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by

David Nualart and *Etienne Pardoux*

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STOCHASTIC DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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David Nualart
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via, 585
08007 Barcelona - Spain.

Etienne Pardoux
Mathématiques, URA 225
Université de Provence
13331 Marseille Cedex 3
France

1. Introduction

The stochastic calculus with anticipating integrands has been recently developed by several authors (see in particular [5,6,9] and the references therein). This new theory allows to study different types of stochastic differential equations driven by a d–dimensional Brownian motion $\{W(t), 0 \leq t \leq 1\}$, where the solutions turn out to be non necessarily adapted to the filtration generated by W. We refer the reader to [12] for a survey of the applications of the anticipating stochastic calculus to stochastic differential equations. In particular one can consider stochastic differential equations of the form

$$dX_t = f(X_t) + \sum_{i=1}^{k} g_i(X_t) \circ dW_t^i, \qquad 0 \leq t \leq 1, \tag{1.1}$$

and, instead of giving the value of the process at time zero, we impose a boundary condition of the form $h(X_0, X_1) = h_0$. In general, the solution $\{X_t, 0 \leq t \leq 1\}$ will not be an adapted process, and the stochastic integral $\int_0^t g_i(X_s) \circ dW_s^i$ is taken in the extended Stratonovich sense. The existence and uniqueness of a solution for an equation of this type has been investigated in some particular cases, and the Markov property of the solution has been studied. More precisely the following particular situations have been considered:

(a) The functions f, g and h are affine. (Ocone–Pardoux [11])
(b) $k = d$ and the function g is a constant equal to the identity matrix. (Nualart–Pardoux [7])
(c) $k = d = 1$, and h is linear. (Donati–Martin [2])
(d) Second order stochastic differential equations in dimension one of the following type

$$\ddot{X}_t + f(X_t, \dot{X}_t) = \dot{W}_t, \qquad 0 \leq t \leq 1, \tag{1.2}$$

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with Dirichlet boundary conditions $X_0 = a$, $X_1 = b$. (Nualart–Pardoux [8]).

The objective of this paper is to describe some of the results obtained in these articles concerning the existence and uniqueness and the Markov property of the solution of the equations (1.1) and (1.2). In order to illustrate the techniques used in the cases (b), (c) and (d) we are going to present a complete study in the particular case of a second order stochastic differential equation of the type (1.2) when the function f depends only on the variable X_t. This hypothesis allows to simplify the arguments used in [8], and more direct computations can be carried out in this case.

The organization of the paper is as follows. In Section 2 we will introduce the basic notations and results of the theory of noncausal stochastic calculus that will be needed later. In particular we will state an extended version of the Girsanov theorem for non necessarily adapted processes which is due to Ramer [13] and Kusuoka [3]. In Section 3 we will study the Markov property of the solution of the second order stochastic differential equation

$$\begin{aligned} &\ddot{X}_t + f(X_t, t) = \dot{W}_t, \qquad 0 \le t \le 1, \\ &X_0 = X_1 = 0. \end{aligned} \tag{1.3}$$

The main result is as follows. Assuming that f has linear growth, admits two continuous partial derivatives with respect to x and $f'_x \le 0$, then the process $\{(X_t, \dot{X}_t)\}$ is Markovian if and only if f is an affine function. The proof of this result is based on the following idea. Denote by $\{Y_t, 0 \le t \le 1\}$ the solution of (1.3) for $f \equiv 0$. Then $\{(Y_t, \dot{Y}_t)\}$ is known to be a Markov process. On the other hand, the law of $\{X_t\}$ is the same as the law of $\{Y_t\}$ under a new probability measure Q which is absolutely continuous with respect to the Wiener measure. Then one can show that for any $t \in (0,1)$ the corresponding Radon–Nikodym derivative cannot be expressed as the product of two factors one of them $\sigma\{(Y_s, \dot{Y}_s), 0 \le s \le t\}$–measurable and the other $\sigma\{(Y_s, \dot{Y}_s), t \le s \le 1\}$–measurable. This lack of factorization prevents for the Markov property to hold.

Section 4 will be devoted to survey some of the results obtained in [2,7,11] on the existence and uniqueness of a solution and its Markov property for different types of first order stochastic differential equations with boundary conditions.

2. Some elements of noncausal stochastic calculus.

Let $\Omega = C_0([0,1])$ be the space of continuous functions on $[0,1]$ which vanish at zero. We denote by P the Wiener measure on Ω, and by $\mathcal{F}$ the Borel σ–field completed with respect to P. Then, $W_t(\omega) = \omega_t$, $0 \le t \le 1$ will represent the Wiener process defined on the canonical space $(\Omega, \mathcal{F}, P)$. The Hilbert space $L^2(0,1)$ will be denoted by H. Let us first recall briefly the notions of derivation on Wiener space and of Skorohod integral. We denote by $\mathcal{S}$ the subset of $L^2(\Omega)$ consisting of those random variables of the form:

$$F = f\Big(\int_0^1 h_1(t)dW_t, \ldots, \int_0^1 h_n(t)dW_t\Big) \tag{2.1}$$

where $n \geq 1$, $h_1, \ldots, h_n \in L^2(0,1)$, and $f \in C_b^\infty(\mathbb{R}^n)$. The random variables of the form (2.1) are called *smooth functionals*. For a smooth functional $F \in \mathcal{S}$ of the form (2.1) we define its derivative DF as the stochastic process $\{D_tF, 0 \leq t \leq 1\}$ given by

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big(\int_0^1 h_1(s)dW_s, \ldots, \int_0^1 h_n(s)dW_s\Big)h_i(t). \tag{2.2}$$

Then D is a closable unbounded operator from $L^2(\Omega)$ into $L^2(\Omega \times [0,1])$. We will denote by $\mathbb{D}^{1,2}$ the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,2}$ defined by

$$\|F\|_{1,2} = \|F\|_2 + \|DF\|_{L^2(\Omega\times[0,1])}, \qquad F \in \mathcal{S}.$$

Similarly, for any fixed $h \in L^2(0,1)$ we can define $D^hF = \int_0^1 D_tFh(t)dt$, if $F \in \mathcal{S}$. Then D^h is closable and we will denote by $\mathbb{D}^{h,2}$ the closure of $\mathcal{S}$ by the norm

$$\|F\|_{h,2} = \|F\|_2 + \|D^hF\|_2.$$

We will denote by δ the adjoint of the derivation operator D. That means, δ is a closed and unbounded operator from $L^2(\Omega \times [0,1])$ into $L^2(\Omega)$ defined as follows: The domain of δ, $\operatorname{Dom}\delta$, is the set of processes $u \in L^2(\Omega \times [0,1])$ such that there exists a positive constant c_u verifying

$$\left|E\int_0^1 D_tFu_tdt\right| \leq c_u\|F\|_2, \tag{2.3}$$

for all $F \in \mathcal{S}$. If u belongs to the domain of δ then $\delta(u)$ is the square integrable random variable determined by the duality relation

$$E\left(\int_0^1 D_tFu_tdt\right) = E\left(\delta(u)F\right), \qquad F \in \mathbb{D}^{1,2}. \tag{2.4}$$

The operator δ is an extension of the Itô integral in the sense that the class L_a^2 of processes u in $L^2(\Omega \times [0,1])$ which are adapted to the Brownian filtration is included in $\operatorname{Dom}\delta$ and $\delta(u)$ is equal to the Itô integral if $u \in L_a^2$. The operator δ is called the Skorohod stochastic integral. We refer the reader to [6] for the proof of the basic facts of the stochastic calculus for the Skorohod integral.

Define $\mathbb{L}^{1,2} = L^2([0,1]; \mathbb{D}^{1,2})$. Then the space $\mathbb{L}^{1,2}$ is included into the domain of δ. The operators D, D^h and δ are *local* in the following sense:

(a) $\mathbf{1}_{\{F=0\}}DF = 0$, for all $F \in \mathbb{D}^{1,2}$,

(b) $\mathbf{1}_{\{F=0\}}D^hF = 0$, for all $F \in \mathbb{D}^{h,2}$,

(c) $\mathbf{1}_{\{\int_0^1 u_t^2dt=0\}}\delta(u) = 0$, for all $u \in \mathbb{L}^{1,2}$.

Then one can define the spaces $\mathbb{D}_{\text{loc}}^{1,2}$, $\mathbb{D}_{\text{loc}}^{h,2}$ and $\mathbb{L}_{\text{loc}}^{1,2}$ by a standard localization method. For instance, $\mathbb{L}_{\text{loc}}^{1,2}$ is the space of processes u such that there exists a sequence $\{(\Omega_n, u_n), n \geq 1\}$ such that $\Omega_n \in \mathcal{F}$, $\Omega_n \uparrow \Omega$, a.s., $u_n \in \mathbb{L}^{1,2}$, and $u_n = u$ on Ω_n for each n. By property (c) the Skorohod integral can be extended to the processes of the class $\mathbb{L}^{1,2}_{loc}$. In a similar way the operators D and D^h can be properly defined on the sets $\mathbb{D}^{1,2}_{loc}$ and $\mathbb{D}^{h,2}_{loc}$, respectively.

The derivation operator D can be used to translate some measurability properties into algebraic conditions. The following lemma is an example of this application of the derivation operator, and it will be used in the next section.

Lemma 2.1. *Fix k functions $h_1, \ldots, h_k \in H$, where we recall that H denotes the Hilbert space $L^2(0,1)$. Let $\mathcal{G}$ be the σ-algebra generated by the Gaussian random variables $\int_0^1 h_1(t)dW_t, \ldots, \int_0^1 h_k(t)dW_t$. Let F be a random variable in the space $\mathbb{D}^{1,2}_{loc}$ such that $F\,\mathbf{1}_G$ is $\mathcal{G}$-measurable for some set $G \in \mathcal{G}$. Then there exist random variables A_i, $1 \le i \le k$ such that*

$$\mathbf{1}_G D_t F = \mathbf{1}_G \sum_{i=1}^{k} A_i h_i(t),$$

for $dP \times dt$ almost all $(\omega, t) \in \Omega \times [0,1]$.

Proof: Consider the subspace K of $H = L^2(0,1)$ spanned by $h_1, \ldots, h_k$. Since we can approximate F by $\varphi_n(F)$ with $\varphi_n \in C_b^\infty(\mathbb{R})$ and $\varphi_n(x) = x$ for $|x| \le n$, it is sufficient to prove the result for $F \in \mathbb{D}^{1,2}_{loc} \bigcap L^2(\Omega)$. Let $h \perp K$. Clearly the conditional expectation $E(F|\mathcal{G})$ belongs to $\mathbb{D}^{h,2}$ and $D^h E(F|\mathcal{G}) = 0$. Moreover the hypotheses of the lemma imply that $F \in \mathbb{D}^{h,2}_{loc}$ and $E(F|\mathcal{G}) = F$ a.s. on G. From the local property of the operator D^h we deduce that

$$D^h F = \int_0^1 h_t D_t F dt = 0, \qquad \text{a.s.} \quad \text{on} \quad G.$$

Then it only remains to choose a countable dense set in the orthogonal of K and we obtain that $\mathbf{1}_G DF \in K$ a.s., which implies the desired result. Q.E.D.

Let us recall the definition of the Stratonovich integral and its relation with the Skorohod integral. Let $\{u_t, 0 \le t \le 1\}$ be a measurable process such that $\int_0^1 u_t^2 dt < \infty$ a.s. Then u is said to be ***Stratonovich integrable*** if

$$\sum_{j=1}^{n-1} \left(\frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} u_s ds \right) (W(t_{j+1}) - W(t_j))$$

converges in probability as $|\pi| \to 0$, where $\pi = \{0 = t_1 < \cdots < t_n = 1\}$ runs over all finite partitions of $[0,1]$ and $|\pi| = \max_j (t_{j+1} - t_j)$. The limit will be called the ***Stratonovich integral*** of u and it will be denoted by $\int_0^1 u_t \circ dW_t$.

Let $\mathbb{L}^{1,2}_C$ denote the set of processes $u \in \mathbb{L}^{1,2}$ such that:

(i) The set of functions $\{s \to D_t u_s, 0 \le s \le t\}$, $t \in [0,1]$, with values in $L^2(\Omega)$ is equicontinuous for some version of Du, and similarly, the set of functions $\{s \to D_t u_s, t \le s \le 1\}$, $t \in [0,1]$, is also equicontinuous for a version (possibly different) of Du.

(ii) $\text{ess sup}_{s,t} E(|D_s u_t|^2) < \infty$.

For a process u in the class $\mathbb{L}_C^{1,2}$ we define

$$D_t^+ u = \lim_{\epsilon \downarrow 0} D_t u_{t+\epsilon} \qquad D_t^- u = \lim_{\epsilon \downarrow 0} D_t u_{t-\epsilon}.$$

Then we have (cf. Theorem 7.3 of [6]):

Proposition 2.2. *If u belongs to $\mathbb{L}_C^{1,2}$ then u is Stratonovich integrable and*

$$\int_0^1 u_t \circ dW_t = \delta(u) + \frac{1}{2}\int_0^1 (D_t^+ u + D_t^- u)dt. \tag{2.5}$$

From the point of view of the stochastic calculus, the Stratonovich integral behaves as an ordinary pathwise integral. We refer to [6] for a detailed discussion of this fact.

The next proposition provides a different kind of sufficient conditions for the existence of the Stratonovich integral.

Proposition 2.3. *Let u be a process in $\mathbb{L}_{\text{loc}}^{1,2}$. Suppose that the integral operator from H into H associated with the kernel $Du(\omega)$ is nuclear for all ω a.s. Then u is Stratonovich integrable and we have*

$$\int_0^1 u_t \circ dW_t = \delta(u) + \operatorname{Tr} Du. \tag{2.6}$$

Proof: From Proposition 6.1 of [9] we know that for any complete orthonormal system $\{e_i, i \geq 1\}$ in H the series

$$\sum_{i=1}^{\infty} \left(\int_0^1 u_t e_i(t)dt\right) \int_0^t e_i(s)dW_s$$

converges in probability to $\delta(u) + \operatorname{Tr} Du$. Then the proposition follows from the results of [10]. Q.E.D.

We will finish this preliminary section with some comments about the noncausal Girsanov theorem. Let $\{u_t, 0 \leq t \leq 1\}$ be a measurable process such that $\int_0^1 u_t^2 dt < \infty$ a.s. Define the transformation $T : \Omega \longrightarrow \Omega$ by

$$T(\omega)_t = \omega_t + \int_0^t u_s(\omega)ds.$$

We know that if u is adapted and if $E(J) = 1$, where $J = \exp\left(-\int_0^1 u_s dW_s - \frac{1}{2}\int_0^1 u_s^2 ds\right)$, then $\{T(\omega)_t\}$ is a Wiener process under a new probability Q given by $dQ/dP = J$. For a non adapted process u we need more restrictive conditions on the process in order to deduce a similar result. Let us first introduce the following definition (see Kusuoka [3]):

Definition 2.4. *Let u be a measurable process such that $\int_0^1 u_t^2 dt < \infty$ a.s. We will say that u is $\mathcal{H}$–C^1 (namely, the mapping $\omega \to u_\cdot(\omega)$ from Ω into H is $\mathcal{H}$–C^1) if there exists a random kernel $Du(\omega) \in L^2([0,1]^2)$ such that:*

(i) $\|u(\omega + \int_0^\cdot h_s ds) - u(\omega) - Du(\omega)(h)\|_H = o(\|h\|_H)$ for all $\omega \in \Omega$ as $\|h\|_H$ tends to zero.

(ii) The mapping $h \longmapsto Du(\omega + \int_0^\cdot h_s ds)$ is continuous from H into $L^2([0,1]^2)$ for all ω.

One can prove (see the proof of Theorem 5.2 in Kusuoka [3]) that if u is $\mathcal{H}$–C^1 then u belongs to $\mathbb{L}^{1,2}_{\text{loc}}$, and the kernel Du verifying the above conditions (i) and (ii) is precisely the operator D applied to u. Furthermore if $u : \Omega \to H$ is continuously differentiable then u is $\mathcal{H}$–C^1, and for every $\omega \in \Omega$, $Du(\omega) \in H \otimes H$ is the derivative of u. Using these defintions, we can state the following result proved by Kusuoka (see Theorem 6.4 of [3]).

Theorem 2.5. *Let $\{u_t, 0 \le t \le 1\}$ be a stochastic process which defines a $\mathcal{H}$–C^1 map from Ω into $L^2(0,1)$. Suppose that:*
(i) *The transformation $T : \Omega \to \Omega$ given by $T(\omega)_t = \omega_t + \int_0^t u_s(\omega)\, ds$ is bijective.*
(ii) *$I + Du(\omega) : H \to H$ is invertible, for all ω a.s.*
Then the process $\{W_t + \int_0^t u_s\, ds\}$ is a Wiener process under the probability Q on $C_0([0,1])$ given by

$$\frac{dQ}{dP} = |d_c(-Du)| \exp\left(- \delta(u) - \frac{1}{2}\int_0^1 u_t^2\, dt\right), \tag{2.7}$$

where $d_c(-Du)$ denotes the Carleman–Fredholm determinant of the square integrable kernel $Du \in L^2([0,1]^2)$.

We refer to [15] and [17] for the definition and main properties of the Carleman–Fredholm determinant of a Hilbert–Schmidt operator. If the integral operator associated with the kernel $Du(\omega)$ is nuclear for each ω, then the Carleman–Ferdholm determinant can be computed as follows (see [15]):

$$d_c(-Du) = \det(I + Du) \exp(-\operatorname{Tr} Du). \tag{2.8}$$

Moreover, from Proposition 2.3 we obtain in this case

$$\frac{dQ}{dP} = |\det(I + Du)| \exp\left(- \int_0^1 u_t \circ dW_t - \frac{1}{2}\int_0^1 u_t^2\, dt\right). \tag{2.9}$$

3. Second order stochastic differential equations with Dirichlet boundary conditions

In this section we are going to study the second order stochastic differential equation

$$\ddot{X}_t + f(X_t, t) = \dot{W}_t, \qquad 0 \le t \le 1, \tag{3.1}$$

with the boundary conditions $X_0 = X_1 = 0$. Here $\{W_t\}$ is a one–dimensional Brownian motion starting at zero, and the function $f : \mathbb{R} \times [0,1] \to \mathbb{R}$, is supposed to be measurable and locally bounded. Equation (3.1) must be regarded as a formal differential version of the integral equation

$$\dot{X}_t + \int_0^t f(X_s s)\, ds = \dot{X}_0 + W_t, \qquad 0 \le t \le 1. \tag{3.2}$$

Before studying the equation (3.1) we will find the solution in the particular case $f=0$. That means, let $\{Y_t\}$ be the solution of

$$\begin{aligned} \ddot{Y}_t &= \dot{W}_t, \qquad 0 \le t \le 1, \\ Y_0 &= Y_1 = 0. \end{aligned} \tag{3.3}$$

The solution to the equation (3.3) is

$$Y_t = -t\int_0^1 W_s ds + \int_0^t W_s ds\,, \tag{3.4}$$

and we also have

$$\dot{Y}_t = W_t - \int_0^1 W_s ds. \tag{3.5}$$

Define the transformation $T: C_0([0,1]) \longrightarrow C_0([0,1])$ by

$$T(\omega)_t = \omega_t + \int_0^t f(Y_s(\omega), s) ds, \tag{3.6}$$

where $\{Y_t\}$ is given by (3.4). Notice that the mapping $\omega \longmapsto Y(\omega)$ from $C_0([0,1])$ into the space $C^1_{0,0}(0,1)$ of continuously differentiable functions y on (0,1) such that $\lim_{t\downarrow 0} y(t) = 0$ and $\lim_{t\uparrow 1} y(t) = 0$ is bijective.

We remark the following two facts:

(I) If $T(\eta) = W$, then the function $X_t = Y_t(\eta)$ is a solution of the equation (3.1). In fact, we have

$$\dot{X}_t = \dot{Y}_t(\eta) = -\int_0^1 \eta_s\, ds + \eta_t = \dot{X}_0 + W_t - \int_0^t f(X_s, s)\, ds\,.$$

(II) Conversely, if we are given a solution $\{X_t\}$ of equation (3.1), then $T\big(Y^{-1}(X)\big) = W$. Indeed, if we set $Y^{-1}(X) = \eta$, then

$$T(\eta)_t = \eta_t + \int_0^t f\big(Y_s(\eta), s\big)\, ds = \eta_t + W_t + \dot{X}_0 - \dot{X}_t = W_t\,.$$

Consequently, if T is a *bijection* then equation (3.1) has the unique solution $X = Y(T^{-1}(W))$ for any continuous function $W \in C_0([0,1])$. In the sequel we will assume the following hypothesis:

(P.1): *The function f is nonincreasing in the variable x, locally Lipschitz and with linear growth, uniformly with respect to the variable t. That means, for every $N > 0$ there exist constants $C > 0$ and $C_N > 0$ such that*

$$|f(x,t) - f(y,t)| \le C_N|x-y|, \qquad \forall \quad |x|, |y| \le N, t \in [0,1],$$

$$|f(x,t)| \le C(1+|x|), \qquad \forall x, t.$$

We will see in the next proposition that this hypothesis is sufficient for the existence and uniqueness of a solution of the equation (3.1).

Proposition 3.1. *Suppose that f verifies (P.1). Then T is a bijection and, consequently, there exists a unique solution to the equation (3.1) for any continuous function W in $C_0([0,1])$.*

Proof: We need to show that for any $\eta \in C_0([0,1])$ there exists a unique function $W \in C_0([0,1])$ such that

$$\dot{\eta}_t = \dot{W}_t + f\left(-t\int_0^1 W_s ds + \int_0^t W_s ds, s\right).$$

Set $V = \eta - W$. Then V satisfies

$$\dot{V}_t = f\left(t\int_0^1 V_s ds - \int_0^t V_s ds + \xi_t, t\right),$$
$$V_0 = 0,$$

where

$$\xi_t = -t\int_0^1 \eta_s ds + \int_0^t \eta_s ds.$$

For any $y \in \mathbb{R}$ we consider the differential equation

$$\dot{V}_t(y) = f\left(ty - \int_0^t V_s(y) ds + \xi_t, t\right)$$
$$V_0(y) = 0.$$

By a comparison theorem for ordinary differential equations and using the monotonicity properties of f we deduce that the mapping $y \longmapsto V_t(y)$ is continuous and nonincreasing for each $t \in [0,1]$. Therefore, $\int_0^1 V_t(y)dt$ is a nonincreasing and continuous function of y, and this implies the existence of a unique real number y such that $\int_0^1 V_t(y)dt = y$. This completes the proof of the proposition. Q.E.D.

We want to study the Markov properties of the process $\{X_t\}$ solution of the equation (3.1). Let us first recall the following types of Markov property:

(1) We say that a d–dimensional stochastic process $\{Z_t, 0 \le t \le 1\}$ is a *Markov process* if for any $t \in [0,1]$ the past and the future of $\{Z_s\}$ are conditionally independent, given the present state Z_t.

(2) We say that $\{Z_t, 0 \le t \le 1\}$ is a *Markov field* if for any $0 \le s < t \le 1$, the values of the process inside and outside the interval $[s,t]$ are conditionally independent, given Z_s and Z_t.

Following the results of Russek [14] we might conjecture that as a solution of a second order stochastic differential equation the process $\{X_t\}$ is a 2–Markov process, that means, the two dimensional process $\{(X_t, \dot{X}_t)\}$ is a Markov process. We first show that this is true for the process $\{Y_t\}$ i.e., when $f \equiv 0$.

Proposition 3.2. *The process $\{(Y_t, \dot{Y}_t), 0 \le t \le 1\}$ defined by the equations (3.4) and (3.5) is a Markov process.*

Proof: Let $\psi(x,y)$ be a real valued bounded and measurable function. Fix $s<t$ and set $\zeta=\int_0^1 W_t\,dt$. We have to compute the conditional expectation

$$
\begin{aligned}
&E(\psi(Y_t,\dot{Y}_t)|(Y_r,\dot{Y}_r),\ 0\le r\le s)\\
&=E(\psi(-t\zeta+\int_0^t W_u du,\ -\zeta+W_t)\,|\,\zeta,\ W_r,\ 0\le r\le s)\\
&=E(\psi(-t\zeta+\int_0^s W_u du+\int_s^t (W_u-W_s)du+(t-s)W_s,\\
&\qquad -\zeta+W_t-W_s+W_s)|\zeta,\ W_r,\ 0\le r\le s)\\
&=\int_{\mathbb{R}^2}\psi(-t\zeta+\int_0^s W_u\,du+x+(t-s)W_s,\ -\zeta+y+W_s)\\
&\cdot N\Big(\Big(\frac{(t-s)^2(3-2s-t)}{2(1-s)^3}\int_s^1 (W_u-W_s)\,du,\\
&\frac{3(t-s)(2-s-t)}{2(1-s)^3}\int_s^1 (W_u-W_s)\,du\Big),\ \Lambda\Big)\,(dx,\,dy)
\end{aligned}
$$

where Λ denotes the conditional covariance matrix of the Gaussian vector $(\int_s^t(W_u-W_s)\,du,\ W_t-W_s)$, given $\int_s^1(W_t-W_s)dt$. Consequently, the above conditional expectation will be a function of the random variables

$$
\begin{aligned}
-t\zeta+\int_0^s W_u\,du+(t-s)\,W_s &=(t-s)\dot{Y}_s+Y_s\,,\\
-\zeta+W_s &=\dot{Y}_s\,,\\
\int_s^1(W_u-W_s)\,du &=-(1-s)\dot{Y}_s-Y_s\,,
\end{aligned}
$$

and this implies the Markov property. Q.E.D.

We will see that, except in the linear case, the Markov property does not hold for the process $\{X_t\}$. One might think that the Markov field property is better adapted to our equation because we impose fixed values at the boundary points $t=0$ and $t=1$. However this is not the case and, as we shall see, the nonlinearity of the function f prevents for any type of Markov property. The main result of this section is the following.

Theorem 3.3. *Let $\{X_t,\ t\in[0,1]\}$ be the solution of equation (3.1) and suppose that the function f has linear growth, f is of class C^2 with respect to x, its partial derivatives f'_x, f''_{xx} are continuous in (x,t), and $f'_x\le 0$. Then if f is an affine function of x, $\{(X_t,\ \dot{X}_t)\}$ is a Markov process, and conversely, if this process is a Markov field, then $f''_{xx}\equiv 0$.*

In order to prove this theorem we are going to introduce a new probability measure Q on $C_0([0,1])$ such that $P=Q\circ T^{-1}$, where T is the mapping defined by (3.6). Then

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$\{T(\omega)_t\}$ will be a Wiener process uner Q, and, consequently, the law of the process $\{X_t\}$ under the probability P coincides with the law of $\{Y_t\}$ under Q. In fact, we have

$$P\{\omega : X(\omega) \in B\} = Q\{\omega : X(T(\omega)) \in B\} = Q\{\omega : Y(\omega) \in B\},$$

for any Borel subset B of $C^1_{0,0}([0,1])$. In this way we will translate the problem of the Markov property of $\{X_t\}$ into the problem of the Markov property of the process $\{Y_t\}$ under a new probability Q. This problem can be handled, provided Q is absolutely continuous with respect to the Wiener measure P and we can compute an explicit expression for its Radon–Nikodym derivative. To do this we will make use of Theorem 2.5. So, before proving Theorem 3.3 we will present some preliminary lemmas. Let us first introduce some notations. Set $\alpha_t = f'_x(Y_t, t)$, and denote by M_t the matrix $\left[\begin{smallmatrix} 0 & -\alpha_t \\ 1 & 0 \end{smallmatrix}\right]$. Let Φ_t be the solution of the linear differential equation

$$\begin{aligned} d\Phi_t &= M_t \Phi_t \, dt \\ \Phi_0 &= I \end{aligned} \tag{3.6}$$

We will also denote by $\Phi(t,s)$ the matrix $\Phi_t \Phi_s^{-1}$.

Lemma 3.4. *Let f be a function satisfying the conditions of Theorem 3.3. Then the process $u_t = f(Y_t, t)$ verifies the conditions of the generalized Girsanov Theorem 2.5 and we have*

$$\frac{dQ}{dP} = Z_1 \exp\left(- \int_0^1 f(Y_t, t) \circ dW_t - \frac{1}{2} \int_0^1 f(Y_t, t)^2 \, dt \right), \tag{3.7}$$

where Z_1 is the solution at time $t = 1$ of the second order differential equation

$$\begin{cases} \ddot{Z}_t + \alpha_t Z_t = 0 \\ Z_0 = 0, \dot{Z}_0 = 1. \end{cases} \tag{3.8}$$

Proof of Lemma 3.4. First we have to check that the process

$$u_t = f(Y_t, t) = f(-t \int_0^1 W_s ds + \int_0^t W_s ds, t)$$

satisfies the conditions of Theorem 2.5. We already know that the mapping $T(\omega)_t = \omega_t + \int_0^t u_s(\omega) ds$ is bijective, due to Proposition 3.1. Furthermore, u is $\mathcal{H}$–C^1 because the mapping $\omega \longmapsto u_\cdot(\omega)$ is continuously differentiable from Ω into H. So, it remains to show that $I + Du$ is invertible a.s. From the properties of the operator D we deduce that

$$D_s Y_t = -t(1-s) + (t-s)^+ = st - s \wedge t, \tag{3.9}$$

and therefore,

$$D_s u_t = \alpha_t (st - s \wedge t). \tag{3.10}$$

From the Fredholm alternative, in order to show that $I + Du$ is invertible, it suffices to check that -1 is not an eigenvalue of $Du(\omega)$, for each $\omega \in \Omega$ a.s. Let $h \in L^2(0,1)$ such that $(I+Du)h=0$. Then

$$h_t+\alpha_t\int_0^1 h_s(st-s\wedge t)\,ds=0,$$

which can be written as

$$h_t+t\alpha_t\int_0^1 s\,h_s\,ds-\alpha_t\int_0^t s\,h_s\,ds-t\alpha_t\int_t^1 h_s\,ds=0.$$

Setting $g_t=\int_0^t h_s\,ds$ and $U_t=\int_0^t g_s\,ds$ we obtain

$$\ddot{U}_t+\alpha_t U_t-t\alpha_t U_1=0\,. \tag{3.11}$$

Then the solution of the second order differential equation (3.11) is given by

$$U_t=U_1\int_0^t \Phi_{21}(t,s)s\alpha_s ds\,, \tag{3.12}$$

which implies $U_1=0$ because $\Phi_{21}\geq 0$ and $\alpha_s\leq 0$.

To finish the proof of the lemma it remains to establish the formula (3.7). To do this we have to compute the Carleman–Fredholm determinant of the kernel (3.10). First observe that this kernel is nuclear and continuous. In fact, the cointinuity is clear. On the other hand, the kernel $st\alpha_t$ is nuclear. Now $\lambda_n=4/[(2n-1)^2\pi^2]$, $n\geq 1$ are the eigenvalues of the integral operator associated with the kernel $\min(s,t)$ and $e_n(t)=\sqrt{2}\sin(2n-1)\frac{\pi t}{2}$ the corresponding eigenfunctions. Then

$$\int_0^1 \alpha_t(s\wedge t)e_n(s)ds=\alpha_t\lambda_n e_n(t),$$

and

$$\sum_{n=1}^{\infty}\|\alpha_t\lambda_n e_n(t)\|_2\leq\|\alpha\|_\infty\sum_{n=1}^{\infty}\lambda_n<\infty,$$

which implies the nuclearity of the kernel $\alpha_t(s\wedge t)$. Therefore we can make use of the formula (2.9), and it suffices to show that $\det(I+Du)$ is equal to Z_1. Note that the absolute value of Z_1 can be omitted because $Z_t\geq 0$. Set $k(s,t)=st-s\wedge t$. From a well-known formula for the determinant of a nuclear operator (see e.g. [15]) we obtain

$$\det(I+Du)=\sum_{n=0}^{\infty}\frac{\gamma_n}{n!},$$

where

$$\gamma_n=\int_{[0,1]^n}\det(D_{t_i}u_{t_j})dt_1\cdots dt_n.$$

We have to compute the coefficients γ_n.

$$\begin{aligned}\gamma_n &= \int_{[0,1]^n} \det(k(t_i,t_j))\alpha(t_1)\cdots\alpha(t_n)dt_1\cdots dt_n \\ &= n! \int_{\{t_1<t_2<\cdots<t_n\}} \det(k(t_i,t_j))\alpha(t_1)\cdots\alpha(t_n)dt_1\cdots dt_n.\end{aligned}$$

The determinant of the matrix $(k(t_i,t_j))$ is equal to

$$\left(\prod_{i=1}^n t_i\right)^2 \det \begin{bmatrix} 1-\frac{1}{t_1} & 1-\frac{1}{t_2} & 1-\frac{1}{t_3} & \cdots & 1-\frac{1}{t_n} \\ 1-\frac{1}{t_2} & 1-\frac{1}{t_2} & 1-\frac{1}{t_3} & \cdots & 1-\frac{1}{t_n} \\ 1-\frac{1}{t_3} & 1-\frac{1}{t_3} & 1-\frac{1}{t_3} & \cdots & 1-\frac{1}{t_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1-\frac{1}{t_n} & 1-\frac{1}{t_n} & 1-\frac{1}{t_n} & \cdots & 1-\frac{1}{t_n} \end{bmatrix}.$$

Substracting the k–th row from the $(k-1)$–th one, $k = 2,\ldots,n$, we get

$$\begin{aligned}\det(k(t_i,t_j)) &= \left(\prod_{i=1}^n t_i\right)^2 \\ &\times \left(\frac{1}{t_2}-\frac{1}{t_1}\right)\left(\frac{1}{t_3}-\frac{1}{t_2}\right)\left(\frac{1}{t_4}-\frac{1}{t_3}\right)\cdots\left(\frac{1}{t_n}-\frac{1}{t_{n-1}}\right)\left(1-\frac{1}{t_n}\right) \\ &= (-1)^n t_1(t_2-t_1)(t_3-t_2)(t_4-t_3)\cdots(t_n-t_{n-1})(1-t_n).\end{aligned}$$

Consequently we obtain

$$\begin{aligned}\det(I+Du) &= \sum_{n=0}^{\infty}(-1)^n \\ &\times \int_{\{t_1<t_2<\cdots<t_n\}} t_1(t_2-t_1)(t_3-t_2)\cdots(t_n-t_{n-1})(1-t_n)\alpha(t_1)\cdots\alpha(t_n)dt_1\cdots dt_n\end{aligned} \tag{3.13}$$

On the other hand the solution of the second order differential equation (3.8) can be expressed as

$$Z_t = t - \int_0^t \int_0^s \alpha_u Z_u du = t - \int_0^t (t-u)\alpha_u Z_u du.$$

Iterating this equality we deduce that $\det(I+Du)$ coincides with Z_1. The proof of the lemma is now complete. Q.E.D.

Lemma 3.5. *Let $\mathcal{G}_t$ be the σ–algebra generated by Y_t, $\dot{Y}_t$ and $\int_0^1 W_s\, ds$, where we recall that Y_t and $\dot{Y}_t$ are defined by (3.4) and (3.5). Let F be a random variable in the space $\mathbb{D}^{1,2}_{\mathrm{loc}}$ such that $F\mathbf{1}_G$ is $\mathcal{G}_t$–measurable for some set $G \in \mathcal{G}_t$. Then there exist random variables A_t, B_t, and C_t such that*

$$\mathbf{1}_G D_\theta F = \mathbf{1}_G\left[A_t\theta + C_t\right]\mathbf{1}_{[0,t]}(\theta) + \mathbf{1}_G B_t(\theta-1)\,\mathbf{1}_{[t,1]}(\theta)\,,$$

for $dP \times d\theta$ almost all $(\omega,\theta) \in \Omega \times [0,1]$.

Proof: Let us first compute

$$D_\theta Y_t = t(1-\theta) + (t-\theta)\mathbf{1}_{[0,t]}(\theta) = \theta(t-1)\mathbf{1}_{[0,t]}(\theta) + (\theta-1)t\,\mathbf{1}_{[t,1]}(\theta)\,, \tag{3.14}$$

$$D_\theta \dot{Y}_t = -(1-\theta) + \mathbf{1}_{[0,t]}(\theta) = \theta\,\mathbf{1}_{[0,t]}(\theta) + (\theta-1)\mathbf{1}_{[t,1]}(\theta)\,, \tag{3.15}$$

and

$$D_\theta\Big(\int_0^1 W_t\,dt\Big) = 1-\theta\,. \tag{3.16}$$

Thus, it suffices to apply Lemma 2.1 to the three–dimensional subspace of H generated by $\theta\mathbf{1}_{[0,t]}(\theta)$, $\mathbf{1}_{[0,t]}(\theta)$ and $(\theta-1)\mathbf{1}_{[t,1]}(\theta)$. Q.E.D.

Proof of Theorem 3.3. Let Q be the probability measure on $C_0([0,1])$ given by Lemma 3.4. The law of the process $\{X_t\}$ under P is the same as the law of $\{Y_t\}$ under Q. Therefore, we can replace the process $\{(X_t,\dot{X}_t)\}$ by $\{(Y_t,\dot{Y}_t)\}$ and the probability P by Q in the statement of the theorem. By Proposition 3.2 $\{(Y_t,\dot{Y}_t)\}$ is a Markov process under P and now we have to study the Markov property with respect to an equivalent probability measure Q. The Radon–Nikodym derivative $J=\frac{dQ}{dP}$ will be given by the formula (3.7). Notice also that $Z_t>0$ for all t in $(0,1]$, and $\dot{Z}_t>0$ for all t in $[0,1]$. For any fixed $t\in(0,1)$ we can factorize the Radon–Nikodym derivative $J=\frac{dQ}{dP}$ as follows

$$J = Z_1 L_t L^t,$$

where

$$L_t = \exp\left(-\int_0^t f(Y_s,s)\circ dW_s - \frac{1}{2}\int_0^t f(Y_s,s)^2\,ds\right),$$

$$L^t = \exp\left(-\int_t^1 f(Y_s,s)\circ dW_s - \frac{1}{2}\int_t^1 f(Y_s,s)^2\,ds\right).$$

We define the σ–algebras
$\mathcal{F}_t = \sigma\left\{(Y_s,\,\dot{Y}_s),\ 0\le s\le t\right\}$
$\mathcal{F}^t = \sigma\left\{(Y_s,\,\dot{Y}_s),\ t\le s\le 1\right\}$, and $\mathcal{F}^t_0 = \mathcal{F}^t\vee\sigma\{Y_0,\dot{Y}_0\} = \mathcal{F}^t\vee\sigma\{\int_0^1 W_t\,dt\}$.
For any random variable ξ integrable with respect to Q we set

$$\wedge_\xi = E_Q(\xi|\mathcal{F}_t) = \frac{E_P(\xi J|\mathcal{F}_t)}{E_P(J|\mathcal{F}_t)} = \frac{E_P(\xi|Z_1|L^t|\mathcal{F}_t)}{E_P(|Z_1|L^t|\mathcal{F}_t)},. \tag{3.17}$$

because L_t is $\mathcal{F}_t$–measurable.

(i) Suppose first that f is an affine function. In that case Z_1 is deterministic and we get

$$\wedge_\xi = \frac{E_P(\xi L^t|\mathcal{F}_t)}{E_P(L^t|\mathcal{F}_t)}\,.$$

Then if ξ is $\mathcal{F}^t$–measurable, using the fact that L^t is also $\mathcal{F}^t$–measurable and applying the Markov property of $\{(Y_t,\dot{Y}_t)\}$ under P we deduce that $\wedge_\xi$ is $\sigma\{Y_t,\dot{Y}_t\}$–measurable and this implies that $\{(Y_t,\dot{Y}_t);\,t\in[0,1]\}$ is a Markov process under Q.

(ii) To prove the converse assertion we suppose that $\{(Y_t, \dot{Y}_t);\, t \in [0,1]\}$ is a Markov field under Q and we have to show that $f''_{xx} = 0$. This assumption implies in particular that for any $t \in (0,1)$ and any $\mathcal{F}_0^t$–measurable random variable ξ, integrable with respect to Q, the conditional expectation $\wedge_\xi = E_Q(\xi|\mathcal{F}_t)$ is $\mathcal{G}_t = \sigma\{Y_t, \dot{Y}_t, \int_0^1 W_t\, dt\}$–measurable. In order to obtain a suitable factorization of the random variable Z_1 we set

$$\begin{bmatrix} \dot{Z}_1 \\ Z_1 \end{bmatrix} = \Phi(1,t) \begin{bmatrix} \dot{Z}_t \\ Z_t \end{bmatrix}, \tag{3.18}$$

that means,

$$Z_1 = \Phi_{21}(1,t)\dot{Z}_t + \Phi_{22}(1,t)Z_t, \tag{3.19}$$

where the process $\Phi(t)$ has been introduced in (3.6). Now define

$$\varphi_t = \frac{\Phi_{21}(1,t)}{\Phi_{22}(1,t)} \qquad \text{and} \qquad \psi_t = \frac{Z_t}{\dot{Z}_t} = \frac{\Phi_{21}(t)}{\Phi_{11}(t)}. \tag{3.20}$$

From the equation $\dot{\widehat{\Phi^{-1}}}(t) = -\Phi^{-1}(t)M_t$, we deduce that $\Phi_{21}(1,t)$ sastisfies the same second order differential equation (3.8) as Z_t. From this fact we deduce that φ_t and ψ_t are continuously differentiable processes on [0,1], $\varphi_1 = 0$, $\psi_0 = 0$, $\varphi_t > 0$ for $t \in [0,1)$ and $\psi_t > 0$ for $t \in (0,1]$. Then we can write

$$Z_1 = \Phi_{22}(1,t)\dot{Z}_t(\varphi_t + \psi_t), \tag{3.21}$$

and we obtain the factorization

$$J = L_t\dot{Z}_t C^t(\varphi_t + \psi_t), \tag{3.22}$$

where $C^t = L^t\Phi_{22}(1,t)$.

In the sequel we will denote by $\overline{F}$ the conditional expectation of the random variable F under P with respect to the σ–algebra $\mathcal{G}_t$. The random variables $\dot{Z}_t$ and ψ_t are $\mathcal{F}_t$–measurable and, on the other hand, $\Phi_{22}(1,t)$, and φ_t are $\mathcal{F}^t$–measurable. Thus, from (3.17) and (3.22) and applying the Markov field property of $\{(Y_t, \dot{Y}_t)\}$ under P we deduce that

$$\wedge_\xi = \frac{\overline{\xi\, C^t}\psi_t + \overline{\xi\, C^t\varphi_t}}{\overline{C^t}\psi_t + \overline{C^t\varphi_t}}$$

and by our hypotheses this expression is $\mathcal{G}_t$–measurable. Therefore we obtain the following equation

$$\left(\wedge_\xi\, \overline{C^t} - \overline{\xi C^t}\right)\psi_t + \left(\wedge_\xi\, \overline{C^t\varphi_t} - \overline{\xi C^t\varphi_t}\right) = 0, \tag{3.23}$$

which is valid for any $\mathcal{F}_0^t$–measurable random variable ξ integrable with respect to Q. Now we choose two particular variables ξ:

$$\xi_1 = (C^t)^{-1} \qquad \text{and} \qquad \xi_2 = (C^t\varphi_t)^{-1}.$$

First we remark that for $t \in (0,1)$, $i = 1,2$, ξ_i is Q–integrable and nonnegative. In fact, this follows easily from the estimates $0 \le \Phi_{21}(1,t)^{-1} \le (1-t)^{-1}$ and $0 \le \Phi_{22}(1,t)^{-1} \le 1$.

Then we define the set

$$G_t = \{ \wedge_{\xi_1} \overline{C^t\varphi_t} = \overline{\xi_1 C^t\varphi_t} \} \cap \{ \wedge_{\xi_2} \overline{C^t\varphi_t} = \overline{\xi_2 C^t\varphi_t} \}.$$

Notice that $G_t \in \mathcal{G}_t$. On the set G_t we have

$$\wedge_{\xi_1} = \frac{\overline{\xi_1 C^t}}{\overline{C^t}} = \frac{\overline{\xi_1 C^t\varphi_t}}{\overline{C^t\varphi_t}},$$
$$\wedge_{\xi_2} = \frac{\overline{\xi_2 C^t}}{\overline{C^t}} = \frac{\overline{\xi_2 C^t\varphi_t}}{\overline{C^t\varphi_t}}.$$

Consequently

$$\overline{\varphi_t}\overline{C^t} = \overline{C^t\varphi_t}$$
$$\overline{\varphi_t^{-1}}\overline{C^t\varphi_t} = \overline{C^t},$$

which implies that $\left(\overline{\varphi_t}\right)^{-1} = \overline{\varphi_t^{-1}}$. By the strict Jensen inequality applied to the measure space $(G_t, \mathcal{F}|_{G_t}, P)$ we get that the random variable $\mathbf{1}_{G_t}\varphi_t$ is $\mathcal{G}_t$–measurable. From the equation (3.23) we deduce that the random variable $\mathbf{1}_{G_t^c}\psi_t$ is $\mathcal{G}_t$–measurable.

It is not hard to show that the random variables φ_t and ψ_t belong to the space $\mathbb{D}^{1,2}_{\text{loc}}$, for any $t \in (0,1)$. Thus by Lemma 3.5 applied to the random variables φ_t and ψ_t and to the sets G_t and G_t^c, there exist random variables $\Gamma_1(t)$ and $\Gamma_2(t)$ such that

$$D_\theta\varphi_t = (\theta - 1)\Gamma_1(t)\,, \tag{3.24}$$

for all $\theta \in [t,1]$, $\omega \in G_t$, a.e., and

$$D_\theta\psi_t = \theta\Gamma_2(t)\,, \tag{3.25}$$

for all $\theta \in [0,t]$, $\omega \in G_tc$, a.e.

On the other hand, from the linear differential equations satisfied by $\Phi(t)$ and $\Phi(1,t)$ we can derive Ricatti type differential equations for φ_t and ψ_t. In fact, differentiating with respect to t the equations

$$\Phi_{21}(1,t) = \varphi_t\Phi_{22}(1,t)$$
$$\Phi_{21}(t) = \psi_t\Phi_{11}(t),$$

we obtain

$$\dot{\varphi}_t + \alpha_t\varphi_t^2 + 1 = 0; \quad \varphi_1 = 0 \tag{3.26}$$
$$\dot{\psi}_t - \alpha_t\psi_t^2 - 1 = 0; \quad \psi_0 = 0 \tag{3.27}$$

Applying the operator D, which commutes with the derivative with respect to the time variable, to the equations (3.26) and (3.27) yields

$$\begin{aligned} &\frac{d}{dt}D_\theta\varphi_t + 2\varphi_t\alpha_t D_\theta\varphi_t + \varphi_t^2 D_\theta\alpha_t = 0, \qquad D_\theta\varphi_1 = 0,\\ &\frac{d}{dt}D_\theta\psi_t - 2\psi_t\alpha_t D_\theta\psi_t - \psi_t^2 D_\theta\alpha_t = 0, \qquad D_\theta\psi_0 = 0. \end{aligned} \tag{3.28}$$

Set $\gamma_{ts} = \exp\left(\int_t^s 2\varphi_r\alpha_r dr\right)$, for any $s, t \in [0,1]$. Notice that

$$D_\theta \alpha_t = \theta(t-1) f''_{xx}(Y_t, t)\mathbf{1}_{[0,t]}(\theta) + (\theta - 1)t f''_{xx}(Y_t, t)\mathbf{1}_{[t,1]}(\theta)\,. \tag{3.29}$$

Then if we solve the linear equation satisfied for $D_\theta\varphi_t$ as a function of t, starting from the point 1 we get

$$\begin{aligned} D_\theta\varphi_t &= \int_t^1 \gamma_{ts}\varphi_s^2 D_\theta \alpha_s ds \\ &= (\theta-1)\int_t^\theta \gamma_{ts} f''_{xx}(Y_s, s) s ds + \theta \int_\theta^1 \gamma_{ts} f''_{xx}(Y_s, s)(s-1) ds \\ &= (\theta-1)\int_t^1 \gamma_{ts} f''_{xx}(Y_s, s) s ds + \theta \int_\theta^1 \gamma_{ts} f''_{xx}(Y_s, s)(s-\theta) ds \end{aligned} \tag{3.30}$$

On the set G_t and for $\theta \in [t,1]$ we know by (3.24) that $D_\theta\varphi_t$ must be a multiple of $(\theta-1)$. Taking into account the equation (3.30), this is only possible if $f''_{xx}(Y_s, s) = 0$ for all s in the interval $[t,1]$ and for all $\omega \in G_t$ a.e. A similar argument, using the process ψ_t, yields that $f''_{xx}(Y_s, s) = 0$ for all s in $[0,t]$ and for all $\omega \in G_t^c$ a.e. The point t being arbitrary in $(0,1)$ we deduce that the process $f''_{xx}(Y_s, s)$ is identically zero. Therefore $f''_{xx}(x) = 0$ for all x, which completes the proof of the theorem. Q.E.D.

Remark

(1) A similar result for a second order differential equation of the form

$$\ddot{X}_t + f(X_t, \dot{X}_t) = \dot{W}_t$$

has been proved in [8]. The proof in this case is more involved and some additional smoothness conditions on the function f are required.

(2) As in [8] one could show that under the conditions of Theorem 3.3, if $\{(X_t, \dot{X}_t)\}$ is a germ Markov field then $f''_{xx} = 0$. We recall that $\{(X_t, \dot{X}_t), 0 \le t \le 1\}$ is a *germ Markov field* if for any $0 \le s < t \le 1$, the values of the process inside and outside the interval $[s,t]$ are conditionally independent, given the germ σ–field $\bigcap_{\epsilon>0}\sigma((X_u, \dot{X}_u), u \in (s-\epsilon, s+\epsilon)\cup(t-\epsilon, t+\epsilon))$.

4. First order stochastic differential equations with boundary conditions

In this section we describe some particular types of stochastic differential equations that have been recently studied using the techniques of noncausal stochastic calculus.

4.1 Bilinear stochastic differential equations of Stratonovich type with boundary conditions.

In [11] Ocone and Pardoux have developed a detailed study of an equation of the form

$$\begin{aligned} dX_t &= (FX_t + a)dt + \sum_{i=1}^k (G_i X_t + b_i)\circ dW_t^i, \qquad 0 \le t \le 1 \\ H_0 X_0 &+ H_1 X_1 = h_0. \end{aligned} \tag{4.1}$$

Here F, G_i, H_0 and H_1 are $d \times d$ matrices, and a, b_i and h_0 are vectors in $\mathbb{R}^d$. We assume that the matrix $[H_0 : H_1]$ has rank d. Particular examples of this type of boundary conditions are:

(1) The first l coordinates of X_0 are given, where $1 \le l \le d$, and the last $d - l$ coordinates of X_1 are also given. The above linear boundary conditions can be reduced to this situation whenever $\mathrm{Im} H_0 \cap \mathrm{Im} H_1 = \{0\}$.

(2) We impose the periodic boundary condition $X_0 = X_1$.

In order to find a solution for the equation (4.1) we introduce the fundamental solution of the corresponding linear system:

$$d\Phi_t = F\Phi_t dt + \sum_{i=1}^{k} G_i \Phi_t \circ dW_t^i, \qquad \Phi_0 = I,$$

and we set $V_t = at + \sum_{i=1}^k b_i W_t^i$. Then the candidate for a solution is

$$X_t = \Phi_t X_0 + \Phi_t \int_0^t \Phi_s^{-1} \circ dV_s. \tag{4.2}$$

Substituting (4.2) into the boundary condition we obtain

$$X_t = \Phi_t \left[(H_0 + H_1 \Phi_1)^{-1} \left(h_0 - H_1 \Phi_1 \int_0^1 \Phi_s^{-1} \circ dV_s \right) \right] + \Phi_t \int_0^t \Phi_s^{-1} \circ dV_s. \tag{4.3}$$

One can show that (4.3) is the unique solution in some space of processes $\mathbb{L}_{\mathrm{loc}}^S$, provided that $\det(H_0 + H_1 \Phi_1) \neq 0$ a.s. We refer the reader to [11] for more details about this class of processes and for the proof of the existence and uniqueness of a solution.

Concerning the Markov property of the solution one can show the following results:

Proposition 4.1. *The solution $\{X_t\}$ of the equation (4.1) is a Markov process under one of the following situations:*
(i) $H_1 = 0$ or $H_0 = 0$.
(ii) $G_i = 0$, $1 \le i \le k$ (Gaussian case) and $\mathrm{Im} H_0 \cap \mathrm{Im} H_1 = \{0\}$.

If $\mathrm{Im} H_0 \cap \mathrm{Im} H_1 \neq \{0\}$ the process X may not be Markovian even in the Gaussian case. On the other hand, in the non Gaussian case, the condition $\mathrm{Im} H_0 \cap \mathrm{Im} H_1 = \{0\}$ does not insure the Markov property. Concerning the Markov field property one has the following result.

Proposition 4.2. *The solution $\{X_t\}$ of the equation (4.1) is a Markov field in the following cases:*
(i) $G_1 = \cdots = G_k = 0$ (Gaussian case)
(ii) $a = b_1 = \cdots = b_k = 0$, and Φ_t is a diagonal matrix for all $t \in [0,1]$ (For instance, this is true in dimension one)

4.2. Non linear stochastic differential equations with boundary conditions.

Consider the following equation

$$\begin{aligned} &dX_t + f(X_t) = dW_t, \qquad 0 \le t \le 1 \\ &h(X_0, X_1) = h_0, \end{aligned} \tag{4.4}$$

where W is a d–dimensional Wiener process, $f : \mathbb{R}^d \to \mathbb{R}^d$, $h : \mathbb{R}^{2d} \to \mathbb{R}^d$, and h_0 is a vector in $\mathbb{R}^d$. The existence and uniqueness of a solution for the equation (4.4) has been discussed in [7]. The Markov property of the process X has also been investigated using the techniques of Section 3. Concerning the Markov property one also has negative results in dimension one when the function f is non linear. Let us describe briefly how one can obtain these results.

Notice first that if we assume a periodic boundary conditions, then the equation (4.4) with $f = 0$ has no solution. For this reason we introduce a decomposition of the form $f(x) = Ax + \overline{f}(x)$, where A is a fixed deterministic $d \times d$–matrix. Then we denote by Y the solution of the corresponding boundary value problem in the case $\overline{f} \equiv 0$. That means

$$\begin{aligned} &dY_t + AY_t = dW_t, \\ &h(Y_0, Y_1) = h_0 \end{aligned} \tag{4.5}$$

The boundary value problem (4.5) is equivalent to the following system

$$\begin{aligned} &Y_t = e^{-At}\left(Y_0 + \int_0^t e^{As} dW_s\right) \\ &h\left(Y_0, e^{-A}\left(Y_0 + \int_0^1 e^{As} dW_s\right)\right) = h_0. \end{aligned} \tag{4.6}$$

In order to solve the equation (4.6) we will impose the following assumption:

(H.1) *For all $z \in \mathbb{R}^d$ the equation $h(y, e^{-A}(y+z)) = h_0$ has a unique solution $y = g(z)$.*

We denote by Ω the space of continuous functions from $[0,1]$ into $\mathbb{R}^d$ which vanish at zero. On the other hand we denote by Σ the space of continuous functions x from $[0,1]$ into $\mathbb{R}^d$ which satisfy the desired boundary condition, that means $h(x_0, x_1) = h_0$. Then, under the hypothesis (H.1) there exists a bijection $\Psi : \Omega \longrightarrow \Sigma$ such that $Y = \Psi(W)$. We define the mapping $T : \Omega \longrightarrow \Omega$ by

$$T(\omega)_t = \omega_t + \int_0^t \overline{f}(\Psi_s(\omega))ds.$$

As in Section 3 one can show that if T is a bijection then the equation (4.4) has the unique solution $X = \Psi(T^{-1}(W))$. The following are sufficient conditions for the transformation T to be bijective:

(1) T is *one–to–one* if:

(i) There exists $\lambda \in \mathbb{R}$ such that $f + \lambda I$ is strictly monotone and

(ii) $e^{\lambda}|g(z) - g(z')| \le |e^{-A}(z - z' + g(z) - g(z'))|$, for all $z, z' \in \mathbb{R}^d$.

(2) T is *onto* if:

(iii) $\overline{f}$ is locally Lipschitz

(iv) $\lim_{a\to\infty} \frac{1}{a} \sup_{|x|\le a} |\overline{f}(x)| = 0$ (sublinear growth)

(v) g has linear growth

Assuming (i), a sufficient condition for (ii) is the following:

(ii') $h(x,y) = h(\bar{x},\bar{y}) = h_0$ implies $e^{\lambda}|x-\bar{x}| \le |y-\bar{y}|$.

We remark that in the periodic case we only need f to be strictly monotone for T to be one–to–one, because we can take $\lambda = 0$ in (1).

We can now study the Markov property of the process X solution of (4.4). By means the extended version of Girsanov theorem (see Theorem 2.5) one can show the following result:

Theorem 4.3. *Suppose that $\overline{f}$ and g are continuously differentiable functions such that the mapping T is bijective. Suppose in addition that*

$$\det\left[I - e^{A}\Phi_1 g'(\xi_1) + g'(\xi_1)\right] \neq 0, \tag{4.7}$$

where $\xi_t = \int_0^t e^{As} dW_s$, and $\Phi_t = I - \int_0^t f'(Y_s)\Phi_s ds$. Then the conditions of Theorem 2.5 are satisfied by the transformation T, and we have

$$\frac{dQ}{dP} = \left|\det\left[I - e^{A}\Phi_1 g'(\xi_1) + g'(\xi_1)\right]\right| \\ \times \exp\left\{\int_0^t \mathrm{Tr}\overline{f}'(Y_t)dt - \int_0^t \overline{f}(Y_t)\circ dW_t - \frac{1}{2}\int_0^t |\overline{f}(Y_t)|^2 dt\right\}.$$

Theorem 4.3 can be used to investigate the Markov properties of the process X. In fact, setting $\overline{W}_t = W_t + \int_0^t \overline{f}(Y_s)ds$, we obtain

$$dY_t + AY_t = dW_t = d\overline{W}_t - \int_0^t \overline{f}(Y_s)ds.$$

Therefore, the process $\{Y_t\}$ is a solution of the equation (4.4) for the process $\overline{W}$, and this implies that the law of Y under the probability Q coincides with the law of X under P. The results one can obtain for the Markov property are the following:

(1) If f is an affine function, then $\{X_t\}$ is always a Markov field. This can be proved by a direct argument.

(2) In dimension one, assuming that f and g are twice continuously differentiable, T is bijective, $g'(x) > -1$, g' is not identically zero, and (i) and (ii') hold, then if $\{X_t\}$ is a Markov field, one necessarily has $f'' \equiv 0$.

(3) Suppose that

$$X_0^{i_k} = a_k; \qquad 1 \le k \le l$$
$$X_1^{j_k} = b_k; \qquad 1 \le k \le d-l$$

and f is triangular, that means, $f^k(x)$ is a function of $x^1, \ldots, x^k$ for all k. In this case, if for each k, f^k verifies a Lipschitz and linear growth condition on the variable x^k, one can show that there exists a unique solution of the equation $dX_t + f(X_t) = dW_t$ with the above boundary conditions, and the solution is a Markov process. This result is also proved by direct methods, and it says that unlike the one-dimensional case the solution of (4.4) can be a Markov process even though f is non linear.

(4) In the following example in dimension 2 one also has a dicothomy similar to the one-dimensional case. Consider the equation

$$\begin{aligned} dX_t + f(X_t) &= dW_t \\ X_1^1 = X_0^2 &= 0 \end{aligned}$$

where $f(x^1, x^2) = (x^1 - x^2, -f_2(x^1))$ and f_2 is a twice continuously differentiable function such that $0 \le f_2'(x) \le K$ for some positive constant K. Then there exists a unique solution which is a Markov field only if $f_2'' \equiv 0$.

(5) In dimension one, and assuming a linear boundary condition of the type $F_0X_0 + F_1X_1 = h_0$, Donati-Martin (cf. [2]) has obtained the existence and uniqueness of a solution for the equation

$$dX_t = \sigma(X_t) \circ dW_t + b(X_t),$$

when the coefficients b and σ are of class C^4 with bounded derivatives, and $F_0F_1 \neq 0$. On the other hand, if σ is linear ($\sigma(x) = \alpha x$), $h_0 \neq 0$, and assuming that b is of class C^2, then one can show that the solution $\{X_t\}$ is a Markov field only if the drift is of the form $b(x) = Ax + Bx \log |x|$, where $|B| < 1$.

For other types of existence results for first order stochastic differential equations with boundary conditions we refer to Dembo and Zeitouni [1] and Huang Zhiyuan [18].

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