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A GENERALIZATION OF WRIGHT'S INEQUALITY

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Let R be a commutative ring with identity, E an R-module and $x_1,\ldots,x_r \in R$ a multiplicity system on E (see [1], p.295). Then the length $\ell_R(E/(x_1^{n_1},\ldots,x_r^{n_r})E)$ is finite for any positive integers $n_1,\ldots,n_r$, and Wright's inequality (see [1] p.296) says

$$\ell_R(E/(x_1^{n_1},\ldots,x_r^{n_r})E) \leq n_1\ldots n_r \cdot \ell_R(E/(x_1,\ldots,x_r)E),$$

for arbitrary $n_1,\ldots,n_r$.

This inequality can be written as

$$\ell_R H_o K(x_1^{n_1},\ldots,x_r^{n_r}|E) \leq n_1\ldots n_r \cdot \ell_R H_o K(x_1,\ldots,x_r|E),$$

where $K(x_1^{n_1},\ldots,x_r^{n_r}|E)$ denotes the Koszul complex defined by E and the elements $x_1^{n_1},\ldots,x_r^{n_r}$.

In this paper we establish that for a Noetherian module similar inequalities hold for the higher Koszul homology modules, i.e., for $i \geq 0$, $\ell H_i K(x_1^{n_1},\ldots,x_r^{n_r}|E) \leq n_1\ldots n_r \cdot \ell H_i K(x_1,\ldots,x_r|E)$. Moreover, the same is true for the higher Euler-Poincaré characteristics of the Koszul complexes $K(x_1^{n_1},\ldots,x_r^{n_r}|E)$, i.e., for $i \geq 0$, we have $\chi_i(x_1^{n_1},\ldots,x_r^{n_r}|E) \leq n_1\ldots n_r \cdot \chi_i(x_1,\ldots,x_r|E)$, where, by definition,

$$\chi_i(x_1,\ldots,x_r|E) = \sum_{j \geq i} (-1)^{j-i}\, \ell H_j K(x_1,\ldots,x_r|E).$$

Actually the inequality for χ_i is an equality if $i=0$ (see [1] p.311).

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We also prove that the functions $\ell H_i K(x_1^{n_1},\dots,x_r^{n_r}|E)$ and $\chi_i(x_1^{n_1},\dots,x_r^{n_r}|E)$ increase with the exponents $n_1,\dots,n_r$.

In what follows R denotes a commutative ring with identity, E is a Noetherian module over R and $x_1,\dots,x_r \in R$ is a system of multiplicity on E. This will ensure that all the lengths which appear are indeed finite, though some of the results would also hold without assuming the lengths to be finite. We denote the length by ℓ or ℓ_R.

Lemma 1. Let E be an R-module and $x_1,\dots,x_r,y$ be elements of R. Then, for any $i \geq 0$,

$$\ell_R H_i K(x_1,\dots,x_r|E) \leq \ell_R H_i K(x_1 y, x_2,\dots,x_r|E).$$

Proof. The inequality follows from the exact sequence (cf. [2] p.IV-2)

$$0 \to \frac{H_i K(x_2,\dots,x_r|E)}{x_1 H_i K(x_2,\dots,x_r|E)} \to H_i K(x_1,x_2,\dots,x_r|E) \to$$

$$\to (0:x_1)_{H_{i-1}K(x_2,\dots,x_r|E)} \longrightarrow 0,$$

and the corresponding one for $H_i K(x_1 y, x_2,\dots,x_r|E)$, by observing that both $(0:x_1)_{H_{i-1}K(x_2,\dots,x_r|E)} \subseteq (0:x_1 y)_{H_{i-1}K(x_2,\dots,x_r|E)}$ and $x_1 H_i K(x_2,\dots,x_r|E) \supseteq x_1 y H_i K(x_2,\dots,x_r|E)$. #

Bearing in mind that the Koszul homology modules do not depend on the order of the elements defining it, we get the following

Proposition 2. For any $i \geq 0$, the mapping from $\mathbb{N}^r$ to $\mathbb{N}$ defined by

$$(n_1,\ldots,n_r) \longmapsto \ell H_i K(x_1^{n_1},\ldots,x_r^{n_r}|E)$$

is increasing, i.e., $n_1 \leq m_1,\ldots,n_r \leq m_r$ imply

$$\ell H_i K(x_1^{n_1},\ldots,x_r^{n_r}|E) \leq \ell H_i K(x_1^{m_1},\ldots,x_r^{m_r}|E). \ \#$$

Lemma 3. If $a \in R$, then we have:

i) $\ell(0 \underset{E}{:} a^n) \leq n \cdot \ell(0 \underset{E}{:} a)$, and

ii) $\ell(E/a^nE) \leq n \cdot \ell(E/aE)$.

Proof. By induction on n. From the exact sequence

$$R/aR \xrightarrow{\ .a^{n-1}\ } R/a^nR \longrightarrow R/a^{n-1}R \longrightarrow 0,$$

if we apply $Hom_R(\ ,E)$, we get i), for $Hom_R(R/a^nR, E) \cong 0 \underset{E}{:} a^n$,

and if we apply $. \otimes_R E$, we get ii), for $(R/a^nR) \otimes_R E \cong E/a^nE$. #

Proposition 4. The following inequality holds for all $i \geq 0$,

$$\ell H_i K(x_1^n,x_2,\ldots,x_r|E) \leq n \cdot \ell H_i K(x_1,x_2,\ldots,x_r|E).$$

Proof. From the exact sequences (see [2] p.IV-2)

$$0 \longrightarrow H_0K(a|H_iK(x_2,\ldots,x_r|E)) \longrightarrow H_iK(a,x_2,\ldots,x_r|E)$$

$$\longrightarrow H_1K(a|H_{i-1}K(x_2,\ldots,x_r|E)) \longrightarrow 0,$$

with $a = x_1$ or x_1^n, we get

$$\ell H_i K(x_1^n,x_2,\ldots,x_r|E) = \ell\left(\frac{H_iK(x_2,\ldots,x_r|E)}{x_1^n H_iK(x_2,\ldots,x_r|E)}\right) + \ell(0:x_1^n)_{H_{i-1}(x_2,\ldots,x_r|E)}$$

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$$\leq n\cdot \ell\left(\frac{H_iK(x_2,\dots,x_r|E)}{x_1H_iK(x_2,\dots,x_r|E)}\right)+n\cdot \ell(0:x_1)_{H_{i-1}K(x_2,\dots,x_r|E)} =$$

$$= n\cdot \ell H_iK(x_1,x_2,\dots,x_r/E),$$

the inequalities being by virtue of lemma 3. #

Again by the independence of the Koszul homology modules with respect to the order of the elements, Proposition 4 yields the following theorem which generalizes Wright's inequality.

<u>Theorem 5.</u> For any $i\geq 0$, and any $n_1,\dots,n_r\geq 0$, we have

$$\ell H_iK(x_1^{n_1},\dots,x_r^{n_r}|E) \leq n_1\dots n_r\cdot \ell H_iK(x_1,\dots,x_r|E).\ \#$$

Let us consider now the higher Euler-Poincaré characteristics. Observe first that $\chi_o(x_1^{n_1},\dots,x_r^{n_r}|E) = n_1\dots n_r\cdot \chi_o(x_1,\dots,x_r|E)$ and that $\chi_o(x_1,\dots,x_r|E)\geq 0$ (cf. [1] p.311). For the higher characteristics we have

<u>Proposition 6.</u> For all $i\geq 0$, the mapping

$$(n_1,\dots,n_r) \longmapsto \chi_i(x_1^{n_1},\dots,x_r^{n_r}|E)$$

from $\mathbb{N}^r$ to $\mathbb{N}$ is increasing, i.e., $n_1\leq m_1,\dots,n_r\leq m_r$, imply

$$\chi_i(x_1^{n_1},\dots,x_r^{n_r}|E)\leq \chi_i(x_1^{m_1},\dots,x_r^{m_r}|E).$$

<u>Proof.</u> By [2] p.IV-56, we have

$$\chi_i(a,x_2,\dots,x_r|E) = \ell H_1K(a|H_{i-1}K(x_2,\dots,x_r|E))+$$

$$+ \chi_o(a|\chi_i(x_2,\ldots,x_r|E)).$$

Setting $a = x_1$ or $x_1 y$ and using the multiplicativity of χ_o (see [1] p.311 Cor.1), we deduce

$$\chi_i(x_1,x_2,\ldots,x_r|E) \leq \chi_i(x_1 y,x_2,\ldots,x_r|E).$$

From this we get, for $n<m$, that

$$\chi_i(x_1^n,x_2,\ldots,x_r|E) \leq \chi_i(x_1^m,x_2,\ldots,x_r|E).$$

Proceeding equally with the other variables (χ_i does not depend on the order of the elements), we get the result. #

We finish with a theorem on higher Euler-Poincaré characteristics similar to theorem 5.

<u>Theorem 7.</u> For any $i \geq 0$, and any $n_1,\ldots,n_r \geq 0$, we have

$$\chi_i(x_1^{n_1},\ldots,x_r^{n_r}|E) \leq n_1\ldots n_r \cdot \chi_i(x_1,\ldots,x_r|E).$$

<u>Proof.</u> It is enough to prove

$$\chi_i(x_1^n,x_2,\ldots,x_r|E) \leq n\cdot\chi_i(x_1,x_2,\ldots,x_r|E),$$

and this can be done by considering the formula used in the proof of the preceding proposition with $a=x_1$ or x_1^n. We get

$$\chi_i(x_1^n,x_2,\ldots,x_r/E) =$$

$$= \ell(0:x_1^n)_{H_{i-1}K(x_2,\ldots,x_r/E)} + \chi_o(x_1^n|\chi_i(x_2,\ldots,x_r|E))$$

$$\leq n\cdot\ell(0:x_1)_{H_{i-1}K(x_2,\ldots,x_r|E)} + n\cdot\chi_o(x_1|\chi_i(x_2,\ldots,x_r|E)) =$$

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$$= n \cdot \chi_i(x_1, x_2, \ldots, x_r | E),$$

the inequality being justified by lemma 3. #

References

[1] Northcott, D.G. Lessons on Rings, Modules and Multiplicities, Cambridge U.P.(1968).

[2] Serre, J-P. Algèbre locale: Multiplicités, Lecture Notes in Math. No.11, Springer, Berlin(1975).

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