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THE DISTRIBUTION OF A DOUBLE STOCHASTIC INTEGRAL WITH RESPECT TO TWO INDEPENDENT BROWNIAN SHEETS

by

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Abstract

Let $\{W_i(z),\ z \in [0,1]^2,\ i=1,2\}$ be two bidimensional independent Wiener processes. We compute the characteristic function of the stochastic integral

$$\int_{[0,1]^2} W_1(z)\, dW_2(z)$$

and we give an expression for its moments. As an application, we present a martingale array, which does not satisfy the Central Limit Theorem.

Key Words: Stochastic integrals, two-parameter Brownian motion, Central Limit Theorem.

1. Introduction

Let $W = \{W(z),\ z \in [0,1]^2\}$ be a bidimensional brownian motion, that means, W is a zero mean gaussian process with covariance function $E[W(s,t)W(s',t')] = (s \wedge s')(t \wedge t')$. Nualart in [8] evaluates the law and the moments of the random variable

$$\int_{[0,1]^2} \int_{[0,1]^2} 1_D\,(z,z')\, dW(z)\, dW(z'),$$

where $D = \{(z,z') \in [0,1]^2 \times [0,1]^2,\quad z = (x,y),\ z' = (x',y');\ x \leq x',\quad y \geq y'\}$ (cf. [4]); which has the same distribution as

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$$\int_{[0,1]^2} W(z)\ dW(z)$$

(see [8]) by an argument of symmetry.

The purpose of this paper is to do a similar study for the integrals

$$\int_{[0,1]^2} W_1(z)\ dW_2(z)$$

and

$$\int_{[0,1]^2}\int_{[0,1]^2} 1_D(z,z')\ dW_1(z)\ dW_2(z'),$$

where $\{W_1(z),\ z \in [0,1]^2\}$ and $\{W_2(z),\ z \in [0,1]^2\}$ are two bidimensional independent brownian motions. In section 2 we compute their characteristic function and in section 3 we give an expression of their moments. As an application, in the fourth section we exhibit a martingale array $\{S_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq n, n \geq 1\}$ such that it verifies the conditional Lindeberg condition, and converges in distribution to the random variable

$$\int_{[0,1]^2} W_1(z)\ dW_2(z)$$

which has a characteristic function of the form $E[\exp(-\frac{t^2}{2}\ \eta)]$, being η a positive random variable. However, the conditional variances converge in law to some limit different from the law of η.

In the one parameter case, Berthued [2] and Yor [9] have studied separately the law of the random variable

$$z = \alpha \int_{[0,1]} W_1(t)dW_2(t) + \beta \int_{[0,1]} W_2(t)dW_1(t)$$

for $\alpha, \beta \in \mathbb{R}$, where $\{W_i(t),\ t \in [0,1]\}$ $i=1,2$ are two independent Wiener processes. They have obtained that:

$$E[e^{itz}] = \begin{cases} [\mathrm{ch}^2(\frac{\alpha-\beta}{2}\ t) + \frac{(\alpha+\beta)^2}{(\alpha-\beta)^2}\ \mathrm{sh}^2\ (\frac{\alpha-\beta}{2}\ t)]^{-\frac{1}{2}} & \text{for } \alpha \neq \beta \\ [1 + \alpha^2 t^2]^{-\frac{1}{2}} & \text{for } \alpha = \beta \end{cases} .$$

As a consequence the characteristic function of

$$\int_{[0,1]} W_1(t)\ dW_2(t)$$

is

$$\Phi(t) = [\mathrm{ch}^2(\tfrac{1}{2}\ t) + \mathrm{sh}^2(\tfrac{1}{2}\ t)]^{-\frac{1}{2}} .$$

Considering again the two-parameter case, from Ito's formula it follows that:

$$\begin{aligned} W_1(z)W_2(z) &= \int_{R_z} W_1(z')\ dW_2(z') + \int_{R_z} W_2(z')\ dW_1(z') \\ &+ \int_{R_z}\int_{R_z} 1_D(z',z'')\ dW_1(z')\ dW_2(z'') \\ &+ \int_{R_z}\int_{R_z} 1_D(z',z'')\ dW_2(z')\ dW_1(z'') \\ &= P_z + M_z + Z_z + L_z \end{aligned} \qquad (1.2)$$

where $R_z = [0,z]$.

We observe that the random variables stP_{11}, P_{st}, stM_{11}, M_{st},

stZ_{11}, Z_{st}, stL_{11} and L_{st} have the same distribution (see [8]) so, it suffices to study the law of P_{11}.

We also remark that $P_{11}+M_{11}$ and $Z_{11}+L_{11}$ can be written as the difference of two independent integrals of the form $\int_{[0,1]^2} W(z)\, dW(z)$. In fact,

$$P_{11} + M_{11} = \int_{[0,1]^2} \frac{W_1(z)+W_2(z)}{\sqrt{2}}\, d\left(\frac{W_1(z)+W_2(z)}{\sqrt{2}}\right)$$
$$- \int_{[0,1]^2} \frac{W_1(z)-W_2(z)}{\sqrt{2}}\, d\left(\frac{W_1(z)-W_2(z)}{\sqrt{2}}\right)$$

and a similar expression holds for $Z_{11}+L_{11}$.

In consequence, from the results of [8], we obtain that the characteristic function of these sums is equal to

$$\Phi(t) = \left(\prod_{k=1}^{\infty} \cosh^2\left(\frac{t}{(2k-1)\pi}\right)\right)^{-1} \left(1 - 16\left(\sum_{k=1}^{\infty} \frac{1}{(2k-1)\pi} \tanh\left(\frac{t}{(2k-1)\pi}\right)\right)^2\right)^{-\frac{1}{2}}.$$

2. Characteristic function

We will use the following facts: $\{\sqrt{2}\, \cos\, (2k-1)\, \frac{\pi x}{2}\}_{k=1}^{\infty}$ is an orthonormal basis formed by the eigenfunctions of the Hilbert-Schmidt operator on $L^2([0,1])$, whose kernel is the function $f(x,x') = (1 - x\vee x')$. Moreover, the series of functions:

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)^2\pi^2} \cos(2k-1)\, \frac{\pi x}{2}\, \cos(2k-1)\, \frac{\pi x'}{2} \qquad (2.1)$$

converges uniformly to $f(x,x')$ in $[0,1]^2$.

Theorem (2.1). The characteristic function of P_{11} is:

$$\Phi(t) = E[e^{itP_{11}}] = \prod_{k\geq 1} \cosh^{-\frac{1}{2}} \left(\frac{2t}{(2k-1)\pi}\right) .$$

Proof: The random variable P_{11} is the limit in $L^2(\Omega,\mathcal{F},P)$ of the sequence

$$T_n = \sum_{k,\ell=1}^{n} W_1\left(\frac{k-1}{n}, \frac{\ell-1}{n}\right) W_2(\Delta_{k\ell}) ,$$

where

$$\Delta_{k\ell} = \left(\left(\frac{k-1}{n}, \frac{\ell-1}{n}\right), \left(\frac{k}{n}, \frac{\ell}{n}\right)\right] ,$$

and

$$W_2(\Delta_{k\ell}) = W_2\left(\frac{k}{n}, \frac{\ell}{n}\right) - W_2\left(\frac{k}{n}, \frac{\ell-1}{n}\right) - W_2\left(\frac{k-1}{n}, \frac{\ell}{n}\right) + W_2\left(\frac{k-1}{n}, \frac{\ell-1}{n}\right) .$$

Then:

$$E[\ell^{itT_n}] = E\left[\exp\left(-\frac{t^2}{2n^2} \sum_{k,\ell=1}^{n-1} W_1 \left(\frac{k-1}{n}, \frac{\ell-1}{n}\right)^2\right)\right] .$$

It holds that

$$\frac{1}{n^2} \sum_{k,\ell=1}^{n-1} W_1\left(\frac{k-1}{n}, \frac{\ell-1}{n}\right)^2$$

$$= \frac{1}{n^2} \sum_{k,\ell=1}^{n-1} \sum_{k',\ell'=1}^{n-1} W_1(\Delta_{k\ell})W_1(\Delta_{k'\ell'})(n-k\vee k')(n-\ell\vee \ell')$$

$$= \sum_{k,\ell=1}^{n-1} \sum_{k',\ell'=1}^{n-1} W_1(\Delta_{k\ell})W_1(\Delta_{k'\ell'}) f\left(\frac{k}{n}, \frac{k'}{n}\right) f\left(\frac{\ell}{n}, \frac{\ell'}{n}\right). \qquad (2.2)$$

Following (2.1), the term (2.2) can be expressed as

$$\sum_{i,j\geq 1} \frac{16}{(2i-1)^2(2j-1)^2\pi^4} \left(2\sum_{k,\ell=1}^{n-1} W_1(\Delta_{k\ell})\cos(2i-1)\frac{\pi k}{2n} \cos(2j-1)\frac{\pi\ell}{2n}\right)^2$$

$$= \sum_{i,j\geq 1} a_{ij} \, (X_{ij}^n)^2$$

where

$$a_{ij} = \frac{16}{(2i-1)^2(2j-1)^2\pi^4} ,$$

and

$$X_{ij}^n = 2 \sum_{k,\ell=1}^{n-1} W_1(\Delta_{k\ell}) \cos(2i-1) \frac{\pi k}{2n} \cos(2j-1) \frac{\pi \ell}{2n} .$$

The sequence X_{ij}^n converges in L^2 to:

$$Y_{ij} = \int_{[0,1]^2} 2\cos(2i-1) \frac{\pi x}{2} \cos(2j-1) \frac{\pi y}{2} \, dW_1(x,y) .$$

$\{Y_{ij}\}_{i,j\geq 1}$ is a sequence of independent random variables with law $N(0,1)$. Therefore, $\sum_{i,j\geq 1} a_{ij}(X_{ij}^n)^2$ will converge in L^1 to $\sum_{i,j\geq 1} a_{ij}Y_{ij}^2$; and we have,

$$\lim_{n\to\infty} E[e^{itT_n}] = E[e^{-t^2/2 \sum_{i,j\geq 1} a_{ij}Y_{ij}^2}] = \prod_{i,j\geq 1} (1+t^2a_{ij})^{-\frac{1}{2}}$$

$$= \prod_{i\geq 1} \cosh^{-\frac{1}{2}}\left(\frac{2t}{(2i-1)\pi}\right) . \quad \square$$

We remark that the stochastic integral

$$P_{11} = \int_{[0,1]^2} W_1(z) \, dW_2(z)$$

has the same distribution as the random variable

$$S = \sum_{i,j\geq 1} \frac{4}{(2i-1)(2j-1)\pi^2} \eta_{ij} \xi_{ij} \quad ,$$

where $\{\eta_{ij}\}_{i,j\geq 1}$ and $\{\xi_{i,j}\}_{i,j\geq 1}$ are independent and identically distributed random variables with law $N(0,1)$.

3. Moments of P_{11}

In this section we will compute the moments of P_{11}.

Proposition (3.1)

$$\forall t \in \mathbb{R}, \; E[e^{tP_{11}}] = \prod_{j\geq 1} \cos^{-\frac{1}{2}} \left(\frac{2t}{(2j-1)\pi}\right) \qquad \text{for } |t| < \pi^2/4 \quad ,$$

$$E[e^{tP_{11}}] = \infty \qquad \text{for} \quad |t| \geq \pi^2/4 \quad .$$

Proof: By the preceding remark we have,

$$E[e^{tP_{11}}] = E[e^{tS}] = \prod_{i,j\geq 1} E[\exp\left(\frac{4t}{(2i-1)(2j-1)} \eta_{ij} \xi_{ij} \right)].$$

Finally,

$$E[\exp\left(\frac{4t}{(2i-1)(2j-1)\pi^2} \eta_{ij} \xi_{ij} \right)] = \begin{cases} = \left(1 - \dfrac{16\, t}{(2i-1)^2(2j-1)^2\pi^4}\right)^{-\frac{1}{2}} \\ \qquad\qquad \text{for } |t| < \dfrac{(2i-1)(2j-1)\pi^2}{4} \quad , \\ = \infty \qquad \text{for } |t| \geq \dfrac{(2i-1)(2j-1)\pi^2}{4} \quad . \square \end{cases}$$

Let $\{ U_t^n, \; t \in [0,1], \; n \geq 1 \}$, $\{ \bar{U}_t^n, \; t \in [0,1], \; n \geq 1 \}$,

$\{V_t^n,\ t \in [0,1],\ n \geq 1\}$ and $\{\bar{V}_t^n,\ t \in [0,1],\ n \geq 1\}$ be independent infinite-dimensional brownian motions. We know (cf. [7]) that

$$\frac{1}{n}\ (\sum_{i=1}^{n} U_s^i\ \bar{U}_t^i\ )\ (\sum_{j=1}^{n} V_s^j\ \bar{V}_t^j\ )$$

converges weakly to $W_1(s,t)\ W_2(s,t)$ in $C([0,1]^2)$. From Ito's formula, we have

$$\frac{1}{n} \sum_{i,j=1}^{n} U_s^i\ \bar{U}_t^i\ V_s^j\ \bar{V}_t^j = \frac{1}{n} \sum_{i,j=1}^{n} \left(\int_0^s U^i\ dV^j + \int_0^s V^j\ dU^i\right) \cdot \left(\int_0^t \bar{U}^i\ d\bar{V}^j + \int_0^t \bar{V}^j\ d\bar{U}^i\right)$$

$$= \frac{1}{n} \sum_{i,j=1}^{n} \int_0^s U^i\ dV^j \int_0^t \bar{U}^i\ d\bar{V}^j + \frac{1}{n} \sum_{i,j=1}^{n} \int_0^s U^i dV^j \int_0^t \bar{V}^j d\bar{U}^i$$

$$+ \frac{1}{n} \sum_{i,j=1}^{n} \int_0^s V^j\ dU^i \int_0^t \bar{U}^i d\bar{V}^j + \frac{1}{n} \sum_{i,j=1}^{n} \int_0^s V^j dU^i \int_0^t \bar{V}^j d\bar{U}^i$$

$$= S_{st}^n + T_{st}^n + \hat{T}_{st}^n + \hat{S}_{st}^n\ .$$

Proposition (3.2)

S_{st}^n, T_{st}^n, $\hat{T}_{st}^n$, $\hat{S}_{st}^n$ converge weakly to P_{st}, Z_{st}, L_{st} and M_{st} in $C([0,1]^2)$, respectively.

Proof: The proof is the same as in the case $W_1 = W_2$, which is given in Lema (2.1) of [8].□

Now, using the approximation of the law of P_{11} by S_{11}^n we are going to deduce a recursive expression for the moments of P_{11}.

Proposition (3.3)

For all $p \geq 1$ $p \in \mathbb{N}$, $E[P_{11}^{2p-1}] = 0$ and $E[P_{11}^{2p}] = \frac{(2p)!}{p!\ 2p} m_p$

where

$$m_p = \sum_{k=1}^{n} \frac{(p-1)!}{(p-k)!} 2^{k-1} \mu_k^2 m_{p-k} \qquad (m_o=1)$$

and

$$\mu_k = \int_{[0,1]^k} (x_1 \wedge x_2)(x_2 \wedge x_3) \ldots (x_k \wedge x_1) dx_1 \ldots dx_k \ .$$

Proof: We compute the moments of S_{11}^n that converge weakly to P_{11} by proposition (3.2).

Set

$$E[(S_{11}^n)^p] = \frac{1}{n^p} \sum_{i,j \in \{1,\ldots,n\}^p} E[\prod_{k=1}^{p} \int_o^1 U^{i_k} dV^{j_k}] \, E[\prod_{k=1}^{p} \int_o^1 \bar{U}^{i_k} d\bar{V}^{j_k}]$$

$$= \frac{1}{n^p} \sum_{i,j \in \{1,\ldots,n\}^p} (E[\prod_{k=1}^{p} \int_o^1 U^{i_k} \, dV^{j_k}])^2$$

where we have written $i=(i_1,\ldots,i_p)$ and $j=(j_1,\ldots,j_p)$.

If there exists a number $h \in \{1,\ldots,n\}$ such that one of the cardinals $\#\{k=1,\ldots,p;\ i_k=h\}$ or $\#\{k=1,\ldots,p\ ;\ j_k=h\}$ is odd, the expectation $E[\prod_{k=1}^{p} \int_o^1 U^{i_k} dV^{j_k}]$ can be decomposed, by Ito's formula, as sum of terms of the form:

$$\frac{1}{2^p} \int_o^1 \int_o^{s_1} \ldots \int_o^{s_{p-1}} E[U_{s_1}^{i_{\ell_1}} U_{s_1}^{i_{m_1}} U_{s_2}^{i_{\ell_2}} U_{s_2}^{i_{m_2}} \ldots U_{s_p}^{i_{\ell_p}} U_{s_p}^{i_{m_p}}] \, \delta_{j_{\ell_1} j_{m_1}}$$

$$\ldots \delta_{j_{\ell_p} j_{m_p}} ds_p \ldots ds_1 \ .$$

Then,

$$E[\prod_{k=1}^{p} \int_0^1 U^{i_k}\, dV^{j_k}] = 0 ,$$

that is, $E[(S_{11}^n)^p] = 0$, if p is odd; so, it suffices to compute $E[(S_{11}^n)^{2p}]$.

Denote by ν_{ij} the number of different integers appearing in the multi-indexes (i,j).

If $i,j \in \{1,\dots,n\}^{2p}$ and $\nu_{ij} = \ell < 2p$ then:

$$\frac{1}{n^{2p}} \sum_{\substack{i,j\in\{1,\dots,n\}^{2p} \\ \nu_{ij}=\ell}} (E[\prod_{k=1}^{2p} \int_0^1 U^{i_k}\, dV^{j_k}])^2 < \frac{1}{n^{2p}} \binom{n}{\ell} \ell^{4p} C^2 \xrightarrow[n\to\infty]{} 0$$

where

$$C = \max_{\substack{r,s\in\{1,2,\dots,n\}^{2p} \\ \nu_{rs} = \ell}} E[\prod_{k=1}^{2p} \int_0^1 U^{r_k}\, dV^{s_k}].$$

By the same argument as before, we have only to consider the multi-indexes $(i,j) \in \{1,\dots,n\}^{2p}$ with $\nu_{ij} = 2p$ and such that there are exactly p different integers in "i".

Denote $\hat{G}_p$ the set of all permutations of $(1,1,2,2,\dots,p,p)$, and G_p the set of permutations of $(1,2,3,\dots,p)$.

Given a permutation $\hat{\imath} = (i_1,i_2,\dots,i_{2p}) \in \hat{G}_p$; we will say that $j \leftrightarrow h$ $(j,h \in \{1,2,\dots,p\})$ if $i_{2j} = i_{2h}$, or $i_{2j} = i_{2h-1}$ or $i_{2j-1} = i_{2h}$ or $i_{2j-1} = i_{2h-1}$. We define a cycle $1 \leftrightarrow j_1 \leftrightarrow j_2 \leftrightarrow \dots \leftrightarrow j_n \leftrightarrow 1$ (where $1,j_1,j_2,\dots,j_n$ are different elements of $\{1,2,\dots,n\}$) in such a way that

j_1 verifies $i_1 = i_{2j_1}$ or $i_1 = i_{2j_1-1}$. Notice that the elements $j_2,\dots,j_n$ are determined, once j_1 is fixed. If this cycle does not contain all the integers $\{1,2,\dots,p\}$ we can form another one starting by the minimum k, of these remaining integers. So, proceeding in the same way, we will have another cycle $k \leftrightarrow k_1 \leftrightarrow \dots \leftrightarrow k_m \leftrightarrow k$ where k_1 is always such that $i_{2k-1} = i_{2k_1}$ or $i_{2k-1} = i_{2k_1-1}$ and $k, k_1, \dots, k_m$ are different elements. We repeat this procedure until we exhaust all numbers from 1 to p. In this way, the permutation $\hat{\imath}$ determines uniquely a permutation of G_p defined as the product of the cycles introduced above.

Let I_σ be the set of all permutations $\hat{\imath} \in \hat{G}_p$ which determine the same permutation $\sigma \in G_p$. Denote by n_σ the number of cycles appearing in the descomposition of σ. Note that:

- If $\hat{\imath} \in I_\sigma$ and $k \in \{1,\dots,p\}$ is not the first element of some cycle, the permutation $(i_1,i_2,\dots,i_{2k},i_{2k-1},\dots,i_{2p})$ belongs to I_σ.
- If $\rho \in G_p$, and $\hat{\imath} \in I_\sigma$, we have $(\rho(i_1),\rho(i_2),\dots,\rho(i_{2p})) \in I_\sigma$.

Therefore $\# I_\sigma = 2^{p-n_\sigma}\, p!$. The expectation

$$E\Big[\prod_{k=1}^{p} \int_0^1 U^{i_{2k}}\, dV^k \int_0^1 U^{i_{2k-1}}\, dV^k\Big]$$

only depens on the permutation σ. Indeed, by Ito's formula:

$$E\Big[\prod_{k=1}^{p}\int_o^1 U^{i_{2k}}\,dV^k \int_o^1 U^{i_{2k-1}}\,dV^k\Big] =$$

$$= \sum_{\substack{h_1,h_2,\dots,h_p=1 \\ h_1,h_2,\dots,h_p \text{ different}}}^{p} \int_o^1\int_o^{s_1} \dots \int_o^{s_{p-1}} E[U_{s_1}^{i_{2h_1}} \cdot U_{s_1}^{i_{2h_1-1}} \dots \cdot U_{s_p}^{i_{2h_p-1}}]\; ds_p \dots ds_1$$

$$= \int_o^1\int_o^{s_1} \dots \int_o^{s_{p-1}} \sum_{\pi\in G_p} E[U_{s_1}^{i_{2\pi(1)}} U_{s_1}^{i_{2\pi(1)-1}} \dots U_{s_p}^{i_{2\pi(p)-1}}]\, ds_p \dots ds_1$$

$$= \int_o^1\int_o^{s_1} \dots \int_o^{s_{p-1}} \sum_{\pi\in G_p} \prod_{i=1}^{p} (s_i \wedge s_{\pi^{-1}(\sigma(\pi(i)))})\; ds_p \dots ds_1 \;.$$

Thus,

$$E[S_{11}^{2p}] = \lim_n E[(S_{11}^n)^{2p}]$$

$$= \lim_n \frac{1}{n^{2p}} \binom{n}{2p}\binom{2p}{p} \sum_{i\in\hat{G}_p} \Big(E\Big[\prod_{k=1}^{p}\int_o^1 U^{i_{2k}}dV^k \int_o^1 U^{i_{2k-1}}dV^k\Big]\Big)^2$$

$$= \frac{(2p)!}{(p!)^2\, 2^p} \sum_{\sigma\in G_p} p!\; 2^{p-n_\sigma} \Big(E\Big[\prod_{k=1}^{p}\int_o^1 U^k dV^k \int_o^1 U^{\sigma(k)} dV^k\Big]\Big)^2 \;.$$

If $\sigma \in G_p$ is irreductible, so is the permutation $\pi^{-1}\circ\sigma\circ\pi$ for all $\pi \in G_p$. In this case,

$$E\Big[\prod_{k=1}^{p}\int_o^1 U^k dV^k \int_o^1 U^{\sigma(k)} dV^k\Big]$$

$$= \int_o^1 \dots \int_o^{s_{p-1}} \sum_{\pi\in G_{\{s_1\dots s_p\}}} (\pi(s_1)\wedge\pi(s_2))(\pi(s_2)\wedge\pi(s_3)) \dots (\pi(s_{p-1})\wedge\pi(s_p))\; ds_p \dots ds_1$$

$$= \int_{[0,1]^p} (s_1 \wedge s_2)(s_2 \wedge s_3) \dots (s_p \wedge s_1)\, ds_p \dots ds_1 = \mu_p \quad .$$

We can decompose any permutation σ into the product of a cycle which contains the element 1 $(1 = j_1 \leftrightarrow j_2 \leftrightarrow \dots \leftrightarrow j_h \leftrightarrow 1)$ and one permutation $\bar{\sigma}$ of the remaining elements. Using this idea, we have

$$E\Big[\prod_{k=1}^{p} \int_0^1 U^k dV^k \int_0^1 U^{\sigma(k)} dV^k\Big]$$

$$= E\Big[\prod_{k=1}^{h} \int_0^1 U^k dV^k \int_0^1 U^{j_k}\, dV^k\Big] \cdot E\Big[\prod_{k=1}^{p-h} \int_0^1 U^k dV^k \int_0^1 U^{\bar{\sigma}(k)} dV^k\Big] .$$

In conclusion, we obtain

$$\sum_{\sigma \in G_p} 2^{p-n_\sigma} \Big(E\Big[\prod_{k=1}^{p} \int_0^1 U^k dV^k \int_0^1 U^{\sigma(k)} dV^k\Big]\Big)^2$$

$$= \sum_{h=1}^{p} \binom{p-1}{h-1} \Big(E\Big[\prod_{k=1}^{p} \int_0^1 U^k dV^k \int_0^1 U^{j_k} dV^k\Big)^2 \, 2^{h-1} (h-1)!$$

$$\cdot \Big(\sum_{\bar{\sigma} \in G_{p-h}} 2^{p-h-n_{\bar{\sigma}}} \Big(E\Big[\prod_{k=1}^{p-h} \int_0^1 U^k dV^k \int_0^1 U^{\bar{\sigma} k} dV^k\Big]\Big)^2\Big)$$

$$= \sum_{h=1}^{p} \binom{p-1}{h-1} 2^{h-1} (h-1)!\, \mu_h^2\, m_{p-h}$$

$$= \sum_{h=1}^{p} \frac{(p-1)!}{(p-h)!} 2^{h-1} \mu_h^2\, m_{p-h} \quad .$$

Observe that in the above expression $n_\sigma = n_{\bar{\sigma}} + 1$. The proof is now complete. □

Corollary (3.4)

The characteristic function of the random variable P_{11} can be expressed as

$$\Phi(t) = \exp\left(\sum_{k=1}^{\infty} \frac{\mu_k^2}{2^k} (-t^2)^k \right) .$$

Proof:

It is known that, for a given probability law, the moments (m_k) and the cumulants (c_k) verify the relation:

$$m_p = \sum_{k=1}^{p} \binom{p-1}{k-1} c_k \, m_{p-k} .$$

By proposition (3.3),

$$E[(P_{11})^{2p}] = \sum_{k=1}^{p} \binom{2p-1}{2k-1} \mu_k^2 (2k-1)! \; E[(P_{11})^{2p-2k}] .$$

Therefore the cumulants of the law of P_{11} are $c_{2k-1}=0$ and $c_{2k} = \mu_k^2(2k-1)!$, and we have the following expression for the characteristic function of P_{11}:

$$\Phi(t) = \exp\left\{\sum_{k=1}^{\infty} \frac{c_{2k}}{2k!} (-t^2)^k \right\} = \exp\left\{\sum_{k=1}^{\infty} \frac{\mu_k^2}{2k} (-t^2)^k \right\} . \square$$

Proposition (3.5)

$$\mu_k = \int_{[0,1]^k} (x_1 \wedge x_2)(x_2 \wedge x_3)\ldots(x_k \wedge x_1) \, dx_1 \ldots dx_k =$$

$$= \sum_{i \geq 1} \left(\frac{2}{(2i-1)\pi}\right)^{2k} = 2^{2k-1} \frac{(2^{2k}-1)}{(2k)!} B_k$$

where B_k is the k-th Bernouilli's number.

Proof: Let S be the random variable defined in (2.2). If $|t| < \pi^2/4$, we can write:

$$\log E[e^{St}] = \sum_{i,j\geq 1} (-\tfrac{1}{2}) \log(1 - \frac{16t^2}{(2i-1)^2(2j-1)^2 \pi^4}) =$$

$$= \sum_{n=1}^{\infty} \frac{t^{2n}}{2n} (\frac{4}{\pi^2})^{2n} (\sum_{i,j \geq 1} \frac{1}{(2i-1)^{2n}})^2 . \qquad (3.1)$$

From (3.1) we conclude that the cumulants of this law are: $c_{2k-1} = 0$ and $c_{2k} = \mu_k^2 (2k-1)!$ which implies the desired result. □

4. Counter-example of the Central Limit Theorem

Nualart in [8] exhibited a martingale array $\{ S_{ni} = \sum_{j=1}^{i} X_{nj}, \mathcal{F}_{ni}, n \geq 1, i=1,\dots,n \}$ such that it verifies the conditional Lindeberg condition,

$$\sum_{i=2}^{n} E(X_{ni}^2 1_{\{|X_{ni}|>\varepsilon\}} / \mathcal{F}_{n,i-1}) \xrightarrow{P} 0 \qquad \forall \varepsilon > 0,$$

and converges in distribution to a non-symmetric random variable, as long as the sequence of conditional variances converges in law. Some other counter-examples for the central limit Theorem have also been proposed by several authors ([1], [5]). Here, we will exhibit a martingale array converging in law to P_{11}. Notice that the law of P_{11} is symmetric and its characteristic function can be expressed as $\Phi(t) = E[e^{-t^2 \eta/2}]$, where

$$\eta = \sum_{i,j\geq 1} \frac{16}{(2i-1)^2(2j-1)^2 \pi^4} \xi_{ij}^2$$

and $\{\xi_{ij}\}_{i,j\geq 1}$ are independent and identically distributed random variables with law $N(0,1)$. However, the sequence of conditional variances converges in distribution to some law different from that of η, and therefore, the central limit Theorem for martingale arrays (cf. [6]) is not satisfied.

Let

$$Y_{ij} = \int_o^1 U^i dV^j \int_o^1 \bar{U}^i d\bar{V}^j + \int_o^1 U^j dV^i \int_o^1 \bar{U}^j d\bar{V}^i$$

for $i,j \geq 1$ $i \neq j$ and

$$Y_{ii} = \int_o^1 U^i dV^i \int_o^1 \bar{U}^i d\bar{V}^i$$

for $i \geq 1$.

We define $X_{nj} = \frac{1}{n} \sum_{i=1}^{j} Y_{ij}$ and $\mathcal{F}_{nj} = \sigma < U^1, \dots, U^j, \bar{U}^1, \dots, \bar{U}^j, V^1, \dots, V^j, \bar{V}^1, \dots, \bar{V}^j >$ for $1 \leq j \leq n$.

Then, $\{S_{ni} = \sum_{k=1}^{i} X_{nk}, \mathcal{F}_{ni}, 1 \leq i \leq n, n \geq 1\}$ is a martingale array. Moreover, by proposition (3.2), S_{nn} converges to P_{11}.

Proposition (4.1)

The martingale array $\{S_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq n, n \geq 1\}$ satisfies the conditional Lindeberg condition.

Proof:

$$E[\sum_{j=2}^{n} X_{nj}^2 \, 1_{\{|X_{nj}| > \varepsilon\}}] \leq \frac{1}{\varepsilon^2} \sum_{j=2}^{n} E[X_{nj}^4]$$

$$= \frac{1}{\varepsilon^2} \sum_{j=2}^{n} \sum_{i_1,i_2,i_3,i_4=1}^{j} E[Y_{i_1 j} Y_{i_2 j} Y_{i_3 j} Y_{i_4 j}]$$

$$\leq \frac{3}{\varepsilon^2 \pi^4} \sum_{j=2}^{n} \sum_{i_1, i_2=1}^{j} E[Y^4_{i_1 j}]^{\frac{1}{2}} E[Y_{i_2 j}]^{\frac{1}{2}} \leq \frac{C}{\varepsilon n} .$$

Therefore,

$$\sum_{i=1}^{n} E[X^2_{n_i} 1_{\{|X_{n_i}|>\varepsilon\}} / \mathcal{F}_{n,i-1}] \xrightarrow[n\to\infty]{P} 0 . \square$$

Proposition (4.2)

The sequence

$$V_n^2 = \sum_{i=2}^{n} E[X^2_{ni} / \mathcal{F}_{n,i-1}]$$

converges in law to the random variable

$$\int_{[0,1]^3} \hat{W}^2_{stu} \, ds\, dt\, du + \int_{[0,1]^3} (\Delta W_{stu})^2 \, ds\, dt\, du ,$$

where $\Delta W_{stu} = W_{11u} - W_{1tu} - W_{s1u} + W_{stu}$, and

$$\{W_{stu}, (s,t,u) \in [0,1]^3\} \quad \text{and} \quad \{\hat{W}_{stu}, (s,t,u) \in [0,1]^3\}$$

are two independent zero mean gaussian processes with covariance function: $E[W_{stu} W_{s't'u'}] = E[\hat{W}_{stu} \hat{W}_{s't'u'}] = (s \wedge s')(t \wedge t')(u \wedge u')$.

Proof:

$$E[X^2_{nj} / \mathcal{F}_{n,j-1}] = \frac{1}{n^2} E[\sum_{i_1,i_2=1}^{j-1} Y_{i_1 j} Y_{i_2 j} / \mathcal{F}_{n,j-1}]$$

$$+ \frac{2}{n^2} E[\sum_{i_1=1}^{j-1} Y_{i_1 j} Y_{jj} / \mathcal{F}_{n,j-1}] + \frac{1}{n^2} E[Y^2_{jj} / \mathcal{F}_{n,j-1}].$$

Note that:

$$E\left[\int_o^1 U^{i_1}dV^j \int_o^1 \bar{U}^{i_1}d\bar{V}^j \int_o^1 U^{i_2}dV^j \int_o^1 \bar{U}^{i_2}d\bar{V}^j \;/\; \mathcal{F}_{n,j-1}\right]$$

$$= \begin{cases} \left(\int_o^1 U_u^{i_1}U_u^{i_2}du\right)\left(\int_o^1 \bar{U}_u^{i_1}\bar{U}_u^{i_2}\,du\right) & \text{for } i_1, i_2 < j \\ = 0 & \text{for } i_1 = j \text{ and } i_2 < j \\ & \text{or } i_2 = j \text{ and } i_1 < j \\ = 1/4 & \text{for } i_1 = i_2 = j \end{cases},$$

$$E\left[\int_o^1 U^{i_1}dV^j \int_o^1 \bar{U}^{i_1}d\bar{V}^j \int_o^1 U^j dV^{i_2} \int_o^1 \bar{U}^j d\bar{V}^{i_2}\right] = 0 \quad \text{for } i_1 \le j \text{ and } i_2 < j \text{ or } i_1 < j \text{ and } i_2 \le j,$$

and

$$E\left[\int_o^1 U^j dV^{i_1} \int_o^1 \bar{U}^j d\bar{V}^{i_1} + \int_o^1 U^j dV^{i_2} \int_o^1 \bar{U}^j d\bar{V}^{i_2} \;/\; \mathcal{F}_{n,j-1}\right] =$$

$$= \begin{cases} \left(\int_o^1 (V_1^{i_1}-V_u^{i_1})(\bar{V}_i^{i_1}-\bar{V}_u^{i_1})du\right)\left(\int_o^1 (V_1^{i_2}-V_u^{i_2})(\bar{V}_1^{i_2}-\bar{V}_u^{i_2})\,du\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } i_1, i_2 < j \\ = 0 \quad \text{for } i_1 = j \text{ and } i_2 < j \text{ or } i_2 = j \text{ and } i_1 < j. \end{cases}$$

Then,

$$E[X_{nj}^2 \;/\; \mathcal{F}_{n,j-1}] = \frac{1}{n^2} \sum_{i_1,i_2=1}^{j-1} \int_o^1 U_u^{i_1}U_u^{i_2}du \int_o^1 \bar{U}_u^{i_1}\bar{U}_u^{i_2}\,du$$

$$+ \frac{1}{n^2} \sum_{i_1,i_2=1}^{j-1} \int_o^1 (V_1^{i_1}-V_u^{i_1})(V_1^{i_2}-V_u^{i_2})du \int_o^1 (\bar{V}_1^{i_1}-\bar{V}_u^{i_1})(\bar{V}_1^{i_2}-\bar{V}_u^{i_2})du + \frac{1}{4n^2}$$

$$= \frac{1}{n^2} \int_o^1\int_o^1 \left[\left(\sum_{i=1}^{j-1} U_u^i\bar{U}_v^i\right)^2 + \left(\sum_{i=1}^{j-1} (V_1^i-V_u^i)(\bar{V}_u^i-\bar{V}_v^i)\right)^2\right] du\,dv + \frac{1}{4n^2}.$$

Define the D_3-valued variables

$$\xi_n(s,t,u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nu]} U_s^i \bar{U}_t^i ,$$

and

$$\zeta_n(s,t,u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nu]} V_s^i \bar{V}_t^i$$

where $D_3 = \{f: [0,1]^3 \longrightarrow \mathbf{R} \ / \ f$ continuous from above, with limits from below $\}$.

Then

$$V_n^2 - \frac{(n-1)}{4n^2} = \int_{[0,1]^3} [\xi_n(u,v,s)^2 + (\zeta_n(1,1,s) - \zeta_n(1,v,s) - \zeta_n(u,1,s) + \zeta_n(u,v,s))^2] \ du \ dv \ ds.$$

By Theorem 6 of Bickel and Wichura [3] we know that ξ_n and ζ_n converge weakly to W_{stu} and $\hat{W}_{stu}$ respectively, and then the proposition follows easily. □

Remark that the random variables η and

$$\int_{[0,1]^3} \hat{W}_{stu} \ ds \ dt \ du + \int_{[0,1]^3} (\Delta W_{stu})^2 \ ds \ dt \ du$$

do not have the same distribution, as it can be cheked by comparing its second order moments.

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