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by

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Abstract

A quasi–bundle is defined to be a morphism from an algebraic surface onto a curve having all smooth fibres connected and isomorphic, and allowing as only singular fibres multiples of smooth curves. When no multiple fibre occurs it is called a fibre bundle. The general fibre F of a quasi–bundle is said to be divisible by an integer k if $(1/k)F$ is still the numerical class of an integral divisor. This paper focuses on the relationship between the divisibility properties of F and the torsion of $H^2(S, \mathbf{Z})$. For fibre bundles, the link between those two notions is established by means of Serre spectral sequence. As for general quasi–bundles, a suitable base change leads back to the fibre bundle case. The results become most explicit for elliptic quasi–bundles, where the action of the monodromy can be fully computed. For any prime number p, the paper contains examples of fibre bundles whose fibre is divisible by p.

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§ 0. INTRODUCTION

The simplest kind of morphisms from an algebraic surface onto a curve are those having all smooth fibres connected and isomorphic to each other, and allowing as only singular fibres multiples of smooth curves. These fibrations will be called quasi–bundles, the fibre bundles being the special case where no singular fibre occurs. The aim of this paper is to study the Picard group of surfaces S endowed with a quasi–bundle fibration. In particular, we are interested in linking two seemingly unrelated notions, namely the divisibility of a fibre in $H^2(S,\mathbb{Z})/$ (torsion) and the torsion of $H^2(S,\mathbb{Z})$.

If $\varphi : S \to C$ is a quasi–bundle with general fibre F, we say that F is divisible by an integer k if there exists $L \in H^2(S,\mathbb{Z})$ such that $F - kL$ is a torsion element of $H^2(S,\mathbb{Z})$ (or zero). One sees easily that the cocycle L must be algebraic as well. Every integer which occurs as multiplicity of some fibre of φ obviously divides F in the abovementioned sense. However, not all the divisibility properties of F are accounted for by the existence of multiple fibres. The action of the monodromy is playing a role, too. As a matter of fact, for each prime number p we are giving examples in § 2 of fibre bundles with fibre divisible by p (despite the absence of singular fibres). Once the possibility of such a phenomenon has been shown it is a question of searching for implications. For a fibre bundle $S \to C$ this is done by means of Serre spectral sequence. In this case we derive a close connection between the divisibility of F and the torsion of $H_1(S,\mathbb{Z})$ (non–canonically isomorphic to the torsion of $H^2(S,\mathbb{Z})$). This is the content of § 2. The results for fibre bundles will be extended in § 3 to general quasi–bundles. The idea now will be to perform a suitable base change, so that the multiple fibres disappear, and the information about the fibre bundle so obtained can be applied to our fibration. Moreover, some results of the author ([12]) about the behaviour of multiple fibres in homology are required. Here, of course, the results will be less precise since one loses information along the process of base-change. Finally, § 4 contains a more detailed study of elliptic quasi-bundles (i.e. $g(F) = 1$). In this context, the results of the preceding sections will become most explicit, inasmuch as the action of the monodromy can be fully described.

The main results of this paper are stated as Theorems 2.2 and 3.6 for general fibre genus, and Theorems 1.10, 4.1, 4.3 and 4.4 for elliptic fibrations.

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§ 1. NOTATION AND PRELIMINARIES

All varieties will be defined over the field of complex numbers. A surface (respectively, a curve) is a projective, irreducible, non-singular scheme of dimension 2 (resp. 1). We shall employ the following terminology:

$\mathcal{O}_X(D) :=$ invertible sheaf associated to a divisor D of the variety X.

$h^i\mathcal{O}_X(D) := \; dim\; H^i\mathcal{O}_X(D)$; $\chi\mathcal{O}_X(D) := \sum_{i=0}^{dim X}(-1)^i h^i\mathcal{O}_X(D)$.

$K_X :=$ canonical divisor of X.

$\pi_1(X) :=$ fundamental group of X.

$tor(G) :=$ torsion of an abelian group G.

$G/(\text{torsion}) := \; G/tor(G)$.

$\mathbb{Z}_m := \; \mathbb{Z}$ modulo $(m)\mathbb{Z}$.

$Pic^\circ(X) :=$ Picard variety (of divisors algebraically equivalent to zero) of X.

The irregularity and geometric genus of a surface S are denoted $q(S) := h^1\mathcal{O}_S$, $p_g(S) := h^2\mathcal{O}_S$ respectively. For a curve or divisor C, $g(C)$ stands for the arithmetic genus of C. The symbol $\sim$ (respectively $\equiv$) represents linear (resp. numerical) equivalence of divisors. If D is a divisor on a surface S, often we will write also D to mean its class in the distinct groups $Pic\, S$, $H^2(S, \mathbb{Z})$, etc., as the context will indicate.

A fibration $\varphi : S \to C$ is a morphism with connected fibres from a surface onto a curve. When the fibre genus is equal to one it is called an elliptic fibration. In general, φ is said to be relatively minimal if no fibre contains a (−1)–curve. We say that φ is the Albanese fibration if the image of the Albanese map $\alpha : S \to Alb(S)$ is a curve isomorphic to C, and $\alpha : S \to \alpha(S)$ coincides with φ. From the universal property of the Albanese variety ([2]) it follows that $q(S) = g(C)$ if and only if either $q(S) = 0$ or φ is the Albanese fibration.

Let $F = \sum_i n_i B_i$ be a fibre of φ, where the $B_i's$ are the irreducible reduced components, and the $n_i's$ their multiplicities. If m denotes the greatest common divisor of the $n_i's$, then we will say that m is the multiplicity of F, and will write $F = m\,D$, where $D = \sum_i (n_i/m) B_i$. Whenever we use the expression "let $m\,D$ be a multiple fibre" we always mean that m is the multiplicity of $m\,D$, and $m \geq 2$.

This paper deals with the simplest types of fibrations, to be defined now:

Definition 1.1. A fibration $\varphi : S \to C$ is called a quasi–bundle if all smooth fibres are isomorphic, and the only singular fibres are multiples of smooth curves. If moreover φ has no singular fibres, then φ is said to be a fibre bundle. For economy of notation, a surface S will also be called a quasi–bundle (respectively, a fibre bundle) if it admits a quasi–bundle (resp., a fibre bundle) fibration.

Remark 1.2. A quasi–bundle surface always admits two distinct quasi–bundle fibrations. In fact, it is proved in [13] that every quasi–bundle surface is the quotient of a product of two curves by the action of a finite group.

Let $\varphi : S \to C$ be a fibration with general fibre F. Denote by $\omega_{S/C} := \mathcal{O}_S(K_S) \otimes \varphi^* \mathcal{O}_C(-K_C)$ the dualising sheaf of φ. The sheaf $\varphi_*(\omega_{S/C})$ is locally free of rank equal to $g(F)$. By relative duality ([1]) one has:

$$\varphi_*(\omega_{S/C}) \simeq (R^1 \varphi_* \mathcal{O}_S)^{\vee}$$

where "$\vee$" denotes dual as $\mathcal{O}_C$–module. We have

$$h^1(R^1 \varphi_* \mathcal{O}_S) = h^\circ\left((\varphi_* \omega_{S/C}) \otimes \mathcal{O}_C(K_C)\right) = h^\circ \mathcal{O}_S(K_S) = p_g(S).$$

The first terms of the Leray spectral sequence $E_2^{p,q} = H^p(R^q \varphi_* \mathcal{O}_S) \Longrightarrow H^{p+q}(\mathcal{O}_S)$ yield an exact sequence ([7]), II 4.17.1 and I 4.5.1)

$$0 \to H^1(\varphi_* \mathcal{O}_S) \to H^1(\mathcal{O}_S) \to H^\circ(R^1 \varphi_* \mathcal{O}_S) \to H^2(\varphi_* \mathcal{O}_S) = 0$$

Since $\varphi_*\mathcal{O}_S = \mathcal{O}_C$ we get $h^\circ(R^1\varphi_*\mathcal{O}_S) = q(S) - g(C)$. Riemann–Roch on C yields

$$\chi(R^1\varphi_*\mathcal{O}_S) = \deg(R^1\varphi_*\mathcal{O}_S) + g(F)(1 - g(C))$$

On the other hand

$$\chi(R^1\varphi_*\mathcal{O}_S) = h^\circ(R^1\varphi_*\mathcal{O}_S) - h^1(R^1\varphi_*\mathcal{O}_S) = 1 - \chi\mathcal{O}_S - g(C).$$

Finally one obtains

Lemma 1.3. $\chi\mathcal{O}_S = \deg(\varphi_*\omega_{S/C}) + \big(g(F) - 1\big)\big(g(C) - 1\big).$ □

Proposition 1.4. With the preceding notation one has $h^1\mathcal{O}_S(-F) = g(C)$ for any fibre F. In particular, the image of $Pic^\circ(S) \to Pic^\circ(F)$ (induced by the inclusion $F \subseteq S$) has dimension $q(S) - g(C)$.

Proof: Let F be the fibre over $p \in C$. Leray spectral sequence yields

$$0 \to H^1\big(\mathcal{O}_C(-p)\big) \to H^1\big(\mathcal{O}_S(-F)\big) \to H^\circ(R^1\varphi_*\varphi^*\mathcal{O}_C(-p)) \to 0$$

We have $R^1\varphi_*\varphi^*\mathcal{O}_C(-p) = (\varphi_*\omega_{S/C})^\vee \otimes \mathcal{O}_C(-p)$. Fujita's decomposition Theorem ([6]) says that $\varphi_*\omega_{S/C} = \mathcal{O}_C^{\oplus h} \oplus E$ with $h = q(S) - g(C)$ and $h^\circ(E^\vee) = 0$. Consequently $R^1\varphi_*\varphi^*\mathcal{O}_C(-p)$ has no global sections, and thus $h^1\mathcal{O}_S(-F) = h^1\mathcal{O}_C(-p) = g(C)$. The tangent spaces at the origin of $Pic^\circ(S), Pic^\circ(F)$ can be identified with $H^1\mathcal{O}_S$ and $H^1\mathcal{O}_F$ respectively. From the sequence

$$0 \to \mathcal{O}_S(-F) \to \mathcal{O}_S \to \mathcal{O}_F \to 0$$

one sees that the kernel of $H^1\mathcal{O}_S \to H^1\mathcal{O}_F$ is isomorphic to $H^1\mathcal{O}_S(F)$, and the last assertion follows. □.

In the next two Lemmas we will restrict our attention to elliptic fibrations.

Lemma 1.5. A relatively minimal elliptic fibration $\varphi : S \to C$ is a quasi–bundle if and only if $\chi\mathcal{O}_S = 0$.

Proof: In view of Theorems 6 and 7 of chapter IV in [14] the topological Euler characteristic $e(S)$ of S vanishes if and only if all singular fibres of φ are multiples of smooth elliptic curves. Noether's formula $12\chi\mathcal{O}_S = e(S)+K_S^2$ combined with $K_S^2 = 0$ shows that $\chi\mathcal{O}_S = 0$ is equivalent to $e(S) = 0$. As for the statement that all smooth fibres of φ are isomorphic, see ([2], VI. 7 and 8). □

A combination of the equality $h^\circ((\varphi_*\omega_{S/C})^\vee) = q(S) - g(C)$ with Lemma 1.3 yields:

Lemma 1.6. Let $\varphi : S \to C$ be a relatively minimal elliptic fibration. One has:

(i) If $\chi\mathcal{O}_S > 0$ then $q(S) = g(C)$.

(ii) If $\chi\mathcal{O}_S = 0$ then $q(S) = g(C)$ if and only if $\varphi_*(\omega_{S/C}) \neq \mathcal{O}_C$. Otherwise $q(S) = g(C) + 1$. □

As a matter of fact, if $\varphi : S \to C$ is an elliptic fibre bundle then $\varphi_*(\omega_{S/C})$ is a torsion line bundle ([1], III 18.3) which is carrying some information about the monodromy of φ. Namely, if $\Psi : R \to B$ is the fibration obtained by base change on φ from the cyclic covering of C determined by $k\varphi_*(\omega_{S/C}) \sim 0$ ($k =$ order of $\varphi_*(\omega_{S/C})$), then $q(R) = g(B) + 1$, i.e., Ψ has trivial monodromy (see § 4).

A central theme of this paper is the divisibility of the fibre of a morphism, where divisibility is understood in the following sense:

Definition 1.7. Let D be a divisor on a surface S. We will say that D is divisible by an integer k if $(1/k)D \in H^2(S,\mathbf{Z})/(\text{torsion})$, that is, if there exists a cocycle $E \in H^2(S,\mathbf{Z})$ such that $D - kE$ is either a torsion element of $H^2(S,\mathbf{Z})$ or zero.

We claim that the cocycle E of Definition 1.7 is also algebraic. The exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0$$

yields the exact piece

$$H^1\mathcal{O}_S^* \xrightarrow{\sigma} H^2(S,\mathbf{Z}) \xrightarrow{\tau} H^2\mathcal{O}_S \qquad (*)$$

Obviously the torsion of $H^2(S,\mathbf{Z})$ lies in the kernel of τ, and so it is algebraic. Now, if D is a divisor and $D - kE \in tor\ H^2(S,\mathbf{Z})$ then $kE \in Im(\sigma)$. But

$H^2(S,\mathbb{Z})/Im(\sigma)$ is embedded in the $\mathbb{C}$-vector space $H^2\mathcal{O}_S$, which has no torsion. Hence $E \in Im(\sigma)$.

Let $Num(S)$ denote the group of numerical equivalence classes of divisors on S. By the sequence (*) above and the Algebraic Index Theorem we see that $Num(S)$ coincides with the image of $H^1\mathcal{O}_S^* \simeq Pic(S)$ in $H^2(S,\mathbb{Z})/(\text{torsion})$. A rephrasing of the previous considerations gives:

Lemma 1.8. Let D be a divisor on S, and $k \in \mathbb{Z}$. Then D is divisible by k in $H^2(S,\mathbb{Z})/(\text{torsion})$ (respectively, in $H^2(S,\mathbb{Z})$) if and only if it is divisible by k in $Num(S)$ (resp., in $Pic(S)$). □

Let $\varphi : S \to C$ be a fibration, and F a general fibre of φ. We are interested in knowing what rational multiples of F are still elements of $Num(S)$. Let $\nu := \max\{k \in \mathbb{Z} \mid k \text{ divides } F \text{ in } Num(S)\}$. It is easy to check that $\lambda F \in Num(S) (\lambda \in \mathbb{Q})$ if and only if $\lambda \in (1/\nu)\mathbb{Z}$, and thus it suffices to determine ν.

On the easy side we have:

Proposition 1.9. Let $m_1 D_1, \dots, m_t D_t$ be the multiple fibres of φ. If μ denotes the least common multiple of $\{m_1, \dots, m_t\}$, then μ divides F in $Num(S)$.

Proof: Since $\mu/m_1, \dots, \mu/m_t$ are relatively prime, there exist integers $\lambda_1, \dots, \lambda_t$ such that $\sum_{i=1}^t (\lambda_i \mu/m_i) = 1$. Moreover, $(1/m_i)F \equiv D_i \in Num(S)$, and we have $(1/\mu)F \equiv \sum_{i=1}^t \lambda_i D_i \in Num(S)$. □

The following result gives a converse of Proposition 1.9 for "most" elliptic fibrations, namely, the ones with $\chi\mathcal{O}_S > 0$. The remaining ones fall into the class of quasi-bundles (Lemma 1.5) and will be the object of § 4.

Theorem 1.10. Let $\varphi : S \to C$ be a relatively minimal elliptic fibration with $\chi\mathcal{O}_S > 0$. Let F be any fibre and μ the least common multiple of the multiplicities of the fibres. Then any integer k which divides F in $Num(S)$ is a divisor of μ.

Proof: Let L be a divisor in the numerical class $(1/k)F$. Riemann–Roch yields $\chi\mathcal{O}_S(L) = \chi\mathcal{O}_S > 0$, so that either $h^\circ\mathcal{O}_S(L) \neq 0$ or $h^\circ\mathcal{O}_S(K_S - L) \neq 0$. From the canonical bundle formula one has $K_S \equiv (\chi\mathcal{O}_S - 2\chi\mathcal{O}_C + \sum_{i=1}^t (m_i - 1)/m_i) F$, where

$\{m_1, \ldots, m_t\}$ are the multiplicities of the multiple fibres of φ. Hence $K \equiv (a/\mu)F$ for some $a \in \mathbb{Z}$. We will now show that if L is effective then $\mu/k \in \mathbb{Z}$. By the previous considerations, the same argument works in case $K - L$ is effective. Let $B = \sum_i d_i B_i$ be a connected component of L whose support is contained in a fibre $F = \sum_i e_i B_i$. If we show that μB is an integral multiple of F then we are done. Since $L^2 = 0$ we get $B^2 = 0$. By ([2], VIII. 4) this implies that $B \equiv (p/q)F$ with p, q relatively prime integers. Hence $\sum_i (q\,d_i - p\,e_i)\,B_i \equiv 0$, and since the $B_i's$ are linearly independant in $(Num(S)) \otimes_{\mathbb{Z}} \mathbb{Q}$ we obtain that $q\,d_i - p\,e_i = 0$ for all i. Thus q divides all the $e_i's$, so that q divides μ as well. $\square$

§ 2. FIBRE BUNDLES

As explained in the Introduction, the main object of this paper is to establish a relationship between the divisibility of the fibre of a quasi–bundle and the torsion of the integral homology of the surface. In this section we will restrict ourselves to fibre bundles.

Given a fibre bundle $\varphi : S \to C$ with fibre F, we can consider Serre's spectral sequence

$$E^2_{p,q} = H_p(C, H_q(F, \mathbb{Z})) \Longrightarrow H_{p+q}(S, \mathbb{Z})$$

where $H_p(C, H_q(F, \mathbb{Z}))$ denotes the p^{th} homology group of C with coefficients in the bundle of abelian groups $\{H_q(\varphi^{-1}(t), \mathbb{Z})\}_{t \in C}$. (see e.g. [15] or [10]). $H_i(F, \mathbb{Z})$ is a $\pi_1(C)$–module by the action of the monodromy, where $\pi_1(C)$ denotes the fundamental group of C. $H_\circ(C, H_1(F, \mathbb{Z}))$ will be denoted $H_1(F)_{\pi_1(C)}$, and by ([15], VI 3.2) it is computed as

$$H_1(F)_{\pi_1(C)} = H_1(F, \mathbb{Z}) / \langle \gamma a - a \,|\, a \in H_1(F, \mathbb{Z}),\ \gamma \in \pi_1(C) \rangle$$

Inasmuch as the map φ is analytically locally trivial ([1], I 10.1), it follows that the action of $\pi_1(C)$ over $H_1(F, \mathbb{Z})$ factors through the action of the group of analytic automorphisms of F, denoted $Aut(F)$. In general, if G is a finite group of order $|G|$, and M a G–module with invariants M^G, then $|G|$ annihilates the kernel of the norm map $N : M_G \to M^G$, defined as $N(x) = \sum_{\gamma \in G} \gamma \cdot x$, where $M_G = M/\langle \gamma a - a \,|\, a \in M,\ \ \gamma \in G \rangle$. If moreover M is a torsion–free abelian group then $|G|$ annihilates the $\mathbb{Z}$–torsion of M_G as well. In our situation we way conclude that the torsion of $H_1(F)_{\pi_1(C)}$ is killed by the order of $Aut(F)$.

The first terms of Serre's spectral sequence yield an exact sequence

$$H_2(S,\mathbb{Z}) \xrightarrow{\varphi_*} H_2(C,\mathbb{Z}) \to H_1(F)_{\pi_1(C)} \to H_1(S,\mathbb{Z}) \xrightarrow{\varphi_*} H_1(C,\mathbb{Z}) \to 0$$

where the homomorphisms φ_* are induced by φ ([10], Thm. 5.8).

The divisibility of F in $H^2(S,\mathbb{Z})/(\text{torsion})$ can be read off from this sequence as follows.

Proposition 2.1. Let $\varphi : S \to C$ be any fibration. Fix an isomorphism $H_2(C,\mathbb{Z}) \simeq \mathbb{Z}$ and assume, with this identification, that $d\mathbb{Z}$ is the image of $\varphi_* : H_2(S,\mathbb{Z}) \to H_2(C,\mathbb{Z}) \simeq \mathbb{Z}$, where d is a positive integer. Then d is the largest integer dividing a fibre of φ in $H^2(S,\mathbb{Z})/(\text{torsion})$.

Proof: Poincaré duality yields isomorphisms

$$\lambda_S : H^i(S,\mathbb{Z}) \xrightarrow{\sim} H_{4-i}(S,\mathbb{Z}),\ \lambda_C : H^i(C,\mathbb{Z}) \xrightarrow{\sim} H_{2-i}(C,\mathbb{Z}).$$

Define $\varphi_! = \lambda_C \circ \varphi_* \circ \lambda_S^{-1}$. Let $p \in H^2(C,\mathbb{Z})$ denote the class of a point, and $F = \varphi^*(p) \in H^2(S,\mathbb{Z})$ the class of a fibre. For $D \in H^2(S,\mathbb{Z})$, the projection formula for cup product ([1], page 11) says.

$$\varphi_!(D \cdot F) = \varphi_!(D \cdot \varphi^*(p)) = \varphi_!(D) \cdot p$$

Multiplication by p defines an isomorphism $H^\circ(C,\mathbb{Z}) \simeq H^2(C,\mathbb{Z}) \simeq \mathbb{Z}$. Moreover, $\varphi_! : H^4(S,\mathbb{Z}) \to H^2(C,\mathbb{Z})$ is an isomorphism too. If we fix the isomorphism $H^4(S,\mathbb{Z}) \simeq \mathbb{Z}$ then the hypothesis imply that $d\mathbb{Z} = \{D \cdot F \,|\, D \in H^2(S,\mathbb{Z})\}$. Applying Poincaré duality as stated in ([8], page 53) one concludes that d is the largest integer dividing F in $H^2(S,\mathbb{Z})/(\text{torsion})$. □

Combining all the above we get:

Theorem 2.2. Let $\varphi : S \to C$ be a fibre bundle, and d the largest integer dividing a fibre F in $H^2(S,\mathbb{Z})/(\text{torsion})$. Then there is an exact sequence

$$0 \to \mathbb{Z}_d \to H_1(F)_{\pi_1(C)} \to H_1(S,\mathbb{Z}) \xrightarrow{\varphi_*} H_1(C,\mathbb{Z}) \to 0$$

In particular $torH_1(S,\mathbb{Z}) \simeq (torH_1(F)_{\pi_1(C)})/\mathbb{Z}_d$, and $torH_1(F)_{\pi_1(C)}$ is annihilated by the order of $Aut(F)$. □

Hopf surfaces provide examples of non–algebraic elliptic fibre bundles whose fibre is homologous to zero (see e.g. [1]). However, here we are only dealing with the algebraic case. At this point one might wonder whether the fibre of an algebraic fibre bundle can be actually divisible by an integer greater than 1, inasmuch as there is no multiple fibre to account for such a phenomenon. The following example shows that this is possible though. Given any prime number p, our aim is to construct a fibre bundle whose fibre is divisible by p. For $p = 2$ some bielliptic surfaces provide the examples sought ([11]), so let us assume $p > 2$.

Example 2.3. Pick any prime number $p > 2$ and any integer $d > 0$ divisible by p. Write $D_1 := \mathbf{P}^1$, and take two Galois finite covers $\rho : D_2 \to D_1$, $\sigma : D_3 \to D_1$ with groups $\mathbf{Z}_d$ and $\mathbf{Z}_p$ respectively, such that $D_2 \simeq \mathbf{P}^1$ and the branch loci $B_\rho, B_\sigma \subseteq D_1$ of ρ and σ are disjoint. The map ρ has exactly two branch points, both with multiplicity d. Let $F := D_2 \times_{D_1} D_3$. We claim that F is an irreducible and smooth curve. Since smoothness of a morphism is preserved by base change we have that

$$X := \left[D_2 - \rho^{-1}(B_\rho)\right] \times_{D_1 - B_\rho} \left[D_3 - \sigma^{-1}(B_\rho)\right] \to D_3 - \sigma^{-1}(B_\rho)$$

$$Y := \left[D_2 - \rho^{-1}(B_\sigma)\right] \times_{D_1 - B_\sigma} \left[D_3 - \sigma^{-1}(B_\sigma)\right] \to D_2 - \rho^{-1}(B_\sigma)$$

are smooth maps. Thus X and Y are both smooth, and $F = X \cup Y$ is smooth as well. On the other hand, F is the preimage of the diagonal divisor by the map $(\rho, \sigma) : D_2 \times D_3 \to D_1 \times D_1 \simeq \mathbf{P}^1 \times \mathbf{P}^1$, so that it is ample and thus connected, as desired. An easy calculation shows that $g(F) = d\big(g(D_3)+p-1\big)-p+1$. Set $G := \mathbf{Z}_d \times \mathbf{Z}_p$. G is acting of F in a way compatible with the actions of $\mathbf{Z}_d$, $\mathbf{Z}_p$ on D_2, D_3 respectively. Consider $\mathbf{Z}_d$ embedded in G as $\mathbf{Z}_d \times \{0\}$; likewise with $\mathbf{Z}_p$. We have a commutative diagram

$$\begin{array}{ccc} F/\mathbf{Z}_p & \longrightarrow & F/G \\ \downarrow\wr & & \downarrow\wr \\ D_2 & \xrightarrow{\rho} & D_1 \end{array}$$

Since the branch points of ρ and σ have multiplicities d and p respectively, the condition $B_\rho \cap B_\sigma = \emptyset$ implies that the only multiplicities of $F \to F/G$ are d and p. Now let E be any elliptic curve, and let G act on E by translations, so that E/G is elliptic too. Denote $S := (E \times F)/G$, where G is acting on $E \times F$ componentwise, and let $\Phi : S \to (E/G)$, $\Psi : S \to (F/G)$ be the two natural projections. All fibres of Φ are isomorphic to F; in particular, Φ is a fibre bundle map. Likewise, Ψ is an elliptic quasi bundle with smooth fibres all isomorphic to E and singular fibres of multiplicities d

and p. Let us denote by $F_\circ$ (respectively, $E_\circ$) the class in $H^2(S,\mathbf{Z})/(\text{torsion})$ of a fibre of Φ (resp., of Ψ). Applying Lemma 1.5 to Ψ one has in turn $e(S)=0$, $K_S^2=0$ and $\chi\mathcal{O}_S=0$, where $e(S)$ stands for the topological Euler characteristic of S. Furthermore:

$$H_1(S,\mathbf{Q}) = H_1(E\times F,\mathbf{Q})^G = H_1(E,\mathbf{Q})^G \times H_1(F,\mathbf{Q})^G$$
$$= H_1(E/G,\mathbf{Q}) \times H_1(F/G,\mathbf{Q}) = H_1(E/G,\mathbf{Q}).$$

We get $2q(S) = dim_{\mathbf{Q}} H^1(S,\mathbf{Q}) = 2$. Combining with $\chi\mathcal{O}_S = 0$ one has $p_g(S)=0$. Finally, from $e(S)=0$ it follows that $H^2(S,\mathbf{Z})/(\text{torsion})$ is of rank 2 and isomorphic to $Num(S)$.

Now we are ready for the statement:

Lemma 2.4. With notations as in the preceding construction, $\{(1/d)E_\circ,\ (1/p)F_\circ\}$ is a basis of $H^2(S,\mathbf{Z})/(\text{torsion})$. In particular, the fibre bundle $\Phi : S \to (E/G)$ has fibres divisible by p in $H^2(S,\mathbf{Z})/(\text{torsion})$.

Proof: Since $\Psi : S \to (F/G) \simeq \mathbf{P}^1$ is not an Albanese fibration we can apply Theorem 4.1 and the fact that p divides d to conclude that d is the largest integer dividing $E_\circ$ in $Num(S)$. By Poincaré duality ([8], page 53), there exists $A \in Num(S)$ with the property that $(1/d)E_\circ A = 1$. Notice that $E_\circ^2 = F_\circ^2 = 0$, $E_\circ F_\circ = dp$. Since $\{E_\circ, F_\circ\}$ is a basis of $H^2(S,\mathbf{Q})$, we can write $A \equiv r\,E_\circ + s\,F_\circ$, $r,s \in \mathbf{Q}$. By the above we have $s = 1/p$. Set $a := A^2 \in \mathbf{Z}$, $b := AF_\circ \in \mathbf{Z}$. From $b/pd = r = a/2d$ we get $ap = 2b$, so that $a = 2a'$, $a' \in \mathbf{Z}$. We have got $A \equiv (a'/d)E_\circ + (1/p)F_\circ$, and by subtracting a' times $(1/d)E_\circ \in Num(S)$ it follows $(1/p)F_\circ \in Num(S)$. Finally, every $L \in Num(S)$ is an integral linear combination of $(1/d)E_\circ$ and $(1/p)F_\circ$, because the intersection product with each one of them is an integer. □

As a conclusion, we can state something a bit stronger:

Proposition 2.5. For each prime number p, there exists a fibre bundle $\varphi : S \to C$ whose fibre is divisible by p in $Pic(S)$.

Proof: For any primer number p we have shown the existence of a fibre bundle $\varphi : S \to C$ with $g(C)=1$, $\chi\mathcal{O}_S = 0$ and fibre divisible by p in $H^2(S,\mathbf{Z})/(\text{torsion})$. A fibre F of φ is linearly equivalent to $pL+M$, for some divisors L, M. If the class of

M in $H^2(S,\mathbb{Z})$ is zero then we are done. Otherwise, M is a torsion element of order k in $H^2(S,\mathbb{Z})$. We may assume $kM \sim 0$, $iM \not\sim 0$ for all $i = 1,\ldots,k-1$. Let $\pi : R \to S$ be the $k^{\underline{th}}$ cyclic étale covering of S determined by the "equation" $kM \sim 0$ ([1], I, 17). One has $\pi^*(M) \sim 0$. From $\pi_*\mathcal{O}_R = \bigoplus_{i=0}^{k-1} \mathcal{O}_S(-iM)$ we get

$$\pi_*\pi^*\mathcal{O}_S(F) = \mathcal{O}_S(F) \otimes \pi_*\mathcal{O}_R = \bigoplus_{i=0}^{k-1} \mathcal{O}_S(F - iM).$$

If $1 \leq i \leq k-1$ one has $H^\circ\mathcal{O}_S(F-iM) = 0$, because all fibres of φ are homologically equivalent and smooth. Therefore $h^\circ(\pi^*\mathcal{O}_S(F)) = h^\circ\mathcal{O}_S(F) = 1$, which implies that $\pi^{-1}(F)$ is connected. Hence $\Psi := \varphi \circ \pi : R \to C$ is a fibration. We have $\chi\mathcal{O}_R = k\chi\mathcal{O}_S = 0$. Recalling that $g(C) = 1$, Lemma 1.3 yields $deg(\Psi_*\omega_{R/C}) = 0$, which implies that Ψ is a fibre bundle map ([1], III 18.2). Finally, a fibre of Ψ is of class $\pi^*(F) \sim \pi^*(pL + M) \sim p\pi^*(L)$, hence divisible by p in $Pic(R)$. □

§ 3. QUASI–BUNDLES

Let $\varphi : S \to C$ be any fibration, $m_1 D_1, \ldots, m_t D_t$ its multiple fibres. We define the group $G(\varphi)$ to be

$$G(\varphi) := Coker\Big(\mathbb{Z} \longrightarrow \bigoplus_{i=1}^{t} \mathbb{Z}_{m_i}\Big)$$
$$1 \longmapsto (\bar{1}, \ldots, \bar{1}).$$

Let F be any smooth fibre. With this generality, the following has been proved in [12]:

Theorem 3.1. There exists an exact sequence

$$H_1(F,\mathbb{Z}) \to H_1(S,\mathbb{Z}) \to H_1(C,\mathbb{Z}) \times G(\varphi) \to 0$$

induced by φ and the inclusion $F \subseteq S$. In particular, if J denotes the image of $H_1(F,\mathbb{Z})$ in $H_1(S,\mathbb{Z})$ one has an exact sequence

$$0 \to tor\ (J) \to tor\ H_1(S,\mathbb{Z}) \to G(\varphi).$$

Furthermore, $tor H_1(S,\mathbb{Z}) \to G(\varphi)$ is surjective provided that $q(S) = g(C)$. □

Remark 3.2. It is proven in ([4], 1.39) that $J = 0$ whenever $\varphi : S \to C$ is an elliptic fibration and $\chi\mathcal{O}_S > 0$.

In this Section we are going to develop the main theme of this paper, namely divisibility of fibres versus torsion of homology, for general quasi–bundles. For this purpose we are going to perform a suitable base change on our fibration which will lead us to a fibre bundle. This will allow us to apply the results of the preceding Section. The main tool, then, is the following

Proposition 3.3. Let $\varphi : S \to C$ be any fibration, and let $m_1 D_1, \ldots, m_t D_t$ be some multiple fibres of φ with D_i smooth for all i. Let μ be a common multiple of $m_1, \ldots, m_t$, and choose any integer $e \geq 1$ such that μ divides $t + e$. Let $\varphi(D_i) = P_i \in C$, $i = 1, \ldots, t$, and pick points $P_{t+1}, \ldots, P_{t+e} \in C$ whose fibres by φ are smooth. Furthermore, let $\mathcal{L}$ be any line bundle on C satisfying $\mathcal{L}^{\otimes\mu} \simeq \mathcal{O}_C(P_1 + \ldots + P_{t+e})$. $\mathcal{L}$ defines a cyclic covering $\varepsilon : B \to C$ of degree μ, totally ramified at $P_1, \ldots, P_{t+e}$. Finally, denote by R the normalization of $S \times_C B$. Then, with these conditions, R is a smooth surface and $q(R) - g(B) = q(S) - g(C)$.

Proof: R is smooth by ([1], III 9.1), and $\pi : R \to S$ is a Galois map with group $G \simeq \mathbb{Z}\mu$, i.e. $S = R/G$. Let $K(R)$ and $K(S)$ be the corresponding fields of rational functions. The long cohomology sequence determined by the G–invariants of

$$0 \to \mathbb{C}^* \to K(R)^* \to K(R)^*/\mathbb{C}^* \to 0$$

yields the piece

$$H^1(G, K(R)^*) \to H^1(G, K(R)^*/\mathbb{C}^*) \to H^2(G, \mathbb{C}^*).$$

The middle term vanishes because $H^1(G, K(R)^*) = 0$ by Hilbert's Theorem 90 (e.g. [9]) and $H^2(G, \mathbb{C}^*) = \mathbb{C}^*/\mu \cdot \mathbb{C}^* = 0$ ([3], page 58). Consider now the G–invariants of

$$0 \to K(R)^*/\mathbb{C}^* \to Div(R) \to Pic(R) \to 0$$

One gets

$$(Div(R))^G \to (Pic(R))^G \to H^1(G, K(R)^*/\mathbb{C}^*) = 0.$$

Hence we have a commutative diagram with exact rows

$$\begin{array}{ccccc} Div(S) & \longrightarrow & Pic(S) & \longrightarrow & 0 \\ \sigma\downarrow & & \tau\downarrow & & \\ (Div(R))^G & \longrightarrow & (Pic(R))^G & \longrightarrow & 0 \end{array}$$

whose vertical maps σ, τ are induced by π. The group $(Div(R))^G$ is generated by divisors of the form $D = \sum_{i=1}^{n} D_i$ where $\{D_1, \dots, D_n\}$ is permuted by the action of G and has no invariant subset. For a given D_i, if $\pi(D_i) = E$ then $\sigma(E) = d \cdot D$, where d is the number of elements in G inducing the identity on D_i. In particular, $\mu \cdot D \in Im(\sigma)$. We conclude that μ annihilates the cokernel of π^*.

The exponential sequences for S and R give rise to a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Pic^\circ(S) & \longrightarrow & Pic(S) & \longrightarrow & H^2(S,\mathbb{Z}) \\ & & \pi^* \big\downarrow & & \tau \big\downarrow & & \lambda \big\downarrow \\ 0 & \longrightarrow & (Pic^\circ(R))^G & \longrightarrow & (Pic(R))^G & \longrightarrow & (H^2(S,\mathbb{Z}))^G \end{array}$$

From ([5], Thm. 2.1) it follows that $Ker(\lambda)$ is annihilated by μ (see proof of Prop. 3.5 below). This fact, combined with the previous considerations, imply that μ^2 annihilates the cokernel of $\pi^* : Pic^\circ(S) \to (Pic^\circ(R))^G$.

Let $F_\circ$ the fibre of φ over P_{t+1}. By construction, π is totally ramified over $F_\circ$, so that $\pi^{-1}(F_\circ) \simeq F_\circ$. Identifying $\pi^{-1}(F_\circ)$ and $F_\circ$ we may consider $h : F_\circ \hookrightarrow R$, $j : F_\circ \hookrightarrow S$ to be the corresponding inclusions, which determine a commutative diagram

$$\begin{array}{ccc} Pic^\circ(S) & \xrightarrow{\pi^*} & Pic^\circ(R) \\ & {}_{j^*}\searrow \quad \swarrow_{h^*} & \\ & Pic^\circ(F_\circ) & \end{array}$$

Pick any $L \in Pic^\circ(R)$ and write $N := \sum_{\gamma \in G} \gamma \cdot L$. We have $N \in (Pic^\circ(R))^G$ and, by the preceding arguments, we have, $\mu^2 \cdot N \in Im(\pi^*)$ so that $h^*(\mu^2 \cdot N) \in Im(j^*)$. But $h^*(\mu^2 \cdot N) = \mu^2 \cdot \sum_{\gamma \in G} \gamma \cdot h^*(L) = \mu^3 \cdot h^*(L)$ since G acts trivially on $\pi^{-1}(F_\circ)$. As a consequence, one gets the following inclusions of abelian varieties:

$$\mu^3(Im(h^*)) \subseteq Im(j^*) \subseteq Im(h^*),$$

which in view of Proposition 1.4 yields the desired equality:

$$q(S) - g(C) = dim\, Im(j^*) = dim\, Im(h^*) = q(R) - g(B). \qquad \square$$

We point out that we map $\varepsilon : B \to C$ of Proposition 3.3 was chosen to ramify at one point other than $P_1, \dots, P_t$ in order to get the equality $q(R) - g(B) = q(S) - g(C)$. Without this precaution, only the inequality $\geq$ holds in general.

We shall apply the results of § 2 by means of the following

Construction 3.4. Let $\varphi : S \to B$ be a quasi–bundle with multiple fibres $m_1 D_1, \ldots, m_t D_t$, and let μ denote the least common multiple of $m_1, \ldots, m_t$. Do as in Proposition 3.3 with the fibres $m_1 D_1, \ldots, m_t D_t$ in order to get a fibration $\Psi : R \to B$ and finite Galois maps $\pi : R \to S$, $\varepsilon : B \to C$, both with group $\mathbb{Z}_\mu$, making commutative the following diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi} & S \\ \Psi \downarrow & & \downarrow \varphi \\ B & \xrightarrow{\varepsilon} & C \end{array}$$

Recall that $q(R) - g(B) = q(S) - g(C)$. One sees that Ψ has no singular fibre ([1], III 9.1), and thus it is a fibre bundle map. Denote by F a smooth fibre of φ, and by $\tilde{F}$ a connected component of $\pi^{-1}(F)$. $\tilde{F}$ is a fibre of Ψ and the restriction $\tilde{F} \to F$ of π is an isomorphism. Let $i : \tilde{F} \to R$, $j : F \to S$ be the two inclusions, and set

$$I := \text{ image of } i_* : H_1(\tilde{F}, \mathbb{Z}) \to H_1(R, \mathbb{Z})$$

$$J := \text{ image of } j_* : H_1(F, \mathbb{Z}) \to H_1(S, \mathbb{Z})$$

By Theorem 3.1 one has exact sequences

$$0 \to I \to H_1(R, \mathbb{Z}) \to H_1(B, \mathbb{Z}) \to 0$$

$$0 \to J \to H_1(S, \mathbb{Z}) \to H_1(C, \mathbb{Z}) \times G(\varphi) \to 0$$

From $\pi \circ i = j \circ (\pi_{|\tilde{F}})$ we deduce that J is a quotient of I. Moreover

$$(\text{rank of } I) = q(R) - g(B) = q(S) - g(C) = (\text{rank of } J)$$

As a consequence, the torsion of J is also a quotient of the torsion of I. This latter fact is one of the crucial points of our construction. □

With the above notation, we now want to compare the divisibility of F with that of $\tilde{F}$.

Proposition 3.5.

(i) If F is divisible by $d\mu$ in $H^2(S,\mathbf{Z})/(\text{torsion})$, $d \in \mathbf{Z}$, then $\tilde{F}$ is divisible by d in $H^2(R,\mathbf{Z})/(\text{torsion})$.

(ii) Conversely, if $\tilde{F}$ is divisible by $d \in \mathbf{Z}$ in $H^2(R,\mathbf{Z})/(\text{torsion})$ then so is F in $H^2(S,\mathbf{Z})/(\text{torsion})$.

Proof: If $F \equiv d\mu M$ then $\mu\tilde{F} \equiv \pi^*(F) \equiv d\mu\pi^*(M)$, so that $\tilde{F} \equiv d\pi^*(M)$. This proves (i). Consider the homomorphism $\pi^* : H^2(S,\mathbf{Z}) \to H^2(R,\mathbf{Z})$. Since $S = R/\mathbf{Z}\mu$, there exists a homomorphism $\delta : H^2(R,\mathbf{Z}) \to H^2(S,\mathbf{Z})$ satisfying $(\delta \circ \pi^*)(x) = \mu \cdot x$ for all $x \in H^2(S,\mathbf{Z})$ ([5], Thm. 2.1). In particular, if $\tilde{F} \equiv dL$ then $\pi^*(F) \equiv \mu dL$ and $\mu F \equiv (\delta \circ \pi^*)(F) \equiv \mu d \cdot \delta(L)$. Thus $F \equiv d \cdot \delta(L)$. □

We reach at last the conclusion we were seeking:

Theorem 3.6. Let $\varphi : S \to C$ be a quasi–bundle with multiple fibres $m_1 D_1, \ldots, m_t D_t$, and denote by μ the least common multiple of $m_1, \ldots, m_t$. Construction 3.4 gives rise to a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi} & S \\ \Psi\big\downarrow & & \big\downarrow\varphi \\ B & \xrightarrow{\varepsilon} & C \end{array}$$

where ε is a finite map of degree μ, R is the normalization of $S \times_C B$, R is smooth and Ψ is a fibre bundle whose general fibre we call $\tilde{F}$. Suppose $d\mu$ is the largest integer dividing a fibre F of φ in $H^2(S,\mathbf{Z})/(\text{torsion})$ (d a positive integer).

Then there exists an integer $a \geq 1$ and an exact sequence

$$0 \to \mathbf{Z}_{ad} \to tor\big(H_1(\tilde{F})_{\pi_1(B)}\big) \to tor\ H_1(S,\mathbf{Z}) \to G(\varphi)$$

Moreover, $tor\, H_1(S,\mathbf{Z}) \to G(\varphi)$ is surjective provided that $q(S) = g(C)$. If $g(F) \geq 2$, then both μ and d are divisors of the order of $Aut(F)$.

Proof: From Proposition 3.5 one knows that the general fibre $\tilde{F}$ of Ψ is divisible by d in $H^2(R,\mathbf{Z})/(\text{torsion})$. Let ad be the largest integer dividing $\tilde{F}$. We may assume $\pi(\tilde{F}) = F$. Recall the definitions of I and J given in Construction 3.4. Theorem 2.2 yields an exact sequence

$$0 \to \mathbf{Z}_{ad} \to tor\, H_1(\tilde{F})_{\pi_1(B)} \to tor\,(I) \to 0$$

On the other hand, from Theorem 3.1 we get

$$0 \to tor(J) \to tor\, H_1(S, \mathbb{Z}) \to G(\varphi)$$

whose right–hand side map is surjective if $q(S) = g(C)$. Since $tor(J)$ is a quotient of $tor(I)$, we can link these two sequences and get the one of the Theorem. Furthermore, d divides the order of $Aut(F)$ by the last sentence of Theorem 2.2. Notice also that if mD is a multiple fibre of φ then F is an unramified Galois cover of D with group $\mathbb{Z}_m$. In particular, m divides the order of $Aut(F)$, and so does μ. $\square$

It is not easy to give explicit instances of quasi–bundles with fibre divisible by an integer larger than the least common multiple of the multiplicities of the fibres. Besides the fibre bundles of Proposition 2.5 we have the following:

Example 3.7. We are going to construct quasi–bundle fibrations with all singular fibres of multiplicity 3 but fibre divisible by 9 in cohomology (modulo torsion). Let B be the elliptic curve $\mathbb{C}/(\mathbb{Z}\cdot\rho\oplus\mathbb{Z})$, where $\rho := (-1/2)+(\sqrt{3}/2)\,i$. Pick any other elliptic curve A. The group $G := \mathbb{Z}_3\times\mathbb{Z}_3$ acts on B as $(\overline{1,0})\cdot x = \rho\cdot x$, $(\overline{0,1})\cdot x = x+(1-\rho)/3$. Let G act on A by translations, and set $S := (A\times B)/G$, where the action on $A\times B$ is componentwise. Denote by $A_\circ$ (respectively, $B_\circ$) a general fibre of the projection $\sigma : S \to (B/G) \simeq \mathbf{P}^1$ (resp., of $\tau : S \to (A/G)$). S is a bielliptic surface ([2]), τ is an elliptic fibre bundle and σ is an elliptic quasi–bundle with three triple fibres, and smooth away from them. Note that $K_S \equiv 0$ and $A_\circ B_\circ = 9$. One can show, as in Lemma 2.4, that $\{(1/3)A_\circ; (1/3)B_\circ\}$ is a basis of $H^2(S,\mathbb{Z})/(\text{torsion})$. Since the fibre of the elliptic fibre bundle τ is divisible by 3, we get from Theorem 4.3 that $H^2(S,\mathbb{Z})$ is torsion–free. It follows that if $B'_\circ$ is a particular fibre of τ then there exists a divisor D such that $3D \sim B'_\circ$. This determines a 3–cyclic covering $\pi : R \to S$ ramified at $B'_\circ$. I claim that $\pi^{-1}(B_\circ)$ is connected. Since $\pi^{-1}(B_\circ) \to B_\circ$ is the 3–cyclic covering defined by the "equation" $3D_{|B_\circ} \sim 0$, it suffices to show that $D_{|B_\circ}$ is not linearly equivalent to zero. For this purpose, note that $0 = h^\circ\mathcal{O}_S(D) = h^\circ\mathcal{O}_S(D-B_\circ) = h^2\mathcal{O}_S(D-B_0)$. By Riemann–Roch $\chi\mathcal{O}_S(D-B_\circ) = 0$, so that $h^1\mathcal{O}_S(D-B_\circ) = 0$. Now the sequence

$$H^\circ\mathcal{O}_S(D) \to H^\circ\mathcal{O}_{B_\circ}(D) \to H^1\mathcal{O}_S(D-B_\circ)$$

yields the vanishing of the middle term, as desired. Set $\Phi := \sigma \circ \pi : R \to (B/G)$, $\Psi := \tau \circ \pi : R \to (A/G)$, and denote by F (respectively, E) the general fibre of Φ (resp., of Ψ). Ψ is an elliptic quasi-bundle with only one singular fibre (of multiplicity 3) over $\tau(B'_\circ)$, and Φ is a quasi-bundle of fibre genus 10 with exactly three triple fibres.

Claim: either E or F is divisible by 9 in $H^2(R, \mathbb{Z})/(\text{torsion})$.

Proof of the Claim: By general theory about cyclic coverings, one has $K_R = \pi^*(K_S + 2D) \equiv (2/3)E$ and $\pi_*\mathcal{O}_R = \bigoplus_{i=0}^{2} \mathcal{O}_S(-i\,D)$. Therefore $p_g(R) = \sum_{i=0}^{2} h^\circ \mathcal{O}_S(K_S + i\,D) = 0$. But $\chi\mathcal{O}_R = 0$ (Lemma 1.5), so that $q(R) = 1$. It follows that $H^2(R, \mathbb{Z})/(\text{torsion})$ is of rank 2. Note that $\{E, F\}$ is a $\mathbb{Q}$–basis of $H^2(S, \mathbb{Q})$, and $EF = 27$. Suppose F is not divisible by 9 in $H^2(R, \mathbb{Z})/(\text{torsion})$. By Poincaré duality ([8], page 53), there must exist a divisor $C \equiv (1/9)E + \alpha F$, $\alpha \in \mathbb{Q}$. From $C^2 = 6\alpha \in \mathbb{Z}$ and $(1/3)E \cdot C = 9\alpha \in \mathbb{Z}$ we get $\alpha \in (1/3)\mathbb{Z}$. Consequently $(1/9)E \in H^2(R, \mathbb{Z})/(\text{torsion})$ and the claim follows. Nevertheless, it seems hard to decide which one of the two, either E or F, is actually divisible by 9. □

§ 4. ELLIPTIC QUASI–BUNDLES

The main statements of the preceding two Sections (Theorems 2.2 and 3.6) will become most significant for elliptic quasi–bundles, since in this case one can give a rather complete description of the groups ocurring there. Some results in [4] have been profitable to me.

Theorem 4.1. Let $\varphi : S \to C$ be an elliptic quasi–bundle with $q(S) = g(C) + 1$, and let μ be the least common multiple of the multiplicities of the fibres. Then:

(i) *tor* $H_1(S, \mathbb{Z})$ is a subgroup of $G(\varphi)$.

(ii) μ is the largest integer dividing a fibre of φ in $H^2(S, \mathbb{Z})/(\text{torsion})$.

Proof: By taking ranks on the sequence of Theorem 3.1

$$H_1(F, \mathbb{Z}) \to H_1(S, \mathbb{Z}) \to H_1(C, \mathbb{Z}) \times G(\varphi) \to 0$$

we get that the left hand side map is one–to–one, and (i) follows. As for (ii), consider the fibre bundle $\Psi : R \to B$ associated to φ by Construction 3.4. If F is divisible by $d\,\mu$ in $H^2(S, \mathbb{Z})/(\text{torsion})$, then a fibre $\tilde{F}$ of Ψ is divisible by d in

$H^2(R,\mathbb{Z})/(\text{torsion})$ (Proposition 3.5). Hence there is an exact sequence

$$0 \to \mathbb{Z}_{ad} \to H_1(\tilde{F})_{\pi_1(B)} \to H_1(R,\mathbb{Z}) \to H_1(B,\mathbb{Z}) \to 0$$

for some $a \in \mathbb{Z}$ (Theorem 2.2). But $q(R)-g(B) = q(S)-g(C) = 1$ and $H_1(\tilde{F})_{\pi_1(B)}$ is a quotient of $H_1(\tilde{F},\mathbb{Z}) \simeq \mathbb{Z}^2$. By rank considerations it follows that $\mathbb{Z}_{ad} = 0$, i.e. $d = 1$. □

Next we turn our attention to elliptic fibre bundles $\varphi : S \to C$ with $q(S) = g(C)$, i.e. with non–trivial monodromy. A fibre F of φ can be written as $\mathbb{C}/(\mathbb{Z}\cdot\omega \oplus \mathbb{Z})$, where $\omega \in \mathbb{C}$ is uniquely determined if its real and imaginary parts satisfy the relations $-1/2 \le Re(\omega) < 1/2$, $Im(\omega) > 0$, $|\omega| \ge 1$ if $Re(\omega) \le 0$, $|\omega| > 1$ if $Re(\omega) > 0$. We are seeking an explicit description of $H_1(F)_{\pi_1(C)}$, which is now a torsion group by reason of Theorem 2.2.

Proposition 4.2. Given an elliptic fibre bundle $\varphi : S \to C$ with $q(S) = g(C)$, the table below is showing the only possible values for $H_1(F)_{\pi_1(C)}$:

ω	$H_1(F)_{\pi_1(C)}$
i	$\mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_2$
$(-1/2) + (\sqrt{3}/2)\,i$	$\mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_3$; 0
otherwise	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Proof: The homomorphism $\pi_1(C) \to Aut\, H_1(F,\mathbb{Z})$ is induced by the monodromy $\pi_1(C) \to Aut\,(F)$. Every automorphism of F is the composite of a translation and multiplication by a complex number which maps the lattice $\mathbb{Z}\cdot\omega \oplus \mathbb{Z}$ into itself. A translation is homotopic to the identity, and thus induces the identity map in homology. Consequently we may restrict our attention to automorphisms of F defined as multiplication by some $\varepsilon \in \mathbb{C}$. Identifying $H_1(F,\mathbb{Z}) \simeq \mathbb{Z}\cdot\omega \oplus \mathbb{Z}$ one sees that the induced automorphism on $H_1(F,\mathbb{Z})$ is also multiplication by ε. We have $\varepsilon\cdot\omega = a\,\omega + b$, $\varepsilon \cdot 1 = c\,\omega + d$ with $a,b,c,d \in \mathbb{Z}$. Hence $\omega = (a\,\omega + b)/(c\,\omega + d)$, so that

$$(*) \qquad c\,\omega^2 + (d-a)\,\omega - b = 0$$

In the ordered basis $\{\omega, 1\}$ of $\mathbb{Z}\cdot\omega \oplus \mathbb{Z}$, the morphism is given by the matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ with determinant ± 1. The discriminant of (*) must be strictly negative because $Im(\omega) \neq 0$. This rules out the possibility $ad - bc = -1$. Solving (*) and using $ad-bc=1$ one gets $\omega = \left(a-d+\sqrt{(a+d)^2-4}\right)/2c$. Hence $a+d=-1,0,1$. Combining this with the restrictions on ω imposed above leads, after some computation, to the following set of posibilities (I stands for the identity matrix):

ω	$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$
i	$\pm I$; $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$(-1/2)+(\sqrt{3}/2)\,i$	$\pm I$; $\pm \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$; $\pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
otherwise	$\pm I$

Now, recalling that

$$H_1(F)_{\pi_1(C)} = H_1(F,\mathbb{Z})/\langle \gamma t - t \,|\, \gamma \in \pi_1(C),\ t \in H_1(F,\mathbb{Z})\rangle$$

one can get the table of Proposition 4.2 by a case by case calculation. For instance, if $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ belongs to the image of $\pi_1(C) \to Aut\, H_1(F,\mathbb{Z})$, then applying $\gamma - I$ to the basis $\{(1,0),\ (0,1)\}$ of $H_1(F,\mathbb{Z}) \simeq \mathbb{Z}^2$ yields that $H_1(F)_{\pi_1(C)}$ is a quotient of $\mathbb{Z}^2/\langle(-1,1),\ (-1,-2)\rangle \simeq \mathbb{Z}_3$. The other cases are similar. □

Taking into account the preceding Proposition together with Theorems 2.2 and 3.6 one gets the two main results of this section, which are stated as follows:

Theorem 4.3. Let $\varphi: S \to C$ be an elliptic fibre bundle with $q(S) = g(C)$, and let d be the largest integer dividing a fibre F in $H^2(S,\mathbb{Z})/(\text{torsion})$. Write $F = \mathbb{C}/(\mathbb{Z}\cdot\omega \oplus \mathbb{Z})$ as before. Then the only possibilities for d and the torsion of $H_1(F,\mathbb{Z})$ are presented in the tables below:

(1) If $\omega \neq i,\ (-1/2)+(\sqrt{3}/2)\,i$, then

d	$tor\, H_1(S,\mathbb{Z})$
1	$\mathbb{Z}_2 \times \mathbb{Z}_2$
2	$\mathbb{Z}_2$

(2) If $\omega = i$, then to the cases listed in (1) we must add

d	$tor\, H_1(S,\mathbb{Z})$
1	$\mathbb{Z}_2$
2	0

(3) if $\omega = (-1/2) + (\sqrt{3}/2)\,i$, then to the cases listed in (1) we must add

d	$tor\, H_1(S, \mathbb{Z})$
1	$\mathbb{Z}_3$; 0
3	0

□

Theorem 4.4. Let $\varphi : S \to C$ be an elliptic quasi–bundle with $q(S) = g(C)$, and let μ be the least common multiple of the multiplicities of the fibres. Suppose $d \cdot \mu$ is the largest integer dividing a fibre F in $H^2(S, \mathbb{Z})/(\text{torsion})$. Write $F \simeq \mathbb{C}/(\mathbb{Z}\,\omega \oplus \mathbb{Z})$ as above. Then there exists an exact sequence

$$0 \to J \to tor\, H_1(S, \mathbb{Z}) \to G(\varphi) \to 0$$

where the only possibilities for d and J are the following:

(1) If $\omega \neq (-1/2) + (\sqrt{3}/2)\,i$, then

d	J
1	$\mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_2$; 0
2	$\mathbb{Z}_2$; 0

(2) If $\omega = (-1/2) + (\sqrt{3}/2)\,i$, then to the cases listed in (1) we must add

d	J
1	$\mathbb{Z}_3$
3	0

□

Remark 4.5. The seven possibilities listed in Theorem 4.3 are actually realized in the seven families of bielliptic surfaces ([11]).

Proposition 4.6. Under the hypothesis of Theorem 4.4, assume furthermore that μ is neither divisible by 2 nor by 3. Then $tor\, H_1(S, \mathbb{Z}) \simeq J \oplus G(\varphi)$, and the possibilities for the pair $(d;\, J)$ coincide with the cases listed in Theorem 4.3 for the pairs $(d;\, tor\, H_1(S, \mathbb{Z}))$ ocurring there.

Proof: First note that the sequence

$$0 \to J \to tor\, H_1(S, \mathbb{Z}) \to G(\varphi) \to 0$$

splits because de orders of J and $G(\varphi)$ are coprime. Let

$$\begin{array}{ccc} R & \xrightarrow{\pi} & S \\ \Psi \downarrow & & \downarrow \varphi \\ B & \xrightarrow{\varepsilon} & C \end{array}$$

be the diagram of Theorem 3.6. Recall that π is Galois with group $\mathbb{Z}_\mu$. The fibre $\tilde{F}$ of Ψ can be divided by at most 3. It follows from Proposition 3.5 that d is the largest integer dividing $\tilde{F}$ in $H^2(R, \mathbb{Z})/(\text{torsion})$. On the other hand, by ([5], Thm. 2.1) there exists a homomorphism $\delta : H^2(R, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ such that if $\pi^* : H^2(S, \mathbb{Z}) \to H^2(R, \mathbb{Z})$ is induced by π then $(\pi^* \circ \delta)(y) = \sum_{\gamma \in \mathbb{Z}_\mu} \gamma \cdot y$, for all y. The condition on μ and the possibilities for $tor\, H^2(R, \mathbb{Z})$ (Theorem 4.3) imply that if $y \in tor\, H^2(R, \mathbb{Z})$ then $\sum_{\gamma \in \mathbb{Z}_\mu} \gamma \cdot y$ is zero only when $y = 0$. Equivalently, $\delta : tor\, H^2(R, \mathbb{Z}) \to tor\, H^2(S, \mathbb{Z})$ is one–to–one. But the image of $tor\, H^2(R, \mathbb{Z})$ in $tor\, H^2(S, \mathbb{Z}) \simeq tor\, H_1(S, \mathbb{Z}) \simeq J \oplus G(\varphi)$ lies inside J, because its order is coprime with that of $G(\varphi)$. Moreover, J is by construction a quotient of $tor\, H^2(R, \mathbb{Z})$. Consequently, $tor\, H^2(R, \mathbb{Z}) \simeq J$. □

REFERENCES

[1] BARTH, W.; PETERS, C.; VAN DE VEN, A.: Compact Complex Surfaces. Springer. Berlin–Heidelberg–New York, 1984.

[2] BEAUVILLE, A.: Surfaces algébriques complexes. Astérisque **54** (1978).

[3] BROWN, K. S.: Cohomology of groups. Grad. Texts in Math., vol. **87**. Springer. Berlin–Heidelberg–New York, 1982.

[4] COX, D. A.; ZUCKER, S.: Intersection numbers of sections of elliptic surfaces. Invent. Math. **53** (1979), 1–44.

[5] FLOYD, E. E.: Periodic maps via Smith theory. In "Seminar on Transformation Groups" by A. Borel and others. Annals of Math. Studies, vol. **46**. Princeton Univ. Press (1961), pp. 35–47.

[6] FUJITA, T.: On Kähler fiber spaces over curves. J. Math. Soc. Japan **30** (1978), 779–794.

[7] GODEMENT, R.: Topologie algébrique et théorie des faisceaux. Hermann, Paris (1964).

[8] GRIFFITHS, P.; HARRIS, J.: Principles of Algebraic Geometry. John Wiley & Sons. New York, 1978.

[9] LANG, S.: Algebra. Addison–Wesley Publ. Co., Reading (Mass.), 1984.

[10] MCCLEARY, J.: User's Guide to Spectral Sequences. Math. Lecture Series, vol. **12**. Publish or Perish, Wilmington (Delaware), 1985.

[11] SERRANO, F.: Divisors of bielliptic surfaces and embeddings in $\mathbf{P}^4$. Preprint.

[12] SERRANO, F.: Multiple fibres of a morphism. Preprint.

[13] SERRANO, F.: The structure of quasi–bundles. Preprint.

[14] SHAFAREVICH, I. R., ET AL.: Algebraic Surfaces. Proc. Steklov Inst. Math. **75** (1965). A. M. S. Translations, Providence R. I. (1967).

[15] WHITEHEAD, G. W.: Elements of Homotopy Theory. Grad. Texts in Math., vol. **61**. Springer. Berlin–Heidelberg–New York, 1982.