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MULTIPLE FIBRES OF A MORPHISM

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Abstract

Let us be given a morphism $\varphi : S \to C$ with connected fibres from a complex surface onto a curve. The aim of this paper is to show that the multiplicities of the fibres can be read off at the level of singular homology. Namely, a suitable exact sequence

$$H_1(F,\mathbb{Z}) \to H_1(S,\mathbb{Z}) \to H_1(C,\mathbb{Z}) \times G(\varphi) \to 0$$

is constructed, where F denotes a general fibre and $G(\varphi)$ is a finite abelian group defined only in terms of the multiplicities $\{m_1,\dots,m_t\}$ of the multiple fibres. More precisely, $G(\varphi) := Coker(f : \mathbb{Z} \to \oplus_i \mathbb{Z}/(m_i))$ where $f(1) = (\bar{1},\dots,\bar{1})$. It is also shown that $\oplus_i \mathbb{Z}/(m_i)$ is invariant under smooth deformations of φ. All this generalizes the already known situation for elliptic surfaces, whose fundamental groups can be explicitely described. Moreover Iitaka has proved that for an elliptic fibration the set of multiplicities $\{m_1,\dots,m_t\}$ is a deformation invariant.

MULTIPLE FIBRES OF A MORPHISM

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§ 0. INTRODUCTION

Let us be given a morphism $\varphi : S \to C$ with connected fibres from a compact complex surface onto a curve. The aim of this paper is to show that the multiplicities of the fibres can be read off at the level of singular homology. Namely, the first homology groups (over $\mathbf{Z}$) of the surface, the base curve and a general fibre F are related by means of an exact sequence (Theorem 1.3):

$$H_1(F,\mathbf{Z}) \to H_1(S,\mathbf{Z}) \to H_1(C,\mathbf{Z}) \times G(\varphi) \to 0$$

Here $G(\varphi)$ denotes a finite abelian group defined only in terms of the multiplicities $\{m_1, \ldots, m_t\}$ of the multiple fibres. More precisely, $G(\varphi) := Coker\,(f : \mathbf{Z} \to \bigoplus_{i=1}^{t} \mathbf{Z}/m_i\mathbf{Z})$ where $f(1) = (\bar{1}, \ldots, \bar{1})$. This generalizes the already known situation for elliptic surfaces, for which the sequence above can be deduced from the explicit description of the fundamental group of the surface ([6]). However, for a larger fibre genus such a description is lacking in general.

Next we will address the question of the variation of $G(\varphi)$ and $\bigoplus_{i=1}^{t} \mathbf{Z}/m_i\mathbf{Z}$ under smooth deformations of φ. It will be shown in § 2 that both groups are actually invariant under deformation. The proof for $G(\varphi)$ relies on the above exact sequence plus the fact that a smooth analytic map is differentiably locally trivial. Then a base change trick will give the invariance of $\bigoplus_i \mathbf{Z}/m_i\mathbf{Z}$. Again for elliptic fibrations, the general picture is neater since Iitaka proved that in this case the set of multiplicities of the fibres is a deformation invariant ([5]).

§ 1. HOMOLOGY GROUPS

We shall always be working over the field of complex numbers. A surface is a compact connected complex manifold of complex dimension 2. A fibration is a proper surjective holomorphic map from a surface onto a smooth connected curve, all of whose fibres are connected. We will also use the following notation:

- $\mathbf{Z}_m :=$ integers $\mathbf{Z}$ modulo $(m)\mathbf{Z}$.
- $tor\, H :=$ torsion of an abelian group H.
- $\pi_1(X) :=$ fundamental group of X.
- $h^i\mathcal{O}_S := dim_{\mathbf{C}} H^i(S, \mathcal{O}_S)$, where $\mathcal{O}_S$ is the structure sheaf of S.

Let $\varphi : S \to C$ be a fibration, and $F = \sum n_i B_i$ a fibre of φ where the $B_i' s$ are the irreducible reduced components of F and the $n_i' s$ are their multiplicities. Let m be the greatest common divisor of the $n_i' s$. We say that m is the multiplicity of F and write $F = m\,D$, where $D = \sum (n_i/m) B_i$. Whenever we say "let $m\,D$ be a multiple fibre" we shall always mean that m is the multiplicity of $m\,D$ and $m \geq 2$.

Let $\varphi : S \to C$ be a fibration and let $m_1 D_1, \ldots, m_t D_t$ be all its multiple fibres.

Definition 1.1.

$$G(\varphi) := Coker\left(\mathbf{Z} \longrightarrow \bigoplus_{i=1}^{t} \mathbf{Z}_{m_i}\right)$$
$$1 \longmapsto (1, \ldots, 1)$$

$$L(\varphi) := \bigoplus_{i=1}^{t} \mathbf{Z}_{m_i}$$

If μ is the least common multiple of $m_1, \ldots, m_t$, by dualizing the sequence

$$0 \to \mathbf{Z}_\mu \to \bigoplus_{i=1}^{t} \mathbf{Z}_{m_i} \to G(\varphi) \to 0$$

we obtain an alternative description of $G(\varphi)$ as:

$$G(\varphi) = Ker\left(\bigoplus_{i=1}^{t} \mathbf{Z}_{m_i} \longrightarrow \mathbf{Z}_\mu\right)$$
$$(a_1, \ldots, a_t) \longmapsto \sum a_i(\mu/m_i)$$

The third characterization that follows will be used later:

Lemma 1.2. Write $\bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \simeq \bigoplus_{j=1}^{k} \mathbb{Z}_{d_j}$ where each d_j divides d_{j+1}. Then

$$G(\varphi) \simeq \bigoplus_{j=1}^{k-1} \mathbb{Z}_{d_j}$$

Proof: Since $\mu/m_1, \dots, \mu/m_t$ are relatively prime, we can find integers $\lambda_1, \dots, \lambda_t$ such that $\sum_{i=1}^{t} (\lambda_i \mu/m_i) = 1$. The homomorphism

$$\begin{aligned} \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} &\longrightarrow \mathbb{Z}_{\mu} \\ (a_1, \dots, a_t) &\longmapsto \sum_{i=1}^{t} a_i(\lambda_i \mu/m_i) \end{aligned}$$

is a retraction of $0 \to \mathbb{Z}_{\mu} \to \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \to G(\varphi) \to 0$, and this sequence splits. If we put $G(\varphi) = \bigoplus_{j=1}^{r} \mathbb{Z}_{e_j}$ with e_j dividing e_{j+1} for all j, then all $e_j's$ divide μ and

$$\bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} = G(\varphi) \oplus \mathbb{Z}_{\mu} = \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{e_j}\right) \oplus \mathbb{Z}_{\mu}$$

Since the $d_i's$ are uniquely determined, it follows that $(d_1, \dots, d_{k-1}, d_k) = (e_1, \dots, e_r, \mu)$
☐

Now it comes the main result of this paper. For elliptic surfaces it can be deduced from the well–known description of their fundamental groups (see [5]). Our proof has been inspired in that of Prop. 1.41 of [2].

Theorem 1.3. Let $\varphi : S \to C$ be a fibration from the surface S onto a smooth curve C.

Denote by $m_1 D_1, \dots, m_t D_t$ all multiple fibres of φ, and let F be any smooth fibre, and $G := G(\varphi)$. Then there exists an exact sequence

$$H_1(F, \mathbb{Z}) \to H_1(S, \mathbb{Z}) \to H_1(C, \mathbb{Z}) \times G \to 0$$

induced by φ and the inclusion of F into S.

Proof: Let $\Omega = \{p \in C \mid \varphi^{-1}(p) \text{ is singular}\}$, $\tilde{C} = C - \Omega$, $\tilde{S} = S - \left(\cup_{p \in \Omega} \varphi^{-1}(p)\right)$.

Consider the following commutative diagram with exact rows and columns, whose homomorphisms come from the obvious inclusions and restrictions:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & M \arrow[r, "\varepsilon"] & H_1(S,\mathbf{Z}) \arrow[r, "\varphi_*"] \arrow[u] & H_1(C,\mathbf{Z}) \arrow[r] \arrow[u] & 0 \\
 & H_1(F,\mathbf{Z}) \arrow[r] \arrow[u, "f"'] & H_1(\tilde{S},\mathbf{Z}) \arrow[r, "\sigma"] \arrow[u, "g"'] & H_1(\tilde{C},\mathbf{Z}) \arrow[r] \arrow[u, "h"'] & 0 \\
 & & N_1 \arrow[r, "\tau"] \arrow[u] & N_2 \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

M, N_1 and N_2 are defined to be the kernels of the corresponding homomorphisms. The second row is exact because $\tilde{S} \to \tilde{C}$ is a C^∞–fibre bundle.

Claim 1: The cokernel of $\tau : N_1 \to N_2$ is a quotient of G.

Proof of Claim 1: Given $p \in \Omega$, denote by γ_p a simple loop around p in $\tilde{C}$. The group N_2 is generated by all the γ_p, $p \in \Omega$, with the single relation $\Pi_{p\in\Omega}\gamma_p = 0$.

If B is a component of multiplicity n of a fibre $\varphi^{-1}(p)$, $p \in \Omega$, then there is a loop α in $\tilde{S}$ around B such that $\alpha \in N_1$ and $\tau(\alpha) = n\gamma_p$. Consequently, if m is the total multiplicity of $\varphi^{-1}(p)$ then $m\gamma_p \in Im(\tau)$, and the claim follows.

Claim 2: There exists an exact sequence:

$$H_1(F,\mathbf{Z}) \xrightarrow{f} M \xrightarrow{\rho} Coker(\tau) \to 0$$

Proof of Claim 2: Define the map $\rho : M \to Coker(\tau)$ as follows. Given $x \in M$, there is $y \in H_1(\tilde{S},\mathbf{Z})$ such that $g(y) = \varepsilon(x)$. Thus $\sigma(y) \in N_2$, and we write $\rho(x)$ as the class of $\sigma(y)$ in $N_2/(Im(\tau))$. An easy diagram–checking shows that the above sequence is exact. This is nothing else than the so–called Snake Lemma, but later we are going to need the explicit description of the map ρ.

Claim 3: There exists a commutative diagram with exact rows and columns as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 & \\
H_1(F,\mathbf{Z}) \arrow[r, "f"] \arrow[dr, "j"'] & M \arrow[r, "\rho"] \arrow[d, "\varepsilon"] & Coker(\tau) \arrow[r] \arrow[u] & 0 \\
 & H_1(S,\mathbf{Z}) \arrow[r, "\lambda"] \arrow[d, "\varphi_*"] & G \arrow[u, "\theta"] & \\
 & H_1(C,\mathbf{Z}) \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

Proof of Claim 3: $\theta : G \to Coker(\tau)$ is the epimorphism of Claim 1, and $j = \varepsilon \circ f$ by definition. We must define λ and prove $\rho = \theta \circ \lambda \circ \varepsilon$. The fundamental group $\pi_1(\tilde{C})$ is generated by elements $\alpha_i, \beta_i, \gamma_p$ (for i from 1 up to genus of C, and $p \in \Omega$) with the unique relation $(\Pi_i \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1})(\Pi_{p\in\Omega}\gamma_p) = 1$. Given $p \in \Omega$ and $m(p) =$ multiplicity of $\varphi^{-1}(p)$, there corresponds to $\varphi^{-1}(p)$ a direct summand $\mathbf{Z}_{m(p)}$ in $\bigoplus_{i=1}^{t} \mathbf{Z}_{m_i}$, with $\mathbf{Z}_{m(p)} = 0$ in case $m(p) = 1$. Define an epimorhism $\pi_1(\tilde{C}) \to G$ by mapping γ_p to the image of $\bar{1} \in \mathbf{Z}_{m(p)} \subseteq \bigoplus_i \mathbf{Z}_{m_i}$ in G, and all α_i, β_i to 0. We get in this fashion a ramified covering $B \to C$, unramified outside Ω and such that the ramification index on points over $p \in \Omega$ divides $m(p)$. If R denotes the normalization of $S \times_C B$ then $R \to S$ is unramified with group G (see [1], III 9.1), and thus it is determined by an epimorphism $\pi_1(S) \to G$ which descends to an epimorphism $\lambda : H_1(S,\mathbf{Z}) \to G$. The preimage of F by $R \to S$ splits into as many components as the order of G, so that the induced map $\pi_1(F) \to G$ is 0. It follows that $\lambda \circ j = 0$. Finally, the commutativity of the diagram of Claim 3 stems from the description of ρ given in Claim 2 combined with the commutativity of the following diagram:

(*)

$$\begin{tikzcd}
H_1(S,\mathbf{Z}) \arrow[r, "\lambda"] & G \arrow[dr, "\theta"] & \\
H_1(\tilde{S},\mathbf{Z}) \arrow[u, "g"] \arrow[r, "\sigma"] & H_1(\tilde{C},\mathbf{Z}) \arrow[u, "q"] & Coker(\tau) \\
 & N_2 \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Claim 4: θ is an isomorphism.

Proof of Claim 4: Since $\lambda \circ j = 0$, one has a commutative diagram

$$\begin{tikzcd}
M/Im(f) \arrow[r, "\sim"] \arrow[dr, "\lambda\circ\bar{\varepsilon}"'] & Coker(\tau) \\
 & G \arrow[u, "\theta"']
\end{tikzcd}$$

In particular, $Coker(\tau)$ is a direct summand of G. Now it suffices to show that $\lambda\circ\bar{\varepsilon}$ is surjective. The class of the loop γ_p in $H_1(\tilde{C},\mathbf{Z})$ maps by $q: H_1(\tilde{C},\mathbf{Z}) \to G$ to the image of $\bar{1} \in \mathbf{Z}_{m(p)} \subseteq \bigoplus_{i=1}^{t} \mathbf{Z}_{m_i}$ in G. By the commutativity of the diagram (*) above, one gets that if $\sigma(x) = \gamma_p$ then $g(x) \in Im(\varepsilon)$, and $(\lambda \circ g)(x)$ is also the image of $\bar{1} \in \mathbf{Z}_{m(p)}$ in G. Consequently $\lambda \circ \bar{\varepsilon}$ is surjective, as we wanted.

Claim 5: The following sequence is exact:

$$H_1(F,\mathbf{Z}) \xrightarrow{j} H_1(S,\mathbf{Z}) \xrightarrow{(\lambda,\varphi_*)} G \times H_1(C,\mathbf{Z}) \to 0$$

Proof of Claim 5: Clearly $Im(j) \subseteq Ker(\lambda,\varphi_*)$. Conversely if $x \in Ker(\lambda,\varphi_*)$ then $x \in M$ and $\rho(x) = 0$, so that $x \in Im(j)$. Let us finally prove the surjectivity of (λ,φ_*). Let $(y,z) \in G \times H_1(C,\mathbf{Z})$. There exists an element $x \in H_1(S,\mathbf{Z})$ such that $\varphi_*(x) = z$. Since $\lambda\circ\varepsilon$ is surjective, one can find $t \in M$ such that $\lambda(\varepsilon(t)) = y - \lambda(x)$. Then $\lambda(x + \varepsilon(t)) = y$ and $\varphi_*(x + \varepsilon(t)) = z$. This ends the proof of Theorem 1.3. ☐

For the remainder of this section we will assume all surfaces to be algebraic.

Remark 1.4. When $g(F) = 1$, that is, when $\varphi : S \to C$ is an elliptic fibration, one has a more accurate information. If φ has a singular fibre other than a multiple of a smooth curve, then the homomorphism $H_1(F,\mathbf{Z}) \to H_1(S,\mathbf{Z})$ is the zero map ([2], 1.39). In particular, $h^1\mathcal{O}_S = h^1\mathcal{O}_C$ in this case. For the other cases, see [9]. In general, the fundamental group of an elliptic surface can be almost completely described ([6]).

Given a fibration $\varphi : S \to C$ we always have $h^1\mathcal{O}_S \geq h^1\mathcal{O}_C$, with equality if and only if either $h^1\mathcal{O}_S = 0$ or φ is the Albanese map. This follows easily from the universal property of the Albanese variety. Denote by $tor(H)$ the torsion of an abelian group H. From Theorem 1.3 one immediately gets

Corollary 1.5. Let J denote the image of $H_1(F,\mathbb{Z})$ in $H_1(S,\mathbb{Z})$. Then there is an exact sequence

$$0 \to tor\, J \to tor\, H_1(S,\mathbb{Z}) \to G$$

Furthermore, $tor\, H_1(S,\mathbb{Z}) \to G$ is surjective provided that $h^1\mathcal{O}_S = h^1\mathcal{O}_C$. □

We recall that $tor\, H_1(S,\mathbb{Z}) \simeq tor H^2(S,\mathbb{Z})$ (non–canonically). The following Proposition describes explicitely some of the elements of $tor\, H^2(S,\mathbb{Z})$ in case $h^1\mathcal{O}_S = h^1\mathcal{O}_C$. Let $m_1D_1,\ldots,m_tD_t$ be the multiple fibres of a fibration $\varphi : S \to C$, and denote μ the least common multiple of $m_1,\ldots,m_t$. Since $\mu/m_1,\ldots,\mu/m_t$ are relatively prime, there exist integers $\lambda_1,\ldots,\lambda_t$ such that $\sum_{i=1}^t (\lambda_i\,\mu/m_i) = 1$ Let $D = \sum_{i=1}^t \lambda_i D_i$. Denote by [E] the class in $H^2(S,\mathbb{Z})$ of a divisor E, and $G := G(\varphi)$.

Proposition 1.6. If $h^1\mathcal{O}_S = h^1\mathcal{O}_C$, then the classes $\{[D_i - (\mu/m_i)D] \mid i = 1,\ldots,t\}$ generate a subgroup of $tor\, H^2(S,\mathbb{Z})$ isomorphic to G.

Proof: First we remark that the subgroup generated by these classes is precisely $\left\{ \sum_{i=1}^t \alpha_i[D_i] \mid \alpha_i \in \mathbb{Z},\ \sum_{i=1}^t (\alpha_i/m_i) = 0 \right\}$.

If F is a general fibre of φ then

$$\begin{aligned} m_i[D_i - (\mu/m_i)D] &= [m_iD_i] - [\mu D] = \\ &= [F] - [F] = 0 \end{aligned}$$

Thus $[D_i - (\mu/m_i)D] \in tor\, H^2(S,\mathbb{Z})$. Define the homomorphisms:

$$\sigma : \mathbb{Z} \to \bigoplus_{i=1}^t \mathbb{Z}_{m_i}, \quad \rho : \bigoplus_{i=1}^t \mathbb{Z}_{m_i} \to tor\, H^2(S,\mathbb{Z})$$

as $\sigma(1) = \sum_{i=1}^t \lambda_i e_i$, $\rho(e_i) = [D_i - (\mu/m_i)D]$, where $e_i = (0,\ldots,0,\bar{1},0,\ldots,0)$, ($\bar{1}$ in the i^{th}–position).

Claim 1: The sequence

$$\mathbb{Z} \xrightarrow{\sigma} \bigoplus_{i=1}^t \mathbb{Z}_{m_i} \xrightarrow{\rho} tor\, H^2(S,\mathbb{Z})$$

is exact.

Proof of Claim 1: First note that

$$\rho\Big(\sum_{i=1}^{t} \lambda e_i\Big) = \Big[\big(\sum_i \lambda_i D_i\big) - \sum_i (\lambda_i\mu/m_i)D\Big] =$$
$$= \big[D - D\big] = 0$$

Hence $Im(\sigma) \subseteq Ker(\rho)$. Now assume $\rho\big(\sum_{i=1}^{t} \gamma_i e_i\big) = 0$, and put $\delta := \sum_i (\gamma_i\mu/m_i)$. From $\big[(\sum_i \gamma_i D_i) - \delta D\big] = 0$ it follows that $(\sum_i \gamma_i D_i) - \delta D$ belongs to the Picard variety of S, denoted $Pic^\circ(S)$. As indicated before, the fact that $h^1\mathcal{O}_S = h^1\mathcal{O}_C$ implies that the Albanese varieties of S and C are isomorphic, hence also their Picard varieties are isomorphic. The symbol $\sim$ is going to denote linear equivalence of divisors. Obviously the restriction $Pic^\circ(C) \to Pic^\circ(D_k)$ is the zero map, and it follows that $\big(\sum_{i=1}^{t} \gamma_i D_i - \delta D\big)_{|D_k} \sim 0$. We know that $(D_i)_{|D_k} \sim 0$ if $i \neq k$, and $(D_k)_{|D_k}$ is torsion of order m_k in $Pic(D_k)$ ([1]; III 8.3). Combining with $D_{|D_k} \sim \lambda_k (D_k)_{|D_k}$ one gets $(\gamma_k - \delta\lambda_k)(D_k)_{|D_k} \sim 0$, which implies that $\gamma_k - \delta\lambda_k$ is a multiple of m_k. Thus $\sum_i \gamma_i e_i = \delta \sum_i \lambda_i e_i \in Im(\sigma)$, as we wanted.

Claim 2: $Ker(\sigma) = (\mu)\mathbb{Z}$

Proof of Claim 2: Let $(\nu)\mathbb{Z} := Ker(\sigma)$. Multiplying the equation $\sum_{i=1}^{t} (\lambda_i\mu/m_i) = 1$ by m_k we obtain that $\lambda_k\mu$ is a multiple of m_k. Hence $\sigma(\mu) = 0$ and one can write $\mu = \nu \cdot d$ for some $d \in \mathbb{Z}$. Since m_i divides $\lambda_i\nu$ we have $\sum_i (\lambda_i\nu/m_i) \in \mathbb{Z}$. On the other hand $1 = \sum_i(\lambda_i\mu/m_i) = d\sum_i(\lambda_i\nu/m_i)$. so that $d = 1$ and Claim 2 follows.

The exact sequence

$$0 \to \mathbb{Z}_\mu \xrightarrow{\bar{\sigma}} \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \to Im(\rho) \to 0$$

splits because $\bar{\sigma}$ admits a retraction τ defined by $\tau(e_i) = \mu/m_i$. Let $Im(\rho) \simeq \bigoplus_{j=1}^{r} \mathbb{Z}_{b_j}$ with b_j dividing b_{j+1} for all j . Since $Im(\rho)$ is a quotient of $\bigoplus_{i=1}^{t} \mathbb{Z}_{m_i}$ we see that b_r divides μ. Hence

$$\bigoplus_{i=1}^{t} \mathbb{Z}_{m_i} \simeq \mathbb{Z}_{b_1} \oplus \ldots \oplus \mathbb{Z}_{b_r} \oplus \mathbb{Z}_\mu$$

The uniqueness of this decomposition together with Lemma 1.2 imply that $Im(\rho) \simeq G$. ☐

§ 2. FAMILIES OF FIBRATIONS

We will consider the following situation. Let X, Y, M be connected complex manifolds, and let $f : X \to Y$, $g : Y \to M$ be surjective, proper, flat holomorphic maps with connected fibres. Write $h := g \circ f$, and suppose that all fibres of g are smooth curves, and the fibres of h are all smooth surfaces. If X_t, Y_t denote the fibres of h and g over $t \in M$, then the induced map $f_t : X_t \to Y_t$ is a fibration.

Definition 2.1. With the hypothesis just stated, we will say that $\{f_t : X_t \to Y_t\}_{t \in M}$ is a family of fibrations. For any $0, t \in M$, f_t is called a smooth deformation of f_0.

Now we ask ourselves how do the groups $L(f_t)$ of Definition 1.1 vary for a family of fibrations $\{f_t\}_{t \in M}$. As a matter of fact, we will see that they are all isomorphic. To begin with, the following Proposition shows the invariance of $G(f_t)$ under smooth deformations. The proof relies on the fact that a smooth holomorphic map is differentiably locally trivial. Then we will recall that $G(f_t)$ is a direct summand of $L(f_t)$ and will do a base change in order to obtain the invariance of $L(f_t)$.

Proposition 2.2. If $\{f_t : X_t \to Y_t\}_{t \in M}$ is a family of fibrations, then the groups $G(f_t)$ are all isomorphic.

Proof: Let (X, Y, M, f, g) be the quintuplet which determines the family $\{f_t : X_t \to Y_t\}$, as defined before. In order to fix ideas, we will choose an element $0 \in M$ and will write $S := X_0$, $C := Y_0$, $\varphi := f_0$. The maps f_t are smooth deformations of $\varphi : S \to C$. A theorem of Ehresmann ([3]; compare with [8], page 19) states that g and $h := g \circ f$ are differentiably locally trivial. In particular, there exists an analytic open neighbourhood U of $0 \in M$ and a commutative diagram

$$\begin{tikzcd}
h^{-1}(U) \arrow[r, "f"] \arrow[d, "p"', "\wr"] & g^{-1}(U) \arrow[d, "q"', "\wr"] \arrow[drr, "g"] & & \\
S \times U \arrow[r] & C \times U \arrow[rr, "(projection)"] & & U \\
(x,t) \arrow[r, mapsto] & (\Psi_t(x), t) & &
\end{tikzcd}$$

where the vertical arrows p, q are diffeomorphisms, and $\Psi_t : S \to C$ a differentiable map. Choose a point $\xi \in C$ such that $F := \varphi^{-1}(\xi)$ is smooth. The map $f : X \to Y$ is also differentiably trivial in a neighbourhood $V \subseteq g^{-1}(U)$ of $q^{-1}(\xi, 0)$ that is, there exists a diffeomorphism $f^{-1}(V) \simeq F \times V$ making commutative the following diagram

$$\begin{tikzcd}
f^{-1}(V) \arrow[r, "\sim"] \arrow[dr, "f"'] & F \times V \arrow[d, "(projection)"] \\
& V
\end{tikzcd} .$$

Put $W := q(V)$. We have a commutative diagram

$$\begin{tikzcd}
F \times W \arrow[r, "(projection)"] \arrow[d] & W \arrow[d] \\
S \times U \arrow[r] & C \times U
\end{tikzcd}$$

working as

$$\begin{tikzcd}
(z;(y,t)) \arrow[r, mapsto] \arrow[d] & (y,t) \arrow[d, mapsto] \\
(\lambda(z,y,t,);t) \arrow[r, mapsto] & ((\Psi_t \circ \lambda)(z,y,t,);t) = (y,t)
\end{tikzcd}$$

The left vertical arrow is a differentiable inmersion, and $\lambda : F \times W \rightarrow S$ is a differentiable map. Let us define $\sigma_t : F \rightarrow S$ $(t \in M)$ by $\sigma_t(z) = \lambda(z, \xi, t)$. Notice that $\sigma_t(F)$ is the fibre of Ψ_t over the point $\xi \in C$. Furthermore the maps σ_t, σ_0 are homotopic to each other for t close enough to 0, and thus they induce the same map in homology. With our identifications and Theorem 1.3 we immediately see that the cokernel of $(\sigma_t)_* : H_1(F, \mathbf{Z}) \rightarrow H_1(S, \mathbf{Z})$ is isomorphic to $H_1(C, \mathbf{Z}) \times G(f_t)$, whose torsion part in $G(f_t)$. Since $(\sigma_t)_* = (\sigma_0)_*$, it follows that $G(f_t) \simeq G(f_0)$ for t near 0. As a matter of fact, we have just proved that the set of $t \in M$ such that $G(f_t) \simeq G(f_0)$ is open. But similar arguments show that it is also closed, and the connectedness of M finishes our proof. □

Theorem 2.3. Let $\{f_t : X_t \rightarrow Y_t\}_{t \in M}$ be a family of fibrations. Then the groups $L(f_t)$ are all isomorphic.

Proof: Let the family be determined by the maps $f : X \rightarrow Y$, $g : Y \rightarrow M$ as described at the beginning of this section. Write $h : g \circ f$, and choose a point $0 \in M$. First we will assume that Y_0 is not rational. Let $\sigma : B \rightarrow Y_0$ be any étale morphism of degree 2. Since g is differentiably locally trivial, there is a neighbourhood U of $0 \in M$

such that $U \times Y_0$ and $g^{-1}(U)$ are diffeomorphic over U. The composite (id, σ) : $U \times B \to U \times Y_0 \approx g^{-1}(U)$ makes $U \times B$ into a topological covering space of $g^{-1}(U)$. Let V denote the space $U \times B$ endowed with the complex structure induced by $g^{-1}(U)$, and set $W := h^{-1}(U) \times_{g^{-1}(U)} V$. The natural projection $\lambda : W \to V$ defines a family of fibrations parametrized by U. Furthermore, each fibre of multiplicity m of $f_t : X_t \to Y_t$, $t \in U$, lifts to a pair of fibres of $\lambda_t : W_t \to V_t$, both with multiplicity m. Thus $L(\lambda_t) \simeq L(f_t) \oplus L(f_t)$. Combining the invariance of $G(\lambda_t)$ asserted in Theorem 2.2 with Lemma 1.2 yields the invariance of $L(f_t)$ for $t \in U$. Now use the connectedness of M to get that $L(f_t)$ is the same for all $t \in M$.

Next let us suppose that Y_0 is rational. Then $Y_t \simeq \mathbf{P}^1$ for all $t \in M$. It follows from [4] that $g : Y \to M$ is analytically locally trivial, so that $g^{-1}(U)$ is analytically isomorphic to $U \times Y_0$ over U, for some neighbourhood U of $0 \in M$. Let $B \to Y_0$ be any double cover which is unramified over the points of Y_0 where $f_0 : X_0 \to Y_0$ fails to be smooth. Making U smaller one may assume that the composite $f : h^{-1}(U) \to g^{-1}(U) \approx U \times Y_0$ is a smooth map over all points (t, x) where x is a branch point of $B \to Y_0$. Set $V := U \times B$ and $W := h^{-1}(U) \times_{g^{-1}(U)} V$. Then W is smooth and the projection $\lambda : W \to V$ defines a family of fibrations. One checkes that $\lambda_t : W_t \to V_t$. has no other multiple fibres than the ones coming from $f_t : X_t \to Y_t$. Hence also $L(\lambda_t) \simeq L(f_t)^{\oplus 2}$ for all t, and one finishes as before. □

Remark 2.4 For elliptic fibrations something stronger than Theorem 2.3 holds, namely, that the set of multiplicities of the fibres is invariant under smooth deformations. This was proved by Iitaka in [5].

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