UNIVERSITAT DE BARCELONA

FACULTAT DE MATEMÀTIQUES

ON THE SUMMATION OF THE SINGULAR SERIES

by

A. ARENAS

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A. ARENAS
Facultat de Matemàtiques. Dpt. d'Àlgebra i Fonaments.
Gran Via de les Corts Catalanes, 585. Universitat de Barcelona
08007 Barcelona, SPAIN.

INTRODUCTION

The singular series has great importance in the study of the number of representations of a rational integer as a sum of integral squares (cf. [3]).

As is known (cf. [3], [8]) the sum of the singular series is just the average number $r(n, \text{gen } I_k)$ of representations of a positive integer n by the genus of the identity quadratic form in k variables.

Bateman [2] calculated the sum of the singular series in the cases $k = 3,4$, following Hardy and Hecke methods. His results can though more easily be obtained by using Siegel's formula for the evaluation of $r(n, \text{gen } I_k)$.

In this paper we derive in some special cases a formula for $r(n, \text{gen } I_k)$, from Siegel's formula, which covers those considered by Bateman. We use Gauss-Weber sums to evaluate the 2-adic densities, which, in Siegel's method, causes the main difficulties.

Our considerations in the case $k = 24$ also yield the celebrated Ramanujan's formula about the number of representations of an integer as sum of 24 squares.

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1. EVALUATION OF $r(n, \text{gen } I_k)$.

The number of representations $r(n,f)$ of a given positive integer by a quadratic form cannot be determined in general. However, this number can be approximated by the average value $r(n, \text{gen } f)$, where gen f stands for the genus of f. Recall (see [8]) that two integral quadratic forms are said to belong to the same genus if they are equivalent over all $\mathbf{Z}_p$ including $p = \infty$.

Given a positive integer n and a positive definite integral quadratic form f of k variables, $k \geq 2$, Siegel's formula (see [8]) asserts that $r(n, \text{gen } f)$ can be evaluated by means of p-adic densities $\partial_p(n,f)$, with p prime or ∞, as follows

$$r(n, \text{gen } f) = \varepsilon\, \partial_\infty (n, f) \prod_p \partial_p (n, f),$$

with $\varepsilon = ½$ if $k = 2$, and $\varepsilon = 1$ if $k > 2$; where

$$\partial_p(n,f) = \begin{cases} \Gamma(k/2)^{-1} (\det f)^{-½} \pi^{k/2} n^{(k-2)/2}, & \text{if } p = \infty . \\ p^{-2\alpha} r_{p^{2\alpha}} (n, f), \quad \text{for all } \alpha \geq 2\beta + 1, \text{ where } p^\beta \| 2n, & \text{if } p \text{ is prime.} \end{cases}$$

($r_q(n, f)$ means the number of representations of n by f modulo q).

Proposition 1. *Let* k *be an odd positive integer, and let* I_k *be the quadratic form* $X_1+...+X_k$. *Then:*

i) $\partial_\infty (n, I_k) = \Gamma (k/2)^{-1} \pi^{k/2} n^{(k-2)/2}$, *for any* $n \in \mathbf{Z}$.

ii) *Let* $\kappa_p (n, b) = \{ (1 - (\frac{(-1)^{(k-1)/2} p^{-2b} n}{p}) p^{(1-k)/2} \}^{-1}$

with $p^{2b} | n$ *but* $p^{2b+2} \nmid n$, $b \geq 0$. *Then for* $p \neq 2$, *and any* $n \in \mathbf{Z}$.

$$\partial_p (n, I_k) = \begin{cases} (1 - p^{1-k})\, \kappa_p (n,0) , & \text{if } p \nmid n \\ (1 - p^{1-k}) \left(\sum_{j=0}^{b-1} p^{j(2-k)} + p^{b(2-k)} \kappa_p(n, b) \right) , & \text{if } p \mid n . \end{cases}$$

iii) $\partial_2 (n, I_k) = 2^{2-k} \left(\sum_{\substack{q=1 \\ q \equiv n \ (\mathrm{mod}\ 4)}}^{k-1} \binom{k}{q} + \nu \right)$, *with* $\nu = \begin{cases} 2 & \text{if } k \equiv n \ (mod\ 8) \\ 0 & \text{in other case} \end{cases}$, *for* n *odd.*

The symbol (·) *always means Legendre's symbol.*

Proof. The statements of i) and ii) come directly from ([8], Hilfssatz 16). iii) By definition, $\partial_2 (n, I_k) = 2^{3(1-k)} r_{2^3} (n, I_3)$ and by elementary combinatoric methods one gets the result. ▪

From proposition 1 and Siegel's formula we can state the following

Proposition 2. *If* k *and* n *are odd positive integers, then the value of* $r(n, \text{gen } I_k)$ *is:*

$$A_k(n)\; n^{(k-2)/2} \sum_{m \text{ odd}} m^{(1-k)/2} \left(\frac{(-1)^{(k-1)/2}\, n}{m} \right) \cdot \prod_{p|n} \left(\sum_{j=0}^{b-1} p^{j(2-k)} + p^{b(2-k)}\, \kappa_p(n, b) \right) ,$$

where $A_k(n) := 2\, \pi^{k/2} \left[(2^{k-1} - 1)\, \Gamma(k/2)\, \zeta(k-1) \right]^{-1} \left(\sum_{\substack{q=1 \\ q \equiv n \ (mod\ 4)}}^{k-1} \binom{k}{q} + \nu \right)$

Here ζ *denotes the Riemann zeta function and* ν *is defined as in proposition* 1. ∎

Proposition 3. *Let* n *be a positive integer such that* $4^a \| n$, $p^{2b} \mid n$, $p^{2b+2} \nmid n$, $a \geq 0$, $b \geq 0$. *Then:*

$$r(n, I_3) = A_3(n) \,.\, n^{1/2} \,.\, L(1, \chi_{-4n}) \prod_{p|n} \left(\sum_{j=0}^{b-1} p^{-j} + p^{-b} \kappa_p(n, b) \right) ,$$

where $L(s, \chi_{-4n})$ *is the* L-*series associated to the character* χ_{-4n}; *and*

$$A_3(n) = \begin{cases} 0 & \text{if } 4^{-a} n \equiv 7 \ (mod\ 8), \\ \pi^{-1}.2^{-a}.16 & \text{if } 4^{-a} n \equiv 3 \ (mod\ 8), \\ \pi^{-1}.2^{-a}.24 & \text{if } 4^{-a} n \equiv 1, 2, 5, 6 \ (mod\ 8), \end{cases}$$

Proof. Since for $k = 3$ there is only one class in the genus of I_3, we have $r(n, \text{gen } I_3) = r(n, I_3)$. Therefore, if n is odd the claim of the proposition follows from proposition 2.

Now, if n is even, to calculate $\partial_2(n, I_3)$ we distinguish two cases:

i) $4 \nmid n$, in which case $r_{2^{t+1}}(n, I_3) = 2^2\, r_{2^t}(n, I_3)$, $t \geq 3$. So, $\partial_2(n, I_3) = 3/2$ if $n \equiv 2,6 \pmod 8$.

ii) $4^a \| n$, $a \geq 1$, in which case $r(n, I_3) = r(4^{-a} n, I_3)$. Therefore, we can apply either the expression given in proposition 1 if $4^{-a}n$ is odd or the one obtained above if $2 \mid 4^{-a}n$. ∎

Next, we calculate $r(n, \text{gen } I_{2k})$ for every n and for k even. As usual, $v_p(n)$ will denote the p-adic valuation of n.

Proposition 4. *Let* n *and* k *be positive integers,* k *even. We have:*

i) $\partial_\infty(n, I_{2k}) = \Gamma(k)^{-1}\pi^k n^{k-1}$.

ii) $\partial_p(n, I_{2k}) = (1-p^{-k})\left(\sum_{j=0}^{v_p(n)} p^{j(1-k)}\right)$, *for* $p \neq 2$.

iii) *If* n *is odd* $\partial_2(n, I_{2k}) = 1$. *If* n *is even, then:*

$$\partial_2(n, I_{2k}) = \begin{cases} 1 + \sum_{j=1}^{v_2(n)-1} 2^{j(1-k)} - 2^{v_2(n)(1-k)}, & \text{if } k \equiv 0 \ (mod\ 4) \\ 1 - \sum_{j=1}^{v_2(n)-1} 2^{j(1-k)} + 2^{v_2(n)(1-k)}, & \text{if } k \equiv 2 \ (mod\ 4) \end{cases}$$

Proof. i) and ii) follow immediately from ([8], Hilf. 16). iii) The evaluation of $\partial_2(n, I_{2k})$ is not covered by Siegel's formulae. So, in order to calculate these densities we define

$$\theta_{2^s}(m, I_{2k}) = \sum_{x \in (Z/2^sZ)^{2k}} \exp((2^{-s}\, 2\pi i\, m\, I_{2k}(x)), \quad \text{for} \quad m \in (Z/2^sZ)^*, \quad \text{and}$$

$$B_s(n, I_{2k}) = \sum_{\substack{\xi \in Q_2/Z_2 \\ v_2(\xi) = -s}} p^{-2ks} \cdot \theta_{2^s}(m, I_{2k}).$$

Then (cf. [1], prop. 3):

$$\partial_2(n, I_{2k}) = \sum_{\substack{\xi \in Q_2/Z_2 \\ v_2(\xi) \leq 0}} B_s(n, I_{2k}).$$

For any $m \in (Z/2^sZ)^*$, we have (cf [4], Ch. 7):

$$\theta_{2^s}(m, I_{2k}) = \begin{cases} 2^{k(s+1)}, & \text{if } k \equiv 0 \ (mod\ 4) \\ 2^{k(s+1)} \exp(2^{-1}\pi i\, m\, k) & \text{if } k \equiv 2 \ (mod\ 4) \end{cases}$$

Taking into account well-known results about the values of the ordinary Gauss sums (Cf. [4], Ch. 7), it is easy to evaluate the sums $B_s(n, I_{2k})$. They are given by:

$$B_s(n, I_{2k}) = \begin{cases} 1 & \text{if} \quad s = 0 \\ 0 & \text{if} \quad s = 1 \text{ or } s > v_2(n) + 1 \\ (-1)^{k/2}\, 2^{-(k-1)(s-1)} & \text{if} \quad 1 < s \leq v_2(n) \\ (-1)^{k/2}\, 2^{-v_2(n)(k-1)} & \text{if} \quad s = v_2(n) + 1,\ v_2(n) > 0 \end{cases}$$

To achieve the asserted results, it suffices to substitute these values in the expression of $\partial_2(n, I_{2k})$. ▪

Let n be a positive integer and write $n = 2^a . m$ with $a \geq 0, 2 \nmid m$. Then, we set

$$\sigma^*_k(n) = (-1)^n [\, 2\,\sigma_k(n/2) - \sigma_k(m)]$$

where, as usual, $\sigma_k(n)$ denotes the sums of the k-th powers of the divisors of n; we agree that $\sigma_k(n/2) = 0$ if n is odd. Next, applying the preceding proposition we get:

Proposition 5. *Let* $n = 2^a m$, *with* $a \geq 0$ *and* $2 \nmid m$ *be a positive integer and* k *an even positive integer. Then:*

i) $r(n, \text{gen } I_{2k}) = C(k)\, \sigma^*_{k-1}(n)$, *if* $k \equiv 0 \ (mod\ 4)$.

ii) $r(n, \text{gen } I_{2k}) = C(k)\, \sigma_{k-1}(m)\ (2^{a(k-1)} - \sum_{j=1}^{(a-1)} 2^{j(k-1)} + 1)$, *if* $k \equiv 2 \ (mod\ 4)$;

where $C(k) = 2k\,[\,(2^k-1)\, B_{k/2}]^{-1}$, *with* $B_{k/2}$ *denoting the Bernoulli numbers* (see [5]).

From the above proposition, in particular, we have:

i) $r(n, \text{gen } I_{24}) = \dfrac{16}{691}\, \sigma^*_{11}(n)$.

ii) $r(n, I_4) = 24\,\sigma(m)$ and $r(n, I_8) = 16\,\sigma_3^*(n)$, since, in the genus of I_4 and I_8 there is only one class.

Next, we apply the preceding considerations to give:

2. RAMANUJAN FORMULA.

Let $\theta(I_{24}, z) = \sum_{n\geq 0} r(n, I_{24})\, q^n$, $q = e^{2\pi i z}$, be the theta series associated to the quadratic form I_{24} and let $\theta(\text{gen } I_{24}, z) = \sum_{n\geq 0} r(n, \text{gen } I_{24})\, q^n$ be the theta series associated to the genus of I_{24}. As is known, $\theta(I_{24}, z)$ belongs to the space $M_{12}(\Gamma_0(4))$ of modular forms of weight 12 with respect to the group $\Gamma_0(4)$ (see [7]). Let $F(z) = (2\pi)^{-12}\,\Delta(z) = \sum_{n\geq 1} \tau(n)\, q^n = \eta^{24}(z)$, be the discriminant modular form [5]. Here, $\tau(n)$ is the Ramanujan function and $\eta(z)$ is the Dedekind eta-function.

Lemma 6. *The space* $S_{12}(\Gamma_0(4))$ *of cusp forms of weight* 12 *with respect to the group* $\Gamma_0(4)$ *admits the following basis:*

$$\{\eta^{24}(z),\ \eta^{24}(z + ½),\ \eta^{24}(2z),\ \theta^{12}(z)\,\eta^{12}(2z)\},$$

being $\theta(z) = \sum_{n\geq 0} q^{n^2}$ *the Jacobi theta-function.*

Proof. As $F(z)$ belongs to $S_{12}(SL_2(\mathbb{Z}))$ is straightforward that $F(2z)$ belongs to $S_{12}(\Gamma_0(2))$. Moreover, as for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ we can write

$$F(\gamma(z) + ½) = F(\gamma_2(z + ½)),\ \text{ with }\ \gamma_2 = 1/4 \begin{pmatrix} 4a + 2c & 4b - c - 2(a-d) \\ 4c & 4d - 2c \end{pmatrix} \in \Gamma_0(4)$$

we have that $F(z + ½)$ is in $S_{12}(\Gamma_0(4))$.

The last function $\theta^{12}(z)\,\eta^{12}(2z)$ belongs also to $S_{12}(\Gamma_0(4))$ because $\theta^{12}(z)$ is in $M_6(\Gamma_0(4))$ and $\eta^{12}(2z)$ in $S_6(\Gamma_0(4))$. It is trivial to test that the four functions are linearly independent. They are a basis because $\dim_{\mathbb{C}} S_{12}(\Gamma_0(4)) = 4$ (Cf. [6], th. 2.23). ∎

Now, taking into account the following data:

n	$\tau(n)$	$r(n, I_{24})$	$691\ r(n, \text{gen } I_{24})$
1	1	48	16
2	- 24	1104	32752
3	252	16192	2834368
4	- 1472	170064	67141616

we see, applying lemma 6, that the function $\theta(I_{24}, z) - \theta(\text{gen } I_{24}, z)$, which lies in $S_{12}(\Gamma_0(4))$, can be expressed in the following way:

$$\theta(I_{24}, z) - \theta(\text{gen } I_{24}, z) = 691^{-1}(-33152\, F(z + \tfrac{1}{2}) - 65536\, F(2z)) .$$

Therefore by proposition 5 we get *Ramanujan formula* to count the number of representations of a positive integer as sum of 24 squares:

$$r(n, I_{24}) = \frac{16}{691}\, \sigma^*_{11}(n) + \frac{128}{691} \{(-1)^n\, 259\, \tau(n) - 512\, \tau(\tfrac{1}{2} n)\} ,$$

where $\tau(y)$ means zero when y is not an integer.

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