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ON THE NUMBER OF GENERATORS OF MODULES
OVER POLYNOMIAL AFFINE RINGS

by

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0. INTRODUCTION

In this paper we give some bounds for the minimal number of generators of a finitely generated module M over a polynomial ring $B = A[X_1, \dots, X_n]$, where A is an affine ring satisfying some regularity conditions.

For $n = 1$, A.Sathaye and N.M.Kumar proved in [S] and [MK] that if B is commutative and noetherian, then:

$$\mu(M) \leq \max \{ \mu(M_{\mathfrak{p}}) + \dim B/\mathfrak{p} \mid \dim B/\mathfrak{p} < \dim B \} \stackrel{def}{=} ee(M)$$

where μ denotes minimal number of generators and \mathfrak{p} runs over the prime ideals of B , solving a conjecture of D.Eisenbud and G.E.Evans Jr.

For $A = k$, an infinite field, G.Lyubeznik has proved in [Ly] the following bound:

$$(0.1) \quad \mu(M) \leq \max \{ \mu(M_{\mathfrak{p}}) + \dim B/\mathfrak{p} \mid \mathfrak{p} \in \text{Spec} B \text{ such that } M_{\mathfrak{p}} \text{ is not free} \}$$

It is easy to see that this bound turns out to be especially sharp when the primes of B at which M is not locally free determine a closed subset of $\text{Spec} B$ of small dimension. As a consequence of our main result (Thm. 2.1), we prove that if A is a regular affine algebra over an infinite field and $M_{\mathfrak{p}}$ is a free $B_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \text{Spec} B$ except a finite number of maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$, then:

$$\mu(M) \leq \max \{ \dim A + r, \mu(M_{\mathfrak{m}_1}), \dots, \mu(M_{\mathfrak{m}_s}) \}$$

where $r = \text{rank}(M)$, thus generalizing for this type of modules the bound (0.1) (see Prop. 3.9).



When $n=1$ we prove that if M is a torsion free B -module and A is a regular affine ring (or is an affine domain such that $\text{Spec}A$ has at worst isolated singularities), then:

$$\mu(M) \leq \max \{ \dim A + r, \mu(M/I_M M) \}$$

where $I_M =$ ideal of definition of the set of primes of B at which M is not locally free. We recall that the ideal I_M can be effectively calculated if a presentation of M is given (see [Br]) and that if A is a domain the Eisenbud - Evans bound can be written as

$$ee(M) = \max \{ \dim A + r, f(M/I_M M) \}$$

where $f(\cdot)$ denotes the bound given by O.Forster in [F] for the number of generators of a module (see Remark 3.6). As there are a number of situations where $f(M/I_M M)$ is not a sharp bound for $\mu(M/I_M M)$, our result could be viewed as a refinement of the Eisenbud - Evans bound in the situation above.

On the other hand, if $M = I$ is an ideal, one knows that $\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1$ and one is interested on knowing when the inequality on the left side becomes an equality. If B is commutative and noetherian and $I \subseteq B$ is an ideal which contains a monic polynomial, S.Mandal has proved in [M] that if $\mu(I/I^2) \geq \dim B + 2$, then $\mu(I) = \mu(I/I^2)$. But, to the extent we know, the only result which holds for general $I \subseteq B$ is that the equality is attained when $\mu(I/I^2) \geq \dim B$ (see [MK]). In Corollary 3.2 we prove that if A is a regular affine ring and $I \subseteq A[X_1, \dots, X_n]$ is an ideal such that $\text{ht}(I) \geq 2n$ and $\mu(I/I^2) > \dim A$, then $\mu(I) = \mu(I/I^2)$. This result follows from the main result in [M] in case I contains a monic polynomial (possibly after a change of variables). However, this may not be so if $\text{ht}(I) \leq \dim A$ (see Remark 3.3).

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1.SOME NOTATION AND LEMMAS

Throughout the paper all rings will be assumed to be commutative and noetherian and all modules will be finitely generated. By an affine ring we will always understand a finitely generated k -algebra, k a field.

If B is a ring and M a B -module, the rank of M is defined as:

$$\text{rank}M = \max \{ \mu(M_{\mathfrak{p}}) \mid \mathfrak{p} \text{ minimal prime of } B \}$$

We recall that given a B -module, the set of primes of B at which M is not locally free determines a closed subset of $\text{Spec}B$. We denote by I_M the (radical) ideal which defines this set.

Definition 1.1 Let B be a ring, M a B -module and $J \subseteq B$ an ideal. We will say that M is free out of J if $M_{\mathfrak{p}}$ is a free $B_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \text{Spec}B$ such that $\mathfrak{p} \not\supseteq J$. It is clear that M is free out of J if and only if $J \subseteq I_M$.

Lemma 1.2 Let B be a ring and let M be a B -module of rank r free out of an ideal $J \subseteq B$. Let $I \subseteq J$ be an ideal and put $N = M/IM$. Suppose $\mu(M/JM) \leq d$ and $d \geq \dim B/I + r$. Then $\mu(N) \leq d$.

PROOF: If $\mu(M/JM) \leq d$ then $\mu(N/JN) \leq d$, i.e. there are $n_1, \dots, n_d \in N$ such that $N = \langle n_1, \dots, n_d \rangle + JN$. By [Pl, Thm.0, see also the remark after the Theorem] in order to get $\mu(N) \leq d$ it is enough to check that if $\mathfrak{p} \in \text{Spec}B$ is such that $\mathfrak{p} \not\supseteq J$ and $\mathfrak{p} \supseteq I$, then:

$$d \geq \mu(N_{\mathfrak{p}/I}) + \dim B/\mathfrak{p}$$

But if $\mathfrak{p} \not\supseteq J$ then $M_{\mathfrak{p}}$ is free and $\text{rank} M_{\mathfrak{p}} \leq r$, so $\mu(N_{\mathfrak{p}/I}) \leq r$ and the result follows from our hypothesis on d . ■

For later use, we have to prove a result (Prop. 1.5) which tell us that if A is an affine domain, $I \subseteq A$ is an ideal and M is an $A[X_1, \dots, X_n]$ -module of rank r , free out of $I[X_1, \dots, X_n]$ and such that:

$$\mu(M/I[X_1, \dots, X_n]M) \leq d, \quad d \geq \dim A + r$$

then one can find a system of generators of $M/I[X_1, \dots, X_n]M$ with some extra good properties. The lemma which follows is close in spirit to some avoidance lemmas proved in [MK] and [M], although its proof and the one of Proposition 1.5 are inspired in the proof of [Ly, Theorem 1]. We keep the same notations as in [Ly], i.e., if B is a ring, $\mathfrak{p} \in \text{Spec}B$, N is a B -module and $J \subseteq B$ is an ideal, put:

$$b_{\mathfrak{p}}(N) = \begin{cases} \mu(N_{\mathfrak{p}}) + \dim B/\mathfrak{p} & \text{if } N_{\mathfrak{p}} \neq 0 \\ 0 & \text{if } N_{\mathfrak{p}} = 0 \end{cases}$$

$$b^J(N) = \max \{ b_{\mathfrak{p}}(N) \mid \mathfrak{p} \in \text{Spec}B \text{ such that } \mathfrak{p} \not\supseteq J \}$$

Lemma 1.3 Let A be a ring, $I \subseteq A$ an ideal. Set $B = A[X_1, \dots, X_n]$ and let M be a B -module of rank r , free out of $I[X_1, \dots, X_n]$. Take $d \geq \dim A + r$ and assume that m_1, \dots, m_t ($t < d$) are given such that:

$$1. \frac{M}{\langle m_1, \dots, m_t \rangle + I[X_1, \dots, X_n]M} \text{ is } (d-t)\text{-generated}$$

2. Putting $M_i = M/(X_i, \dots, X_n)M$ for $1 \leq i \leq n$ and $M_{n+1} = M$ we have:

$$b^{I[X_1, \dots, X_{i-1}]}(M_i / \langle m_1, \dots, m_t \rangle) \leq b^{I[X_1, \dots, X_{i-1}]}(M_i) - t$$

for $1 \leq i \leq n+1$. Then we can find $m_{t+1} \in M$ such that:

$$1. \frac{M}{\langle m_1, \dots, m_{t+1} \rangle + I[X_1, \dots, X_n]M} \text{ is } (d-t-1)\text{-generated}$$

$$2. b^{I[X_1, \dots, X_{i-1}]}(M_i / \langle m_1, \dots, m_{t+1} \rangle) \leq b^{I[X_1, \dots, X_{i-1}]}(M_i) - t - 1 \text{ for } 1 \leq i \leq n+1$$

PROOF: Set, for $1 \leq i \leq n+1$

$$N_i = M_i / \langle m_1, \dots, m_t \rangle$$

Take m'_{t+1} belonging to a system of $d-t$ generators of $N_{n+1}/I[X_1, \dots, X_n]N_{n+1}$ and put $m_{t+1} = m'_{t+1} + c\lambda y$. We have to find $y \in M$, $\lambda \in I[X_1, \dots, X_n]$ and $c \in A[X_1, \dots, X_n]$ such that:

$$(0.2) \quad b^{I[X_1, \dots, X_{i-1}]}(N_i / \langle m_{t+1} \rangle) \leq b^{I[X_1, \dots, X_{i-1}]}(N_i) - 1 \text{ for } 1 \leq i \leq n+1$$

As it is observed in [Ly], it follows from the proof of the main theorem of [F] that, in order to get (0.2), it is enough to make m_{t+1} basic for N_i at certain primes $\mathfrak{p}_{i,1}, \dots, \mathfrak{p}_{i,j_i}$ of $A[X_1, \dots, X_{i-1}]$, no one of them containing $I[X_1, \dots, X_{i-1}]$. But we have that if $\mathfrak{q} \in \text{Spec}A[X_1, \dots, X_{i-1}]$ and $m \in M$ is basic for N_{n+1} at $(\mathfrak{q}, X_i, \dots, X_n)$, then m is basic for N_i at \mathfrak{q} (One only needs to check that if $m \in \mathfrak{q}(N_i)_{\mathfrak{q}}$, then $m \in (\mathfrak{q}, X_i, \dots, X_n)(N_{n+1})_{(\mathfrak{q}, X_i, \dots, X_n)}$).

Therefore, we have to make m_{t+1} basic for N_{n+1} at a finite set of primes:

$$\mathcal{P} = \{\mathfrak{p}_{n+1,1}, \dots, \mathfrak{p}_{n+1,j_{n+1}}, (\mathfrak{p}_{n,1}, X_n), \dots, (\mathfrak{p}_{n-1,1}, X_{n-1}, X_n), \dots\}$$

no one of them containing $I[X_1, \dots, X_n]$. For this purpose, we make the following claim:

Claim: Let N be a finitely generated B -module and $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \subseteq \text{Spec}B$. If $x, y \in M$ and y is basic for M at each \mathfrak{p}_i , then there is a $c \in B$ such that $x + cy$ is also basic for M at all the primes \mathfrak{p}_i .

PROOF(OF THE CLAIM): We can assume that \mathfrak{p}_j is minimal in $\{\mathfrak{p}_1, \dots, \mathfrak{p}_j\}$ and we make induction on s , the case $s = 1$ being trivial. By induction hypothesis there is a $c' \in B$ such that $x + c'y \notin \mathfrak{p}_i M_{\mathfrak{p}_i}$ for $1 \leq i \leq s-1$. Suppose $x + c'y \in \mathfrak{p}_s M_{\mathfrak{p}_s}$, and take $\alpha \in \bigcap_{i=1}^{s-1} \mathfrak{p}_i - \mathfrak{p}_s$. If we put $c = c' + \alpha$, then $x + cy$ is basic for M at $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. ■

Now take $y \in M$ basic at all $\mathfrak{p} \in \mathcal{P}$ (this is possible by [F,Hilfssatz]) and $\lambda \in I[X_1, \dots, X_n] - \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$. By the claim, applied to m'_{i+1} and λy , there is a $c \in A[X_1, \dots, X_n]$ such that m_{i+1} has the required properties. ■

From this lemma we get:

Corollary 1.4 Let A be a ring and $I \subseteq A$ an ideal. Set $B = A[X_1, \dots, X_n]$ and let M be a B -module of rank r , free out of $I[X_1, \dots, X_n]$ and such that:

$$\mu(M/I[X_1, \dots, X_n]M) \leq d \quad , \quad d \geq \dim A + r$$

Then there are $m_1, \dots, m_d \in M$ such that:

- a. m_1, \dots, m_d generate $M/I[X_1, \dots, X_n]M$
- b. $b^{I[X_1, \dots, X_{i-1}]}(M_i / \langle m_1, \dots, m_d \rangle) \leq b^{I[X_1, \dots, X_{i-1}]}(M_i) - d$ for $1 \leq i \leq n + 1$

where the M_i are defined as in lemma 1.3.

Proposition 1.5 Let A be an affine domain over a field k and $I \subseteq A$ an ideal. Put $B = A[X_1, \dots, X_n]$ and let M be a B -module of rank r , free out of $I[X_1, \dots, X_n]$ and such that:

$$\mu(M/I[X_1, \dots, X_n]M) \leq d \quad , \quad d \geq \dim A + r$$

Then there are $m_1, \dots, m_d \in M$ such that:

1. m_1, \dots, m_d generate $M/I[X_1, \dots, X_n]M$
2. Putting

$$J_i = \text{Ann}(M_i / \langle m_1, \dots, m_d \rangle) \subseteq A[X_1, \dots, X_{i-1}]$$

we have $\text{ht}(J_i) > \dim A$ for $1 \leq i \leq n + 1$

3. $J_i + I[X_1, \dots, X_{i-1}] = A[X_1, \dots, X_{i-1}]$ for $1 \leq i \leq n + 1$

PROOF: We can assume that we have $m_1, \dots, m_d \in M$ verifying conditions a. and b. from Corollary 1.4. From a. we get:

$$(0.3) \quad M_i = \langle m_1, \dots, m_d \rangle + I[X_1, \dots, X_{i-1}]M_i \quad \text{for } 1 \leq i \leq n + 1$$

Localizing (0.3) at the multiplicative system $1 + I[X_1, \dots, X_n]$ and applying Nakayama's lemma we get a $s_i \in I[X_1, \dots, X_{i-1}]$ such that $1 + s_i \in \text{Ann}(M_i / \langle m_1, \dots, m_d \rangle) = J_i$. Thus $J_i + I[X_1, \dots, X_{i-1}] = A[X_1, \dots, X_{i-1}]$ and we have 3..

Also, if $\mathfrak{p} \in \text{Spec}A[X_1, \dots, X_{i-1}]$ is such that $\mathfrak{p} \not\supseteq I[X_1, \dots, X_{i-1}]$ then $(\mathfrak{p}, X_i, \dots, X_n) \not\supseteq I[X_1, \dots, X_n]$, hence $M_{(\mathfrak{p}, X_i, \dots, X_n)}$ is free and the isomorphism:

$$(M_i)_{\mathfrak{p}} \simeq \frac{M_{(\mathfrak{p}, X_i, \dots, X_n)}}{(X_i, \dots, X_n)M_{(\mathfrak{p}, X_i, \dots, X_n)}}$$

gives $\mu((M_i)_{\mathfrak{p}}) \leq r$. Therefore:

$$(0.4) \quad b^{I[X_1, \dots, X_{i-1}]}(M_i) \leq \dim A[X_1, \dots, X_{i-1}] + r$$

Putting together (0.4) and b. from Corollary 1.4, we get:

$$b^{I[X_1, \dots, X_{i-1}]}(L_i) \leq \dim A + i - 1 + r - d \leq i - 1$$

where we denote by L_i the quotient module $M_i / \langle m_1, \dots, m_d \rangle$. Now take a minimal prime over-ideal of J_i , say \mathfrak{p} . From 3. we have that $\mathfrak{p} \not\supseteq I[X_1, \dots, X_{i-1}]$, so $b_{\mathfrak{p}}(L_i) \leq i - 1$, that is:

$$\dim \frac{A[X_1, \dots, X_{i-1}]}{\mathfrak{p}} \leq i - 1 - \mu((L_i)_{\mathfrak{p}})$$

But $\mu((L_i)_{\mathfrak{p}}) \geq 1$, hence:

$$\dim \frac{A[X_1, \dots, X_{i-1}]}{J_i} \leq i - 2$$

Now from our hypothesis on A and this inequality we get that $\text{ht}(J_i) > \dim A$, so we are done. ■

Remark 1.6 It is clear from the proof that the proposition above remains valid under much weaker assumptions on the ring A (for example, it would be enough to assume that it is an universally catenary equicodimensional Jacobson ring). However, the later use will involve only affine domains.

Finally we need a lemma from [Li2] which will be important on proving the main result. We give a different proof from the one in [Li2].

Lemma 1.7 Let A be any commutative ring, $J \subseteq A[X]$ an ideal which contains a monic polynomial, $I \subseteq A$ an ideal. If $J + I[X] = A[X]$ then $J \cap A + I = A$.

PROOF: The extension $A/J \cap A \rightarrow A[X]/J$ is integral, because J contains a monic polynomial. Thus, it induces a surjective map on the spectra. Suppose $J \cap A + I \subseteq \mathfrak{m}$, \mathfrak{m} a maximal ideal of A . Take $\mathfrak{n} \in \text{Spec}A[X]$ such that $J \subseteq \mathfrak{n}$ and $\mathfrak{n} \cap A = \mathfrak{m}$. Then $I \subseteq \mathfrak{m} \Rightarrow I[X] \subseteq \mathfrak{m}[X] = (\mathfrak{n} \cap A)[X] \subseteq \mathfrak{n}$, so $J + I[X] \subseteq \mathfrak{n}$, which is impossible. ■

2.MAIN RESULTS

In this section we prove our main results.

Theorem 2.1 Let A be a regular affine domain and $I \subseteq A$ an ideal. Put $B = A[X_1, \dots, X_n]$ and let M be a B -module of rank r , free out of $I[X_1, \dots, X_n]$ and such that:

$$\mu(M/I[X_1, \dots, X_n]M) \leq d, \quad d \geq \dim A + r$$

Then $\mu(M) \leq d$.

PROOF: Due to technical reasons we are going to prove something a bit more precise, namely that if A, I, M are as in the theorem one has $\mu(M) \leq d$ and, moreover, given $m_1, \dots, m_d \in M$ verifying the conditions **1.,2.,3.** of Prop. 1.5, there is an $s \in A[X_1, \dots, X_n]$ such that $(s, I[X_1, \dots, X_n]) = A[X_1, \dots, X_n]$ and a system of d generators of M , say n_1, \dots, n_d , such that $n_i = m_i$ in M_s .

The proof is by induction on n , the case $n = 0$ being a consequence of Prop. 1.5. If $n > 0$, take $m_1, \dots, m_d \in M$ which verify the conditions **1.,2.,3.** of Prop. 1.5.

In particular,

$$(0.5) \quad J_{n+1} + I[X_1, \dots, X_n] = A[X_1, \dots, X_n]$$

and $\text{ht}(J_{n+1}) > \dim A$. There is a change of variables of the form:

$$\begin{aligned} A[X_1, \dots, X_n] &\xrightarrow{\varphi} A[X_1, \dots, X_n] \\ X_i &\longrightarrow X_i + X_n^{r_i} \quad \text{for } 1 \leq i \leq n-1 \\ X_n &\longrightarrow X_n \end{aligned}$$

such that $\varphi(J_{n+1})$ contains a monic polynomial in X_n , so from (0.5) we have:

$$\varphi(J_{n+1}) + I[X_1, \dots, X_n] = A[X_1, \dots, X_n]$$

and then from lemma 1.7 we get:

$$\varphi(J_{n+1}) \cap A[X_1, \dots, X_{n-1}] + I[X_1, \dots, X_{n-1}] = A[X_1, \dots, X_{n-1}]$$

Take $s_1 \in \varphi(J_{n+1}) \cap A[X_1, \dots, X_{n-1}]$ such that $(s_1, I[X_1, \dots, X_{n-1}]) = A[X_1, \dots, X_{n-1}]$ and denote by M^* the module M with a new B -module structure given by restricting scalars by means of φ^{-1} (i.e., $b * m = \varphi^{-1}(b) \cdot m$). Notice that, denoting "modulo X_n " by a "bar", we have:

$$\text{i) } \overline{M^*} = \overline{M}$$

$$\text{ii) } \text{Ann}(M^*/\langle m_1, \dots, m_d \rangle) = \varphi(J_{n+1})$$

$$\text{iii) } M^* \text{ is free out of } I[X_1, \dots, X_n]$$

By the induction hypothesis applied to \overline{M} and $\overline{m}_1, \dots, \overline{m}_d$ (observe that they verify the conditions of Prop 1.5 and that \overline{M} is free out of $I[X_1, \dots, X_{n-1}]$) there are $\overline{n}_1, \dots, \overline{n}_d$ which generate \overline{M} and there is an $s_2 \in A[X_1, \dots, X_{n-1}]$ such that:

$$\text{iv) } \overline{n}_i = \overline{m}_i \text{ in } \overline{M}_{s_2}$$

$$\text{v) } (s_2, I[X_1, \dots, X_{n-1}]) = A[X_1, \dots, X_{n-1}]$$

Put $s = s_1 s_2$ and take $s' \in I[X_1, \dots, X_{n-1}]$ such that $s + s' = 1$. $M_{s'}^*$ is an $A[X_1, \dots, X_{n-1}]_{s'}[X_n]$ projective module, so from Lindel's theorem (see [Li]) it will be extended, i.e. $M_{s'}^* \simeq \overline{M}_{s'}^*[X_n] = \overline{M}_{s'}[X_n]$. Thus the \overline{n}_i give us a system of generators l_1, \dots, l_d of $M_{s'}^*$ such that $\overline{l}_i = \overline{n}_i$ for $1 \leq i \leq d$.

On the other hand, M_s^* will be generated by m_1, \dots, m_d because $s \in \varphi(J_{n+1})$ and $\overline{m}_i = \overline{n}_i$ in \overline{M}_s^* , since s_2 divides s .

Consider the exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & B_s^{\oplus d} & \longrightarrow & M_s^* \longrightarrow 0 \\ & & & & e_i & \longrightarrow & m_i \end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & L' & \longrightarrow & B_{s'}^{\oplus d} & \longrightarrow & M_{s'}^* \longrightarrow 0 \\ & & & & e_i & \longrightarrow & l_i \end{array}$$

Then $L_{s'}$ and L'_s are $A[X_1, \dots, X_{n-1}]_{s s'}[X_n]$ - projective modules, so they are extended (again by Lindel's theorem). Applying [Pl, Proposition 2], we get $\mu(M^*) \leq d$. Moreover, if the generators of M^* obtained in this way are x_1, \dots, x_n one has that $x_i = m_i$ in M_s^* , so $x_i = m_i$ in $M_{\varphi^{-1}(s)}$ and $(\varphi^{-1}(s), I[X_1, \dots, X_n]) = A[X_1, \dots, X_n]$. ■

Remark 2.2 The preceding result extends to a regular affine k -algebra A because we will have $A = A_1 \times \dots \times A_r$, where the A_i are regular affine domains and we can work on each component separately.

When the number of variables is restricted to one, we have been able to relax the hypothesis of regularity on the coefficient ring A . We have:

Theorem 2.3 Let A be an affine domain such that $\text{Spec}A$ has at worst isolated singularities, $I \subseteq A$ an ideal. Let M be an $A[X]$ -module of rank r , free out of $I[X]$.

If $\mu(M/I[X]M) \leq d$ and $d \geq \dim A + r$, then $\mu(M) \leq d$.

PROOF: In what follows we denote "modulo X " by a "bar". Prop. 1.5 gives that there are $m_1, \dots, m_d \in M$ such that:

- i) m_1, \dots, m_d generate $M/I[X]M$
- ii) $\bar{m}_1, \dots, \bar{m}_d$ generate \bar{M} (from 2. taking $i = 1$)
- iii) If $J = \text{Ann}(M / \langle m_1, \dots, m_d \rangle)$, then $\text{ht}(J) = \dim A[X]$ (taking $i = 2$)
- iv) $J + I[X] = A[X]$

From iii) we get that J contains a monic polynomial and from iv) and lemma 1.7 one has $J \cap A + I = A$. Take $s \in I$, $s' \in J \cap A$ with $s + s' = 1$. Then M_s is an $A_s[X]$ -projective module of rank r , \bar{M}_s can be generated by $\bar{m}_1, \dots, \bar{m}_d$, and $d \geq \dim A_s + r$. Applying [Pl, Theorem 4] there are n_1, \dots, n_d which generate M_s and $\bar{n}_i = \bar{m}_i$ for $1 \leq i \leq d$. Therefore we have exact sequences:

$$0 \longrightarrow L \longrightarrow A_s[X]^{\oplus d} \longrightarrow M_s \longrightarrow 0$$

$$e_i \longrightarrow n_i$$

$$0 \longrightarrow L' \longrightarrow A_{s'}[X]^{\oplus d} \longrightarrow M_{s'} \longrightarrow 0$$

$$e_i \longrightarrow m_i$$

We observe that:

- i) $L_{s'}$ can be assumed to be extended.

Put $S_1 = 1 + A_s$. $S_1 \subseteq A_s$ is a multiplicative system and the regularity hypothesis on A implies that $S_1^{-1}A_s$ is a regular domain. Let \mathfrak{m} be a maximal ideal of $S_1^{-1}A_s$. Then $(S_1^{-1}L)_{\mathfrak{m}}$ will be a stably free $(S_1^{-1}A_s)_{\mathfrak{m}}[X]$ -module of rank $d - \text{rank}(S_1^{-1}M_s)_{\mathfrak{m}} \geq d - r \geq \dim A \geq \dim(S_1^{-1}A_s)_{\mathfrak{m}}[X]$, so from [Pl, Theorem 1] it will be free.

From this and Quillen's extendability criterion we get that $S_1^{-1}L$ is extended. Let $s_1 \in S_1$ be such that L_{s_1} is extended. Replacing s' by $s_1 s'$ we have $(s, s_1 s') = A$ and all above works the same. In what follows we assume that this replacement has already been made.

- ii) L'_s can be assumed to be extended.



Put $S_2 = 1 + As' \subseteq A_{ss'}$. Just like before, $S_2^{-1}(A_{ss'})$ is a regular ring and $\dim A \geq \dim S_2^{-1}(A_{ss'})[X]$, so there is a $s_2 \in S_2$ such that L'_{ss_2} is extended. Replace s by ss_2 .

Now, by [Pl, Proposition 2], we are done. ■

Remark 2.4 Same conclusion holds, without regularity hypothesis on A , if there is a monic polynomial $f \in A[X]$ such that M_f is free. The hypothesis on A is only necessary in order to assert that $(S_1^{-1}L)_m$ and $(S_2^{-1}L'_s)_m$ are stably free, but if M_f is free, f monic, then $(S_1^{-1}M_s)_m$ is also free (it is projective and free after inverting a monic polynomial) and then we are through.

3. CONSEQUENCES

Proposition 3.1 Let A be a regular affine ring and set $B = A[X_1, \dots, X_n]$. Let $J \subseteq B$ be an ideal such that $\text{ht}(J) \geq 2n$ and let M be a B -module of rank r , free out of J .

If $\mu(M/JM) \leq d$ and $d \geq \dim A + r$, then $\mu(M) \leq d$.

PROOF: Put:

$$\begin{aligned} I &= J \cap A & J' &= J/I[X_1, \dots, X_n] \\ B' &= (A/I)[X_1, \dots, X_n] & M' &= M/I[X_1, \dots, X_n]M \end{aligned}$$

From the hypothesis on the height of J one easily deduces that $\dim B' \leq \dim A$, hence $d \geq \dim B' + r$. Applying lemma 1.2 we get that $\mu(M') \leq d$, and then we are done because of Thm.2.1. ■

Taking $M = J$ we have:

Corollary 3.2 Let A be a regular affine ring and $J \subseteq A[X_1, \dots, X_n]$ an ideal such that $\text{ht}(J) \geq 2n$ and $\mu(J/J^2) > \dim A$. Then $\mu(J) = \mu(J/J^2)$.

Remark 3.3 Suppose that A is a ring which satisfies the requirements of corollary 3.2. and $J \subseteq A[X_1, X_2]$ has height 4. If $\dim A = 5$ and $\mu(J/J^2) = 6$ from 3.2 one gets $\mu(J) = 6$. But $\dim A > \text{ht}(J)$, so in general J will not contain a monic polynomial (not even after a change of variables) and the main theorem of [M] could have not been applied.

Proposition 3.4 Let A be a regular affine ring (or an affine domain such that $\text{Spec} A$ has at worst isolated singularities) and let $J \subseteq A[X]$ be an ideal such that $J \cap A$ contains a non-zero divisor. Let M be an $A[X]$ -module of rank r , free out of J .

If $\mu(M/JM) \leq d$ and $d \geq \dim A + r$, then $\mu(M) \leq d$.

PROOF: If we see that $\mu(M/(J \cap A)[X]M) \leq d$ then we have finished in view of Thm. 2.3. But this follows from lemma 1.2 taking $B = (A/J \cap A)[X]$, $N = M/(J \cap A)[X]M$. ■

Corollary 3.5 Let A be a regular affine ring and let M be a torsion-free $A[X]$ -module of rank r . Let I_M be the ideal of definition of the set of primes at which M is not locally free. Then:

$$\mu(M) \leq \max\{ \dim A + r , \mu(M/I_M M) \}$$

PROOF: We only have to observe that $I_M \cap A$ contains a non-zero divisor. Set $S = \{a \in A \mid a \text{ is not a zero-divisor}\}$. Then $S^{-1}M$ is a torsion-free module over $(S^{-1}A)[X] \simeq (k_1 \times \dots \times k_s)[X]$, hence it is free. This gives that $I_M \cap S \neq \emptyset$ and then from Prop. 3.4 we obtain the desired bound. ■

Remark 3.6 1. In the situation of corollary 3.5, if M has constant rank the Eisenbud-Evans bound for $\mu(M)$ can be written as follows:

$$(0.6) \quad ee(M) = \max\{ \dim A + r , f(M/I_M M) \}$$

where for N a module over a ring B , $f(N)$ is the Forster bound, defined as:

$$f(N) = \max\{ \mu(N_{\mathfrak{p}}) + \dim B/\mathfrak{p} \mid \mathfrak{p} \in \text{Spec} B \}$$

It is known (see [F]) that $\mu(N) \leq f(N)$ and for general B, N this is the best possible bound. However, there are a number of situations where it can be sharpened.

For the proof of (0.6) observe that:

$$f(M/I_M M) = \max\{ \mu(M_{\mathfrak{p}}) + \dim A[X]/\mathfrak{p} \mid \mathfrak{p} \in \text{Spec} A[X] \text{ such that } M_{\mathfrak{p}} \text{ is not free} \}$$

Take $\mathfrak{p} \in \text{Spec} A[X]$ such that $\dim A[X]/\mathfrak{p} < \dim A[X]$.

If $M_{\mathfrak{p}}$ is not free, then $\mu(M_{\mathfrak{p}}) + \dim A[X]/\mathfrak{p} \leq f(M/I_M M)$.

If $M_{\mathfrak{p}}$ is free, then $\mu(M_{\mathfrak{p}}) + \dim A[X]/\mathfrak{p} \leq r + \dim A$.

So, $ee(M) \leq \max\{ \dim A + r , f(M/I_M M) \}$. For the other inequality, take $\mathfrak{p} \in \text{Spec} A[X]$ such that $\dim A[X]/\mathfrak{p} = \dim A$. Then:

$$ee(M) \geq \mu(M_{\mathfrak{p}}) + \dim A[X]/\mathfrak{p} \geq r + \dim A[X]/\mathfrak{p} = r + \dim A$$

2. Let B be a ring and M a B -module of rank r , free out of an ideal $J \subseteq B$. G.Lyubeznik proves in [Ly] that in case $B = k[X_1, \dots, X_n]$, k an infinite field, one has:

$$(0.7) \quad \mu(M/JM) \leq d \quad , \quad d \geq 1 + r + \dim B/J \Rightarrow \mu(M) \leq d$$

and from this he deduces (see [Ly, Theorem 2]) that:

$$\mu(M) \leq \max\{ \mu(M_{\mathfrak{p}}) + \dim B/\mathfrak{p} \mid \mathfrak{p} \in \text{Spec} B \text{ such that } M_{\mathfrak{p}} \text{ is not free} \} \stackrel{def}{=} \eta(M)$$

He also points out that if $\dim B \leq 2$, then every module is the surjective image of a projective module of rank $\eta(M)$ (so it is $\eta(M)$ -generated if B -projectives are free).

We study now the case $\dim B = 3$, B a polynomial ring. As a consequence of our result, we get that the bound $\mu(M) \leq \eta(M)$ also holds for modules over $\mathbb{F}_p[X_1, X_2, X_3]$ (\mathbb{F}_p a finite field).

Proposition 3.7 Let A be a regular affine ring of pure dimension two such that projective $A[X]$ -modules are free. Then (0.7) holds for $B = A[X]$.

PROOF: We distinguish three cases:

a. $\text{ht} J \leq 1$. Then the inequality $d \geq 1 + r + \dim A[X]/J$ gives $d \geq r + \dim A[X]$ and the result follows from lemma 1.2.

b. $\text{ht}(J) = 2$. Then one has $d \geq r + \dim A$ and $J \cap A \neq (0)$, so we can apply Prop. 3.4.

c. $\text{ht}(J) = 3$. If $d > r + 1$ then $d \geq r + \dim A$ and we can use again 3.4. If $d = r + 1$ one can find $s, s' \in A[X]$ such that M_s is projective of rank r and $M_{s'}$ is $(r + 1)$ -generated, and then apply the last proposition of [Ly]. ■

Finally, we are going to give a bound for the number of generators of a finitely generated $A[X_1, \dots, X_n]$ -module which is locally free everywhere except at a finite number of maximal ideals. First we have to observe that the main result in [Ly] (Theorem 1) can be extended to rings of the form $(k_1 \times \dots \times k_r)[X_1, \dots, X_n]$ where the k_i are extensions of an infinite field k . The only difficulty could appear on the geometric argument used to prove $I \cap A + J \cap A = (1)$ (in the notations of [Ly]). But one has the following easy lemma:

Lemma 3.8 Let $V, W \subseteq \text{Spec}(k_1 \times \dots \times k_r)[X_1, \dots, X_n]$ closed subschemes of dimensions d, d' . Suppose that k_1, \dots, k_r are extensions of an infinite field k . If $V \cap W = \emptyset$ and $d + d' + 1 < n$, then, after a generic linear change of variables, the projections of V, W on $\text{Spec}(k_1 \times \dots \times k_r)[X_1, \dots, X_{n-1}]$ are disjoint closed subschemes.

PROOF: One has $\text{Spec}(k_1 \times \dots \times k_r)[X_1, \dots, X_n] = \mathbf{A}_{k_1}^n \cup \dots \cup \mathbf{A}_{k_r}^n$, $V = V_1 \cup \dots \cup V_r$, $W = W_1 \cup \dots \cup W_r$ with $V_i, W_i \subseteq \mathbf{A}_{k_i}^n$ and $\dim V_i \leq d$, $\dim W_i \leq d'$ for $1 \leq i \leq r$. A generic linear change of variables with coefficients in k makes that if we denote by $p_i : \mathbf{A}_{k_i}^n \rightarrow \mathbf{A}_{k_i}^{n-1}$ the natural projection, then $p_i(V_i), p_i(W_i)$ are disjoint closed subschemes (see [Ly, proof of Theorem 1]). Now a general projection will work in every $\mathbf{A}_{k_i}^n$. ■

Then we get:

Proposition 3.9 Let A be a regular affine ring over an infinite field k . Let M be an $A[X_1, \dots, X_n]$ -module of rank r , locally free at every prime except at a finite number of maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$. Then:

$$\mu(M) \leq \max \{ \dim A + r, \mu(M_{\mathfrak{m}_1}), \dots, \mu(M_{\mathfrak{m}_s}) \}$$

PROOF: We can assume that A is a domain. Denote by d the bound above and set:

$$J = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s$$

$$\mathfrak{n}_i = \mathfrak{m}_i \cap A \quad \text{for } 1 \leq i \leq s$$

$$I = \mathfrak{n}_1 \cap \dots \cap \mathfrak{n}_s$$

Observe that $A/I = \prod_{i=1}^s k(\mathfrak{n}_i)$, where $k(\mathfrak{n}_i) = A/\mathfrak{n}_i \supseteq k$. Put:

$$B' = (A/I)[X_1, \dots, X_n]$$

$$J' = J/I[X_1, \dots, X_n]$$

$$M' = M/I[X_1, \dots, X_n]M$$

Then M' is a B' -module of rank r , free out of J' and such that:

$$\mu(M'/J'M') = \mu(M/JM) = \max\{\mu(M_{\mathfrak{m}_1}), \dots, \mu(M_{\mathfrak{m}_s})\} \leq d$$

So, from Lyubeznik's result (observe that the requirement $d \geq 1 + \text{rank} M' + \dim B'/J'$ is fulfilled), we get $\mu(M') \leq d$. Now the result follows from Thm. 2.1. ■

Remark 3.9 Using Prop. 3.7 it is easy to see that if $n \leq 3$, the result above remains true when k is a finite field.

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