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THE SURFACE C–C ON JACOBI VARIETIES AND 2nd ORDER THETA FUNCTIONS

by

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Introduction

In their preprint [3], B. van Geemen and G. van der Geer stated four conjectures dealing with the modular significance of the surface C–C on a Jacobi variety. The first of these conjectures can be rephrased as follows:

(0.1) Conjecture ([3]). Let X be the jacobian of an irreducible non-singular algebraic curve C over $k=\mathbb{C}$, of genus $g \geq 1$. Let Γ_{00} be the vector space of sections of $\mathcal{O}_X(2\,\Theta)$ (Θ a symmetric theta divisor) having a zero of multiplicity at least 4 at $0 \in X$, and write $F_X = \{ x \in X \mid s(x)=0 \text{ for all } s \in \Gamma_{00} \}$. Then $F_X = \{x-y \mid x,y \in C\}$.

In [3] the above authors give several partial results in this direction. Quite simultaneously, R.C. Gunning considered also this question in his preprint [7], getting partial results, too (cf. also (2.1) below). Thirdly, in his book [12], D. Mumford asked (we change some notations):

(0.2) Question ([12], p. 3.238). If D is a divisor class of degree 0 on C such that for all divisors E of degree g-1 for which $|E|$ is a pencil, then either $|D+E| \neq \emptyset$ or $|-D+E| \neq \emptyset$, then does it follow that $D \equiv a-b$ for some $a,b \in C$?

By standard reasons (cf. Section 2), a positive answer to (0.2) would imply (0,1) (Actually, the answer to (0.2) is known to be negative if C is a trigonal curve).

In this connection it is natural to ask also:

(0.3) Question. If D is a divisor class of degree 0 on C such that for all divisors E of degree g-1 for which $|E|$ is a pencil, then $|D+E| \neq \emptyset$, then does it follow that $D \equiv a-b$ for some $a,b \in C$?

For example, if $W^1_{g-1}(C)$ is irreducible, Questions (0,2) and (0.3) are the same. Now, in [14] M. Teixidor has shown that, except for trigonal curves, superelliptic curves and some curves of genus 5, $W^1_{g-1}(C)$ is irreducible. In this way, the seemingly more accesible Question (0.3) almost dominates the picture.

In the present paper we give a complete answer to Conjecture (0.1) and Questions (0.2) and (0.3). In Section 1 we show that (0.3) is true if $g \geq 5$ (cf. Theorem (1.1)). The proof is cohomological and uses ideas of M. Green ([4]). In Section 2 the relation between (0.1), (0.2) and (0.3) is discussed. Section 3 deals with superelliptic curves, completing the answer to (0.2) (cf. Theorem (2.4)). Finally, Section 4 is devoted to the study of trigonal curves, completing the proof of (0.1) for $g \geq 5$ (Corollary (2.5)). In Proposition (4.14) we discuss the case in which (0.1) turns out to be false (g=4).

(0.4) Convention. Throughout, when speaking of a trigonal curve, it will be assumed implicitly that it is non-hyperelliptic.

§1.

Let C be an irreducible smooth complete curve of genus $g \geq 5$ over $k=\mathbb{C}$. Let $\Theta \subset JC$ be a copy of the theta divisor of the polarized jacobian of C, and denote by $C-C \subset JC$ the surface consisting of the differences $x-y \in JC$ for all $x, y \in C$.

(1.1) Theorem. (We assume $g \geq 5$). The following equality holds in JC:

$$C-C = \{ a \in JC \mid a + \mathrm{Sing}\,\Theta \subset \Theta \}.$$

(1.2) Remark. There are canonical models of Θ and $\mathrm{Sing}\,\Theta$ in $\mathrm{Pic}^{g-1}(C)$, given respectively by W^o_{g-1} and W^1_{g-1} (Riemann Parametrization Theorem and Riemann Singularity Theorem, cf. [8]). The natural scheme structure of W^o_{g-1} and W^1_{g-1} given by Brill-Noether Theory is reduced (for W^1_{g-1} this holds because of the condition $g \geq 5$, cf. e.g. [14]). Therefore, in writing $a+\mathrm{Sing}\,\Theta \subset \Theta$, it makes no difference to consider this as a set-theoretical statement or a scheme-theoretical one.

(1.3) Remark. The statement of (1.1) thus reads: $W^o_1 - W^o_1 = \{ a \in \mathrm{Pic}^o(C) \mid a + W^1_{g-1} \subset W^o_{g-1} \}$. This is reminiscent of the well-known identities (cf. e.g. [9]): $\{ a \in \mathrm{Pic}^{d'}(C) \mid a + W^o_{d-d'} \subset W^o_d \} = W^o_{d'}$, $(0 \leq d' \leq d \leq g-1)$. It would be quite interesting to know if more general equalities of this type hold between other W^r_d's (at least as long as the Brill-Noether number remains non-negative). For example, one could ask for a comparison between $W^o_k - W^o_k$ and $\{ a \in \mathrm{Pic}^o(C) \mid a + W^r_d \subset W^{r-k}_d \}$, $0 \leq k \leq r$. We shall not consider these questions here.

Proof of (1.1). Clearly $C-C \subset \{ a \in JC \mid a + \mathrm{Sing}\,\Theta \subset \Theta \}$.

(1.4) If C is hyperelliptic, the result is easy: $W^1_{g-1} = g^1_2 + W^o_{g-3}$, hence, if $a+\mathrm{Sing}\,\Theta \subset \Theta$, we have $(a+g^1_2)+W^o_{g-3} \subset W^o_{g-1}$. Therefore, as recalled in (1.3), $a+g^1_2 \in W^o_2$. From this one concludes $a \in W^o_1 - W^o_1$.

(1.5) Although we shall not need to make this distinction, we give an independent proof of (1.1) for a trigonal curve C, because it is elementary, too. Here ([1]) $W^1_{g-1} = (g^1_3+W^o_{g-4}) \cup (K-g^1_3-W^o_{g-4})$. If $a + \mathrm{sing}\,\Theta \subset \Theta$ one deduces as above that $a+g^1_3 \in W^o_3$ and that $-a+g^1_3 \in W^o_3$. Writing $a = D_3 - g^1_3 = g^1_3 - D'_3$ with $D_3, D'_3 \in W^o_3$, it follows that $D_3+D'_3 = 2g^1_3$. Now $h^o(2g^1_3) \geq 3$, but $h^o(2g^1_3) = 4$ would imply that C has a g^3_6, which is impossible (Clifford) unless $g=4$ (and $2g^1_3=K$) or C is hyperelliptic. Forgetting about these cases, it follows that $D_3+D'_3 = A+B$ with $A,B \in g^1_3$. Thus either D_3 or D'_3 contain two points of a member of the g^1_3, hence $a \in W^o_1 - W^o_1$, as claimed.

If $g=4$, and C is non-hyperelliptic, and $2g^1_3=K$ (i.e. C has a vanishing Thetanullwert) the set $\{a \in \mathrm{Pic}^o(C) \mid a+\mathrm{Sing}\,\Theta \subset \Theta\}$ equals $W^o_3 - g^1_3$: the point is that here the "right" scheme structure for $\mathrm{Sing}\,\Theta$ is non-reduced.

For completeness sake : the statement of (1.1) is rather meaningless if $g=1,2$ or if $g=3$ and C is non-hyperelliptic, since $\mathrm{Sing}\,\Theta$ is empty in these cases. If $g=3$ and C is hyperelliptic, the proof in (1.4) goes through.

(1.6) In the rest of § 1 it will be assumed that C is a non-hyperelliptic curve of genus $g \geq 5$. We shall use ideas of M. Green ([4]).

The variety W^1_{g-1} is of pure dimension g-4 ([1]) and, as recalled above, it is reduced. Define the subscheme $Z \subset C^{(g-1)}$ by the pull-back diagram

$$\begin{tikzcd} C^{(g-1)} \arrow[r] & \mathrm{Pic}^{g-1}(C) \\ Z \arrow[u, hook] \arrow[r] & W^1_{g-1} \arrow[u, hook] \end{tikzcd} .$$

We shall write $\widetilde{\Theta} = W^o_{g-1} \subset \mathrm{Pic}^{g-1}(C)$ and, for all $b \in JC = \mathrm{Pic}^o(C)$: $\widetilde{\Theta}_b = W^o_{g-1}+b \subset \mathrm{Pic}^{g-1}(C)$. Also, when using the symbols $\widetilde{\Theta}$, $\widetilde{\Theta}_b$ in connection with other varieties (e.g. $C^{(g-1)}$) they will mean the divisor classes on these varieties gotten by pullback of $\widetilde{\Theta}$, $\widetilde{\Theta}_b$.

(1.7) Lemma. Let $b \in \mathrm{Pic}^o(C)$, $b \neq 0$. One has:

$$H^o\mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b) \cong k; \qquad H^i\mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b) = 0 \qquad \text{for} \quad i > 0.$$

Proof. The map $C^{(g-1)} \longrightarrow \widetilde{\Theta} \subset \mathrm{Pic}^{g-1}(C)$ is a rational resolution ([8]), hence $H^i\mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b) \cong H^i\,\mathcal{O}_{\widetilde{\Theta}}(\widetilde{\Theta}_b)$ for all i. It suffices then to use the exact sequence on $\mathrm{Pic}^{g-1}(C)$:

$$0 \longrightarrow \mathcal{O}(\widetilde{\Theta}_b - \widetilde{\Theta}) \longrightarrow \mathcal{O}(\widetilde{\Theta}_b) \longrightarrow \mathcal{O}_{\widetilde{\Theta}}(\widetilde{\Theta}_b) \longrightarrow 0,$$

plus the fact ([10]) that $H^i\mathcal{O}(\widetilde{\Theta}_b - \widetilde{\Theta}) = 0$ for all $i \geq 0$, Q.E.D.

(1.8) Assume from now on that $b \in \mathrm{Pic}^o(C)$, $b \neq 0$, satisfies $W^1_{g-1} \subset \widetilde{\Theta}_b$. We aim to show that $b \in C-C$.

From (1.7) and the exact sequence, on $C^{(g-1)}$,

$$0 \longrightarrow \mathcal{Y}_Z(\widetilde{\Theta}_b) \longrightarrow \mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b) \longrightarrow \mathcal{O}_Z(\widetilde{\Theta}_b) \longrightarrow 0,$$

we conclude that the assumption is stated equivalently by asking that $H^0\mathcal{Y}_Z(\widetilde{\Theta}_b) \neq 0$ ($\cong k$, in fact).

(1.9) From [4], §1, we recall that $\omega_{C^{(g-1)}} \cong \mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta})$ and that there is an exact sequence of sheaves on $C^{(g-1)}$:

$$0 \longrightarrow T_{C^{(g-1)}} \otimes \omega^{\vee}_{C^{(g-1)}} \longrightarrow (\omega^{\vee}_{C^{(g-1)}})^g \longrightarrow \mathcal{Y}_Z \longrightarrow 0.$$

This implies an exact sequence

$$(1.10)\quad 0 \longrightarrow T_{C^{(g-1)}} \otimes \mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b - \widetilde{\Theta}) \longrightarrow \mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b - \widetilde{\Theta})^g \longrightarrow$$

$$\longrightarrow \mathcal{Y}_Z(\widetilde{\Theta}_b) \longrightarrow 0.$$

Imitating the proof of Lemma (1.7) one finds that $H^i\mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b - \widetilde{\Theta}) \cong H^i\,\mathcal{O}_{\widetilde{\Theta}}(\widetilde{\Theta}_b - \widetilde{\Theta}) \cong H^{i+1}\,\mathcal{O}(\widetilde{\Theta}_b - 2\widetilde{\Theta})$. But, on $\mathrm{Pic}^{g-1}(C)$: $\widetilde{\Theta}_b - 2\widetilde{\Theta} \equiv -\widetilde{\Theta}_{-b}$, by the Theorem of the Square. Hence the latter vector space is isomorphic with $H^{i+1}\mathcal{O}(-\widetilde{\Theta}_{-b}) \cong$ $\cong (H^{g-i-1}\,\mathcal{O}(\widetilde{\Theta}_{-b}))^{\vee}$, by Kodaira-Serre duality. Using again [10] one obtains therefore:

$$(1.11)\quad H^i\mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b - \widetilde{\Theta}) = 0 \quad \text{if} \quad i \neq g-1, \qquad \cong k \quad \text{if} \quad i = g-1.$$

Using this together with (1.10), the assumption (1.8) can be stated equivalently as

$$(1.12)\quad H^1(T_{C^{(g-1)}} \otimes \mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}_b - \widetilde{\Theta})) \neq 0 \qquad (\cong k, \ \text{in fact}).$$

Consider now the diagram:

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] \arrow[dr, "p"'] & C^{(g-1)} \times C \arrow[r, "q"] \arrow[d] & C \\ & C^{(g-1)} & \end{tikzcd} ,$$

where $\mathcal{D}$ is the "universal divisor" for the Hilbert scheme $C^{(g-1)}$, i.e. $\mathcal{D} = \{(D_{g-1}, x) \mid x \leq D_{g-1}\}$. By the general theory of Hilbert schemes ([6]) there is an isomorphism of sheaves

$$(1.13) \quad T_{C^{(g-1)}} \cong R^{o}_{p}\,\mathcal{O}_{\mathcal{D}}(\mathcal{D}).$$

The morphism p being finite, we have, by the Projection Formula:

$$(1.14) \; H^1(T_{C^{(g-1)}} \otimes \mathcal{O}_{C^{(g-1)}}(\tilde{\theta}_b - \tilde{\theta})) \cong H^1\,\mathcal{O}_{\mathcal{D}}\,(\mathcal{D} + \tilde{\theta}_b - \tilde{\theta}).$$

(In agreement with an earlier convention, $\tilde{\theta}_b$ and $\tilde{\theta}$ in the right-hand side of (1.14) mean the divisor classes obtained on $\mathcal{D}$ by pullback via p; when occurring -in a moment- over $C^{(g-1)} \times C$, they are understood as being obtained by means of the projection map of this product space onto the first factor). Consider the exact sequence on $C^{(g-1)} \times C$:

$$(1.15) \; 0 \longrightarrow \mathcal{O}_{C^{(g-1)} \times C}(\tilde{\theta}_b - \tilde{\theta}) \longrightarrow \mathcal{O}_{C^{(g-1)} \times C}(\mathcal{D} + \tilde{\theta}_b - \tilde{\theta}) \longrightarrow$$

$$\longrightarrow \mathcal{O}_{\mathcal{D}}(\mathcal{D} + \tilde{\theta}_b - \tilde{\theta}) \longrightarrow 0.$$

By Künneth one has:

$$h^i\mathcal{O}_{C^{(g-1)} \times C}(\tilde{\theta}_b - \tilde{\theta}) = \sum_{j=o}^{i} h^j(\mathcal{O}_{C^{(g-1)}}(\tilde{\theta}_b - \tilde{\theta}) \cdot h^{i-j}\,\mathcal{O}_C \; .$$

Therefore one gets, by (1.11):

$$(1.16)\quad H^i\mathcal{O}_{C^{(g-1)}\times C}(\widetilde{\Theta}_b-\widetilde{\Theta}) = 0 \quad \text{if} \quad i \le g-2,$$
$$\cong k \quad \text{if} \quad i = g-1,$$
$$\cong k^g \quad \text{if} \quad i = g.$$

We deduce from (1.15) and (1.16) that

$$(1.17)\quad H^1\,\mathcal{O}_{\mathcal{D}}(\mathcal{D}+\widetilde{\Theta}_b-\widetilde{\Theta}) \;\cong\; H^1\mathcal{O}_{C^{(g-1)}\times C}(\mathcal{D}+\widetilde{\Theta}_b-\widetilde{\Theta}).$$

Applying the Leray Spectral sequence for the map q and the sheaf $\mathcal{O}_{C^{(g-1)}\times C}(\mathcal{D}+\widetilde{\Theta}_b-\widetilde{\Theta})$:

$$H^iR^j_q := H^iR^j_q\mathcal{O}_{C^{(g-1)}\times C}(\mathcal{D}+\widetilde{\Theta}_b-\widetilde{\Theta}) \;\Rightarrow\; H^{i+j}\,\mathcal{O}_{C^{(g-1)}\times C}(\mathcal{D}+\widetilde{\Theta}_b-\widetilde{\Theta}),$$

we obtain an exact sequence

$$(1.18)\quad 0 \longrightarrow H^1R^o_q \longrightarrow H^1\mathcal{O}_{C^{(g-1)}\times C}(\mathcal{D}+\widetilde{\Theta}_b-\widetilde{\Theta}) \longrightarrow H^oR^1_q \longrightarrow 0.$$

Combining (1.8), (1.12), (1.14), (1.17) and (1.18) we conclude that either $H^oR^1_q\neq 0$ or $H^1R^o_q\neq 0$.

For any $x\in C$, we shall denote by U_x, E_x the following divisors of $C^{(g-1)}$:

$$U_x = x + C^{(g-2)} = \{D_{g-1} \mid D_{g-1} \ge x\},$$
$$E_x = \{D_{g-1} \mid h^o(x+D_{g-1}) \ge 2\} = \{D_{g-1} \mid D_{g-1} \le |K-x|\}.$$

It is a standard fact (of easy proof) that, if $x,y\in C$, $x\neq y$, one has in $C^{(g-1)}$:

$$|\widetilde{\Theta}_{x-y}| = \{U_x + E_y\}$$

(cf. Lemma (1.7)). Taking limits as (x,y) tends to the diagonal of $C \times C$ it follows, for all $x \in C$, in $C^{(g-1)}$:

(1.19) $\quad \widetilde{\Theta} \equiv U_x + E_x.$

(Alternatively, to get (1.19) one could have used the fact that $\mathcal{O}_{C^{(g-1)}}(\widetilde{\Theta}) \cong \omega_{C^{(g-1)}}$ (cf. (1.9)), plus the description of the canonical divisors of $C^{(g-1)}$ as $\{D_{g-1} \mid D_{g-1} \leq \Lambda\}$, when $\Lambda \subset |K_C|$ runs through the codimension one subsystems of the canonical system of C).

(1.20) __Lemma__. The sheaf $R^o_q \mathcal{O}_{C^{(g-1)} \times C}(\mathcal{D} + \widetilde{\Theta}_b - \widetilde{\Theta})$ is concentrated at a finite set of points of C, and $H^1 R^o_q = 0$.

(Actually, by (1.18) (cf. (1.8)), if non-zero, this sheaf is concentrated at a single point of C and has stalk isomorphic with k there).

__Proof__. Since $\mathcal{D}$ intersects the fibre of q above $x \in C$ giving U_x, it is sufficient to show that $H^o \mathcal{O}_{C^{(g-1)}}(U_x + \widetilde{\Theta}_b - \widetilde{\Theta}) = 0$ for general $x \in C$.

Assume that, for some $x \in C$, $H^o \mathcal{O}_{C^{(g-1)}}(U_x + \widetilde{\Theta}_b - \widetilde{\Theta}) \neq 0$. Fixing any $D_{g-2} \in C^{(g-2)}$, the restriction of $\mathcal{O}_{C^{(g-1)}}(U_x + \widetilde{\Theta}_b - \widetilde{\Theta})$ to the curve $D_{g-2}+C$ equals $\mathcal{O}_C(x+b)$. Hence $H^o \mathcal{O}_C(x+b) \neq 0$, which implies $b=y-x$ for some $y \in C$.

It follows that, if $H^o \mathcal{O}_{C^{(g-1)}}(U_x + \widetilde{\Theta}_b - \widetilde{\Theta}) \neq 0$ for all $x \in C$, one would have $b + W^o_1 \subset W^o_1$, hence, as recalled in (1.3), $b=0$. But, by assumption, $b \neq 0$, Q.E.D.

(1.21) The above now implies, together with (1.18), that $H^oR^1_q \neq 0$. Thus $R^1_q \neq 0$ and, a fortiori, there exists $x \in C$ such that $H^1\mathcal{O}_{C^{(g-1)}}(U_x + \tilde{\Theta}_b - \tilde{\Theta}) \neq 0$. We shall show that this implies $H^o\mathcal{O}_{C^{(g-1)}}(U_x + \tilde{\Theta}_b - \tilde{\Theta}) \neq 0$. By the reasoning made in the proof of (1.20), this will imply finally that $b \in C-C$, thereby ending the proof of (1.1).

We start recalling that, by (1.19), $\mathcal{O}_{C^{(g-1)}}(U_x + \tilde{\Theta}_b - \tilde{\Theta}) \cong$ $\cong \mathcal{O}_{C^{(g-1)}}(\tilde{\Theta}_b - E_x)$. Consider the exact sequence, on $C^{(g-1)}$:

$$0 \longrightarrow \mathcal{O}_{C^{(g-1)}}(\tilde{\Theta}_b - E_x) \longrightarrow \mathcal{O}_{C^{(g-1)}}(\tilde{\Theta}_b) \longrightarrow \mathcal{O}_{E_x}(\tilde{\Theta}_b) \longrightarrow 0 .$$

By Lemma (1.7) we obtain an exact sequence

$$(1.22) \quad 0 \longrightarrow H^o\mathcal{O}_{C^{(g-1)}}(\tilde{\Theta}_b - E_x) \longrightarrow H^o\mathcal{O}_{C^{(g-1)}}(\tilde{\Theta}_b) \xrightarrow{\rho}$$
$$\longrightarrow H^o\mathcal{O}_{E_x}(\tilde{\Theta}_b) \longrightarrow H^1\mathcal{O}_{C^{(g-1)}}(\tilde{\Theta}_b - E_x) \longrightarrow 0,$$

and $\dim H^o\mathcal{O}_{C^{(g-1)}}(\tilde{\Theta}_b) = 1$. It suffices to show that the restriction map ρ is zero.

Supppose that this were not the case. Then, in particular, $E_x \not\subset \tilde{\Theta}_b$ in $C^{(g-1)}$. If we show that, under these assumptions, $\dim H^o\mathcal{O}_{E_x}(\tilde{\Theta}_b) \leq 1$, then, in view of (1.22), this will contradict the fact that $H^1\mathcal{O}_{C^{(g-1)}}(U_x + \tilde{\Theta}_b - \tilde{\Theta}) \neq 0$.

Now, E_x is the pullback to $C^{(g-1)}$ of $K-x-W^o_{g-2} \subset \mathrm{Pic}^{g-1}(C)$. The map $\mathrm{Pic}^{g-2}(C) \longrightarrow \mathrm{Pic}^{g-1}(C)$ given by $\xi \longmapsto K-x-\xi$ is an isomorphism, and the inverse image of $\tilde{\Theta}_b$ under this map is $K-x-(\tilde{\Theta}+b) = \tilde{\Theta}_{-x-b}$. Since $E_x \not\subset \tilde{\Theta}_b$, it follows that

$W^o_{g-2} \not\subset \widetilde{\Theta}_{-x-b}$.

We recall that, for all $\zeta \in \mathrm{Pic}^{g-2}(C)$, $\mathcal{O}_C(\widetilde{\Theta}_{-\zeta}) = \mathcal{O}_C(K-\zeta)$ (Jacobi Inversion) and that $C \subset \widetilde{\Theta}_{-\zeta}$ is equivalent to $h^o\mathcal{O}_C(K-\zeta) \geq 2$, hence to $h^o\mathcal{O}_C(\widetilde{\Theta}_{-\zeta}) \geq 2$. Thus, if $D_{g-3} \in C^{(g-3)}$ is such that $D_{g-3}+C \not\subset \widetilde{\Theta}_{-x-b}$, we get $\dim H^o\mathcal{O}_{D_{g-3}+C}(\widetilde{\Theta}_{-x-b}) = 1$. A fortiori, $H^o\mathcal{O}_{C^{(g-2)}}(\widetilde{\Theta}_{-x-b}) \cong k$ hence (cf. [8])

$$H^o\mathcal{O}_{E_x}(\widetilde{\Theta}_b) \cong H^o\mathcal{O}_{K-x-W^o_{g-2}}(\widetilde{\Theta}_b) \cong H^o\mathcal{O}_{W^o_{g-2}}(\widetilde{\Theta}_{-b-x}) \cong$$
$$\cong H^o\mathcal{O}_{C^{(g-2)}}(\widetilde{\Theta}_{-b-x})$$

has dimension 1, as claimed. This ends the proof of Theorem (1.1), Q.E.D.

§2.

(2.1) In their preprint [3] Van Greemen and Van der Geer notice in particular that for a 2^{nd} order theta function (corresponding to a choice of a Riemann matrix for C) to have a zero of multiplicity ≥ 4 at the origin is equivalent with the fact of vanishing identically along the canonical locus C–C of JC (cf. also (4.8) below). Motivated by a series of partial results, they conjecture that the locus C–C coincides with the set of common zeroes of the above functions.

We shall use (1.1) and a recent result of M. Teixidor ([14]) to show that the conjecture of Van Geemen and Van der Geer (Conjecture 1 of [3]) holds true if $g \geq 5$. (For g=3 it is known to be true -cf. [3], also recalled in (4.17) below-; for g=4 it is false, in general - cf. (4.14)).

This kind of questions have been considered also by R.C. Gunning in [7]. The reader will find there partial results in this direction, as well as concerning the deeper question of the scheme-theoretical intersection of the divisors determined by the above functions.

We shall prove also a result intermediate between (1.1) and the above one, which answers Question (0.2) of the Introduction. (cf. (2.4), (2.5) and (2.6), for the main statements).

(2.2) Theta functions of second order (with zero characteristics) correspond to sections of the line bundle associated with the divisor $2\,\Theta$, where Θ is any symmetric theta divisor. The image of the (irreducible principally polarized) abelian variety by the corresponding map into projective space, $\mathbb{P}^{2^g-1}$, is the associated Kummer-Wirtinger variety. We would like therefore to call the system $|2\,\Theta|$ the Kummer-Wirtinger system.

We keep the notations of Section 1, recalling in particular that

$$\widetilde{\Theta} = \{\zeta_{g-1} \in \mathrm{Pic}^{g-1}(C) \mid h^0\mathcal{O}_C(\zeta_{g-1}) \geq 1\} \subset \mathrm{Pic}^{g-1}(C)$$

is a canonical model of the theta divisor of JC. The different translates of the theta divisor of JC are obtained by taking $\widetilde{\Theta}_{-\xi}$, as ξ varies in $\mathrm{Pic}^{g-1}(C)$. Notice that, if $\xi \in \mathrm{Pic}^{g-1}(C)$ and $\xi' := K-\xi \in \mathrm{Pic}^{g-1}(C)$ (K being, as before, the canonical class of C), then $\widetilde{\Theta}_{-\xi'}$ is the image of $\widetilde{\Theta}_{-\xi}$ under the symmetry of JC.

Therefore, by the Theorem of the Square, the divisors $D = \widetilde{\Theta}_{-\xi} + \widetilde{\Theta}_{-\xi'}$ belong to the Kummer system.

One has, for the multiplicity of D at the origin:

$$\mu_o(D) = \mu_\xi(\widetilde{\Theta}) + \mu_{\xi'}(\widetilde{\Theta}) = 2h^o\mathcal{O}_C(\xi),$$

by the Riemann Singularity Theorem and Riemann-Roch. Thus $\mu_o(D) \geq$ ≥ 4 if and only if $\xi \in \mathrm{Sing}(\widetilde{\Theta}) = W^1_{g-1}$. Consequently (cf. also below, (4.8), for the first inclusion):

$$\text{(2.3)} \quad C-C \subseteq \Big(\bigcap_{\substack{D\in|2\Theta| \\ \mu_o(D)\geq 4}} D\Big) \subseteq \bigcap_{\xi\in W^1_{g-1}} (\widetilde{\Theta}_{-\xi} + \widetilde{\Theta}_{-\xi'}).$$

We can state now:

(2.4) <u>Theorem</u>. Assume $g \geq 5$. Then

$$C - C = \bigcap_{\xi\in W^1_{g-1}} (\widetilde{\Theta}_{-\xi} + \widetilde{\Theta}_{-\xi'})$$

except if C is trigonal; in the latter case the right-hand side member equals $(W^o_3 - g^1_3) \cup (g^1_3 - W^o_3)$.

(2.5) <u>Corollary</u>. Assume $g \geq 5$. Then

$$C - C = \bigcap_{\substack{D\in|2\Theta| \\ \mu_o(D)\geq 4}} D .$$

(2.6) <u>Remarks</u>. (i) If $g=1,2$, (2,4) and (2,5) are meaningless -as stated here.

(ii) If $g=3$, then (2.4) makes sense iff C is hyperelliptic, and in this case the identity holds (cf. below). As for (2.5), it has been proved in this case in [3] (cf. (4.17)).

(iii) If $g=4$, the statement of (2.4) holds verbatim, provided one applies strictly Convention (0.4), i.e. reading "trigonal" as "non-hyperelliptic". The symbol g_3^1 then means any one of the two (possibly coincident) series of this type on C. Corollary (2.5) is true except if C is a non-hyperelliptic curve of genus 4 without vanishing Thetanullwert (cf (4.14)).

As announced earlier, the main ingredient to derive (2.4) from (1.1) is a result by M. Teixidor ([14]). By using ideas of Fulton and Lazarsfeld ([2]), it is proved in Loc. Cit.:

(2.7) **Theorem** ([14]). Let C be a smooth algebraic curve, irreducible, of genus $g \geq 5$, over $k=\mathbb{C}$. Then $W^1_{g-1}=\mathrm{Sing}(\tilde{\Theta})$ is reduced, and it is irreducible except in the following cases:

(a) C is trigonal;

(b) C is superelliptic;

(c) C is an étale double cover of a non-hyperelliptic genus 3 curve (hence $g=5$ in this case).

Let us see how (2.7) implies (2,4), except for the cases (b) and (c). Write $X_1,\ldots,X_r$ for the irreducible components of $W^1_{g-1} = \mathrm{Sing}(\tilde{\Theta})$. Reflection with respect to $K \in \mathrm{Pic}^{2g-2}(C)$ permutates these components. We write $X_i' = K-X_i$. Then, since $z \in \tilde{\Theta}_{-\xi}+\tilde{\Theta}_{-\xi'}$ is equivalent to $\xi+z \in \tilde{\Theta}$ or $\xi'+z \in \tilde{\Theta}$, one has:

(2.8) $\bigcap_{\xi \in W^1_{g-1}} (\tilde{\Theta}_{-\xi} + \tilde{\Theta}_{-\xi'}) =$

$= \{ z \in JC \mid \text{for all } i: \quad X_i + z \subset W^o_{g-1} \quad \text{or} \quad X'_i + z \subset W^o_{g-1} \}.$

It C does not belong to the types (a), (b), (c) of (2.7) one has $i=1$ and $X_1 = X'_1 = W^1_{g-1}$, hence (2.4) reduces to (1.1), which has been proved already. If C is trigonal, we know by (1.5) that $i=2$ and $X'_1 = X_2$, hence the right-hand side of (2.8) equals

$$\{ z \in JC \mid X_1 + z \subset W^o_{g-1} \} \cup \{ z \in JC \mid X_2 + z \subset W^o_{g-1} \}.$$

By (1.3) and (1.5), the conclusion of Theorem (2.4) holds in this case.

It remains to consider cases (b), (c) of (2.7). We shall devote Section 3 to their study. We shall see in particular that:

(2.9) Fact. In cases (b), (c) of (2.7), every irreducible component of W^1_{g-1} is fixed by the reflection with respect to K.

Thus the right hand side of (2.8) equals the right hand side of (1.1), and this will finish the proof of (2.4).

(2.10) Corollary (2.5) then follows, by (2.3), except if C is a trigonal curve. In this case we shall need a better understanding of second order theta divisors on jacobians. We shall study this in Section 4. The idea is that, although we dont know explicitly any other divisors of $|2\Theta|$ except those of type $\tilde{\Theta}_{-\xi} + \tilde{\Theta}_{-\xi'}$, we become more rich in geometrical descriptions when looking at the traces of the divisors of $|2\Theta|$ on $C^{(d)}$, $d < g$, and particular-ly for d=g-1. We shall get enough insight to show that the system cut out by $|2\,\Theta|$ on $W_3^o - g_3^1$ has precisely C-C as its basis locus. By symmetry, this will finish the proof of Corollary (2.5).

§ 3.

In this section we study the irreducible components of $W_{g-1}^1 = \mathrm{Sing}\,\tilde{\Theta}$ for superelliptic curves. (Some aspects have been considered already in [13]). We shall settle in particular Fact (2.9), thereby finishing the proof of Theorem (2.4). We keep the assumption $g \geq 5$.

(3.1) Let $\pi : C \longrightarrow E$ be a (2:1) morphism of smooth curves, with E an elliptic curve. By Zeuthen-Hurwitz, the discriminant divisor Δ of π is a divisor on E of degree 2g-2 and, moreover, the branch divisor B is a canonical divisor of C.

Once E is given, the curve C is determined by Δ and a (unique) element $\alpha \in \mathrm{Pic}^{g-1}(E)$ satisfying $2\alpha \equiv \Delta$. In the language of schemes, $C = \mathrm{Spec}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-\alpha))$, where the $\mathcal{O}_E$-algebra structure for $\mathcal{O}_E \oplus \mathcal{O}_E(-\alpha)$ is determined by the map $\mathcal{O}_E(-\alpha) \otimes \mathcal{O}_E(-\alpha) \cong \mathcal{O}_E(-\Delta) \longrightarrow \mathcal{O}_E$ given by multiplication with an equation for Δ. We shall write i for the superelliptic involution of C. Also, if D is a divisor on C, we write $\bar{D}$ its image in E.

(3.2) Let D_{g-1} be an effective divisor of degree g-1 on C. We may write it in a unique way as

$$D_{g-1} = \pi^{-1}(\bar{D}_a) + D_b, \qquad 2a+b = g-1,$$

where D_b does not contain inverse images of divisors of E. One may view sections of $\mathcal{O}_C(D_{g-1})$ as sections of $\mathcal{O}_C(\pi^{-1}(\bar{D}_a+\bar{D}_b))$ vanishing at iD_b. By the Leray Spectral Sequence one has:

(3.3) $H^o\mathcal{O}_C(\pi^{-1}(\bar{D}_a+\bar{D}_b)) = \pi^{-1}H^o\mathcal{O}_E(\bar{D}_a+\bar{D}_b) \oplus (\pi^{-1}H^o\mathcal{O}_E(\bar{D}_a+\bar{D}_b-\alpha))T$,
where $T \in H^o\mathcal{O}_C(B)$ is an equation for B (Note that $B \equiv \pi^*\alpha$).

(3.4) The class $\bar{D}_a+\bar{D}_b-\alpha$ has degree $-a$. Therefore, by (3.3), if $a>0$, $H^o\mathcal{O}_C(D_{g-1})$ can be identified with the space of sections of $\pi^{-1}H^o\mathcal{O}_E(\bar{D}_a+\bar{D}_b)$ vanishing at iD_b, that is, with $H^o\mathcal{O}_E(\bar{D}_a)$. Therefore, if $a>0$, one has $\dim|D_{g-1}| \geq 1$ if and only if $a \geq 2$.

On the other side, if $a=0$ ($\bar{D}_a=0$), the space $H^o\mathcal{O}_E(\bar{D}_a+\bar{D}_b-\alpha)$ is zero unless $\bar{D}_a+\bar{D}_b-\alpha \equiv 0$. If this vector space is zero, one may identify again $H^o\mathcal{O}_C(D_{g-1})$ with the space of sections of $\pi^{-1}H^o\mathcal{O}_E(\bar{D}_a+\bar{D}_b)$ vanishing at iD_b, getting: $|D_{g-1}| = \{D_{g-1}\}$.

(3.5) Consider again the subscheme $Z \subset C^{(g-1)}$ of (1.6). It is of pure dimension g-3 and, as shown in [4], it is reduced. We shall treat it therefore as a variety. Write

$$Z' = \{\pi^{-1}\bar{D}_2 + D_{g-5} \mid \bar{D}_2 \in E^{(2)},\quad D_{g-5} \in C^{(g-5)}\}.$$

This is an irreducible subvariety of $C^{(g-1)}$, of dimension g-3, thus it is an irreducible component of Z. By cohomological reasons, Z has other irreducible components (cf. [13]): If $D_{g-3} \in C^{(g-3)}$ is general, then $D_{g-3}+C^{(2)}$ intersects Z' with multiplicity $\frac{1}{2}(g-3)(g-4)$; on the other side, by Brill-Noether theory ([5]) $(D_{g-3}+C^{(2)})\cdot Z = \frac{1}{2}g(g-3)$. Since we are assuming $g \geq 5$, the claim follows.

(3.6) Proposition. If $g \geq 6$, Z has precisely two irreducible components: Z' and $Z'' = \{D_{g-1} \mid \bar{D}_{g-1} \equiv \alpha, \ \dim |D_{g-1}| \geq 1\}$.

Proof. Write $Z = Z' \cup Z''$ with Z'' the union of the remaining irreducible components of Z. By (3.4), Z'' is contained in the divisor

$$Y'' = \{D_{g-1} \mid \bar{D}_{g-1} \equiv \alpha\} \quad \subset \quad C^{(g-1)}.$$

On the other side, Z' is contained in the divisor

$$Y' = \{\pi^{-1}\bar{x} + D_{g-3} \mid \bar{x} \in E, \ \ D_{g-3} \in C^{(g-3)}\} \subset C^{(g-1)}.$$

Call A the subvariety of $C^{(g-1)}$ gotten by intersecting set-theoretically Z' and Y'', i.e.:

$$A = \{\pi^{-1}\bar{D}_2 + D_{g-5} \mid 2\bar{D}_2 + \bar{D}_{g-5} \equiv \alpha\} .$$

(3.7) Lemma. If $g \geq 6$ then A is irreducible (of dimension $g-4$).

Proof. Given $\bar{D}_2 \in E^{(2)}$ and $D_{g-6} \in C^{(g-6)}$ there exists a unique point $\bar{x} \in E$ such that $2\bar{D}_2 + \bar{D}_{g-6} + \bar{x} \in |\alpha|$. This defines a morphism $f\colon \ E^{(2)} \times C^{(g-6)} \longrightarrow E$. Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C \arrow[d, "\pi"] \\ E^{(2)} \times C^{(g-6)} \arrow[r, "f"'] & E \end{tikzcd}$$

The image of P in $E^{(2)} \times C^{(g-5)}$ maps onto A. It suffices therefore to show that P is irreducible. Now, P is a (2:1) covering of $E^{(2)} \times C^{(g-6)}$ and a necessary condition for it to be reducible is that every component of $f^*\Delta$ has even multiplicity. By direct inspection, however, this is found to be not the case. (If $g=6$ this is immediate, if $g \geq 7$ one can use e.g. restriction of f to curves $\bar{D}_2 \times (D_{g-7}+C)$). This proves (3.7), Q.E.D.

(3.8) Lemma. If $g \geq 6$, every irreducible component of Z'' contains A.

Proof. It follows from (3.4) that $Z \cap Y' = Z'$, hence that $Z \cap Y' \cap Y'' = A$. Let Z''_i be an irreducible component of Z''. If $Z''_i \cap Y' \neq \emptyset$, then $\dim Z''_i \cap Y' = g-4$ (Intersection Formula) and therefore $Z''_i \cap Y' \subset Z \cap Y'' \cap Y' = A$ implies $Z''_i \cap Y' = A$, by (3.7).

It remains to see that $Z''_i \cap Y' = \emptyset$ is impossible. Suppose that $Z''_i \cap Y' = \emptyset$. In the first place, using (3.3) and arguing like in (3.4), one shows easily that, if $D_{g-1} \in Y''$, then $\dim|D_{g-1}| \geq 2$ if and only if $D_{g-1} = \pi^{-1}\bar{D}_3 + D_{g-7}$. The hypothesis $Z''_i \cap Y' = \emptyset$ therefore implies that, for all $D_{g-1} \in Z''_i$, $\dim|D_{g-1}| = 1$.

Hence: the image of Z''_i in $\mathrm{Pic}^{g-1}(C)$ is an irreducible component $\bar{Z}''_i$ of W^1_{g-1}, and Z''_i is its inverse image in $C^{(g-1)}$. Since the subvariety A of $C^{(g-1)}$ too is the inverse image of its image $\bar{A}$ in W^1_{g-1}, it follows that $\bar{Z}''_i$ would not meet $\bar{A}$. But $\bar{A}$ is the only codimension 1 component of the singular locus of W^1_{g-1} (cf. e.g. [14], Lemma 1), and, following Remark (1.8) of [2], we derive a contradiction with the connectivity result of that paper. Namely, since the complement $W^1_{g-1} \setminus \bar{A}$ is Cohen-Macaulay and non-singular in codimension one, its connected components coincide with its irreducible components. So $\bar{Z}''_i$ ought to be a connected component of $W^1_{g-1} \setminus \bar{A}$ and, not meeting $\bar{A}$, it would be a connected component of W^1_{g-1}. But W^1_{g-1} is connected ([2]) and contains at least one more irreducible component other than $\bar{Z}''_i$, namely the image $\bar{Z}'$ of Z'. This contradiction concludes the proof of (3.8). Q.E.D.

To finish the proof of (3.6) it suffices, in view of (3.7) and (3.8), to show that Z has two branches at a general point of A.

One of these branches will correspond to Z'; the remaining ares correspond to the irreducible components of Z'' and ly therefore in Y''. Now, recall that Z is the basis locus of the (canonical) system of $C^{(g-1)}$, consisting of the divisors $E_\Lambda = \{ D_{g-1} \mid D_{g-1} \leq \Lambda \}$ as Λ runs through the codimension one subsystems of the canonical system of C (cf Section 1, immediately after (1.19)). We shall be done, therefore, by showing that, given a general point $D_{g-1} \in A$ there exists Λ such that E_Λ meets Y'' transversally at D_{g-1} (because then Z can have at most one branch at that point, contained in Y'').

This is an infinitesimal computation, like in Section 1 of [4]. Let $D_{g-1} = \pi^{-1}\bar{D}_2 + D_{g-5}$ be a fixed (general) point of A (thus $\bar{D}_{g-1} = 2\bar{D}_2+\bar{D}_{g-5} \in |\alpha|$). The divisor Y'' of $C^{(g-1)}$ is obtained by means of a pullback diagram

$$\begin{array}{ccc} Y'' & \hookrightarrow & C^{(g-1)} \\ \downarrow & & \downarrow \pi \\ |\alpha| & \hookrightarrow & E^{(g-1)} \end{array} .$$

Taking cotangent spaces and using standard deformation theory ([6]) we have a diagram

$$\begin{tikzcd}
 & & H^0\mathcal{O}_{D_{g-1}}(\omega_C \otimes \mathcal{O}_{D_{g-1}}) \arrow[d, equal] & & \\
 & T^\vee_{Y''}(D_{g-1}) & T^\vee_{C(g-1)}(D_{g-1}) \arrow[l] & & \\
0 & T^\vee_{|\alpha|}(\bar{D}_{g-1}) \arrow[l] \arrow[u] & T^\vee_{E(g-1)}(\bar{D}_{g-1}) \arrow[l] \arrow[u] \arrow[d, equal] & H^0\omega_E \arrow[l] & 0, \arrow[l] \\
 & & H^0\mathcal{O}_{\bar{D}_{g-1}}(\omega_E \otimes \mathcal{O}_{\bar{D}_{g-1}}) & &
\end{tikzcd}$$

with the square a pushout diagram. As the image of $H^0\omega_E$ in $H^0\mathcal{O}_{D_{g-1}}(\omega_C \otimes \mathcal{O}_{D_{g-1}})$ is non-zero (A non-zero element of $H^0\omega_E$ pulls back to a 1-form on C having B as divisor of zeroes), we deduce an exact sequence:

$$(3.9)\quad 0 \longleftarrow T^\vee_{Y''}(D_{g-1}) \longleftarrow H^0\mathcal{O}_{D_{g-1}}(\omega_C \otimes \mathcal{O}_{D_{g-1}}) \longleftarrow H^0\omega_E \longleftarrow 0.$$

This shows that Y'' is smooth at D_{g-1} and computes its cotangent space there.

As noticed earlier, the sheaves ω_C and $\pi^*\mathcal{O}_E(\alpha) \cong \mathcal{O}_C(B)$ are isomorphic. One must keep in mind, however, that their natural i-linearizations are opposite: The space of invariant 1-forms is $\langle\mu\rangle = \pi^{-1}H^0\omega_E$, while the antiinvariant subspace of $H^0\pi^*\mathcal{O}_E(\alpha)$ is $\langle T\rangle$, T being (cf above) an equation for the divisor B. Therefore the vector space of antiinvariant 1-forms, $(H^0\omega_C)^-$, corresponds with $(H^0\pi^*\mathcal{O}_E(\alpha))^+ = \pi^{-1}H^0\mathcal{O}_E(\alpha)$ under this isormorphism.

Using this, one finds that one may choose g-2 (linearly independent) elements of $(H^{o}\omega_C)^{-}$, $\lambda_1,\ldots,\lambda_{g-3}, \lambda$, such that, writing $\bar{D}_2 = Q_1+Q_2$, $\pi^{-1}Q_i = Q_i'+Q_i''$, $D_{g-5} = \sum_{i=1}^{g-5} P_i$:

(i) For $i=1,\ldots,g-5$: $\lambda_i(P_i)\neq 0$; $\lambda_i(x)=0$ if $x\in \mathrm{Supp}(D_{g-1})$, $x\neq P_i$;

(ii) $\lambda_{g-4}(Q_1')$, $\lambda_{g-4}(Q_1'')\neq 0$; $\lambda_{g-4}(x)=0$ if $x\in \mathrm{Supp}(D_{g-1})$, $x\neq Q_1',Q_1''$;

(iii) $\lambda_{g-3}(Q_2')$, $\lambda_{g-3}(Q_2'')\neq 0$; $\lambda_{g-3}(x)=0$ if $x\in \mathrm{Supp}(D_{g-1})$, $x\neq Q_2',Q_2''$;

(iv) $\lambda(x)=0$ for all x $\mathrm{Supp}(D_{g-1})$, and λ has a zero of order 1 either at Q_1',Q_1'' or at Q_2',Q_2'' (in fact: at all four points, a fortiori).

The equation $\lambda_1\wedge\ldots\wedge\lambda_{g-3}\wedge\lambda\wedge\mu = 0$ defines a divisor E_Λ of $|\mathcal{O}_{C^{(g-1)}}(E_{|K|})|$. Let z_1,z_2 be local coordinates of E at Q_1 and Q_2, and let z_1', z_1'' and z_2', z_2'' be the induced local coordinates of C at Q_1',Q_1'' and Q_2',Q_2'' respectively. Put $\mu(Q_i') = c_i dz_i'$ ($c_1,c_2\neq 0$, cf before (3.9)).

A straightforward computation shows that the cotangent space of E_Λ at D_{g-1} is given by the quotient of $H^{o}\mathcal{O}_{D_{g-1}}(\omega_C\otimes\mathcal{O}_{D_{g-1}}) = \bigoplus_{x\in \mathrm{Supp}(D_{g-1})} \omega_C(x)$ defined by the element:

$$c_1\frac{d\lambda}{dz_2'}(Q_2') + c_1\frac{d\lambda}{dz_2''}(Q_2'') - c_2\frac{d\lambda}{dz_1'}(Q_1') - c_2\frac{d\lambda}{dz_1''}(Q_1'').$$

Thus only four entries in $\oplus\,\omega_C(x)$ are involved, and at least one of these is non-zero. Since $(\mu(x))_{x\in \mathrm{Supp}(D_{g-1})}$ has all its entries non-zero, and $g\geq 6$, the claimed transversality follows, finishing the proof of (3.6). Q.E.D.

We conclude from this, by recalling again that $K_C \equiv \pi^*\alpha$:

(3.10) <u>Corollary</u>. Let C be a superelliptic curve of genus $g \geq 6$. Then W^1_{g-1} has exactly two irreducible components, and these are left fixed by the reflection with respect to $K \in \mathrm{Pic}^{2g-2}(C)$.

Finally:

(3.11) <u>Proof of Fact (2.9)</u>. By (3.10) it suffices to consider the case of a curve C of genus g=5, non-trigonal and non-hyperelliptic. In this case (cf [14]), e.g., for details) W^1_4 is a curve which is an admissable (2:1) covering (in the sense of Beauville) of a plane quintic Γ with ordinary double points (at worst). The superelliptic structures of C correspond with lines contained in Γ (if existent), and the curve belongs to Case (c) of (2.7) if and only if Γ contains a smooth conic. By the admissability of the covering map $W^1_4 \to \Gamma$, the irreducible components of W^1_4 are the inverse images of the irreducible components of Γ. The conclusion then follows from the fact that the map induced on W^1_4 by the reflection with respect to K coincides with the covering involution for $W^1_4 \longrightarrow \Gamma$, Q.E.D.

§ 4.

(4.1) The present section is devoted to the study of a few general facts about 2^{nd} order theta divisors on Jacobi varieties (quite well-known, but not easy to refer to), with special regard to the case of trigonal curves. Our aim is to prove Corollary (2.5) of Section 2 for trigonal curves, which is the only case left (cf (2.10)).

We keep the notations of Section2 and introduce furthermore the following one: Let $d > 0$ be fixed, and let Λ be a linear system on C, of dimension $\geq d-1$. We shall write E_Λ for any divisor of $C^{(d)}$ obtained as $\{ D_d \mid D_d \leq \Lambda' \}$, where $\Lambda' \subset \Lambda$ is some subsystem of Λ, of dimension d-1. More precisely, given $\Lambda' \subset \Lambda$, let $s_1, \ldots, s_d$ be a basis for the corresponding vector space of equations. Then the divisor E_Λ determined by Λ' is defined scheme-theoretically by the equation $s_1 \wedge \ldots \wedge s_d = 0$ on $C^{(d)}$. It is clear that the linear equivalence class of E_Λ in $C^{(d)}$ depends only on the linear equivalence class of the divisors of Λ in C.

(4.2) Fix a general element $\zeta_d \in \mathrm{Pic}^d(C)$, and consider the map $\mu_d: C^{(d)} \longrightarrow JC$ sending D_d to $D_d - \zeta_d$. Let $\xi, \xi' \in \mathrm{Pic}^{g-1}(C)$ be such that $\xi + \xi' \equiv K$, but otherwise general. We compute the inverse image $\mu_d^{-1}(\widetilde{\Theta}_{-\xi} + \widetilde{\Theta}_{-\xi'})$. One has: $\mu_d(D_d) \in \widetilde{\Theta}_{-\xi}$ iff $h^0(D_d - \zeta_d + \xi) > 0$, which is $D_d + \xi \equiv D_{g-1} + \zeta_d$ for some D_{g-1}. If $D_{g-1} + D'_{g-1} \equiv K$, we may write this equivalently as: $D_d + D'_{g-1} + \xi \equiv$ $\equiv K + \zeta_d$, i.e.: $D_d \leq |\xi' + \zeta_d|$. We deduce: $\mu_d^{-1}(\widetilde{\Theta}_{-\xi}) = E_{|\xi' + \zeta_d|}$. Similarly, $\mu_d^{-1}(\widetilde{\Theta}_{-\xi'}) = E_{|\xi + \zeta_d|}$. Hence, for all $\zeta_d \in \mathrm{Pic}^d(C)$ and all $\xi \in \mathrm{Pic}^{g-1}(C)$ $(\xi' := K - \xi)$ we obtain:

$$\text{(4.3)} \quad \mu_d^* \mathcal{O}_{JC}(2\Theta) = \mathcal{O}_{C^{(d)}}(E_{|\xi + \zeta_d|} + E_{|\xi' + \zeta_d|}).$$

In a similar way one gets the following (well-known) fact: If $\delta: C \times C \longrightarrow JC$ denotes the difference map, it is

$$\text{(4.4)} \quad \delta^* \mathcal{O}_{JC}(2\Theta) = \mathcal{O}_{C \times C}(K_1 + K_2 + 2\Delta) ,$$

where $K_1 = K \times C$, $K_2 = C \times K$, K being a copy of the canonical divisor of C.

Let now $\widetilde{JC}$ be the blowing up of JC at the origin, and call E the exceptional divisor, $E = \mathbf{P}\, T_{JC}(0) = \mathbf{P}^{g-1}$. The map δ extends to a morphism $\tilde{\delta} : C \times C \longrightarrow \widetilde{JC}$, and $\Delta = \tilde{\delta}^{-1}(E)$. Writing M_o for the line bundle corresponding with $\mathcal{O}_{\widetilde{JC}}(2\Theta - 2E)$, one has, by (4.4),

$$\tilde{\delta}^* M_o \cong \mathcal{O}_{C\times C}(K_1 + K_2).$$

Note that $H^o(\widetilde{JC}, M_o)$ can be identified with the subspace $H^o\,\mathcal{O}_{JC}(2\Theta)_o$ of $H^o\mathcal{O}_{JC}(2\Theta)$ consisting of those sections which vanish at the origin (hence, being even sections -invariant under the symmetry of JC-, they vanish doubly there).

Furthermore, $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbf{P}^{g-1}}(-1)$, hence $\mathcal{O}_E(M_o) \cong \mathcal{O}_{\mathbf{P}^{g-1}}(2)$, and the restriction map

$$(4.5)\quad H^o\mathcal{O}_{JC}(2\Theta)_o = H^o(\widetilde{JC}, M_o) \longrightarrow H^o\mathcal{O}_E(M_o) = H^o\mathcal{O}_{\mathbf{P}^{g-1}}(2) = S^2 H^o\omega_C$$

translates geometrically into the map sending a second onder theta divisor passing through the origin of JC with multiplicity 2 to its projectivized tangent cone (In terms of theta functions, it sends a 2^{nd} onder theta function having a zero at $0 \in JC$ to the (initial) term of degree 2 of its Taylor expansion).

In particular, the kernel of the morphism (4.5) is the vector space $H^o\mathcal{O}_{JC}(2\Theta)_{oo}$ (notation of [3]) of sections having a zero of multiplicity ≥ 4 at $0 \in JC$.

On the other side, taking inverse images by $\tilde{\delta}$ gives a map

$$(4.6)\quad H^o\mathcal{O}_{JC}(2\Theta)_o = H^o(\widetilde{JC}, M_o) \longrightarrow H^o\mathcal{O}_{C\times C}(K_1 + K_2) = \otimes^2 H^o\omega_C.$$

But, since under $\tilde{\delta}$ the symmetry of $\widetilde{JC}$ (inherited from the symmetry of JC) corresponds to the symmetry of $C \times C$, and that, secondly, the sections of $H^o \mathcal{O}_{JC}(2\,\Theta)$ are invariant under the symmetry of JC, the image of the map (4.6) lies actually in the subspace $\bar{S}^2 H^o \omega_C \subset \otimes^2 H^o \omega_C$ of the symmetric tensors. Thus (4.6) actually is:

$$(4.7) \quad H^o \mathcal{O}_{JC}(2\,\Theta)_o \longrightarrow \bar{S}^2 H^o \omega_C .$$

We claim that, under the natural identification $\bar{S}^2 H^o \omega_C \cong$ $\cong S^2 H^o \omega_C$, the maps (4.5) and (4.7) become identified, at least up to multiplication with a non-zero constant.

To see this, we check that they induce the same rational map of projective spaces.

Observe first that $\widetilde{\Theta}_{-\xi} + \widetilde{\Theta}_{-\xi'}$ with $\xi \in \widetilde{\Theta} \setminus \mathrm{Sing}\ \widetilde{\Theta}$ span the subsystem $|2\,\Theta|_o$ of $|2\,\Theta|$ consisting of the divisors passing through the origin. In fact, if

$$\psi : JC \longrightarrow |2\,\Theta|^{\vee} = \mathbf{P}^{2^g - 1}$$

denotes the Kummer-Wirtinger map, there exists exactly one hyperplane H in $\mathbf{P}^{2^g-1}$ such that $\Theta \subset \psi^{-1}(H)$ (actually: $\psi^{-1}(H) = 2\Theta$). By Wirtinger duality ([11]) this hyperplane corresponds with the subsystem $|2\,\Theta|_o \subset |2\,\Theta|$, hence $|2\,\Theta|_o$ is spanned by the divisors as above, with $\xi \in \widetilde{\Theta}$. Clearly we may drop Sing $\widetilde{\Theta}$, getting the same result.

Since $\widetilde{\Theta} \setminus \mathrm{Sing}\,\widetilde{\Theta}$ spans $|2\,\Theta|_o$ and it is positive dimensional, it suffices to see that the two maps in question coincide at this set. Write $\xi=[D_{g-1}]$, $\xi'=[D'_{g-1}]$, hence $h^o(D_{g-1}) = h^o(D'_{g-1})=1$, and $D_{g-1}+D'_{g-1} = K$. The image of $\tilde{\Theta}_{-\xi}+\tilde{\Theta}_{-\xi'}$ in $|\mathcal{O}_{\mathbf{P}^{g-1}}(2)|$ is the double hyperplane 2K, since the projectivized tangent spaces of $\tilde{\Theta}_{-\xi}$ and $\tilde{\Theta}_{-\xi'}$ at the origin are both equal to $K \in |\mathcal{O}_{\mathbf{P}^{g-1}}(1)| = |\omega_C|$ (Riemann-Kempf singularity Theorem, cf [8]).

On the other side, $\tilde{\Theta}_{-\xi}$ and $\tilde{\Theta}_{-\xi'}$ cut out on $C \times C$ the divisors $\Delta+D'_{g-1}\times C + C\times D_{g-1}$ and $\Delta+D_{g-1}\times C + C\times D'_{g-1}$ respectively. Hence the divisor of $|\mathcal{O}_{C\times C}(K_1+K_2)|$ obtained from $\tilde{\Theta}_{-\xi}+\tilde{\Theta}_{-\xi'}$ equals $K\times C + C\times K$, thereby ending this proof.

We deduce the following fact, which is proved also in J. Fay's preprint: "On the Even-Order Vanishing of Jacobian Theta Functions", as well as in [3]. (The latter authors attribute this result (essentially) to Frobenius (Loc. Cit.)).

(4.8) Proposition. Let $D \in |2\Theta|$ be a second order theta divisor. Then $\mu_o(D) \geq 4$ is equivalent to $(C-C) \subset D$.

Next we shall restrict ourselves to trigonal curves, aiming towards the proof of (2.5) in this case. We know that the right hand side of (2.5) is contained in the union $(W^o_3-g^1_3)\cup(g^1_3-W^o_3)$, by Theorem (2.4) and (2.3). Since the intersection appearing in (2.5) is a symmetric subvariety of JC, it is sufficient to prove that the divisors $D\in|2\Theta|$ with $\mu_o(D)\geq 4$ cut out on $W^o_3-g^1_3$ precisely the locus C-C.

Consider the map $\mu_3: C^{(3)} \longrightarrow JC$ sending D_3 to $D_3 - g_3^1$. Write $S \subset C^{(3)}$ for the (set theoretical) inverse image of $C-C$. One has (cf (1.5) and (4.1), and also (0.4)):

(4.9) $S = E_{|2g_3^1|}$.

Proposition (4.8) implies that $D \in |2\Theta|$ satisfies $\mu_0(D) \geq 4$ iff $\mu_3^{-1}(D)$ contains the surface S. Theorem (2.5) will be proved for trigonal curves by showing that for every point in the complement $C^{(3)} \setminus S$ there exists a $D \in |2\Theta|$ sucht that $\mu_3^{-1}(D) \supset S$ but $\mu_3^{-1}(D)$ not containing that point.

Let $D_{g-4} \in C^{(g-4)}$ be a general element, fixed from now on for a while. Consider the map μ_{g-1}: $C^{(g-1)} \longrightarrow JC$ sending D_{g-1} to $D_{g-1} - D_{g-4} - g_3^1$. By (4.3) applied to $\zeta_{g-1} = D_{g-4} + g_3^1$ we have, for any $\xi, \xi' \in \mathrm{Pic}^{g-1}(C)$ such that $\xi + \xi' \equiv K$:

(4.10) $\mu_{g-1}{}^*\mathcal{O}_{JC}(2\Theta) \cong \mathcal{O}_{C^{(g-1)}}(E_{|\xi + D_{g-4} + g_3^1|} + E_{|\xi' + D_{g-4} + g_3^1|})$.

Secondly, as in the proof of Lemma (1.7), we obtain a surjection

(4.11) $H^0\mathcal{O}_{JC}(2\Theta) \longrightarrow H^0(C^{(g-1)}, \mu_{g-1}^*\mathcal{O}_{JC}(2\Theta))$.

This implies that, for every divisor of $|\mathcal{O}_{C^{(g-1)}}(E_{|\xi + D_{g-4} + g_3^1|} + E_{|\xi' + D_{g-4} + g_3^1|})|$, its restriction to $C^{(3)}$ by the inclusion

$$C^{(3)} \xhookrightarrow{+\,D_{g-4}} C^{(g-1)}$$

yields an inverse image $\mu_3^{-1}(D)$ for some $D \in |2\Theta|$.

Taking in particular $\xi = D_{g-4} + g^1_3$, the sheaf (4.10) is $\mathcal{O}_{C^{(g-1)}}(E_{|2D_{g-4}+2g^1_3|} + E_{|K|})$. Since $D_{g-4} \in C^{(g-4)}$ is general, $\dim |2D_{g-4} + 2g^1_3| = g-2$. According to our conventions (cf (4.1)), the symbol $E_{|2g^1_3+2D_{g-4}|}$ therefore stands for a unique divisor on $C^{(g-1)}$. We claim that, writing $D_{g-4} = \Sigma P_i$,

$$(4.12) \quad E_{|2D_{g-4}+2g^1_3|} \cap (D_{g-4}+C^{(3)}) = (\cup (P_i+C^{(2)})) \cup S.$$

In fact, $D_{g-4}+D_3 \leq |2D_{g-4}+2g^1_3|$ is equivalent to $h^0(D_{g-4} + 2g^1_3 - D_3) \geq 1$. But, since $h^0(2g^1_3) = 3$ (cf (1.5)) and that D_{g-4} is general in $C^{(g-4)}$, we have $h^0(D_{g-4}+2g^1_3) = 3$. Therefore the left hand side of (4.12) equals the divisor of $C^{(3)}$ given by $E_{|D_{g-4}+2g^1_3|} = \Sigma (P_i+C^{(2)}) + E_{|2g^1_3|}$, as claimed.

(4.13) Secondly (cf §3, after Lemma (3.8)), the basis locus of $|\mathcal{O}_{C^{(g-1)}}(E_{|K|})|$ equals the locus Z, in the terminology of Loc. Cit. The intersection of this basis locus with $D_{g-4}+C^{(3)}$ consists of the elements $D_3 \in C^{(3)}$ such that $h^0(D_3+D_{g-4}) \geq 2$.

Now we allow $D_{g-4} \in C^{(g-4)}$ to vary. Let $D_3 \in C^{(3)} \setminus S$. A general choice of D_{g-4} implies $D_3 \notin \cup (P_i+C^{(2)})$. On the other side, $D_3 \notin S$ implies $h^0(D_3)=1$ hence, for general D_{g-4}, $h^0(D_3+D_{g-4}) = 1$ too. Making a common choice of D_{g-4} with respect to these conditions, one obtains a divisor in

$$|\mathcal{O}_{C^{(g-1)}}(E_{|2D_{g-4}+2g^1_3|} + E_{|K|})|$$

containing S and not containing D_3. This finishes the proof of Theorem (2.5), Q.E.D.

It remains to consider the cases $g=3, 4$ (cf (2.6)).

(4.14) <u>Proposition</u>. Let C be a non-hyperelliptic curve of genus 4. Call g^1_3 and h^1_3 its (possibily coincident) series of degree 3. One has:

$$\bigcap_{\substack{D\in|2\Theta| \\ \mu_o(D)\geq 4}} D = (C-C) \cup \{\pm(g^1_3-h^1_3)\}.$$

Therefore, if $g^1_3 \neq h^1_3$ (i.e., if C has no vanishing Thetanullwert) this locus exhibits two isolated points, besides the surface $C-C$.

<u>Proof</u>. We begin as after (4.8), taking account of Remark (2.6)(iii). The left-hand side member of the above equality is contained in $(W^o_3-g^1_3)\cup(g^1_3-W^o_3) = (W^o_3-g^1_3)\cup(W^o_3-h^1_3)$. By symmetry, it suffices to compute its intersection with $W^o_3-g^1_3$. As in (4.11), we have a surjection

$$(4.15) \qquad H^o\mathcal{O}_{JC}(2\Theta) \longrightarrow\!\!\!\!\!\rightarrow H^o(C^{(3)}, \mu_3{}^*\mathcal{O}_{JC}(2\Theta)).$$

In analogy with the previous argument, we choose $\xi=g^1_3$, getting

$$(4.16) \qquad \mu_3{}^*\mathcal{O}_{JC}(2\Theta) = \mathcal{O}_{C^{(3)}}(S+E_{|K|}).$$

By (4.15) and (4.16), the intersection with $W^o_3-g^1_3$ we are looking for is the image (by μ_3) of $S\cup$(Basis Locus of $|E_{|K|}|$). The basis locus of $|E_{|K|}|$ is $g^1_3\cup h^1_3\subset C^{(3)}$, hence we obtain $(C-C)\cup\{h^1_3-g^1_3\}$. This proves the first statement in (4.14).

As for the second one, if $h^1_3 \neq g^1_3$ then $h^1_3 - g^1_3 \equiv x-y$ for some $x, y \in C$ would imply that $h^1_3 + y \equiv x + g^1_3$. As C has no g^2_4, this implies that g^1_3 and h^1_3 have members sharing two of their three points. But, looking at C as the intersection of a (non-degenerate) quadric and a cubic in $\mathbf{P}^3$, the two series are cut out by the two rulings of the quadric. So the above is impossible, and $h^1_3 - g^1_3 \notin C-C$, proving (4.14), Q.E.D.

Finally, we recall Van Greemen and Van der Geer's proof of (2.5) for $g=3$.

(4.17) Proposition ([3]). If $g=3$ then

$$\bigcap_{\substack{D \in |2\Theta| \\ \mu_o(D) \geq 4}} D = C-C \ .$$

Proof. We may assume that C is non-hyperelliptic (cf (2.6) (ii)). The map (4.5) is surjective (This goes back to Wirtinger), hence $\dim H^o \mathcal{O}_{JC}(2\Theta)_{oo} = \dim H^o \mathcal{O}_{JC}(2\Theta)_o - \dim S^2 H^o \omega_C = 2^g - 1 - \binom{g+1}{2}$. For $g=3$ this yields 1. There exists therefore (cf (4.8)) a unique divisor in $|2\Theta|$ containing $C-C$. As the cohomology class of $C-C$ is $[C]*[C] = [2\Theta]$, they coincide, Q.E.D.

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