UNIVERSITAT DE BARCELONA
FACULTAT DE MATEMÀTIQUES

ON THE SUMMATION OF THE SINGULAR SERIES

by

A. ARENAS

PRE-PRINT N.º 45
JUNY- 1986
ON THE SUMMATION OF THE SINGULAR SERIES

A. ARENAS
Facultat de Matemàtiques. Dpt. d'Àlgebra i Fonaments.
Gran Via de les Corts Catalanes, 585. Universitat de Barcelona
08007 Barcelona, SPAIN.

INTRODUCTION

The singular series has great importance in the study of the number of representations of a rational integer as a sum of integral squares (cf. [3]).

As is known (cf. [3], [8]) the sum of the singular series is just the average number \( r(n, \text{gen } I_k) \) of representations of a positive integer \( n \) by the genus of the identity quadratic form in \( k \) variables.

Bateman [2] calculated the sum of the singular series in the cases \( k = 3, 4 \), following Hardy and Hecke methods. His results can though more easily be obtained by using Siegel's formula for the evaluation of \( r(n, \text{gen } I_k) \).

In this paper we derive in some special cases a formula for \( r(n, \text{gen } I_k) \), from Siegel's formula, which covers those considered by Bateman. We use Gauss-Weber sums to evaluate the 2-adic densities, which, in Siegel's method, causes the main difficulties.

Our considerations in the case \( k = 24 \) also yield the celebrated Ramanujan's formula about the number of representations of an integer as sum of 24 squares.

I wish to thank Professor P. Bayer for her encouragement in doing this paper.

1. EVALUATION OF \( r(n, \text{gen } I_k) \).

The number of representations \( r(n, f) \) of a given positive integer by a quadratic form cannot be determined in general. However, this number can be approximated by the average value \( r(n, \text{gen } f) \), where \( \text{gen } f \) stands for the genus of \( f \). Recall (see [8]) that two integral quadratic forms are said to belong to the same genus if they are equivalent over all \( \mathbb{Z}_p \) including \( p = \infty \).

Given a positive integer \( n \) and a positive definite integral quadratic form \( f \) of \( k \) variables, \( k \geq 2 \), Siegel's formula (see [8]) asserts that \( r(n, \text{gen } f) \) can be evaluated by means of \( p \)-adic densities \( \partial_p(n, f) \), with \( p \) prime or \( \infty \), as follows.
\[ r(n, \text{gen } f) = \varepsilon \partial_\infty (n, f) \prod_{p} \partial_p (n, f), \]

with \( \varepsilon = 1 \) if \( k = 2 \), and \( \varepsilon = -1 \) if \( k > 2 \); where

\[
\partial_p(n, f) = \left\{ \begin{array}{ll}
\Gamma \left( \frac{k}{2} \right) (\det f)^{\frac{k}{2}} \pi^{\frac{k}{2}} n^{(k-2)/2}, & \text{if } p = \infty, \\
p^{-2\alpha} r_{p2\alpha} (n, f), & \text{for all } \alpha \geq 2 \beta + 1, \text{ where } p^{\beta} \nmid 2n, \text{ if } p \text{ is prime.}
\end{array} \right.
\]

\((r_q(n, f)) \) means the number of representations of \( n \) by \( f \) modulo \( q \).

**Proposition 1.** Let \( k \) be an odd positive integer, and let \( l_k \) be the quadratic form \( X_1^2 + \ldots + X_k \). Then:

i) \( \partial_\infty (n, l_k) = \Gamma \left( \frac{k}{2} \right) \pi^{\frac{k}{2}} n^{(k-2)/2}, \) for any \( n \in \mathbb{Z} \).

ii) Let \( \kappa_p(n, b) = \left\{ (1 - \frac{1}{p^{1-k}}) \right\}^{-1} \left\{ \left( \frac{1}{p^{(1-k)/2}} \right) \right\}^{b-1} \), with \( p^{2b} \nmid n \) but \( p^{2b+2} \nmid n, b > 0 \). Then for \( p \neq 2 \), and any \( n \in \mathbb{Z} \).

\[
\partial_p(n, l_k) = \begin{cases}
(1 - p^{1-k}) \kappa_p(n, 0), & \text{if } p \nmid n, \\
(1 - p^{1-k}) \left( \sum_{j=0}^{b-1} p^{j(2-k)} + p^{b(2-k)} \kappa_p(n, b) \right), & \text{if } p \mid n.
\end{cases}
\]

iii) \( \partial_2(n, l_k) = 2^{2-k} \left( \sum_{q=1}^{k-1} \left( \frac{r_q(n)}{q} \right)^2 + v \right), \) with \( v = \begin{cases}
2 & \text{if } k \equiv n \pmod{8}, \\
0 & \text{in other case}
\end{cases} \), for \( n \) odd.

The symbol \((-\)\) always means Legendre's symbol.

**Proof.** The statements of i) and ii) come directly from ( [8], Hilfssatz 16). iii) By definition, \( \partial_2(n, l_k) = 2^{3(1-k)} r_2^3(n, l_k) \) and by elementary combinatoric methods one gets the result.

From proposition 1 and Siegel's formula we can state the following

**Proposition 2.** If \( k \) and \( n \) are odd positive integers, then the value of \( r(n, \text{gen } l_k) \) is:
$A_k(n) = 2 \pi^{(k-1)/2} \sum_{m \text{ odd}} m^{(1-k)/2} \left( \frac{(-1)^{(k-1)/2} n}{m} \right) \prod_{p \mid m} \left( \sum_{j=0}^{b-1} p^{j(2-k)} + p^{b(2-k)} \zeta_{p}(n, b) \right)$,

where $A_k(n) = 2 \pi^{k/2} \left[ \frac{(2k-1)}{(k/2)} \Gamma(k/2) \zeta(k-1) \right]^{-1} \left( \sum_{q=1}^{n \equiv 0 \pmod{4}} \frac{\zeta}{q} \right)$.

Here $\zeta$ denotes the Riemann zeta function and $\nu$ is defined as in proposition 1.

Proposition 3. Let $n$ be a positive integer such that $4 \nmid n$, $p^2 \mid n$, $p^{2b+2} \nmid n$, $a \geq 0$, $b \geq 0$. Then:

$$r(n, I_3) = A_3(n) \cdot n^\nu \cdot L(1, \chi_{4n}) \prod_{p \mid n} \left( \sum_{j=0}^{b-1} p^{j} + p^{b} \zeta_{p}(n, b) \right),$$

where $L(s, \chi_{4n})$ is the $L$-series associated to the character $\chi_{4n}$; and

$$A_3(n) = \begin{cases} 0 & \text{if } 4^n n \equiv 7 \pmod{8}, \\ \pi^{-1.2^{a}.16} & \text{if } 4^n n \equiv 3 \pmod{8}, \\ \pi^{-1.2^{a}.24} & \text{if } 4^n n \equiv 1, 2, 5, 6 \pmod{8}. \end{cases}$$

Proof. Since for $k = 3$ there is only one class in the genus of $I_3$, we have $r(n, \text{gen } I_3) = r(n, I_3)$. Therefore, if $n$ is odd the claim of the proposition follows from proposition 2.

Now, if $n$ is even, to calculate $r(n, I_3)$ we distinguish two cases:

i) $4 \nmid n$, in which case $r_2(t+1, (n, I_3)) = 2^2 r_2(t, (n, I_3))$, $t \geq 3$. So, $r_2(n, I_3) = 3/2$ if $n \equiv 2, 6 \pmod{8}$.

ii) $4^n n$, $a \geq 1$, in which case $r(n, I_3) = r(4^n n, I_3)$. Therefore, we can apply either the expression given in proposition 1 if $4^n n$ is odd or the one obtained above if $2 \mid 4^n n$.

Next, we calculate $r(n, \text{gen } I_2)$ for every $n$ and for $k$ even. As usual, $\nu_p(n)$ will denote the $p$-adic valuation of $n$.

Proposition 4. Let $n$ and $k$ be positive integers, $k$ even. We have:
i) \( \varphi_\pi(n, l_{2k}) = \Gamma(k)^{-1} \pi^k n^{k-1} \).

\[ v_p(n) \]

ii) \( \varphi_p(n, l_{2k}) = (1-p^{-k}) \left( \sum_{j=0}^{l_{2k}} \frac{p^j}{(1-k)^j} \right) \), for \( p \neq 2 \).

iii) If \( n \) is odd \( \varphi_\pi(n, l_{2k}) = 1 \). If \( n \) is even, then:

\[ \varphi_\pi(n, l_{2k}) = \begin{cases} 
\frac{v_2(n)-1}{2} & \text{if } k \equiv 0 \pmod{4} \\
1 + \sum_{j=1}^{v_2(n)-1} 2^{j(1-k)} - 2^{v_2(n)(1-k)} & \text{if } k \equiv 2 \pmod{4}
\end{cases} \]

Proof. i) and ii) follow immediately from ( [8], Hilf. 16 ). iii) The evaluation of \( \varphi_\pi(n, l_{2k}) \) is not covered by Siegel's formulae. So, in order to calculate these densities we define

\( \varphi_\pi(m, l_{2k}) = \sum_{x \in (\mathbb{Z}/2^k\mathbb{Z})^{2k}} \exp(2^{s} 2 \pi i m l_{2k}(x)) \), for \( m \in (\mathbb{Z}/2^k\mathbb{Z})^{*} \), and

\[ B_\pi(n, l_{2k}) = \sum_{\xi \in \mathbb{Q}/Z_2} p^{\cdot 2k} \cdot \varphi_\pi(m, l_{2k}) \cdot v_2(\xi) = -1 \]

Then (cf. [11], prop. 3):

\[ \varphi_\pi(n, l_{2k}) = \sum_{\xi \in \mathbb{Q}/Z_2} B_\pi(n, l_{2k}) \cdot v_2(\xi) \leq 0 \]

For any \( m \in (\mathbb{Z}/2^k\mathbb{Z})^{*} \), we have (cf [4], Ch. 7):

\[ \varphi_\pi(m, l_{2k}) = \begin{cases} 
2^{k(s+1)} & \text{if } k \equiv 0 \pmod{4} \\
2^{k(s+1)} \exp(2^{-1} \pi i m k) & \text{if } k \equiv 2 \pmod{4}
\end{cases} \]
Taking into account well-known results about the values of the ordinary Gauss sums (Cf. [4], Ch. 7), it is easy to evaluate the sums $B_s(n, I_{2k})$. They are given by:

$$B_s(n, I_{2k}) = \begin{cases} 
1 & \text{if } s = 0 \\
0 & \text{if } s = 1 \text{ or } s > \nu_2(n) + 1 \\
(-1)^{k/2} 2^{-(k-1)(s-1)} & \text{if } 1 < s \leq \nu_2(n) \\
(-1)^{k/2} 2^{-\nu_2(n)(k-1)} & \text{if } s = \nu_2(n) + 1, \ \nu_2(n) > 0
\end{cases}$$

To achieve the asserted results, it suffices to substitute these values in the expression of $\partial_2 (n, I_{2k})$.

Let $n$ be a positive integer and write $n = 2^a \cdot m$ with $a \geq 0$, $2 \nmid m$. Then, we set

$$\sigma_k^* (n) = (-1)^a \left[ 2 \sigma_k (n/2) - \sigma_k (m) \right]$$

where, as usual, $\sigma_k (n)$ denotes the sum of the $k$-th powers of the divisors of $n$; we agree that $\sigma_k (n/2) = 0$ if $n$ is odd. Next, applying the preceding proposition we get:

**Proposition 5.** Let $n = 2^a \cdot m$, with $a \geq 0$ and $2 \nmid m$ be a positive integer and $k$ an even positive integer. Then:

i) $r (n, \text{gen } I_{2k}) = C(k) \sigma_{k-1}^* (n)$, if $k \equiv 0 \pmod{4}$.

ii) $r (n, \text{gen } I_{2k}) = C(k) \sigma_{k-1} (m) \left( 2^{a(k-1)} \cdot \sum_{j=1}^{k-1} 2^{j(k-1)+1} \right)$, if $k \equiv 2 \pmod{4}$;

where $C(k) = 2k \left[ (2^{k-1}) B_{k/2} \right]^{-1}$, with $B_{k/2}$ denoting the Bernoulli numbers (see [5]).

From the above proposition, in particular, we have:

i) $r (n, \text{gen } I_{24}) = \frac{16}{691} \sigma_{11}^* (n)$.

ii) $r (n, I_4) = 24 \sigma (m)$ and $r (n, I_8) = 16 \sigma_3^* (n)$, since, in the genus of $I_4$ and $I_8$ there is only one class.
Next, we apply the preceding considerations to give:

2. RAMANUJAN FORMULA.

Let \( \vartheta (l_{24}, z) = \sum_{n \geq 0} r(n, l_{24}) q^n \), \( q = e^{2 \pi i z} \), be the theta series associated to the quadratic form \( l_{24} \) and let \( \vartheta (\text{gen } l_{24}, z) = \sum_{n \geq 0} r(n, \text{gen } l_{24}) q^n \) be the theta series associated to the genus of \( l_{24} \). As is known, \( \vartheta (l_{24}, z) \) belongs to the space \( M_{12}(\Gamma_0(4)) \) of modular forms of weight 12 with respect to the group \( \Gamma_0(4) \) (see [7]). Let \( F(z) = (2\pi)^{-12} \Delta(z) = -\sum_{n\geq 1} \tau(n) q^n = \eta^{24}(z) \), be the discriminant modular form [5]. Here, \( \tau(n) \) is the Ramanujan function and \( \eta(z) \) is the Dedekind eta-function.

Lemma 6. The space \( S_{12}(\Gamma_0(4)) \) of cusp forms of weight 12 with respect to the group \( \Gamma_0(4) \) admits the following basis:

\[
\{ \eta^{24}(z), \eta^{24}(z + \frac{1}{2}), \eta^{24}(2z), \vartheta^{12}(z) \eta^{12}(2z) \},
\]

being \( \vartheta(z) = \sum_{n \geq 0} q^n \) the Jacobi theta-function.

Proof. As \( F(z) \) belongs to \( S_{12}(\text{SL}_2(\mathbb{Z})) \) is straightforward that \( F(2z) \) belongs to \( S_{12}(\Gamma_0(2)) \). Moreover, as for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \) we can write

\[
F(\gamma((z + \frac{1}{2}) = F(\gamma_2(z + \frac{1}{2})), \quad \text{with} \quad \gamma_2 = 1/4 \begin{pmatrix} 4a + 2c & 4b - c - 2(a-d) \\ 4c & 4d - 2c \end{pmatrix} \in \Gamma_0(4)
\]

we have that \( F(z + \frac{1}{2}) \) is in \( S_{12}(\Gamma_0(4)) \).

The last function \( \vartheta^{12}(z) \eta^{12}(2z) \) belongs also to \( S_{12}(\Gamma_0(4)) \) because \( \vartheta^{12}(z) \) is in \( M_6(\Gamma_0(4)) \) and \( \eta^{12}(2z) \) in \( S_6(\Gamma_0(4)) \). It is trivial to test that the four
functions are linearly independent. They are a basis because \( \dim_C S_{12}(\Gamma_0(4)) = 4 \) (Cf. [6], th. 2.23).

Now, taking into account the following data:

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( \tau(n) )</th>
<th>( r(n, 1_{24}) )</th>
<th>( 691 r(n, \text{gen } 1_{24}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>48</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>-24</td>
<td>1104</td>
<td>32752</td>
</tr>
<tr>
<td>3</td>
<td>252</td>
<td>16192</td>
<td>2834368</td>
</tr>
<tr>
<td>4</td>
<td>-1472</td>
<td>170064</td>
<td>67141616</td>
</tr>
</tbody>
</table>

we see, applying lemma 6, that the function \( \theta(I_{24}, z) - \theta(\text{gen } I_{24}, z) \), which lies in \( S_{12}(\Gamma_0(4)) \), can be expressed in the following way:

\[
\theta(I_{24}, z) - \theta(\text{gen } I_{24}, z) = 691^{-1} (-33152 F(z + \frac{1}{2}) - 65536 F(2z)).
\]

Therefore by proposition 5 we get Ramanujan formula to count the number of representations of a positive integer as sum of 24 squares:

\[
r(n, I_{24}) = \frac{16}{691} \sigma_{11}(n) + \frac{128}{691} \{(-1)^n 259 \tau(n) - 512 \tau(\frac{1}{2}n)\},
\]

where \( \tau(y) \) means zero when \( y \) is not an integer.

REFERENCES
