameters of A is regular on M (i.e. any s. of p. of A is an M-sequence). Balanced big Cohen-Macaulay modules are a particular class of big Cohen-Macaulay modules and their existence for any equicharacteristic local ring follows from the well known Hochster's construction of a big Cohen-Macaulay module for such a ring (see [2], [5], [7] and [8]).

Big Cohen-Macaulay modules are not, in general, finitely generated, so that their properties related to M-sequences are not the same that for finitely generated modules (see [2] and [5]).

However Sharp in [11] and [12] shows that balanced big Cohen-Macaulay modules verify some of the properties of finitely generated Cohen-Macaulay modules. In particular he proofs that for a catenary local domain balanced big Cohen-Macaulay modules localize and he conjectures that this property holds for any local ring.

The aim of this paper is to give an affirmative answer to the above-mentioned question. In part 1 we generalize Griffith's Proposition 1.4 of [6] (Proposition 1.1). It follows from this result that the

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ring A may be assumed to be complete and so catenary. Finally we see that for any catenary local ring balanced big Cohen-Macaulay modules localize. This is part 2.

1.

Let $A \xrightarrow{f} B$ be an extension of local rings (A,n), (B,m). Suposse that:

(i) Given any family of elements $x_1,\ldots,x_r \in f(A)$ such that they form part of a system of parameters of B there exist $f^{-1}(x_1),\ldots,f^{-1}(x_r)$ such that they form part of a system of parameters of A.

(ii) m is the only prime ideal of B lying over n.

1.1. <u>Proposition</u>. If M is a B-module such that as an A-module is b.b.C.M. then M is a b.b.C.M. B-module.

<u>Proof</u>: by (ii) it is clear that $mM \neq M$. We must show that any system of parameters of B $x_1,\ldots,x_d$ is regular on M. Let r be the number of elements in the system of parameters not belonging to f(A). The proof follows by induction on r.

r=0. By (1) $x_1,\ldots,x_d$ is regular on M.

$r>0$. Assume that $x_1 \notin f(A)$ (if not we may consider an adequate quotient). Take the extension

$$A/(x_2,\ldots,x_d)^c \xrightarrow{\quad f \quad} B/(x_2,\ldots,x_d).$$

$\mathrm{Dim}(B/(x_2,\ldots,x_d)) = 1$ hence $x, x_2,\ldots,x_d$, $x \in B$ is a system of parameters of B if and only if $\bar{x} \notin \bar{P}_1 \cup \ldots \cup \bar{P}_s$, where $\bar{P}_1,\ldots,\bar{P}_s$ are the minimal prime ideals of $B/(X_2,\dots,X_d)$. On the other hand by (ii) we have that $(\bar{x}_1)^c \not\subset \bar{P}_1^c \cup \dots \cup \bar{P}_s^c$ so there exists $y \in A$ such that $f(y) = bx_1$, $b \in B$, and $y \notin \bar{P}_1^c \cup \dots \cup \bar{P}_s^c$. Hence $\overline{f(y)} = \bar{f}(\bar{y}) \notin \bar{P}_1 \cup \dots \cup \bar{P}_s$ and $f(y)$, x_2, $\dots, x_d$ is a system of parameters of B.

By induction $f(y), x_2,\dots,x_d$ is regular on M and so also is x_1.

Now suposse that $\sum_{i=1}^{t>2} x_i m_i = 0$, $m_i \in M$ $\forall i$. Multiplying this equality by b we get that $\sum_{i=1}^{t>2} bx_i m_i = 0$. But $f(y),x_2,\dots,x_d$ is regular on M so $bm_t \in (f(y),x_2,\dots,x_{t-1})M$ and $bm_t = n_1 f(y) + \sum_{i=2}^{t-1} n_i x_i$, $n_i \in M$ $\forall i$. Multiplying this by x_1 we obtain that $f(y)m_t = n_1 x_1 f(y) + \sum_{i=2}^{t-1} n_i x_i x_1$. Since $f(y),x_2,\dots,x_d$ is regular on M we get that $f(y) \notin Z_A(M/(x_2,\dots,x_{t-1})M)$ so $m_t - n_1 x_1 \in (x_2,\dots,x_{t-1})M$ and $m_t \in (x_1,\dots$ $\dots,x_{t-1})M$. Thus, certainly, $x_1,\dots,x_d$ is regular on M and the proof is complete.

1.2. <u>Corollary</u>. Let $A \xrightarrow{f} B$ be a faithfully flat extension of local rings verifying conditions (i) and (ii). Then, if M is a b.b.C.M. A-module $M \otimes_A B$ is a b.b.C.M. B-module.

<u>Proof</u>: take M a b.b.C.M. A-module. Since $A \xrightarrow{f} B$ is a faithfully flat extension $M \otimes_A B$ is a b.b.C.M. A-module, hence by Proposition 1.1. $M \otimes_A B$ is a b.b.C.M. B-module.

1.3. <u>Lemma</u>. If $A \subset B$ is an integral extension of local rings then (i) and (ii) hold. If $A \xrightarrow{f} B$ is a faithfully flat extension of local rings such that dim A = dim B then (i) holds. In particular $A \subset \hat{A}$ and $A \subset A^h$ verify (i) and (ii).

<u>Proof</u>: Let $A \subset B$ be an integral extension and let $x_1,\dots,x_r$ be a family of elements of A such that they form part of a system of parameters of

B, i.e., $\dim(B/(x_1,\ldots,x_r)B) = \dim B-r$. By ([9], Theorem 5), $\dim(A/(x_1,\ldots,x_r)A) = \dim(B/(x_1,\ldots,x_r)B) = \dim B-r = \dim A-r$. Thus $x_1,\ldots,x_r$ form part of a system of parameters of A and condition (i) holds. Since $A\subset B$ is an integral extension condition (ii) holds too.

Now assume that $A \xrightarrow{f} B$ is a faithfully flat extension such that dim A = dim B. Let $x_1,\ldots,x_r$ be a family of elements of f(A) such that they form part of a system of parameters of B, i.e., $\dim(B/(x_1,\ldots,x_r))=$ $= \dim B-r$. Choose $f^{-1}(x_1),\ldots,f^{-1}(x_r)$ and consider $A/(f^{-1}(x_1),\ldots$ $\ldots,f^{-1}(x_r))$. By ([9], Theorem 3 and Theorem 4), $\dim(A/(f^{-1}(x_1),\ldots$ $\ldots,f^{-1}(x_r))\leq \dim(B/(x_1,\ldots,x_r)) = \dim B-r = \dim A-r$. But ever $\dim(A/(f^{-1}(x_1),\ldots,f^{-1}(x_r))\geq \dim A-r$, hence the equality holds and $f^{-1}(x_1),\ldots,f^{-1}(x_r)$ form part of a system of parameters of A.

1.4.<u>Corollary</u>. Let A be a local ring and M be a b.b.C.M. A-module. Then $M \otimes_A \hat{A}$ $(M \otimes_A A^h)$ is a b.b.C.M. $\hat{A}$-module (A^h-module).

1.5.<u>Corollary</u>. Let $A\subset B$ be a finite, free extension of local rings. If M is a b.b.C.M. A-module then $M \otimes_A B$ is a b.b.C.M. B-module.

<u>Remark 1</u>. Corollary 1.5 has been proved by Riley (see [10] (2.2)).

<u>Remark 2</u>. Let A be a local ring and M and A-module. $\hat{M}$, the separated completion of M, is an $\hat{A}$-module. On the other hand if M is a big Cohen-Macaulay module $\hat{M}$ is a b.b.C.M. A-module (see [2], (1.7)). So $\hat{M}$ is also a b.b.C.M. $\hat{A}$-module. However, if M is not finitely generated $\hat{M}$ and $M \otimes_A \hat{A}$ don't coincide necessarily.

If (A,m,k) is a local ring the equality $\operatorname{depth}_A(M)=\inf\{i \mid \operatorname{Ext}_A(k,M)=$ $=\mu^i(m,M)\neq 0\}$ may be taken as a definition of depth for any A-module M, in such a way that if $x_1,\dots,x_r$ is an M-sequence and $m\in \operatorname{Ass}_A(M/(x_1,\dots,x_r)M)$ then $\operatorname{depth}_A(M) = r$ (see [3], part 1). Therefore if M is a big Cohen-Macaulay module and $\dim A = n$ we get that $\operatorname{depth}_A(M) = n$. On the other hand let $\operatorname{supp}_A(M)$ be the set of prime ideals of A such that $\operatorname{depth}_{A_p}(M_p)<\infty$, i.e., the set of prime ideals of A such that $\mu^i(p,M)\neq 0$ for some i (see [4], part 7). For a b.b.C.M. A-module the set $\operatorname{supp}_A(M)$ may be characterized as those prime ideals for wich there exist an M-sequence $x_1,\dots,x_r$ such that $p\in \operatorname{Ass}(M/(x_1,\dots,x_r)M)$, $r=\operatorname{ht}(p)$ (see [11], 3.2).

We want to proof that b.b.C.M. A-modules localize, what means that if M is a b.b.C.M. A-module and $p\in\operatorname{supp}_A(M)$ then M_p is a b.b.C.M. A_p-module.

2.1.<u>Proposition</u>. Let (A,m) be a local ring. Then if b.b.C.M. $\hat{A}$-modules localize also b.b.C.M. A-modules do.

<u>Proof</u>: suppose that M is a b.b.C.M. A-module and take $p\in\operatorname{supp}_A(M)$; then $p\in \operatorname{Ass}_A(M/(x_1,\dots,x_r)M)$ for some M-sequence $x_1,\dots,x_r$, $r=\operatorname{ht}(p)$. But $\operatorname{Ass}_A(M/(x_1,\dots,x_r)M) = \operatorname{Ass}_{\hat{A}}(M\otimes_A\hat{A}/(x_1,\dots,x_r)M\otimes_A\hat{A})^c$ ([9], 9.B, Corollary), so there exists $q\in \operatorname{Ass}_{\hat{A}}(M\otimes_A\hat{A}/(x_1,\dots,x_r)M\otimes_A\hat{A})$ such that $q^c = p$; moreover $q\in\operatorname{supp}_{\hat{A}}(M\otimes_A\hat{A})$ and $\operatorname{ht}(p) = \operatorname{ht}(q) = r$ since $M\otimes_A\hat{A}$ is a b.b.C.M $\hat{A}$-module and $x_1,\dots,x_r$ is also an $M\otimes_A\hat{A}$ -sequence.

It is easy to see that the extension $A\longrightarrow\hat{A}$ is faithfully flat and that every system of parameters of A_p is a system of parameters of

$\hat{A}_q$. Thus if b.b.C.M. $\hat{A}$-modules localize given that $M_p \otimes_{A_p} \hat{A}_q = (M \otimes_A \hat{A})_q$ we have that M_p is a b.b.C.M. A-module.

Any complete ring is catenary, thus we may assume that A is catenary.

2.2.Lemma. Let (A,m) be a catenary, local ring and $p \in \mathrm{Spec}(A)$ such that $\dim A = \dim A_p + \dim A/p$. Assume that $x_1,\ldots,x_r \in A$ is a system of parameters of A_p. Then there exists $y \in A$ such that $y \in x_1 A_p$, y forms part of a system of parameters of A and $y,x_2,\ldots,x_r$ is a system of paramters of A_p.

Proof: We are looking for $y \in I = (X_1 A_p) \cap A$ such that (1) $y \notin q$ $\forall q \in \mathrm{Spec}(A)$ with $\dim A/q = \dim A$ (i.e. y forms part of a system of parameters of A) and (2) $y \notin q$ $\forall q \in \mathrm{Spec}(A)$ such that $(x_2,\ldots,x_r) \subset q \subset p$, $q \neq p$ (i.e. $y,x_2,\ldots,x_r$ is a system of parameters of A_p). Both conditions (1) and (2) concern to a finite number of prime ideals, namely $q_1,\ldots,q_s$, and we seek that $I \not\subset q_1 \cup \ldots \cup q_s$, so it's sufficient to show that $I \not\subset q_i$ $\forall i$.

Let q be a prime ideal such that $\dim(A/q) = \dim A$. A is catenary, hence if $q \subset p$ we get that $ht(p/q) = \dim A_p$. So $x_1 \notin q$ and therefore $I \not\subset q$. Assume now that $q \not\subset p$ and take $B = A/(x_1)$; if $x_1 \in q$ then q is a minimal prime ideal of B, hence B_q is an artinian local ring. Thus for $t \in q$, $t \notin p$ there exist n such that $An_B(\bar{t}^n) \not\subset q$ and, consequently, $((X_1):t^n) \not\subset q$. But $t^n \notin p$ and $I = \bigcup_{s \notin p} ((x_1):s)$. Therefore $I \not\subset p$.

Finally if $q \in \mathrm{Spec}\, A$ is such that $(x_2,\ldots,x_r) \subset q \subset p$ and $q \neq p$ given that $(x_1,\ldots,x_r)$ is a system of parameters of A_p and $(x_1,\ldots,x_r) \subset \subset (I,x_2,\ldots,x_r)$ it's clear that $I \not\subset p$.

2.3.<u>Proposition</u>. Let (A,m) be a catenary, local ring. Let M be a b.b.C.M. A-module and $p \in \mathrm{supp}_A(M)$. Then M_p is a b.b.C.M. A_p-module.

<u>Proof</u>: first we must show that $pM_p \neq M_p$. By ([11], (3.5)) we have that $\dim A = ht(p)+\dim A/p$, so we may find a chain of prime ideals $p_o \subset \ldots$ $\ldots \subset p_r = p \subset \ldots \subset p_n = m$, $r=ht(p)$, $n=\dim A$. Then use ([1] (1.11)) to choose $x_1,\ldots,x_r \in A$ such that $x_1,\ldots,x_r$ is a system of parameters of A_p and $x_1,\ldots,x_r$ form part of a system of parameters of A. Given that M is a b.b.C.M. A-module we get that $x_1,\ldots,x_r$ is an M-sequence, hence $p \in \mathrm{Ass}_A(M/(x_1,\ldots,x_r)M)$. In particular $M_p/(x_1,\ldots,x_r)M_p \neq 0$ and $pM_p \neq M_p$.

Assume now that $x_1,\ldots,x_r$ is a system of parameters of A_p. We must show that $x_1,\ldots,x_r$ is regular on M_p thus we may suppose that $x_1,\ldots$ $\ldots,x_r \in A$. Let k be the greatest number such that $x_1,\ldots,x_k$ form part of a system of parameters of A. The proof follows by induction on $s=r-k$.

$s=0$. Then $x_1,\ldots,x_r$ is a system of parameters of A so $x_1,\ldots,x_r$ is regular on M and also regular on M_p.

$s>0$. Taking an adequate quotient and using ([11], (2.3)) we may assume that $n=r$. By Lemma 2.2 there exists $y \in A$ such that $y \in x_1A_p$, y forms part of a system of parameters of A and $y,x_2,\ldots,x_r$ is a system of parameters of A_p. By induction $y,x_2,\ldots,x_r$ is regular on M_p so, similarly to Proposition 1.1 we get that $x_1,x_2,\ldots,x_r$ is regular on M_p.

2.4.<u>Corollary</u>. Let (A,m) be a local ring. Let M be a b.b.C.M. A-module and $p \in \mathrm{supp}_A(M)$. Then M_p is a b.b.C.M. A_p-module.

<u>Proof</u>: Use Propositions 2.1. and 2.3.

Remark 1. Proposition 2.3 has been proved by Foxby (private comunication).

Remark 2. Proposition 2.3 has been proved by Sharp when in adition A is a domain ([12], (4.3)).

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