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by

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Abstract: Bielliptic surfaces (also called "hyperelliptic surfaces") are defined to be minimal algebraic surfaces of Kodaira dimension 0 and irregularity 1. They play a special role in the birational classification of surfaces. The first part of this paper gives an explicit description of the cohomology group $H^2(S, \mathbf{Z})$ for a bielliptic surface S. In the second part the author proves the existence of smooth bielliptic surfaces in $\mathbf{P}^4$. The proof relies on Reider's criterion for very-ampleness. In fact, a complete characterization of polarized bielliptic surfaces in $\mathbf{P}^4$ is given. These surfaces add to the very short list of known irregular surfaces in $\mathbf{P}^4$, the other two being the abelian surfaces of Horrocks–Mumford and the elliptic quintic scrolls.

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§ 0. INTRODUCTION

The aim of this paper is to describe the Picard group of the bielliptic surfaces and to show that some of these surfaces can be embedded into $\mathbf{P}^4$.

Bielliptic surfaces are defined to be minimal algebraic surfaces of Kodaira dimension 0 and irregularity 1. They play a special role in the birational classification of surfaces. Often they are also called "hyperelliptic surfaces" (as in [1]) although we prefer to use the name "bielliptic" as proposed by Beauville in [2]. It is already classical the distribution of the bielliptic surfaces into seven families.

In Section 1 we give an explicit description of the cohomology group $H^2(S, \mathbf{Z})$ for a bielliptic surface S. A pleasant reward of such a study is the construction of smooth models in $\mathbf{P}^4$ for some of these surfaces, for a very special class indeed. This is carried out in Section 2. We will also provide a complete characterization of the polarized bielliptic surfaces in $\mathbf{P}^4$. Surfaces in $\mathbf{P}^4$ are in general quite hard to study, and several important questions remain unaswered. In particular, it is still a mystery whether there is a universal bound for their irregularity. Until now, to this author's knowledge, there were only two known families of smooth irregular surfaces in $\mathbf{P}^4$, namely the abelian surfaces of Horrocks–Mumford and the elliptic quintic scrolls. The family of bielliptic surfaces we are constructing here provides a third example. Hence their interest. The proof relies on the description of $Pic(S)$ combined with Reider's criterion for very–ampleness.

§ 1. PICARD GROUP OF BIELLIPTIC SURFACES

Let us set up the notation first. Our surfaces will always be smooth, irreducible, projective schemes of dimension 2 defined over the field of complex numbers $\mathbb{C}$. If D is a divisor on a surface S we will denote:

- $\mathcal{O}_S(D) :=$ the invertible sheaf associated to D.
- $h^i\mathcal{O}_S(D) := dim\ H^i\mathcal{O}_S(D)$.
- $\chi\mathcal{O}_S(D) := \sum_{i=0}^{2}(-1)^i\ h^i\mathcal{O}_S(D)$.
- $K_S :=$ the canonical divisor on S.
- $g(D) := \frac{1}{2}(D^2 + DK_S) + 1$, the genus of D.

Given two divisors D, E, we will write $D \equiv E$ (respectively, $D \sim E$) if they are numerically (resp., linearly) equivalent. The group of divisors on S modulo numerical equivalence is denoted $Num(S)$; it is a free abelian group.

A bielliptic surface (also called a hyperelliptic surface) is a minimal surface of Kodaira dimension zero with $h^1\mathcal{O}_S = 1$, $h^2\mathcal{O}_S = 0$. It satisfies $K_S \equiv 0$. Its structure can be described as follows (see [2]):

Proposition 1.1. Given a bielliptic surface S, there exist two elliptic curves A, B, and an abelian group G acting on A and on B such that:

(i) A/G is elliptic and $B/G \simeq \mathbf{P}^1$.

(ii) $S \simeq (A \times B)/G$, where G acts on $A \times B$ componentwise. □

The curve B is isomorphic to $\mathbb{C}/(\mathbf{Z} \cdot \omega \oplus \mathbf{Z})$, where $\omega \in \mathbb{C}$ is uniquely determined if it is chosen in the fundamental region $-1/2 \leq Re\,\omega < 1/2$, $Im\ \omega > 0$, $|\omega| \geq 1$ if $Re\,\omega \leq 0$, $|\omega| > 1$ if $Re\,\omega > 0$. Write $\rho := (-1/2) + (\sqrt{3}/2)i\cdot$ The group G acts on A by translations and the action over B is shown in the following table. This is an already classical result given by Bagnera and de Franchis at the beginning of the century. Here $\mathbf{Z}_n$ stands of $\mathbf{Z}$ modulo $(n)\mathbf{Z}$.

Proposition 1.2 (see [2]). There are seven types of bielliptic surfaces, which are described as follows:

Type	ω	G	Action of G on $B \simeq \mathbb{C}/(\mathbb{Z}\omega \oplus \mathbb{Z})$
1	any	$\mathbb{Z}_2$	$x \longmapsto -x$
2	any	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$x \longmapsto -x$ $x \longmapsto x + \varepsilon$ with $2\varepsilon = 0$
3	i	$\mathbb{Z}_4$	$x \longmapsto i\,x$
4	i	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$x \longmapsto i\,x$ $x \longmapsto x + (1+i)/2$
5	ρ	$\mathbb{Z}_3$	$x \longmapsto \rho x$
6	ρ	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$x \longmapsto \rho x$ $x \longmapsto x + (1-\rho)/3$
7	ρ	$\mathbb{Z}_6$	$x \longmapsto -\rho x$

□

We will denote by $\Phi : S \to (A/G)$, $\Psi : S \to (B/G)$ the two natural projections. Since $A \to (A/G)$ is étale, all fibres of Φ are smooth. The fibre of Ψ over a point $P \in B/G$ is a multiple of a smooth elliptic curve, the multiplicity being the one of P by the finite map $B \to (B/G)$. Inasmuch as all smooth fibres of Ψ (respectively of Φ) are isomorphic to A (resp., to B), in the sequel we will also denote by A or B the class (in $Num(S)$, $H^2(S,\mathbb{Z})$ or $H^2(S,\mathbb{Q})$) of a fibre of Ψ or Φ respectively. This notation is very economic and produces no ambiguity. Finally, let γ be the order of the group G .

The terminology introduced so far will be used throughout this paper without further notice.

From the exponential sequence one obtains the exact sequence

$$H^1(S,\mathcal{O}_S^*) \to H^2(S,\mathbb{Z}) \to H^2(S,\mathcal{O}_S) = 0$$

Consequently, the group $Num(S)$ coincides with $H^2(S,\mathbb{Z})$ modulo torsion (use Poincaré duality as in [4], page 53), and $\mathbb{Q} \otimes_{\mathbb{Z}} Num(S) \simeq H^2(S,\mathbb{Q})$.

Let $e(S)$ denote the topological Euler characteristic of S. From Noether's formula $12\chi\mathcal{O}_S = K_S^2 + e(S)$ and $\chi\mathcal{O}_S = K_S^2 = 0$ one gets $e(S) = 0$. Having into account that $dim\, H^1(S,\mathbb{Q}) = 2 \cdot h^1\mathcal{O}_S = 0$ we obtain $dim\, H^2(S,\mathbb{Q}) = 2$. The classes A, B intersect (in $H^2(S,\mathbb{Q})$ or in $Num(S)$) as:

$$A^2 = B^2 = 0 \quad ; \quad AB = \gamma \quad , \quad (\gamma = \text{order of } G)$$

Clearly $\{A, B\}$ is a basis of $H^2(S,\mathbb{Q})$.

Let us put together some useful observations:

Lemma 1.3. Let D be a divisor of numerical class $\alpha A + \beta B$ with $\alpha, \beta \in \mathbb{Q}$. Then:

(i) $\chi \mathcal{O}_S(D) = \alpha\beta\gamma$.

(ii) D is ample if and only if $\alpha > 0,\ \beta > 0$.

(iii) If D is ample then $h^0\mathcal{O}_S(D) = \chi\mathcal{O}_S(D)$.

(iv) If $H^0\mathcal{O}_S(D) \neq 0$ then $\alpha \geq 0,\ \beta \geq 0$.

Proof: (i) and (ii) are immediate consequences of Riemann–Roch and Nakai's criterion respectively. If D is ample then $D - K_S \equiv D$ is ample too, and Kodaira vanishing gives $h^1\mathcal{O}_S(D) = h^1\mathcal{O}_S\big(-(D-K)\big) = 0$, so we have (iii), As for (iv), if D is effective then $DA \geq 0,\ DB \geq 0$. $\square$

Our aim now is to compute a basis of $Num(S)$ over $\mathbb{Z}$:

Theorem 1.4. The following table yields the multiplicities $\{m_1, \ldots, m_t\}$ of the singular fibres of $\Psi : S \to (B/G) \simeq \mathbb{P}^1$ and a basis of $Num(S)$ for the seven types of bielliptic surfaces:

Type	$\{m_1, \ldots, m_t\}$	Basis of $Num(S)$
1	$\{2,2,2,2,\}$	$\{(1/2)A;\ B\}$
2	$\{2,2,2,2,\}$	$\{(1/2)A;\ (1/2)B\}$
3	$\{2,4,4,\}$	$\{(1/4)A;\ B\}$
4	$\{2,4,4,\}$	$\{(1/4)A;\ (1/2)B\}$
5	$\{3,3,3,\}$	$\{(1/3)A;\ B\}$
6	$\{3,3,3,\}$	$\{(1/3)A;\ (1/3)B\}$
7	$\{2,3,6,\}$	$\{(1/6)A;\ B\}$

The values of $\{m_1, \ldots, m_t\}$ are easily computed by applying Hurwitz formula to the map $B \to (B/G) \simeq \mathbb{P}^1$ and recalling the action of G on B described in Proposition 1.2. In order to determine a basis of $Num(S)$ we need an auxiliary result:

Lemma 1.5. Let μ be the least common multiple of $\{m_1, \ldots, m_t\}$. Then μ is the largest integer d such that $(1/d)A$ belongs to $Num(S)$.

Proof of the Lemma: This claim holds in general for any morphism, such as our Ψ, which is an elliptic quasi–bundle fibration different from the Albanese map (see [9]).

Nevertheless we are going to give here an ad hoc simpler proof for bielliptic surfaces. On the one hand, it is clear that $(1/\mu)A \in Num(S)$. Let $\alpha \geq 1$ be an integer such that $(1/\alpha)A \in Num(S)$. Note that $\gamma/\alpha = (1/\alpha)AB \in \mathbb{Z}$. Checking the values of μ for the seven types of bielliptic surfaces we observe that $\alpha > \mu$ implies $\alpha = \gamma$. Hence it is enough to show that if $(1/\gamma)A \in Num(S)$ then $\gamma = \mu$. Let D be a divisor of class $(1/\gamma)A + B$. By Riemann–Roch $\chi\mathcal{O}_S(D) = 1 > 0$. Since $K_S - D \equiv (-1/\gamma)A - B$ cannot be effective, we can choose D to be effective. Assume D to be irreducible. From $DB = 1$ it follows that the restriction of $\Phi : S \to (A/G)$ to D is an isomorphism, and so $g(D) = g(A/G) = 1$. But adjunction formula yields $g(D) = 2$, a contradiction. Hence D must be reducible, and the only way this can be true is if there exists an effective divisor $E \equiv (1/\gamma)A$ (use Lemma 1.3). But $EA = 0$ implies that E is contained in fibres of $\Psi : S \to (B/G)$. Therefore Ψ has a fibre of multiplicity γ, and $\mu = \gamma$. □

Proof of Theorem 1.4: The main point to be checked is that the cocycles listed in the Theorem do actually belong to $Num(S)$. Once we have seen this, a straightforward calculation shows that the given pairs define basis.

For surfaces of types 1, 3, 5, and 7 we know that $(1/\gamma)A \in Num(S)$. Suppose that we are in cases 2, 4 or 6. We claim that $(1/\gamma)A + \delta B$ does not belong to $Num(S)$ for any $\delta \in \mathbb{Q}$. Otherwise, if $D \equiv (1/\gamma)A + \delta B$ is a divisor, then $\delta = \chi\mathcal{O}_S(D) \in \mathbb{Z}$. Thus $D - \delta B \equiv (1/\gamma)A$ belongs to $Num(S)$, contradicting Lemma 1.5. As a consequence of the claim we obtain that for all $D \in Num(S)$:

$$DB \in (2)\mathbb{Z} \qquad \text{for types } 2 \text{ and } 4;$$

$$DB \in (3)\mathbb{Z} \qquad \text{for type } 6.$$

Applying Poincaré duality ([4]), page 53) we conclude:

$$(1/2)B \in Num(S) \qquad \text{for types } 2 \text{ and } 4;$$

$$(1/3)B \in Num(S) \qquad \text{for type } 6. \qquad \square$$

Remark 1.6. For bielliptic surfaces of types 2, 4 and 6, B is divisible in $Num(S)$ by an integer ≥ 2 despite the fact that $\Phi : S \to (A/G)$ is a smooth fibration, so that the divisibility of B is not accounted for by the existence of multiple fibres of Φ. As it is pointed out in [9], Serre's spectral sequence gives a relationship between the divisibility of the fibre of a fibre bundle S and the torsion of $H_1(S,\mathbb{Z})$. For a bielliptic surface, this torsion has been computed ([5], [10], [11], [9]) and is shown in the following table:

Type	Torsion of $H_1(S,\mathbb{Z})$
1	$\mathbb{Z}_2 \times \mathbb{Z}_2$
2, 3	$\mathbb{Z}_2$
4, 6, 7	0
5	$\mathbb{Z}_3$

In general, given an algebraic elliptic fibre bundle $\Phi : S \to C$ with $h^1\mathcal{O}_S = g(C)$, there are only seven possibilities for the pair $(d;$ torsion of $H_1(S,\mathbb{Z}))$, where d is the largest integer dividing a fibre of Φ in $Num(S)$ (see [9]). It turns out that each one of these cases in realized in one type of bielliptic surface, and thus the seven are actually possible.

The torsions of $H_1(S,\mathbb{Z})$ and $H^2(S,\mathbb{Z})$ are always isomorphic (non–canonically). For bielliptic surfaces of types 4 and 6 the group $H^2(S,\mathbb{Z})$ is torsion–free and thus B is divisible in $H^2(S,\mathbb{Z})$ (not only in $Num(S) = H^2(S,\mathbb{Z})/$ (torsion)) by 2 or 3 respectively. Things are not so clear for type 2. Now we want to find out whether in this case B is or is not divisible by 2 in $H^2(S,\mathbb{Z})$.

Proposition 1.7. If S is a bielliptic surface of type 2 then B is divisible by 2 in $H^2(S,\mathbb{Z})$, i.e. $B = 2L$ in $H^2(S,\mathbb{Z})$ for some L.

Proof: As we know, $B = \mathbb{C}/(\mathbb{Z}\omega \oplus \mathbb{Z})$ and G acts on B as $X \longmapsto -X$, $X \longmapsto X + \bar{\varepsilon}$ where $\bar{\varepsilon}$ is the class of $\varepsilon \in \Gamma = \{1/2;\ (1/2)\omega;\ (1/2)\omega + (1/2)\}$. Choose $\delta \in \Gamma$, $\delta \neq \varepsilon$, and denote by $\tau : B \to B$ the translation by $\bar{\delta}$. We see that $\tau \notin G$ and τ commutes with the elements of G. The curve $E := B/(\tau)$ is elliptic and G acts on E so that no non–zero subgroup of G is acting trivially. The surface $T := (A \times E)/G$ is bielliptic of type 2 and the map $B \to E$ induces an étale map $f : S \to T$ of degree 2. By general theory (e.g. [3], (2.6)), there exists a divisor $M \in Pic(T)$ such that f is the 2–cyclic covering determined by the "equation" $2M \sim 0$ (see [1], I. 17). Consider the commutative diagram

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\Phi"'] & & T \arrow[dl, "h"] \\ & A/G & \end{tikzcd}$$

Let E_0 be a smooth fibre of h such that $B_0 := f^{-1}(E_0)$ is a smooth fibre of Φ. Inasmuch as $f_{|B_0} : B_0 \to E_0$ is the 2–cyclic covering defined by $2M_{|E_0} \sim 0$, we get that $M_{|E_0}$ is not linearly equivalent to 0 (otherwise $f^{-1}(E_0)$ would have two connected components). The Albanese map of T is h because $h^1\mathcal{O}_T = g(A/G)$. Hence $Alb(E_0) \to Alb(T)$ is the zero map, and thus $Pic^0(T) \to Pic^0(E_0)$ is also zero, since Picard and Albanese varieties are dual to each other. It follows that the class $\overline{M}$ of M in $H^2(T,\mathbb{Z})$ is the only torsion element (of order 2). The class of E_0 in $H^2(T,\mathbb{Z})$ is either of the form $2D$ or $2D+\overline{M}$. But $f^*(\overline{M}) \sim 0$, so that $B_0 = f^*(D)$ in $H^2(S,\mathbb{Z})$. □

§ 2. BIELLIPTIC SURFACES IN $\mathbb{P}^4$

In this section we are going to construct smooth bielliptic surfaces of degree 10 in $\mathbb{P}^4$ and give a complete characterization of them. Their existence contradicts the statement made in ([8], page 172), which is due to an obvious arithmetical mistake. We emphasize that the bielliptic surfaces which admit an embedding into $\mathbb{P}^4$ are somewhat of the most special kind; in particular, B must have j–invariant equal to 0. Once we have a complete description of the Picard group, provided by Theorem 1.4, the essential part of the proof will be based on Reider's criterion for very–ampleness, which is stated as follows:

Theorem 2.1 [7]. Let L be a divisor on a surface S such that $L^2 \geq 10$ and $LC \geq 0$ for all curves C on S. Then $L+K_S$ is very ample unless there exists an effective divisor E on S satisfying one of the following conditions:

(i) $LE = 0$, $E^2 = -1$ or -2

(ii) $LE = 1$, $E^2 = 0$ or -1

(iii) $LE = 2$, $E^2 = 0$ □

With the notation introduced so far one has:

Theorem 2.2. Let S be a bielliptic surface of type 6 and L a divisor on S of numerical class $(1/3)A+(5/3)B$. Then L is very ample and provides an embedding of S into $\mathbb{P}^4$ as a surface of degree 10. Conversely, any smooth bielliptic surface in $\mathbb{P}^4$ is of type 6 and has a hyperplane divisor of the class indicated.

Proof: Suppose first that S is of type 6 and $L \equiv (1/3)A+(5/3)B$. We have $L^2 = 10$ and $h^0\mathcal{O}_S(L) = 5$ (Lemma 1.3). The only irreducible curves E in S with $E^2 \leq 0$

are of numerical class $(1/3)A$, A or B, and so $LE \geq 3$. Theorem 2.1 shows that L is very ample.

Conversely, let S be a smooth bielliptic surface embedded in $\mathbf{P}^4$. Since S cannot lie in $\mathbf{P}^3$ nor can be the projection of a surface in $\mathbf{P}^5$, we may as well assume that $h^0\mathcal{O}_S(L) = 5$, where L is the hyperplane divisor. We can write $L \equiv (\alpha/\gamma)A + (\beta/\gamma)B$ with $\gamma =$ order of G and α, β positive integers.

From $\alpha\beta/\gamma = \chi\mathcal{O}_S(L) = h^0\mathcal{O}_S(L)$ (Lemma 1.3) we get

(*) $\alpha\beta/\gamma = 5$

We will divide the proof into three cases:

Case I: Assume that S is of type 1, 3, 5 or 7.

The morphism $\Psi : S \to (B/G)$ has a fibre of multiplicity γ. The reduced component of this fibre is an elliptic curve of class $(1/\gamma)A$. Hence $\beta/\gamma = (1/\gamma)A \cdot L \geq 3$. B is also elliptic and thus $\alpha = B \cdot L \geq 3$. Hence $\alpha\beta/\gamma \geq 9$, which contradicts (*).

Case II: Let S be of type 2 or 4.

On the one hand we note that $\alpha = 2\delta$ with $\delta \in \mathbf{Z}$. Also the map $\Psi : S \to (B/G)$ has a multiple fibre whose reduced component is of class $(2/\gamma)A$. Write $\eta = 2\beta/\gamma$. Then $\eta = (2/\gamma)AL \geq 3$ and $2\delta = BL \geq 3$, so that $6 \leq \eta\delta = \alpha\beta/\gamma$, against (*).

Case III: Suppose that S is of type 6.

Now we have $\alpha = 3\delta$ with $\delta \in \mathbf{Z}$. Write $\eta = 3\beta/\gamma \in \mathbf{Z}$. Then $\eta = (3/\gamma)AL \geq 3$ and $3\delta = BL \geq 3$. From $5 = \alpha\beta/\gamma = \eta\delta$ we obtain $\delta = 1$, $\eta = 5$ as the only possibility. Thus $L \equiv (1/3)A + (5/3)B$. $\square$

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