ON POLYNOMIAL BOUNDS FOR THE KOSZUL HOMOLOGY OF CERTAIN MULTIPlicitY SYSTEMS

by

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1. Introduction

In this paper \( R \) will be a commutative Noetherian local ring.

Let \( M \) be a finitely generated \( R \)-module and \( x_1, \ldots, x_r \in R \) a multiplicity system on \( M \), which means that

\[ \text{length}_R(M/IM) \leq \text{length}_R(M/I^nM) \]

where \( I \) is the \( R \)-ideal generated by \( x_1, \ldots, x_r \). We will tacitly assume that no \( x_i \) is a unit, for, otherwise the functions to be considered in what follows would be all zero.

It is well-known that the function

\[ n \mapsto \text{length}_R(M/IM) \]

is polynomial of degree equal to the dimension \( d \) of \( M \), for \( n \) large ( \( d \) is necessarily less than or equal to \( r \)).

As \( I^n \subseteq (x_1^n, \ldots, x_r^n)R \subseteq I^n \), we see that the function

\[ n \mapsto \text{length}_R(M/(x_1^n, \ldots, x_r^n)M) \]

is bounded above and below by polynomial functions in \( n \) of degree \( d \). However \( M/(x_1^n, \ldots, x_r^n)M \) is just the zeroth homology module of the Koszul complex \( K(x_1^n, \ldots, x_r^n|M) \), so part of the above assertion says that the function

\[ n \mapsto \text{length}_R H_0 K(x_1^n, \ldots, x_r^n|M) \]

is bounded above by a degree \( d \) polynomial in \( n \).

In this paper we prove that a similar statement is true for the higher homology modules of the Koszul com-
plex, i.e., that \( \text{length}_{R} H_{1}K(x_{1}^{n}, \ldots, x_{r}^{n}|M) \) is bounded above by a polynomial in \( n \) of degree \( d \), for any \( i \geq 0 \).

I am indebted to Dr. D. Kirby for his helpful suggestions.

12. The higher Koszul homology modules. We start with two technical lemmas.

**Lemma 1.** Let \( M \) be an \( R \)-module and \( x_{1}, \ldots, x_{r}, y \) be elements of \( R \). Then, for any \( i \geq 0 \),

\[
\text{length}_{R} H_{1}K(x_{1}, x_{2}, \ldots, x_{r}|M) \geq \text{length}_{R} H_{1}K(x_{1}^{n}, \ldots, x_{r}^{n}|M).
\]

**Remark.** It is not necessary for the lengths involved to be finite.

**Proof.** The inequality follows from the exact sequence (cf. [4] p.IV-2)

\[
\begin{array}{cccccc}
0 & \rightarrow & H_{1}K(x_{2}, \ldots, x_{r}|M) & \rightarrow & H_{1}K(x_{1}, x_{2}, \ldots, x_{r}|M) & \rightarrow \\
& & x_{1}H_{1}K(x_{2}, \ldots, x_{r}|M) & \rightarrow & \rightarrow (0:x_{1}) & \rightarrow \rightarrow 0,
\end{array}
\]

and the corresponding one for \( H_{1}K(x_{1}y, x_{2}, \ldots, x_{r}|M) \), by observing that both \((0:x_{1}) \subseteq H_{1-1}K(x_{2}, \ldots, x_{r}|M) \), \((0:x_{1}y) \subseteq H_{1-1}K(x_{2}, \ldots, x_{r}|M) \) and \( x_{1}H_{1}K(x_{2}, \ldots, x_{r}|M) \supseteq x_{1}yH_{1}K(x_{2}, \ldots, x_{r}|M) \). #
Lemma 2. If $x$ is a non-zero-divisor on $M$, we have

$$\text{length}_{R} H_{1}K(y_{1}, \ldots, y_{r}|_{M} / x^{n}M) \leq n \cdot \text{length}_{R} H_{1}K(y_{1}, \ldots, y_{r}|_{M} / xM).$$

Proof. Apply the functor $H_{1}K(y_{1}, \ldots, y_{r}|_{*})$ to the sequence

$$0 \rightarrow M / x^{n-1} \rightarrow M / x^{n} \rightarrow M / xM \rightarrow 0,$$

which is exact because $x$ is $M$-regular, and use induction on $n$. #

Next, the goal of this paper.

Theorem 3. Let $M$ be a finitely generated $R$-module of dimension $d$ and $x_{1}, \ldots, x_{r} \in R$ a multiplicity system on $M$. Then, for each $i \geq 0$, the function

$$n \mapsto \text{length}_{R} H_{1}K(x_{1}^{n}, \ldots, x_{r}^{n}|M)$$

is bounded above by a polynomial in $n$ of degree not greater than $d$.

Proof. By induction on $d$, the case $d=0$ being trivial.

Assume $d>0$. As $x_{1}, \ldots, x_{r}$ is a multiplicity system on $M$,

$$(x_{1}, \ldots, x_{r})R \notin \bigcup_{P \in \text{Ass}(M)} P .$$

$P$ non-maximal

Thus, by Theorem 124 of [1], there exist $\lambda_{2}, \ldots, \lambda_{r}$ in $R$

such that $x=x_{1}^{\lambda_{2}}x_{2}^{\lambda_{2}} \ldots + x_{r}^{\lambda_{r}} \notin \bigcup_{P \in \text{Ass}M} P$. In particular

$P$ non-maximal

$\dim(0:x^{n})<d$ for all $n$, since

$M$
Ass(0:x^n) = AssHom(R/\langle x^n \rangle, M) = V(x^n) \cap AssM = V(x) \cap AssM,

and if P \in \text{Supp}(M) has coheight d, then x \notin P.

Now, from the expression for x, we observe that 

\((x_1, \ldots, x_r)_R = (x, x_2, \ldots, x_r)_R\),

so that \(x, x_2, \ldots, x_r\) is also a multiplicity system on \(M\) and that \(x^n = \nu_1 x_1^n + \cdots + \nu_r x_r^n\)

for some \(\nu_1, \ldots, \nu_r\) (depending on \(n\)). But we have

\[ \mu H_1 K(x_1^n, \ldots, x_r^n | M) \leq \mu H_1 K(\nu_1 x_1^n, x_2^n, \ldots, x_r^n | M) = \mu H_1 K(x_1^n, x_2^n, \ldots, x_r^n | M), \]

the inequality by virtue of lemma 1 and the equality due to the fact that the invertible matrix

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
\nu_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\nu_r & 0 & \cdots & 1
\end{pmatrix}
\]

together with its exterior powers establish an isomorphism between \(K.(x_1^n, x_2^n, \ldots, x_r^n | M)\) and \(K.(\nu_1 x_1^n, x_2^n, \ldots, x_r^n | M)\) and so it is enough to prove the theorem with \(x_1^n\) replaced by \(x^n\).

Take \(t\) large enough so that in the exact sequence

\[ 0 \to 0:x^{rt}_M \to M_M \to M/(0:x^{rt}_M) = \overline{M}_M \to 0 \]

\(x^n\) be \(\overline{M}\)-regular. From the Koszul homology sequence

\[ H_1 K(x_1^n, x_2^n, \ldots, x_r^n | 0:x^{rt}_M) \to \]

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\[ \rightarrow H_1 K(x^n_1, x^n_2, \ldots, x^n_r|\overline{M}) \rightarrow H_1 K(x^n_2, \ldots, x^n_r|\overline{M}) \]

we have, for \( n \geq t \),

\[ \leq H_1 K(x^n_2, \ldots, x^n_r|\overline{M}) \leq \]

\[ \leq H_1 K(x^n_2, \ldots, x^n_r|0:x^r t) + H_1 K(x^n_2, \ldots, x^n_r|\overline{M}). \]

The first summand on the right is bounded by induction by a polynomial in \( n \) of degree at most \( d-1 \). As to the second, since \( x \) is \( \overline{M} \)-regular, we have

\[ \leq H_1 K(x^n_2, \ldots, x^n_r|\overline{M}) = H_1 K(x^n_2, \ldots, x^n_r|\overline{M}/x^n_r) \quad \text{(by [3]p.368)} \]

\[ \leq n \cdot H_1 K(x^n_2, \ldots, x^n_r|\overline{M}/x^n_r) \quad \text{(by lemma 2),} \]

and as \( \dim(\overline{M}/x^n_r) < \dim(\overline{M}) = \dim(M) \), by induction also,

\[ \leq H_1 K(x^n_2, \ldots, x^n_r|\overline{M}/x^n_r) \leq \text{polynomial in } n \text{ of degree strictly less than } d. \]

This concludes the proof. #

Remarks. 1. In case \( r=d \), for instance, if \( x_1, \ldots, x_d \) is a system of parameters for \( M \), then a stronger result is obtained from [2]. There it is proved that \( H_1 K(x^n_1, \ldots, x^n_r|M) \) is bounded by a polynomial in \( n \) of degree at most \( r-1 \). So the importance of the theorem above becomes clear when \( r \) takes large values.

2. A proof by double induction on \( d \) and \( r \) can be given to theorem 3 if we bear in mind that [2] ensures the case \( r=d \).
Corollary 4. Let $M$ and $x_1, \ldots, x_r$ be as in the theorem. Then the functions

$$n \mapsto x_i^n(x_1^n, \ldots, x_r^n|M)$$

where $x_i$ stands for the $i$'th Euler-Poincaré characteristic (cf. [4] App. II), are bounded above by polynomials in $n$ of degree not exceeding the dimension of $M$. #

Remark. The results above carry over almost immediately to the case of a semilocal ring by means of the formulae

$$\text{length}_R(N) = \sum_{\text{maximal } m} \text{length}_{R_m}(N_m),$$

for any module $N$ of finite length, and $\dim_R M = \sup_{\text{maximal } m} (\dim_{R_m}(M_m))$, together with the fact that taking homology commutes with localisation, i.e., in our case $H_i K(x_1^n, \ldots, x_r^n|M) = H_i K(x_1^n, \ldots, x_r^n|M_m)$. In fact, if the Hilbert-Samuel polynomial function associated with $(M,I)$ has degree $g$ (where $g \leq d$), then, for each $i \geq 0$, both $H_i K(x_1^n, \ldots, x_r^n|M)$ and $x_i^n(x_1^n, \ldots, x_r^n|M)$ are bounded above by polynomials in $n$ of degree at most $g$.

REFERENCES


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Key words, Hilbert polynomial functions, multiplicity system, Koszul complex.