CAIXA 31.19

UNIVERSITAT DE BARCELONA
FACULTAT DE MATEMÀTIQUES

LOCALIZATION OF FIBRATIONS WITH NILPOTENT FIBRE

by

Irene Llerena

BIBLIOTECA DE LA UNIVERSITAT DE BARCELONA
0701570606

PRE–PRINT N.º 22
març 1984

LOCALIZATION OF FIBRATIONS WITH NILPOTENT FIBRE

by

Irene Llerena

§1.- Introduction

The theory of localization for nilpotent spaces, parallel to that for nilpotent groups, has proved to be a powerful tool in algebraic topology. Recently, P. Hilton has developed a localization theory for relative nilpotent groups [5] and for nilpotent crossed-modules [6] which represents a first step in the construction of a localization theory for nilpotent fibrations and for fibrations with nilpotent fibre (weak nilpotent fibrations). A fibre map $f:E \longrightarrow B$ is said to be nilpotent if $\pi_1 E$ acts nilpotently on the homotopy groups of the fibre. Nilpotent fibre maps turn out to be the right relativization of nilpotent spaces, because their Moore-Postnikov systems admit principal refinements [7]. In [8] we constructed a localization theory for nilpotent fibrations. In this note we consider the more general situation of fibrations with nilpotent fibre and develop a localization theory for such fibrations.

The paper is organized as follows: in §2 we recall some basic facts and definitions; in §3 we define a P-localization for fibrations with nilpotent fibre and prove its existence; the proof of the expected universal property justifying the term P-localization is divided into two parts: in §4-5 we prove such a universal property for fibrations with fibre an Eilenberg-MacLane space of type (G,1) and in §6 for the general case. As a consequence we obtain the uniqueness up to a homotopy equivalence of the P-localization of a fibration with nilpotent fibre.

I am indebted to Peter Hilton for suggesting me this work and spending his time in helpful conversations with me.

UNIVERSITAT DE BARCELONA CRAI

§2.- Some basic definitions and results

We work in the pointed homotopy category of path-connected spaces having the homotopy type of a CW-complex. A space is nilpotent if $\pi_1 X$ is nilpotent and acts nilpotently on $\pi_n X$ for all $n \geqslant 2$. A fibre map $f:E \longrightarrow B$ is nilpotent if $\pi_1 E$ acts nilpotently on the homotopy groups of the fibre. Nilpotent spaces as well as nilpotent fibrations are characterized by the fact that their Postnikov decompositions admit principal refinements [7;II.2.4].

Let P be a given arbitrary collection of rational primes. A space is P-local if all its homotopy groups are P-local. A fibre map is P-local if its fibre is P-local. By a theory of P-localization in a category with P-local objects we mean the following: For every object X there is a P-local object Y and a morphism $f:X \longrightarrow Y$ satisfying the following universal property: for every P-local object Z,

$$f^*: [Y,Z] \cong [X,Z],$$

where $[A,B]$ stands for the set of morphisms from A to B. A theory of P-localization for nilpotent groups and spaces is developed in detail in [7]. A comprehensive treatment of a process more general than localization but executed in the semi-simplicial category is given in [2].

A crossed-module is a right exact sequence of groups $N \xrightarrow{\rho} G \xrightarrow{\kappa} Q$ together with an action of G on N satisfying i) $\rho(x.a)=x(\rho a)x^{-1}$, ii) $\rho(a).b=aba^{-1}$, $a,b \in N$, $x \in G$. We refer to it briefly as the crossed module κ.

If N is nilpotent and $e:N \longrightarrow N_P$ is a P-localization there always exists a crossed-module $N_P \longrightarrow G_{(P)} \twoheadrightarrow Q$ and a morphism from κ to it inducing e; this gives a P-localization theory in the category of crossed-modules κ with N nilpotent [6].

Let G be a non abelian group and let $K(G,1)$ be an Eilenberg-MacLane space of type $(G,1)$. There is a universal classifying fibration, hereafter referred to as $K(G,1) \longrightarrow E_G \xrightarrow{q} B$, where E_G is a $K(\mathrm{Aut}\, G,1)$, B is B_H with H the H-space of homotopy equivalences of $K(G,1)$ and the homotopy sequence for q reduces to the natural one $ZG \rightarrowtail G \longrightarrow \mathrm{Aut}\, G \twoheadrightarrow \mathrm{Out}\, G$ [3].

Now let G be an abelian group, Q a CW-complex of type $K(\mathrm{Aut}\ G,1)$ and Q^* the universal cover of Q. Let

$$\hat{K}(G,n) = \frac{K(G,n) \times Q^*}{\mathrm{Aut}\ G}$$

be the quotient of $K(G,n) \times Q^*$ by the diagonal action of Aut G. The CW-complex $\hat{K}(G,n)$ is a classifying space for $K(G,n-1)$-fibrations and

$$K(G,n-1) \longrightarrow P \xrightarrow{p} \hat{K}(G,n)$$

is a universal fibration, where P is the space of unbased paths in $\hat{K}(G,n)$ which have initial point in Q ($Q \hookrightarrow \hat{K}(G,n)$ is induced by the canonical map $Q^* \longrightarrow K(G,n) \times Q^*$) and which lie entirely within some fibre of $q:\hat{K}(G,n) \longrightarrow Q$ (induced by $K(G,n)\times Q^* \longrightarrow Q^*$) [10].

Let G be a local coefficient system on X and $e:X \longrightarrow Q$ a map inducing $\epsilon:\pi_1 X \longrightarrow \pi_1 Q = \mathrm{Aut}\ G$, where ϵ is the action of $\pi_1 X$ on the local system G. If $[X,\hat{K}(G,n)]_e$ denotes the set of fibrewise homotopy classes of maps $\phi:X \longrightarrow \hat{K}(G,n)$ satisfying $q\phi=e$, we have

$$H^n(X;G) = [X,\hat{K}(G,n)]_e.$$

The results on homotopy pullbacks we need are all included in [9]. In particular, a topological pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

where f is a fibration is a homotopy pullback. Homotopy pullbacks satisfy the pullback property, i.e. for every space X and any maps $u:X \longrightarrow B$, $v:X \longrightarrow A$ such that $gu \simeq fv$, there is a map ϕ such that the diagram

$$\begin{tikzcd} X \arrow[dr, "\phi"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & P \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

is homotopy commutative and "essentially unique".

§3.- Existence of a P-localization

All spaces are (pointed) path-connected and have the homotopy type of a CW-complex. If $f:E \longrightarrow B$ is a fibration with nilpotent fibre F and P is an arbitrary collection of rational primes, we say that f is P-local if F is P-local [7].

3.1. Definition. *Let $f:F \longrightarrow B$ be a fibration with nilpotent fibre F. A map β from f to a P-local fibration $f_{(P)}:E_{(P)} \longrightarrow B$ is a P-localization if it induces a P-localization on the fibres*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "e"'] & E \arrow[r, "f"] \arrow[d, "\beta"'] & B \arrow[d, equal] \\ F_P \arrow[r] & E_{(P)} \arrow[r, "f_{(P)}"] & B \end{tikzcd}$$

The following theorem asserts the existence of a P-localization for fibrations with nilpotent fibre.

3.2. Theorem. *Every fibration $f:E \longrightarrow B$ with nilpotent fibre F admits a P-localization.*

Proof. We argue step-wise on the Moore-Postnikov factorization of f

$$E \to \dots \to E_n \xrightarrow{p_{n-1}} E_{n-1} \to \dots \to E_1 \xrightarrow{p_1} B,$$

where each p_n is a fibration with fibre $K(\pi_n F,n)$.

First step: Let $G=\pi_1 F$. p_1 is induced from a universal fibration $K(G,1) \longrightarrow E_G \longrightarrow B_H$ by a map $\phi:B \longrightarrow B_H$.

We know that every automorphism $\psi:G \longrightarrow G$ determines uniquely an automorphism $\psi_P:G_P \longrightarrow G_P$ such that $e\psi = \psi_P e$, where $e:G \longrightarrow G_P$ is a P-localization. Hence if H and $\bar{H}$ are the H-spaces of self homotopy equivalences of $K(G,1)$ and $K(G_P,1)$ respectively, by Lemma 3.3 of [4] applied to the pair (B_H,E_G), there is a map between the corresponding universal fibrations

$$\begin{tikzcd} K(G,1) \arrow[r] \arrow[d] & E_G \arrow[r] \arrow[d] & B_H \arrow[d, "\alpha"] \\ K(G_P,1) \arrow[r] & E_{G_P} \arrow[r] & B_{\bar{H}} \end{tikzcd}$$

such that on the fibres is a P-localization.

Now we can define $p_{1(P)}:E_{1(P)} \longrightarrow B$ as the fibration induced from $K(G_P,1) \longrightarrow E_{G_P} \longrightarrow B_{\bar{H}}$ by $\alpha\phi: B \longrightarrow B_{\bar{H}}$. The universal property of the pullback allows us to define a map $\beta_1: E_1 \longrightarrow E_{1(P)}$ over B inducing a P-localization on the fibres.

n^{th} step ($n \geq 2$): Assume we have constructed

$$\begin{tikzcd} E \arrow[r] & \cdots \arrow[r] & E_n \arrow[r, "p_n"] & E_{n-1} \arrow[r, "p_{n-1}"] \arrow[d, "\beta_{n-1}"] & E_{n-2} \arrow[r, "p_{n-2}"] \arrow[d, "\beta_{n-2}"] & \cdots \arrow[r] & E_1 \arrow[r, "p_1"] \arrow[d, "\beta_1"] & B \arrow[d, equal] \\ & & & E_{n-1(P)} \arrow[r, "p_{n-1(P)}"] & E_{n-2(P)} \arrow[r] & \cdots \arrow[r] & E_{1(P)} \arrow[r] & B \end{tikzcd}$$

such that the map of fibrations $p_i \longrightarrow p_{i(P)}$, $i \leq n-1$, induces a P-localization on the fibres. In particular, the fibre of $p_{i(P)}$ is $K(\pi_i F_P, i)$.

Note that if F^i, $\bar{F}^i$ are the fibres of $E_i \longrightarrow B$, $E_{i(P)} \longrightarrow B$ respectively, the square of fibrations

$$\begin{tikzcd} E_{i+1} \arrow[r, "p_{i+1}"] \arrow[d, "\beta_{i+1}"'] & E_i \arrow[d, "\beta_i"] \\ E_{i+1(P)} \arrow[r, "p_{i+1(P)}"] & E_{i(P)} \end{tikzcd}$$

induces on the fibres a diagram

$$\begin{tikzcd} K(\pi_{i+1}F, i+1) \arrow[r] \arrow[d] & F^{i+1} \arrow[r] \arrow[d] & F^i \arrow[d] \\ K(\pi_{i+1}F_P, i+1) \arrow[r] & \bar{F}^{i+1} \arrow[r] & \bar{F}^i \end{tikzcd}$$

and, arguing by induction, it turns out that $F^{i+1} \longrightarrow \bar{F}^{i+1}$ is a P-localization.

Let $\rho: P \longrightarrow \hat{K}(\pi_n F, n+1)$ and $\bar{\rho}: \bar{P} \longrightarrow \hat{K}(\pi_n F_P, n+1)$ be the universal fibrations for $K(\pi_n F, n)$- and $K(\pi_n F_P, n)$-fibrations respectively (see §2) and let $\phi^n: E_{n-1} \longrightarrow \hat{K}(\pi_n F, n+1)$ the map which classifies p_n. We define $p_{n(P)}: E_{n(P)} \longrightarrow E_{n-1(P)}$ as the fibration induced from $\bar{\rho}$ by a map

$$\psi^n: E_{n-1(P)} \longrightarrow \hat{K}(\pi_n F_P, n+1)$$

such that
$$\beta^*_{n-1}(\psi^n) = e_*(\phi^n),$$

where

$$H^n(E_{n-1(P)},\pi_n F_P) \xrightarrow{\beta^*_{n-1}} H^n(E_{n-1},\pi_n F_P) \xrightarrow{e_*} H^n(E_{n-1},\pi_n F)$$

are induced by β_{n-1} and e respectively. The existence of a ψ^n uniquely determined up to homotopy follows from Lemma 3.3 below. The desired map $\beta_n: E_n \longrightarrow E_{n(P)}$ over B inducing P-localization on the fibres follows then from the universal property of the pullback.

3.3. **Lemma**. *Let β be a map of fibrations*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "e"'] & E \arrow[r, "f"] \arrow[d, "\beta"] & B \arrow[d, equal] \\ F_P \arrow[r] & E_{(P)} \arrow[r, "f_{(P)}"] & B \end{tikzcd}$$

inducing a P-localization on the fibres. Let A be a local coefficient system on $E_{(P)}$ with P-local abstract group, such that the induced action of $\pi_1 F_P$ on A is nilpotent. Then

$$\beta^*:\ H^n(E_{(P)};A) \longrightarrow H^n(E;A)$$

is an isomorphism for all $n\geq 0$.

Proof. A simple generalization of the algebraic case [7;I.4.14] to spaces assures that for nilpotent X acting nilpotently on the P-local group A

$$e^*:\ H^n(X_P;A) \cong H^n(X;A) \quad \text{for all } n\geq 0.$$

Hence $e: H^n(F_P;A) \cong H^n(F;A)$ and therefore the homomorphism induced by β on the Serre spectral sequence with local coefficients

$$\begin{tikzcd} H^p(B;H^q(F_P;A)) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(E_{(P)};A) \arrow[d] \\ H^p(B;H^q(F;A)) \arrow[r, Rightarrow] & H^{p+q}(E;A) \end{tikzcd}$$

is an isomorphism at the E_2-level. Thus the homomorphism $\beta^*: H^*(E_{(P)};A) \longrightarrow H^*(E;A)$ is an isomorphism and the proof of Lemma 3.3 is completed.

If the Moore-Postnikov factorization of $f:E \longrightarrow B$ is finite, the above procedure yields the desired P-localization. In the general case let $E_{(P)}$ the geometric realization of the singular complex of $\varprojlim E_{n(P)}$. Then there is a map $\beta: E \longrightarrow E_{(P)}$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & \varprojlim E_n \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle \varprojlim \beta_n} \\ E_{(P)} & \longrightarrow & \varprojlim E_{n(P)} \end{array}$$

is homotopy commutative. Recall [2;p.254] that there exists a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 \pi_{i+1} E_n \longrightarrow \pi_i(\varprojlim E_n) \longrightarrow \varprojlim \pi_i E_n \longrightarrow 0.$$

But in our case the $\varprojlim^1$-term vanishes because $\pi_{i+1}(p_n)$ is an isomorphism for all $n \neq i, i+1$ and an epimorphism for $n = i+1$. Therefore $\varphi: E \longrightarrow \varprojlim E_n$ is a weak homotopy equivalence.

Analogously it turns out that

$$\pi_i E_{(P)} \cong \pi_i(\varprojlim E_{n(P)}) \cong \varprojlim \pi_i E_{n(P)} \cong \pi_i E_{N(P)}$$

for N large $(N \geq i+1)$.

Consider now the diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{f} & B \\ \downarrow & & {\scriptstyle\beta}\downarrow & & \| \\ \bar{F} & \longrightarrow & E_{(P)} & \xrightarrow{f_{(P)}} & B \end{array},$$

where $f_{(P)}$ is induced by the natural map $\varprojlim E_{n(P)} \longrightarrow B$ and $\bar{F}$ is the homotopy fibre of $f_{(P)}$. The commutative square over B

$$\begin{array}{ccc} E & \longrightarrow & E_n \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta_n} \\ E_{(P)} & \longrightarrow & E_{n(P)} \end{array}$$

induces on the homotopy fibres a square

$$\begin{array}{ccc} F & \longrightarrow & F^n \\ \downarrow & & \downarrow \\ \bar{F} & \longrightarrow & \bar{F}^n \end{array},$$

where $F^n \longrightarrow \bar{F}^n$ is a P-localization.

Given an integer i we can take n large enough such that the horizontal arrows induce isomorphisms on the homotopy groups.

Hence $\pi_i F \longrightarrow \pi_i \bar{F}$ is a P-localization for all i and therefore $F \longrightarrow \bar{F}$ is a P-localization. This completes the proof of Theorem 3.2.

§4.- A universal property for fibrations with fibre a K(,1)

In this paragraph we want to prove a special case of the main theorem 6.1.

4.1. Theorem. *Let $f:E \longrightarrow B$ be a fibration with nilpotent fibre $K(G,1)$ and let $\beta : f \longrightarrow f_{(P)}$ be a P-localization. For every map $\tau : f \longrightarrow \bar{f}$ from f to a P-local fibration $\bar{f}:\bar{E} \longrightarrow B$ with nilpotent fibre $K(\bar{G},1)$ there exists a map $\delta : f_{(P)} \longrightarrow \bar{f}$ uniquely determined up to homotopy over B such that $\delta\beta \simeq \tau$ over B.*

Proof. Let $K(\bar{G},1) \longrightarrow E_{\bar{G}} \xrightarrow{q} \bar{B}$ be the universal fibration for $K(\bar{G},1)$-fibrations and let $\alpha : B \longrightarrow \bar{B}$ be the classifying map corresponding to $\bar{f}$. Look at the diagram

$$\begin{tikzcd}
K(G,1) \arrow[r] \arrow[d] \arrow[dd, bend right] & E \arrow[r, "f"] \arrow[d, "\beta"] \arrow[dd, bend right, "\tau"'] & B \arrow[d, equal] \\
K(G_P,1) \arrow[r] & E_{(P)} \arrow[r, "f_{(P)}"] \arrow[d, dashed, "\delta"] \arrow[dd, dashed, bend left, "\varphi"] & B \arrow[d, equal] \\
K(\bar{G},1) \arrow[r] \arrow[d, equal] & \bar{E} \arrow[r, "\bar{f}"] \arrow[d, "\rho"'] & B \arrow[d, "\alpha"] \\
K(\bar{G},1) \arrow[r] & E_{\bar{G}} \arrow[r, "q"] & \bar{B}
\end{tikzcd}$$

We shall construct a map $\varphi : E_{(P)} \longrightarrow E_{\bar{G}}$ with $q\varphi = \alpha f_{(P)}$ so that it induces a map $\delta : E_{(P)} \longrightarrow \bar{E}$ satisfying the desired conditions by applying Lemma 4.2 below which generalizes Lemma 3.3 of [4].

We use the following notation: Z_f, $Z_{f_{(P)}}$, Z_q denote the mapping cylinders of $f, f_{(P)}, q$ respectively, $f=hf'$, $f_{(P)}=h'f'_{(P)}$, $q=\bar{h}q'$ the natural decompositions into a cofibration and a homotopy equivalence and $j:B \longrightarrow Z_f$, $j':B \longrightarrow Z_{f_{(P)}}$, $\bar{j}:\bar{B} \longrightarrow Z_q$ the canonical imbeddings.

4.2. Lemma. *Let (X,A) be a 1-connected CW-pair, where X is path-connected and let $B \overset{i}{\hookrightarrow} X$ a sub-CW-complex with $A \cap B = \emptyset$ which is a deformation retract of X. Let $\chi : \pi_1 A \longrightarrow \mathrm{Aut}\,\bar G$ and $\nu : \pi_2(X,A) \longrightarrow \bar G$ be homomorphisms rendering commutative the diagram*

$$\begin{tikzcd}
\pi_2(X,A) \arrow[r] \arrow[d, "\nu"'] & \pi_1 A \arrow[d, "\chi"] \\
\bar G \arrow[r] & \mathrm{Aut}\,\bar G.
\end{tikzcd}$$

Let $\sigma : \pi_2 X \longrightarrow Z\bar G$ and $\psi : \pi_1 X \longrightarrow \mathrm{Out}\,\bar G$ be the homomorphisms induced by ν and χ. If $\alpha : B \longrightarrow \bar B$ is a map which induces ψ and σ on π_1 and π_2 respectively, then there exists a unique homotopy class over α of maps

$$\varphi :\ (X,A) \longrightarrow (Z_q, E_{\bar G})$$

such that $\chi = (\varphi|A)_* : \pi_1 A \longrightarrow \pi_1 E_{\bar G}$ and $\nu = \varphi_* : \pi_2(X,A) \longrightarrow \pi_2(Z_q, E_{\bar G})$.

The proof of the Lemma is given in §5.

Take as (X,A) the pair $(Z_{f_{(P)}}, E_{(P)})$ and as $\alpha : B \longrightarrow \bar B$ the classifying map of $\bar f$ and consider the diagram of homotopy exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi_2 E \arrow[r] \arrow[d] \arrow[dd, bend right] & \pi_2 B \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, "e"] \arrow[dd, bend right] & \pi_1 E \arrow[r] \arrow[d, "\beta_*"] \arrow[dd, bend right] & \pi_1 B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_2 E_{(P)} \arrow[r] \arrow[d, dashed] & \pi_2 B \arrow[r] \arrow[d, equal] & G_P \arrow[r, "\tau_*"] \arrow[d, dashed, "\nu"] & \pi_1 E_{(P)} \arrow[r] \arrow[d, dashed, "\theta"] & \pi_1 B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_2 \bar E \arrow[r] \arrow[d] & \pi_2 B \arrow[r] \arrow[d] & \bar G \arrow[r] \arrow[d, equal] & \pi_1 \bar E \arrow[r] \arrow[d, "\rho_*"] & \pi_1 B \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & Z\bar G \arrow[r] & \bar G \arrow[r] & \mathrm{Aut}\,\bar G \arrow[r] & \mathrm{Out}\,\bar G \arrow[r] & 0
\end{tikzcd}$$

Since $\bar G$ is a P-local group the map $G \longrightarrow \bar G$ factors through the P-localization $e : G \longrightarrow G_P$ making the corresponding square commutative. The existence (and uniqueness) of a homomorphism $\theta : \pi_1 E_{(P)} \longrightarrow \pi_1 \bar E$ which makes the diagram commutative follows from the fact that $\bar G \longrightarrow \pi_1 \bar E \twoheadrightarrow \pi_1 B$ is a P-local crossed-module and

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "e"'] & \pi_1 E \arrow[r, two heads] \arrow[d, "\beta_*"] & \pi_1 B \arrow[d, equal] \\
G_P \arrow[r] & \pi_1 E_{(P)} \arrow[r, two heads] & \pi_1 B
\end{tikzcd}$$

is a P-localization of crossed-modules [6].

UNIVERSITAT DE BARCELONA CRAI

Consider now $\chi = \rho_* \theta$ and $\nu : \pi_2(Z_{f_{(P)}}, E_{(P)}) = G_P \longrightarrow \bar{G}$. Applying Lemma 4.2 it follows that there exists a unique map

$$\bar{\varphi} : (Z_{f_{(P)}}, E_{(P)}) \longrightarrow (Z_q, E_{\bar{G}})$$

which extends α and induces χ and ν. The restriction $\bar{\varphi}_1 = \bar{\varphi} | E_{(P)}$ then satisfies

$$q\bar{\varphi}_1 = \bar{h}q'\bar{\varphi}_1 = \bar{h}\bar{\varphi}f'_{(P)} \simeq \bar{h}\bar{\varphi}j'h'f'_{(P)} = \alpha h'f'_{(P)} = \alpha f_{(P)} .$$

But, since q is a fibre map, there exists a map $\varphi_1 : E_{(P)} \longrightarrow E_{\bar{G}}$ homotopic to $\bar{\varphi}_1$ such that

$$q\varphi_1 = \alpha f_{(P)} ,$$

which can be extended to a map

$$\varphi : (Z_{f_{(P)}}, E_{(P)}) \longrightarrow (Z_q, E_{\bar{G}})$$

by

$$\begin{aligned} \varphi(x,t) &= (\varphi_1 x, t) && \text{if } (x,t) \in E_{(P)} \times I \\ \varphi(b) &= \alpha(b) && \text{if } b \in B. \end{aligned}$$

(Observe that φ satisfies the conditions in Lemma 4.2 and moreover $\bar{h}\varphi = \alpha h'$.)

Look now at the diagram

$$\begin{tikzcd}
E \arrow[r, "f'"] \arrow[d, "\beta"] \arrow[dd, bend right=60, "\rho\tau"'] & Z_f \arrow[d, "\eta"] \arrow[dd, bend left=60, "\gamma"] \\
E_{(P)} \arrow[r, "f'_{(P)}"] \arrow[d, "\varphi_1"] & Z_{f_{(P)}} \arrow[d, "\varphi"] \\
E_{\bar{G}} \arrow[r, "q'"] & Z_q
\end{tikzcd}$$

where η and γ are defined as follows:

$$\begin{aligned} \eta(x,t) &= (\beta x, t) && \text{if } (x,t) \in E \times I \\ \eta(b) &= b && \text{if } b \in B, \end{aligned}$$

$$\begin{aligned} \gamma(x,t) &= (\rho\tau x, t) && \text{if } (x,t) \in E \times I \\ \gamma(b) &= \alpha(b) && \text{if } b \in B. \end{aligned}$$

Since the maps γ and $\varphi\eta$ both induce the same homomorphisms $\chi\beta_* = \rho_*\tau_* : \pi_1 E \longrightarrow \text{Aut}\ \bar{G}$, $\nu e : \pi_2(Z_f, E) = G \longrightarrow \bar{G}$ and both extend the map α , by Lemma 4.2 they are homotopic over α; in other words, there is a homotopy

$$H:\ (Z_f\times I, E\times I) \longrightarrow (Z_q, E_{\bar G})$$

such that $H_t j = \bar{j}\alpha$ for every t.

Since

$$\begin{tikzcd} \bar{E} \arrow[r, "\bar{f}"] \arrow[d, "\rho"'] & B \arrow[d, "\alpha"] \\ E_{\bar G} \arrow[r, "q"] & \bar{B} \end{tikzcd}$$

is a pullback, φ_1 and $f_{(P)}$ induce a map $\delta: E_{(P)} \longrightarrow \bar{E}$ such that $\bar{f}\delta = f_{(P)}$ and $\rho\delta = \varphi_1$ (see the diagram at the beginning of §4). But this pullback is also a homotopy pullback because q is a fibre map. Consider now the diagram

$$\begin{tikzcd} E \arrow[dr, "\tau"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\rho\tau"'] & & \\ & \bar{E} \arrow[r, "\bar{f}"] \arrow[d, "\rho"] & B \arrow[d, "\alpha"] \\ & E_{\bar G} \arrow[r, "q"] & \bar{B} \end{tikzcd} \qquad \begin{tikzcd} E \arrow[dr, "\delta\beta"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\rho\tau"'] & & \\ & \bar{E} \arrow[r, "\bar{f}"] \arrow[d, "\rho"] \arrow[dl, phantom, "M"] & B \arrow[d, "\alpha"] \\ & E_{\bar G} \arrow[r, "q"] & \bar{B} \end{tikzcd}$$

where all possible squares and triangles are commutative except that $\rho\delta\beta = \varphi_1\beta \simeq \rho\tau$. Let M denote the restriction of H to ExI, which satisfies $qM \sim \alpha f$; i.e. there exists a map

$$G:\ (E\times I)\times I \longrightarrow \bar{B}$$
$$(x,s,r) \longmapsto \bar{h}H((x,r),s)$$

such that

$$\begin{aligned} G(x,s,0) &= \bar{h}H((x,0),s) = \bar{h}H_s jf(x) = \bar{h}\bar{j}\alpha f(x) = \alpha f(x) \\ G(x,s,1) &= \bar{h}H((x,1),s) = qM(x,s) \\ G(x,0,r) &= \bar{h}H((x,r),0) = \bar{h}\gamma(x,r) = \alpha f(x) \\ G(x,1,r) &= \bar{h}H((x,r),1) = \bar{h}\bar{j}\alpha f(x) = \alpha f(x). \end{aligned}$$

This means that the second diagram is homotopy commutative and therefore, by the universal property of homotopy pullbacks [9], $\tau \simeq \delta\beta$ by a homotopy $N: E\times I \longrightarrow E$ such that $\bar{f}N \sim f$. But, since $\bar{f}$ is a fibre map, there exists a homotopy $\bar{N}$ from τ to $\delta\beta$ such that $\bar{f}\bar{N} = f$. Hence τ and $\delta\beta$ are fibrewise homotopic and the first part of Theorem 4.1 is proved.

In order to show the uniqueness of δ suppose that $\delta': f_{(P)} \longrightarrow f$ also satisfies the condition in Theorem 4.1 and extend $\rho\delta'$ to a map $\varphi': (Z_{f_{(P)}}, E_{(P)}) \longrightarrow (Z_q, E_{\bar G})$ over α. By the uniqueness in Lemma 4.2, φ and φ' are homotopic over α and arguing as before it follows that δ and δ' are fibrewise homotopic.

§5.- Proof of Lemma 4.2

Let X^*, Z_q^* be the universal cover of X, Z_q respectively and let A^*, $E_{\bar{G}}^*$, B^*, $\bar{B}^*$ the restrictions to A, $E_{\bar{G}}$, B, $\bar{B}$ respectively. The homotopy classes of maps $(X,A) \longrightarrow (Z_q, E_{\bar{G}})$ which induce $\psi:\pi_1 X \longrightarrow \pi_1 Z_q = \text{Out}\,\bar{G}$ correspond one to one to the based ψ-equivariant homotopy classes of maps $(X^*,A^*) \longrightarrow (Z_q^*, E_{\bar{G}}^*)$. Therefore it suffices to show that there exists a unique ψ-equivariant homotopy class inducing

$$\nu:\pi_2(X,A) = \pi_2(X^*,A^*) \longrightarrow \bar{G} = \pi_2(Z_q^*,E_{\bar{G}}^*) = \pi_2(Z_q,E_{\bar{G}})$$

and

$$\tau:\ \chi|\pi_1 A^*:\ \pi_1 A^* \longrightarrow \text{In}\,\bar{G} = \pi_1 E_{\bar{G}}^*.$$

B^* and $\bar{B}^*$ are the universal covers of B and $\bar{B}$, hence we can lift $\alpha:B \longrightarrow \bar{B}$ to a map $\alpha^*:B^* \longrightarrow \bar{B}^*$. Let $\varphi_1:A \longrightarrow E_{\bar{G}}$ be a map inducing χ on the fundamental groups and let φ_1^* be a lifting of φ_1. B^* is a deformation retract of X^* and the imbedding and retraction $B^* \underset{r^*}{\overset{i^*}{\rightleftarrows}} X^*$ are lifting maps of the imbedding and retraction $B \underset{r}{\overset{i}{\rightleftarrows}} X$ respectively. Analogously, $\bar{B}^* \underset{\bar{h}^*}{\overset{\bar{j}^*}{\rightleftarrows}} Z_q^*$ are an imbedding and a retraction which lift $\bar{B} \underset{\bar{h}}{\overset{\bar{j}}{\rightleftarrows}} Z_q$.

Now define

$$g:\ X^* \longrightarrow Z_q^*$$

by

$$g = \bar{j}^*\alpha^*r^*.$$

Then, by Lemma 5.1. of [4], in the diagram

$$\begin{array}{ccc} A^* & \xrightarrow{\ c\ } & X^* \\ {\scriptstyle \varphi_1^*}\downarrow & & \downarrow{\scriptstyle g} \\ E_{\bar{G}}^* & \xrightarrow{\ q'^*\ } & Z_q^* \end{array}$$

we have $gc = q'^*\varphi_1^*$. Since c is a cofibration there exists a a map $g_1:X^* \longrightarrow Z^*_q$ homotopic to g which makes the diagram commutative and such that the restriction to B^* is equal to α. Therefore

$$g_1 c = q'^* \varphi_1^* \quad \text{and} \quad g_1 i^* = \bar{j}^* \alpha^* .$$

g_1 coincides with φ_1^* and α^* on A^* and B^* respectively and thus it is a ψ-equivariant map. From the facts that $\pi_1 X$ acts freely on the cells of X^*-A^* (we can assume that (X,A) has no relative cells of dimensions <2, since (X,A) is 1-connected) and that $\nu:\pi_2(X^*,A^*)=\pi_2(X,A) \longrightarrow \bar{G}=\pi_2(Z^*_q,E^*_{\bar{G}})$ is ψ-equivariant we can construct, cell by cell, a homotopy relative to $A^*\cup B^*$ from g_1 to an ψ-equivariant $\bar{\varphi}$.

If $\bar{\varphi}_1:(X^*,A^*) \longrightarrow (Z^*_q,E^*_{\bar{G}})$ is another ψ-equivariant map which induces $\alpha^*:B^* \longrightarrow \bar{B}^*$, $\nu:\pi_2(X^*,A^*) \longrightarrow \bar{G}=\pi_2(Z^*_q,E^*_{\bar{G}})$ and $\tau = \chi|\pi_1 A^*:\pi_1 A^* \longrightarrow \mathrm{In}\,\bar{G}$ then

$$\bar{\varphi}_1|B^* = \bar{\varphi}|B^* \quad \text{and} \quad \bar{\varphi}_1|A^* \simeq \bar{\varphi}|A^*$$

and, since both maps induce the same homomorphism on $\pi_2(X^*,A^*)$, we can extend the last homotopy to a homotopy rel α on the 2-skeleton of X^* and hence on X^* because all obstructions vanish. Furthermore there is no obstruction to deforming this homotopy into an equivariant one rel α. This completes the proof of Lemma 4.2.

§6.- The universal property

In this paragraph we prove a universal property which justifies the definition of P-localization of a fibration with nilpotent fibre introduced in §3.

6.1. Theorem. *Let $f:E \longrightarrow B$ be a fibration with nilpotent fibre F and let $\beta:f \longrightarrow f_{(P)}$ a P-localization. For every P-local fibration $\bar{f}:\bar{E} \longrightarrow \bar{B}$ with nilpotent fibre $\bar{F}$ and every map $\tau:f \longrightarrow \bar{f}$ there exists a map $\delta:f_{(P)} \longrightarrow \bar{f}$, uniquely determined up to fibrewise homotopy, such that $\delta\beta \simeq \tau$ (fibrewise).*

Proof. Let $\alpha:B \longrightarrow \bar{B}$ be the map induced by τ on the base spaces and let $\hat{f}$ be the fibre map induced by $\bar{f}$ and α.

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "\beta"] \arrow[dd, bend right=60, "\tau'"'] \arrow[ddd, bend right=80, "\tau"'] & B \arrow[d, equal] \\
E_{(P)} \arrow[r, "f_{(P)}"] \arrow[d, dashed, "\delta_2"'] \arrow[dd, dashed, bend left, "\delta_1"] & B \arrow[d, equal] \\
\hat{E} \arrow[r, "\hat{f}"] \arrow[d, "\alpha'"'] & B \arrow[d, "\alpha"] \\
\bar{E} \arrow[r, "\bar{f}"] & \bar{B}
\end{tikzcd}$$

Denote by $\tau':f \longrightarrow \hat{f}$ the map induced by τ (by the pullback property). The fibrewise homotopy classes of maps δ_1 are in one to one correspondence with those of maps δ_2, because, since $\bar{f}$ is a fibre map, the bottom square is a homotopy pullback [9] and all the diagrams are homotopy commutative. So we have only to prove the theorem in the special case $\bar{B}=B$ (so that τ is a map over B).

Consider then the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "e"] \arrow[dd, bend right] & E \arrow[r, "f"] \arrow[d, "\beta"] \arrow[dd, bend right, "\tau"'] & B \arrow[d, equal] \\
F_P \arrow[r] \arrow[d] & E_{(P)} \arrow[r, "f_{(P)}"] \arrow[d, dashed, "\delta"] & B \arrow[d, equal] \\
\bar{F} \arrow[r] & \bar{E} \arrow[r, "\bar{f}"] & B
\end{tikzcd}$$

Because of the naturality of the Moore-Postnikov decompositions $\{E_n,h_n\}$, $\{E_{(P)n},h'_n\}$, $\{\bar{E}_n,\bar{h}_n\}$of f, $f_{(P)}$, $\bar{f}$, respectively [1], β and τ determine up to homotopy maps $\beta_1:E_1 \longrightarrow E_{(P)1}$ and $\tau_1:E_1 \longrightarrow \bar{E}_1$ over B such that

$$\beta_1 h_1 \simeq h'_1\beta \quad \text{and} \quad \tau_1 h_1 \simeq \bar{h}_1\tau \text{ over B.}$$

The desired map δ determines up to fibrewise homotopy a map $\delta'_1:E_{(P)1} \longrightarrow \bar{E}_1$ such that $\delta'_1\beta_1 \simeq \tau_1$ over B and $\delta_1 = \delta'_1 h'_1$ is the first lifting of $f_{(P)}$ in

$$\begin{tikzcd}
E \arrow[rr, "\tau"] \arrow[dd, "\beta"'] & & \bar{E} \arrow[dl, "\bar{h}_1"'] \arrow[dd, "\bar{f}"] \\
 & \bar{E}_1 \arrow[dr] & \\
E_{(P)} \arrow[ur, dashed, "\delta_1"] \arrow[rr, "f_{(P)}"'] & & B
\end{tikzcd}$$

Now $\bar{p}_1:\bar{E}_1 \longrightarrow B$, $p'_1:E_{(P)1} \longrightarrow B$ and $p_1:E_1 \longrightarrow B$ are fibrations with fibres $K(\pi_1\bar{F},1)$, $K(\pi_1F_P,1)$ and $K(\pi_1F,1)$ respectively, β_1 induces P-localization on the fibres and $K(\pi_1\bar{F},1)$ is a P-local space by assumption. Theorem 4.1 then guarantees the existence of a δ_1', unique up to homotopy, and therefore of a unique lifting $\delta_1 : E_{(P)} \longrightarrow \bar{E}_1$ of $f_{(P)}$.

The higher obstructions to the lifting of δ_1 lie in the group $H^{n+1}(E_{(P)},E;\mathcal{T}_n\bar{F})$, where $\mathcal{T}_n\bar{F}$ is the local coefficient system determined by the action of $\pi_1E_{(P)}$ on $\pi_n\bar{F}$ via the map $\delta_{1*} : \pi_1E_{(P)} \longrightarrow \pi_1\bar{E}_1 = \pi_1\bar{E}$ and the action of $\pi_1\bar{E}$ on $\pi_n\bar{F}$ is determined by the fibration $\bar{f}$. The action of $\pi_1E_{(P)}$ induces a nilpotent action of π_1F_P on $\pi_n\bar{F}$. For consider the diagram

$$
\begin{tikzcd}
\pi_1F_P \arrow[r, "\epsilon_1"] & \pi_1E_{(P)} \arrow[r, "f_{(P)*}"] \arrow[d, "\delta_{1*}"'] & \pi_1B \arrow[d, equal] \\
\pi_1\bar{F} \arrow[r, "\epsilon_2"] & \pi_1\bar{E} \arrow[r, "\bar{f}"] & \pi_1B
\end{tikzcd}
$$

For $\xi \epsilon \pi_n\bar{F}$ and $\beta \epsilon \pi_1F_P$ we have

$$\xi^{\beta} = \xi^{\epsilon_1(\beta)} = \xi^{\delta_{1*}\epsilon_1(\beta)}$$

Since $\bar{f}_*\delta_{1*}\epsilon_1 = f_{(P)*}\epsilon_1 = 0$, there is a $\zeta \epsilon \pi_1\bar{F}$ such that $\epsilon_2(\zeta) = \delta_{1*}\epsilon_1(\beta)$ and thus

$$\xi^{\beta} = \xi^{\epsilon_1(\zeta)} = \xi^{\zeta} .$$

But $\bar{F}$ is nilpotent and therefore the action of $\pi_1\bar{F}$ on $\pi_n\bar{F}$ is nilpotent. From the last equality it follows that the action of π_1F_P on $\pi_n\bar{F}$ is also nilpotent.

Since $\pi_n\bar{F}$ are P-local groups we deduce from Lemma 3.3 that $H^{n+1}(E_{(P)},E;\mathcal{T}_n\bar{F}) = 0$ and hence there is a unique lifting of δ_1 to E. This completes the proof of the theorem.

From this universal property it follows by a standard argument

6.2. Corollary. *The P-localization of a fibration with nilpotent fibre is uniquely determined up to a homotopy equivalence.*

UNIVERSITAT DE BARCELONA CRAI

References

[1] H.J. Baues, *Obstruction theory*, Lect. Notes in Math 628, Springer, 1977.

[2] A.K. Bousfield-D.M. Kan, *Homotopy Limits, Completions and Localizations*, Lect. Notes in Math 304, Springer, 1972.

[3] R.O. Hill Jr, *Geometric interpretation of a classical group cohomology obstruction*, Proc. Amer. Math. Soc. 54 (1976), 405-412.

[4] R.O. Hill Jr, *Moore-Postnikov towers for fibrations in which π_1(fibre) is non-abelian*, Pac. J. Math. 62 (1976), 141-148.

[5] P. Hilton, *Relative nilpotent groups*, Lect. Notes in Math. 915, Springer, 1982, 136-147.

[6] P. Hilton, *Localization of crossed-modules*, Preprint, 1982.

[7] P. Hilton-G. Mislin-J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Math. Studies 15, 1975.

[8] I. Llerena, *Localization of nilpotent fibre maps*, Collect. Math. 33 (1982), 177-185.

[9] M. Mather, *Pullbacks in homotopy theory*, Can. J. Math. 28 (1976), 225-263.

[10] C.A. Robinson, *Moore-Postnikov systems for non-simple fibrations*, Illinois J. Math. 16 (1972), 234-242

Universitat de Barcelona
Barcelona, Spain

publicacions
edicions
universitat
de barcelona

Dipòsit Legal B.: 9913-1984
BARCELONA – 1984