W--ALGEBRAS WHICH ARE BOOLEAN PRODUCTS OF MEMBERS OF SR[1] AND CW--ALGEBRAS

by

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Abstract: We show that the class of all isomorphic images of Booleans products of members of SR[1] is the class of all Archimedean W-algebras. And the class of all isomorphic images of CW-algebras is the class of all W-algebras such that the family of all minimal prime implicative filters is the family of all Stone ultrafilters.

INTRODUCTION AND PURPOSES

W-algebras (or Wajsberg algebras) are the algebraic models of \( \mathbb{N}_5 \)-valued Lukasiewicz's Propositional calculus. Indeed, see [7], they are equivalent to MV-algebras introduced by C.Chang in [4] and used in [5] to show the completeness of Lukasiewicz's Propositional Calculus. The advantage of to use W-algebras is that they are defined with the operations "implication" (\( \rightarrow \)) and "negation" (\( \sim \)), which have a clear logic signification.

The class of all W-algebras is a variety generated by the W-algebra \( R[1] \), defined in I.D., and it has the property that every simple W-algebra is isomorphic to a subalgebra of it. In the other hand, every W-algebra is isomorphic to a subdirect product of CW-algebras (or W-algebras which are chains with the associated partial order defined in (1.12)). The purpose of this paper is to give a characterization of the W-algebras which are isomorphic to a boolean product of a subalgebras of \( R[1] \), or isomorphic to a boolean product of CW-algebras.
In section 1, we give a several well known definitions and results, without proof, on W-algebras which we will need is the paper. In section 2, we define the topological Spectrum of any W-algebra, which is a Bounded Stone Space, and we show that it is a Boolean Space if and only if the W-algebra is Archimedean. In section 3, we see that the class of all isomorphic images of boolean products of members of SR[1] is the class of all Archimedean W-algebras, moreover this representation is unique. Finally, in section 4, we prove that the class of all isomorphic images of boolean products of CW-algebras is the class of all W-algebras such that the family of all minimal prime implicative filters is the family of all Stone ultrafilters, and we see that this representation is unique.

To show the last result we could have used the results of R.Cignoli, see[6], we give a direct proof because the proof using Cignoli's results is very large. A complet study of W-algebras has been made in [8], unfortunately this work has not published, any way in [7] can be found the properties used in this paper. For the definition and properties of boolean product see [2], and for its connection with boolean sheaf spaces see [3].

1. W-ALGEBRAS: DEFINITIONS AND PROPERTIES

Along all the paper A = (A, +, ~, u) represents an algebraic structure of type (2,1,0), we write it \( A \in K(2,1,0) \).

1.A. Let \( A \in K(2,1,0) \) we say that \( A \) is a W-algebra provided that it satisfies the following equations:

(1.1) \( u + x = x \)

(1.2) \( (x + y) + y = (y + x) + x \)

(1.3) \( (x + y) + ((y + z) + (x + z)) = u \)

(1.5) \( (\sim x + \sim y) + (y + x) = u \)
By definition the class of all W-algebras is a Variety, we denote it \( W \).

To short we write \( 0 = \neg u \), \( x^0 + y = y \), and for any \( n < \omega \), \( x^n + y = x + (x^{n-1} \to y) \). If \( P \) is a property, \( W \models P \) denote that \( P \) is valid in all \( W \).

It satisfies:

(1.5) \( W \models x + x = u \)

(1.6) \( W \models x + u = u \)

(1.7) \( W \models x \to (y + z) = y + (x \to z) \), for any \( n, m < \omega \).

(1.8) \( W \models x + 0 = \neg x \)

(1.9) \( W \models \neg x + \neg y = y + x \)

(1.10) \( W \models \neg (\neg x) = x \)

1.B. In any W-algebra we can define a lattice structure in the next way. We set:

(1.11) \( x \vee y = (x + y) + y \) and \( x \wedge y = \neg (\neg x \vee \neg y) \).

Then for any \( A \in W \), \( \langle A, \wedge, \vee, \neg, 0, u \rangle \) is a De Morgan algebra, where \( \wedge \) is the meet, \( \vee \) is the join, \( \neg \) is the negation, \( 0 \) is the lower bound and \( u \) is the upper bound. Moreover the lattice partial order is given by:

(1.12) \( x \leq y \) if and only if \( x + y = u \)

To show the above results it is necessary to see the following properties:

(1.13) \( W \models x^n + y \leq x^m + y \), for any \( 0 \leq n \leq m < \omega \)

(1.14) \( W \models (x \wedge y) + z = (x + z) \vee (y + z) \)

(1.15) \( W \models (x \vee y) + z = (x + z) \wedge (y + z) \)

(1.16) \( W \models (x + y) \vee (y + x) = u \)

(1.17) \( W \models x \to (y \vee z) = (x \to y) \vee (x \to z) \)

(1.18) \( W \models x \to (y \wedge z) = (x + y) \wedge (x \to z) \)

1.C. \( W \) is an arithmetical Variety with 2/3 minority term: \( m(x, y, z) = ((x + y) + z) \wedge (z \to y) + x) \Lambda (x \vee y) \), hence it is Congruence distributive and Congruence permutative.
1.D. The Variety \( W \) is generated by the following algebra:

\[ R[1] = ([0,1], +, \sim, 1), \]

where \([0,1]\) is the unit interval of the totally ordered additive group of real numbers and \( a + b = \inf \{1, 1-a+b\}, \sim a = 1-a \).

1.E. Let \( A \in W \) and \( f \subseteq A \) we say that \( f \) is an implicative filter when:

\( u \in f \); for any \( a, b \in A \), \( a \in f \) and \( a + b \in f \) implies \( b \in f \). The family of all implicative filters of \( A \), which we denote \( \mathcal{F}_i(A) \), is an algebraic closure system and it is a subfamily of the family of all lattice filters of the De Morgan algebra defined in 1.B. Hence \( \mathcal{F}_i(A) \) is an algebraic lattice where the meet is the set-theoretic intersection, the join is: \( f_1 \vee f_2 = F_i(f_1 \cup f_2) \) (\( F_i \) is the associated closure operator to \( \mathcal{F}_i(A) \)), \( A \) is the upper bound and \( \{u\} \) is the lower bound. To short we write \( F_i(a) = F_i([a]) \) and \( F_i(X, a) = F_i(X \cup [a]) \).

From the properties of \( \mathcal{F}_i(A) \) we quote the following:

(1.19) (Deduction principle).

\[ F_i(X, a) = \{b \in A: a^n + b \in F_i(X), \text{ for some } n < \omega \}. \]
\[ F_i(a) = \{b \in A: a^n + b = u, \text{ for some } n < \omega \}. \]

(1.20) \( \mathcal{F}_i(A) \) has the family of prime implicative filters (prime as lattice filters) as a basis, hence every proper implicative filter is characterized by the set of all prime implicative filters which contain it.

(1.21) If \( f \subseteq \mathcal{F}_i(A) \), \( f \) is a proper maximal if and only if for any \( a \notin f \) there exists \( n < \omega \) such that \( a^n + 0 \notin f \).

(1.22) For every prime implicative filter there exists a prime implicative filter contained in it which is minimal in the partial ordered set of all prime implicative filters.
1.F. Let $A \in W$ and let $C(A)$ be the algebraic lattice of all congruence relations of $A$, then the map:

$$\theta : \mathcal{F}_1(A) \rightarrow C(A) : f \mapsto \theta_f = \{(a,b) \in A \times A/ (a+b) \wedge (b+a) \in f\}$$

is an order isomorphism and its inverse is:

$$f : C(A) \rightarrow \mathcal{F}_1(A) : \theta \mapsto f_\theta = \{a \in A/ (a,1) \in \theta\}.$$

Hence $\mathcal{F}_1(A)$ is a distributive algebraic lattice.

Now we give several properties more:

(1.23) If $A \in W$ and $\theta \in C(A)$, the quotient algebra $A/\theta$ is a chain, with the partial order of $L.B.$, if and only if $f_\theta$ is prime implicative filter.

We call *CW-algebra* to a *W*-algebra which is chain, and we denote by $\mathcal{W}$ the class of all CW-algebras.

(1.24) If $A \in W$ and $\theta \in C(A)$, then $A/\theta$ is simple if and only if $f_\theta$ is a proper maximal implicitative filter, or equivalently, $A/\theta$ is a subalgebra of $\mathbb{R}[1]$.

2. TOPOLOGICAL SPECTRUM OF A *W*-ALGEBRA

2.A. Let $A \in W$, we consider:

$$\text{Sp } A = \{p \in \mathcal{F}_1(A)/ p \text{ is prime }\},$$

and for any $a \in A:

$$S(a) = \{p \in \text{Sp } A/ a \in p\} = \{p \in \text{Sp } A/ F_i(a) \subseteq p\}$$

**Lemma 1.** For any $A \in W$ it satisfies:

(2.1) $S(a) = S(b)$ implies $F_i(a) = F_i(b)$, for any $a,b \in A$

(2.2) $S(a \wedge b) = S(a) \cap S(b)$, for any $a,b \in A$

(2.3) $S(a \lor b) = S(a) \cup S(b)$, for any $a,b \in A$

(2.4) $S(u) = \text{Sp } A$ and $S(0) = \emptyset$.

**Proof.** (2.1) is a consequence of (1.20)
(2.2) and (2.3) are consequences of the fact that every $p \in \text{Sp}_\mathcal{A}$ is a prime lattice filter. (2.4) is trivial.

It is clear that the family $(S(a)/a \in A)$ is a basis for a topology of open sets on $\text{Sp}_\mathcal{A}$. This topological space is called the **topological spectrum** of $\mathcal{A}$ and we represent it for $\text{Sp}_t\mathcal{A}$.

2. B. In order to determine the properties of $\text{Sp}_t\mathcal{A}$ we consider the set of all principal implicative filters of $\mathcal{A}$, or the compact elements of $\mathcal{O}_1(A)$, which are denoted by $F^A_i$. $F^A_i$ is the universe of a sublattice of $\mathcal{O}_1(A)$, because we have, for any $a, b \in A$:

$$F^A_i(a \vee b) = F^A_i(a) \cap F^A_i(b) \text{ and } F^A_i(a \wedge b) = F^A_i(a) \vee F^A_i(b).$$

Hence $(F^A_i, \cap, \vee)$ is a distributive lattice, moreover it has a lower bound $\{u\} = F^A_i(u)$ and upper bound $A = F^A_i(0)$.

Let $\text{Sp}^*_t\mathcal{A}$ be the topological space defined on the set of all prime lattice filters of $F^A_i$, which are denoted by $\text{Sp}^*_t\mathcal{A}$, and as a basis of open sets the family $(F^A_i(a) = \{p \in \text{Sp}^*_t\mathcal{A} / F^A_i(a) \notin p \} / a \in A)$.

It is easy to see that $\text{Sp}^*_t\mathcal{A}$ is a Bounded Stone Space (in the sense of [1], pág 79).

**Theorem 2.** For any $\mathcal{A} \in \mathcal{U}$, $\text{Sp}_t\mathcal{A}$ and $\text{Sp}^*_t\mathcal{A}$ are homeomorphic

**Proof.** Let $h$ be the correspondence defined:

$$h : \text{Sp}_t\mathcal{A} \longrightarrow \text{Sp}^*_t\mathcal{A} : p \longmapsto h(p) = \{F^A_i(a) / a \notin p\}.$$ 

That $h$ is a map is a simple comprovasion, trivially is one to one.

To see that it is onto, for any $P \in \text{Sp}^*_t\mathcal{A}$ we define $p = \{a \in A / F^A_i(a) \notin P\}$ then $p$ is implicative filter, because if $a, a+b \in p$, then $F^A_i(a), F^A_i(a+b) \notin P$, since $P$ is prime $F^A_i(a) \vee F^A_i(a+b) \notin P$, that is $F^A_i(a \wedge (a+b)) \notin P$, by definition of implicative filter $b \in F^A_i(a \wedge (a+b))$, that implies $F^A_i(b) \notin F^A_i(a \wedge (a+b))$ and hence $F^A_i(b) \notin P$, and $b \in p$. 

In the other hand if \( a \vee b \in p \), then \( F(a) \cap F(b) \notin p \), that is \( F(a) \notin p \) or \( F(b) \notin p \), hence \( a \in p \) or \( b \in p \). Then \( p \) is prime and \( p \in \text{Sp} A \). Moreover it is easy to see that \( h(p) = p \).

Finally we have to see that \( h \) is homeomorphism:

\[ P \in h(S(a)) \text{ iff } h^{-1}(P) \in S(a) \text{ iff } a \in h^{-1}(P) \text{ iff } F_i(a) \notin p \text{ iff } p \in F_i(a), \]

thus \( h(S(a)) = F_i(a) \). Similarly we could be shown that \( h^{-1}(F_i(a)) = S(a) \).

**COROLLARY.** If \( A \in W \), then \( \text{Sp} A \) is a Bounded Stone Space.

2.C. To characterize the clopen sets of \( \text{Sp} A \) we need to define a special elements of the \( W \)-algebras. Let \( A \in W \) and \( a \in A \), we say that \( a \) is \textit{archimedean} when there exists \( n < \omega \) such that \((a^n + 0)v a = u\); and we say that \( a \) is \textit{boolean} when it has complement, is this case this is \( \sim a \). The set of boolean elements of \( A \) is denoted by \( B(A) \).

Now we give a previous result:

**LEMMA 3.** If \( A \in W \) and \( a \in A \), then for any \( n < \omega \), \( F_i(a) = F_i(\sim(a^n + 0)) \).

**PROOF.** We use (1.19). From (1.5), (1.7) and (1.8) we have that for any \( n < \omega \):

\[ a^n \sim(\sim(a^n + 0)) = a^n \sim((a^n + 0) + 0) = (a^n + 0) + (a^n + 0) = u \]

this implies that \( (a^n + 0) \in F_i(a) \). In the other hand using (1.10) (1.9) (1.8) and (1.13):

\[ \sim(a^n + 0) + a = \sim a + (a^n + 0) = (a + 0) + (a^n + 0) = u \]

this shows that \( F_i(a) \leq F_i(\sim(a^n + 0)) \), and hence the Lemma is true.

**THEOREM 4.** For any \( A \in W \), the following conditions are equivalent:

(i) \( N \) is a clopen subset of \( \text{Sp} A \)

(ii) \( N = S(a) \) for some \( a \in A \) archimedean

(iii) \( N = S(b) \) for some \( b \in B(A) \).
PROOF. (i) \(\rightarrow\) (ii). Let \(N\) be a clopen subset of \(\text{Sp}_N\), then \(N\) and \(N^c\) are compact open, since \(\text{Sp}_N\) is Bounded Stone space, there exist \(a, c \in A\) such that \(N = S(a)\) and \(N^c = S(c)\). We will show that \(a\) is archimedean. It is clear that \(\emptyset = S(a) \cap S(c) = S(a \land c)\), hence \(S(0) = S(a \land c)\), by (2.1) we have \(F_i(a \land c) = F(0)\), that is, \(0 \in F_i(a \land c) = F_i(\{a, c\})\), hence by (1.19) there exists \(n < \omega\) such that \(a^n + 0 \in F_i(c)\) this implies that \(F_i(a^n + 0) \subseteq F_i(c)\). Thus \(F_i(a \lor (a^n + 0)) = F_i(a) \land F_i(a^n + 0) \subseteq F_i(a) \land F_i(c) = F_i(a \lor c)\). But from \(S(a \lor c) = S(a) \cup S(c) = \text{Sp}_N\), we deduce that \(F_i(a \lor c) = \{u\}\), hence \(F_i(a \lor (a^n + 0)) = \{u\}\) then \(a \lor (a^n + 0) = u\). This shows that \(a\) is archimedean.

(ii) \(\Rightarrow\) (iii). We suppose (ii), then there exists \(n < \omega\) such that \(a \lor (a^n + 0) = u\). We will show that \(a^n + 0\) is boolean, then we will have, by Lemma 3, \(S(a) = S(\neg(a^n + 0))\), with \(\neg(a^n + 0)\) boolean.

Using Lemma 3, we have:

\[
\{u\} = F_i(a \lor (a^n + 0)) = F_i(a) \land F_i(a^n + 0) = F_i(\neg(a^n + 0)) \land F_i(a^n + 0) = F_i(\neg(a^n + 0) \lor (a^n + 0)).
\]

Hence \(\neg(a^n + 0) \lor (a^n + 0) = u\), Moreover since \((A, \land, \lor, \neg, 0, u)\) is a De Morgan algebra, we have that \((a^n + 0) \land \neg(a^n + 0) = 0\).

(iii) \(\Rightarrow\) (i) is trivial.

2.D. Given \(A \in \mathcal{W}\) we say that \(A\) is archimedean \(\mathcal{W}\)-algebra, when for any \(a \in A\) \(a\) is archimedean. \(\mathcal{A}\) denotes the class of all Archimedean \(\mathcal{W}\)-algebras.

THEOREM 5. Let \(A \in \mathcal{W}\), then the following conditions are equivalent:

(i) \(A \in \mathcal{A}\)

(ii) \(\text{Sp}_A\) is Boolean Space

(iii) \(\text{Sp}_A\) is Hausdorff

(iv) \(\text{Sp}_A\) is \(T_1\)
PROOF. (i)$\iff$(ii), (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(iv) are immediate consequences from Theorem 4. (iv)$\Rightarrow$(ii) is satisfied because any Bounded Stone space $T_1$ is Boolean space.

CROLLARY. Let $A \in W$, then $A \in W_a$ if and only if every prime implicative filter is a proper maximal implicative filters, that is, the family of all proper maximal implicative filters is justly $Sp A$.

PROOF. By Theorem 5, $A \in W_a$ iff $Sp A$ is $T_1$, this is equivalent to every two prime implicative filters are equal or non comparables, that is any prime implicative filter is a proper maximal. The fact that any proper maximal implicative filter is prime concludes the proof.

We remark that the class $W_a$ is not the class $W$, because $W_a$ is not definable by means of generalized implications (see [8]) since the direct product of members of $W_a$ is not necessarily in $W_a$. In the other hand, every Archimedean $W$-algebra is semisimple, but there exist semisimple $W$-algebras which are not in $W_a$, since the class of all semisimple $W$-algebra is definable by means of generalized implications (see [7]).


$SR[1]$ will denotes the class of all isomorphic images of subalgebras of $R[1]$ that is, $SR[1]$ is the class of all simple $W$-algebras.

3.A. Given a class of algebraic structures $K$, and algebra $A$ of the same type, we say that $A$ is Boolean product of members of $K$, when there exists a family $(A_x/x \in X)$ of $K$ such that :

$X$ can be endowed with a Boolean Space topology
(3.2) $\mathbb{A}$ is a subdirect product of $(A_x/\{x \in X\})$

(3.3) The equalizers are clopen subsets of $X$. That is if $\pi_x: A_x \twoheadrightarrow A_x$ is the canonical projection, then for any $a, b \in A$ the set $[a = b] = \{x \in X / \pi_x(a) = \pi_x(b)\}$ is clopen subset of $X$.

(3.4) For any $a, b \in A$ and $N$ clopen subset of $X$ we define: $a/N \cup b/Nc$

the element of $\otimes(A_x/\{x \in X\})$ given by:

$$\pi_x(a/N \cup b/Nc) = \begin{cases} \pi_x(a) & \text{if } x \in N \\ \pi_x(b) & \text{if } x \notin N \end{cases}$$

To represent that $\mathbb{A}$ is Boolean product of the family $(A_x/\{x \in X\})$ we will write $A \subseteq_{bp} \otimes(A_x/\{x \in X\})$. Given a class $\mathfrak{K}$ we will denote by $\Gamma^a(\mathfrak{K})$ the class of all isomorphic images of Boolean products of members of $\mathfrak{K}$.

3.B. The main result of this section gives the relation between $\mathbb{W}_a$ and $\Gamma^a(\mathfrak{SR}[1])$.

**Theorem 6.** $\mathbb{W}_a = \Gamma^a(\mathfrak{SR}[1])$.

**Proof.** $\mathbb{W}_a \subseteq \Gamma^a(\mathfrak{SR}[1])$.

Let $\mathfrak{A} \in \mathbb{W}_a$, we consider $X = \text{Sp} A$ with the spectral topology, which is Boolean space. By (1.20) $\cap X = \{a\}$, hence $\cap(\theta_x/\{x \in X\})$ is the diagonal congruence relation. This implies that $\mathbb{A}$ is isomorphic to a subdirect product of the family $(A/\theta_x/\{x \in X\})$ (to short we will write $A_x = A/\theta_x$, and $[a]_x$ the class of a modul $\theta_x$). We suppose that the isomorphism is the following:

$$\vartheta : A \longrightarrow \otimes(A_x/\{x \in X\}): a \longmapsto \vartheta(a) = ([a]_x)_{x \in X}$$

We need on to see that it satisfies (3.3) and (3.4).

(3.3) is immediate because for any $a, b \in A$ \[a = b] = S((a + b) \land (b + a))
(3.4) Let \( c \in B(A) \), and \( a, b \in A \), we consider \( d = (c + a) \land (\neg c + b) \), we will show that \( \partial(d) = \partial(a)/S(c) \cup \partial(b)/S(c) \).

Let \( x \in S(c) \) we need to see that \( [d]_x = [a]_x \), that is \( (d + a) \land (a + d) \in x \).

We write to the next of equalities and of inequalities the properties used.

\[
(a + d) \land (d + a) = [a + ((c + a) \land (\neg c + b))] \land [(c + a) \land (\neg c + b)] =
\]

(1.18)(1.14) / = 

\[
[(a + (c + a)) \land (a + (\neg c + b))] \land [(c + a) \land a] 
\]

(1.13)(1.12)(1.11)/ = 

\[
(a + (\neg c + b)) \land (c \lor a) \geq (1.10)(1.9) / \geq c \land (c \lor a) = c
\]

Since \( x \in S(c) \) then \( c \in x \). \( x \) is a lattice filter then \( (a + d) \land (d + a) \in x \).

Similarly, it can be obtained that for any \( x \in S(c) = S(\neg c) \),

\[ [d]_x = [b]_x \]

\( \Gamma^3(SR[1]) \subseteq W_3 \):

Let \( A \in \Gamma^3(SR[1]) \) we suppose that the isomorphism which gives a Boolean product representation is:

\[ \partial: A \stackrel{\sim}{\longrightarrow} \Omega(A_x/ x \in X) : a \longrightarrow \partial(a) = (a_x) \]

where \( A \cong \partial(A) \), \( \partial(A) \subseteq_{bp} \Omega(A_x/ x \in X) \) and \( A_x \in SR[1] \).

First we observe that \( A \in W_3 \) because \( W \) is a variety and \( SR[1] \subseteq W \).

We need to see that any \( a \in A \) is archimedean. Fixed \( a \in A \), since \( A_x \) is a simple \( W \)-algebra for any \( x \in X \), if \( a_x \neq u_x \), then for any \( x \in X \) such that \( a_x \neq u_x \) there exists \( n(x) \) such that \( u_x = a_x^n = 0 \), that is \( \pi_x(u) = \pi_x(a^n_x = 0) \), let \( X = \{x \in X/ a_x \neq u_x \} \), then we have

\[ X = \{u_x, (a^n_x = 0 = u) \cup [\partial(a) = u] \}
\]

since \( X \) is compact, there exist \( x_1, \ldots, x_r \in X \) such that \( X = [\partial(a)^{n(x_1)} = 0] \cup \ldots \cup [\partial(a)^{n(x_r)} = 0] \cup [\partial(a) = u] \)

If \( n = \sup \{n(x_1) \ldots n(x_r)\} \), then by (1.13) \( \partial(a)^{n(x_i)} = 0 \leq \partial(a)^n = 0 \)
hence \( X = \{ \partial(a)^n = u \} \cup \{ \partial(a) = u \} \). Thus for any \( x \in X \) we have:

\[
\pi_x((\partial(a)^n + 0) v (\partial(a))) = \pi_x(\partial(a)^n + 0) v \pi_x(\partial(a)) = u,\]

this implies that \((\partial(a)^n + 0) v \partial(a) = u\), as \(\partial\) is isomorphism, then \((a^n + 0) v a = u\). That is, \(a\) is archimedean.

3.C. In this part we will show that the representation of archimedean \(W\)-algebras by means Boolean products is a good representation in the sense that every \(A \in W_a\) is obtained by Boolean products with stalks in \(SR[1]\) in unique way.

**Theorem 7.** Let \( A \subseteq \mathbb{B} \otimes \mathbb{X}(A/x \in X) \), where \( A \subseteq SR[1] \) for any \( x \in X \). Then there exist a homeomorphism \( h: X \rightarrow Sp A \) such that \( A \cong A/\mathcal{G} h(x) \).

**Proof.** We define \( h: X \rightarrow Sp A : x \mapsto h(x) = \{ a \in A : x \in [a = u] \} \).

Since: \( x \in [a = u] \) iff \( \pi_x(a) = \pi_x(u) \) iff \((a, u) \in \ker \pi_x\), thus \( A \subseteq SR[1] \) implies that \( h(x) = f_{\ker \pi_x} \in Sp A \), hence \( h \) is well defined.

\( h \) is one to one, because if \( x, y \in X \) and \( x \neq y \), then there exist \( N \) clopen subset of \( X \) such that \( x \in N \) and \( y \notin N \), let \( a = u/N \cup 0/N^c \), then \( x \in [a = u] = N \) and \( y \notin [a = u] \), hence \( a \notin h(x) \) and \( a \notin h(y) \).

We suppose that \( h \) is not onto, then there exists \( q \in Sp A \), such that for any \( x \in X \) there exists \( b^x \in h(x) \setminus q \). Thus

\[
X = U \{ [b^x = u] / x \in X \}. \]

By compactness \( X = [b^{x_1} = u] \cup \ldots \cup [b^{x_r} = u] \) for some \( x_1, \ldots, x_r \in X \), since \( b^{x_i} \subseteq b^{x_1} v \ldots v b^{x_r} \), we have

\[
X = [b^{x_1} v \ldots v b^{x_r} = u], \]

hence \( b^{x_1} v \ldots v b^{x_r} = u \in q \). Since \( q \in Sp A \), \( b^{x_i} \in q \), for some \( i \in \{ 1, \ldots, r \} \), that contradicts the assumption, and \( h \) is onto.

\( h \) is homeomorphism; if \( N \) is clopen subset of \( X \), then \( N = [a = u] \) with \( a = u/N \cup 0/N^c \), hence we have:

\[
q \in h([a = u]) \iff h^{-1}(q) \in [a = u] \iff a \in q, \]

this implies that \( h([a = u]) = S(a) \).
x \in h^{-1}(S(a)) \iff h(x) \in S(a) \iff a \in h(x) \iff x \in \{a = u\} \text{ hence } h^{-1}(S(a)) = \{a = u\}.

In the other hand, since \( h(x) = e \ker \pi_x \), we have \( A_x \cong A/\theta_x \).

4. BOOLEAN PRODUCTS OF CW-ALGEBRAS

To give the main results of this section we have to analyze the Stone filters in W-algebras.

4.A. Let \( A \in W \), then \( B(A) \) is the universe of a Boolean algebra with the operations of \( A \). We represents by \( \mathfrak{B}(A) \) the family of all lattice filters of \( B(A) \). Given \( f \subseteq A \) we say that \( f \) is a Stone filter when \( f \) is a lattice filter generated by a member of \( \mathfrak{B}(A) \). \( \mathfrak{F}_{\Sigma}(A) \) will denote the family of all Stone filters. An Stone ultrafilter is a proper maximal element of \( \mathfrak{F}_{\Sigma}(A) \). \( U_{\Sigma}(A) \) denotes the family of all Stone ultrafilters of \( A \).

**LEMMA 8.** If \( A \in W \), then it satisfies:

(4.1) Every Stone filter is an implicative filter, i.e. \( \mathfrak{F}_{\Sigma}(A) \subseteq \mathfrak{L}_{\Sigma}(A) \)

(4.2) \( f \in U_{\Sigma}(A) \) if and only if \( f \in \mathfrak{F}_{\Sigma}(A) \) and \( f \cap B(A) \) is an ultrafilter of \( B(A) \).

**PROOF.** (4.1): Let \( f \in \mathfrak{F}_{\Sigma}(A) \), we suppose \( a \in f \) and \( a + b \in f \), then there exist \( c_1, c_2 \in B(A) \cap f \) such that \( c_1 \leq a \) and \( c_2 \leq a + b \). Then \( c = c_1 \land c_2 \in f \cap B(A) \) and \( c \leq a \) and \( c \leq a + b \), by (1.12) \( c + (a + b) = u \), by (1.7) \( a + (c + b) = u \), hence \( c \leq a \leq c + b \), that is \( c + (c + b) = c + b = u \) (because \( c \) is boolean iff \( c^2 = d = c + d \), for any \( \varepsilon A \)), then \( c \leq b \) and \( b \in f \). This show that \( f \in \mathfrak{L}_{\Sigma}(A) \).

(4.2) It deduces from the fact that if \( f \in U_{\Sigma}(A) \) iff for any \( c \in B(A), c \in f \) iff \( \neg c \notin f \), and \( f \in \mathfrak{F}_{\Sigma}(A) \).
We denote by $\text{Spm}_A$ the family of all minimal prime implicative filters, in the sense of (1.22). The relation between $U_S(A)$ and $\text{Spm}_A$ is given by the next result.

**THEOREM 9.** If $A \in \mathcal{W}$, then the following conditions are equivalent:

(i) $U_S(A) \subseteq \text{Sp}_A$

(ii) $U_S(A) = \text{Spm}_A$

**PROOF.** (ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (ii): Let $f \in U_S(A) \subseteq \text{Sp}_A$, if $p \in \text{Spm}_A$ is such that $p \nleq f$ (it exists by (1.22)), then $(f \setminus p) \cap B(A) \neq \emptyset$, because if this is not true, for any $c \in f \cap B(A)$, $c \leq p$ and $f \nleq p$. Let $c \in (f \setminus p) \cap B(A)$, since $c \wedge c = u \in p$ and $p \in \text{Sp}_A$, $c \nleq f$, that is not possible. Hence $p = f$. This shows that $U_S \subseteq \text{Spm}_A$.

If $q \in \text{Spm}_A$, let $f = F_1(q \cap B(A))$, it is clear that $f \nleq q$ and $f \in U_S(A)$ hence $f \in \text{Spm}_A$, that is $f = q$. This shows that $U_S(A) = \text{Spm}_A$.

4.B. The main result of this section characterizes the algebras which are Boolean products of CW-algebras.

**THEOREM 10.** $A \in \text{fin}^d(CW)$ if and only if $A \in \mathcal{W}$ and $\text{Spm}_A = U_S(A)$.

**PROOF.** $\Rightarrow$ We suppose that $A \subseteq \text{bp} \otimes(A_x/ x \in X)$, where $A_x \in CW$ for any $x \in X$.

Since $CW \subseteq \mathcal{W}$ and $\mathcal{W}$ is a Variety, then $A \in \mathcal{W}$. Let $x \in X$, we consider $p_x = f_\theta \in \text{Ker} \pi_x$ by Theorem 9, it suffices to see that $p_x = \{p_x/ x \in X\} = U_S(A)$.

$P_x \subseteq U_S(A)$: Let $x \in X$ and $a \in p_x$, then $x \in \{a = u\}$. If $c = U/a = u \cup 0/[a \neq u]$, then $c \in B(A)$ and $c \leq a$ and $[a = u] = [c = u]$ hence $c \in p_x \cap B(A)$.

Then $p_x \in \text{Sp}_A$. Since $p_x \in \text{Sp}_A$, $p_x \cap B(A)$ is an ultrafilter of $B(A)$, then $p_x \in U_S(A)$. This shows that $U_S(A) \supseteq P_x$.

$U_S(A) \nleq P_x$: if it is not satisfied, then there exists $f \in U_S(A)$
such that for any \( x \in X \), \( f \neq p_x \). Then for any \( x \in X \), there exists 
\[
 c^x \in (p_x \setminus f) \cap B(A).
\]
Thus \( X = U \left( \left[ c^x = u \right] / x \in X \right) \), since \( X \) is Boolean space \( X = \left[ c^x = u \right] U ... U \left[ c^n = u \right] \), for some \( x_1, \ldots, x \in X \). Since \( x_i \leq x^1 v \ldots v x^n \) for any \( i \in \{1, \ldots, n\} \), we have 
\[
 X = \left[ c^1 v \ldots v c^n = u \right]
\]
hence \( c^1 v \ldots v c^n = u \in f \cap B(A) \), since \( f \cap B(A) \) is ultrafilter of \( B(A) \), then there exists \( r \in \{1, \ldots, n\} \), such that \( c^r \in f \), that contradicts the assumption. This shows that \( P_x \supset U_{S(A)} \).

\[\begin{align*}
\text{Let } A & \in \mathcal{W}, \text{ such that } U_{S(A)} = SpmA. \text{ First we will see that} \\
SpmA \text{ with the induced topology by the Spectral topology is a Boolean Space.} \\
\text{We write } Sm(a) = S(a) \cap SpmA, \text{ for any } a \in A. \text{ It is clear that for any} \\
c \in B(A), Sm(c) \text{ is a clopen subset of SpmA. Now we will show that for any} \\
a \in A \text{ there exist } c \in B(A) \text{ such that } Sm(a) = Sm(c_a). \text{ If } Sm(a) = \emptyset \\
\text{then } c_a = 0. \text{ We suppose that } Sm(a) \neq \emptyset, \text{ then } \bigcap Sm(a) \in \mathcal{P}(A), \text{ and } a \in \bigcap Sm(a), \\
\text{hence there exist } c \in Sm(a) \text{ such that } c \leq a, \text{ then we have:} \\
x \in Sm(a) \text{ implies } c \leq x, \text{ hence } x \in Sm(c_a). \text{ This shows that } Sm(a) \leq Sm(c_a) \\
x \in Sm(c_a) \text{ implies } c \leq x, \text{ since } c \leq x, \text{ then } a \in x \text{ and } x \in Sm(a). \text{ This shows} \\
\text{that } Sm(a) = Sm(c_a). \text{ Thus for any } a \in A, \text{ Sm(a) is clopen subset of Spm(A).} \\
\end{align*}\]

Let SpB(A) be the topological Spectrum of the boolean algebra \( B(A) \). WE consider the map \( h : SpmA \rightarrow SpB(A) : f \mapsto f \cap B(A) \), by Lemma 8 this map is one to one and onto, moreover is easy to see that 
\[
h(Sm(c)) = \mathcal{S}_{B(A)}(c) \text{ and } h^{-1}(\mathcal{S}_{B(A)}(c)) = Sm(c) \text{ for any } c \in B(A), \text{ hence} \\
h \text{ is homeomorphism}. \text{ Then SpmA is Boolean space with the topology induced} \\
\text{by the spectral topology of } SpA.. \\

From (1.23) it is easy to see that \( \bigcap SpmA = \{u\} \), hence A is 
\text{isomorphic to a subdirect product of} \( \left( A/\theta \times x \in SpmA \right) \). 
\text{This isomorphism is given by:} 
\[
\vartheta : A \rightarrow \mathcal{C}(A/\theta \times x \in SpmA) : a \mapsto \vartheta(a) = ([a]_x \times x \in SpmA). 
\]
\text{Since } \vartheta(a) = \vartheta(b) \text{ is clopen subset of SpmA.
and \( \partial(a)/\text{Sm}(c) \cup \partial(b)/\text{Sm}(\neg c) = \partial((c \cdot c_a) \wedge (\neg c \cdot c_b)) \), then

\[
\partial(A) \subseteq \bigotimes_{bp} (A/\theta_x / x \in X), \text{ hence } A \in \Gamma^a(CW)
\]

4.C. **THEOREM 11.** Let \( A \in \mathcal{W} \) and \( A \subseteq_{bp} (A_x / x \in X) \), where \( A \in \mathcal{CW} \), then there exists \( h : X \rightarrow \text{SpmA} \) homeomorphism such that \( A \cong A/\theta h(x) \).

**PROOF.** Let \( h : X \rightarrow \text{SpmA} \) \( : x \mapsto h(x) = p_x = \{ a \in A / x \in \{ a = u \} \} \), by the proof of first part of the Theorem 10 \( h \) is one to one and onto.

By definition \( h(\{ a = u \}) = \text{Sm}(a) \) and \( h^{-1}(\text{Sm}(a)) = \{ a = u \} \) hence \( h \) is homeomorphism. Since \( h(x) = f^\lambda_{\ker \pi_x} \) we have \( A \cong A/\theta h(x) \).

**Remark:** The archimedean \( \mathcal{W} \)-algebras are special cases of \( \mathcal{W} \)-algebras representables by means of Boolean products of \( \mathcal{CW} \)-algebras. They are the limit case because the Stone ultrafilters are all prime implicative filters.

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