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0. Introduction

In [13], Skorohod introduced a stochastic integral of non–adapted random processes with respect to a Gaussian measure with orthogonal increments. The Skorohod integral is an extension of the classical Itô integral and coincides with the adjoint of the derivative operator on the Wiener space (see [5]).

The relation between the Skorohod integral and the Malliavin calculus has been analyzed by Nualart and Zakai in [8]. More recently, a generalized or anticipating stochastic calculus based on the Skorohod integral has been developed by Nualart and Pardoux [9] (see also [12, 14, 15]). We also refer to [10] for an exposition of the basic ideas of this theory.

The anticipating calculus has some special features. One of them is that the "indefinite" Skorohod integral does not have the martingale property. However it possesses an orthogonality property (see Proposition 5.1 (i) in [9]) which can be formulated as follows: Assume that $u = \{u_t,\, 0 \le t \le 1\}$ is a process such that the Skorohod integral $M_t = \int_0^1 u_s 1_{[0,t]}(s) dW_s$ exists for any $t \in [0,1]$. Then

$$E\{M_t - M_s \,|\, \mathcal{F}_{[s,t]^c}\} = 0, \tag{0.1}$$

for all $s < t$, where $\mathcal{F}_{[s,t]^c}$ denotes the σ–field generated by the increments of the Brownian motion W on $[s,t]^c$.

On the other hand, if $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^2, under suitable hypotheses, and with the same notations as before, it follows from the extended Itô formula that $f(X_t)$ can be written as the sum of a process M_t satisfying (0.1), and a process of bounded variation. This gives the feeling that the property (0.1) plays the role of the martingale property in the non–adapted case.

The aim of this paper is to study a class of processes $X = \{X_t,\, t \in [0,1]\}$ for which a generalized *Doob–Meyer decomposition* holds that means $X_t = M_t + A_t$, with M satisfying (0.1) and A a process of bounded variation.

In the first section we introduce the notion of S–quasimartingale. These processes are the analogue of the quasimartingales of [4] (see also [3], [6], [11]) in the non–adapted case. We also include the essential tools of the anticipating calculus which are needed in the development of our work. In the second section we give a sufficient condition for a S–quasimartingale to have a Doob–Meyer decomposition. Notice that, due to the lack of adaptedness, it is not clear how to define the analogue of class D (see [3]) in our situation. Our sufficient condition has been inspired by the work of Brennan ([1]), where the problem of the Doob–Meyer decomposition for two-parameter quasimartingales is considered. No ingredient of anticipating calculus is needed to prove the results of this section. In section 3, using the anticipating calculus, we give a necessary and sufficient condition ensuring the Doob–Meyer decomposition of some classes of S–quasimartingales, and we also prove uniqueness.

1. Preliminaries

Let $X = \{X_t,\, t \in I\}$ be a real stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ indexed by a Borel subset I of $[0, +\infty)$ containing 0. Assume we are given a family of sub σ–fields of $\mathcal{F}$, $\{\mathcal{F}_A,\ A$ measurable subset of $I\}$ which are complete with respect to the probability P, and such that $\mathcal{F}_A \subset \mathcal{F}_B$ whenever $A \subset B$. Given $s, t \in I,\ s < t$, we define $(s,t] = \{u \in I : s < u \le t\}$, and $(s,t]^c = I - (s,t]$.

Let $\pi = \{0 = t_0 < t_1 < \ldots < t_n\}$ be a finite sequence of elements of I. If $X \subset L^1(\Omega)$ we define

$$Q(X) = \sup_\pi E \sum_{i=0}^{n-1} \left| E\left\{X_{t_{i+1}} - X_{t_i} \,\middle|\, \mathcal{F}_{(t_i,t_{i+1}]^c}\right\}\right| . \tag{1.1}$$

Then, by analogy with the adapted case, we can introduce the notion of S–quasimartingale as follows.

Definition 1.1. An integrable process X is called an S–quasimartingale if and only if $Q(X) < +\infty$.

The class of S–quasimartingales is nonempty. Indeed, let us call *S–martingale* any integrable process $X = \{X_t,\, t \in I\}$ such that

$$E\left\{X_t - X_s \,|\, \mathcal{F}_{(s,t]^c}\right\} = 0 , \tag{1.2}$$

for any $s, t \in I,\ s \le t$.

Notice that $Q(X) = 0$ is equivalent to the fact that X is an S–martingale. The process $\{M_t,\, t \in [0,1]\}$ given in the introduction is an S–martingale, and therefore an S–quasimartingale.

As in the adapted case a S–quasimartingale can be characterized by means of a Doléans–Föllmer measure. Assume for instance that $I = [0,1]$. We introduce the class of sets

$$R = \{(s,t] \times F;\ 0 \le s < t,\ F \in \mathcal{F}_{(s,t]^c}\} .$$

Let $\mathfrak{a}$ be the algebra generated by the finite unions of elements in R. Given an integrable process $X = \{X_t,\, t \in [0,1]\}$ we define

$$\lambda_X\left((s,t] \times F\right) = E\left\{1_F(X_t - X_s)\right\},$$

for any $(s,t] \times F \in R$. Then λ_X is a finitely additive measure on $\mathfrak{a}$, and it is easy to check that the following properties are satisfied:

(i) $\lambda_X \equiv 0$ if and only if $Q(X) = 0$.

(ii) $Q(X) < \infty$ if and only if λ_X is of bounded variation on $\mathfrak{a}$, and in this case $|\lambda_X|\left((0,1] \times \Omega\right) = Q(X)$, where $|\lambda_X|$ is the total variation of λ_X.

From now on, we assume in this section that $I = [0,1]$, $(\Omega, \mathcal{F}, P)$ is the canonical space associated with a standard Brownian motion on I, $\{W_t,\, t \in I\}$, and $\mathcal{F}_A$, $A \in \mathcal{B}(I)$, is the completion with respect to P of the σ–algebra generated by $\{W(B),\, B \subset A\}$. In the sequel we will call this situation "the Brownian case". Every random variable defined on this probability space is called a Brownian functional. We will denote by $\mathcal{S}$ the class of Brownian functionals of the from

$$F = f\left(W(t_1), \ldots, W(t_n)\right), \tag{1.3}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is an infinitely differentiable function such that f and all its derivatives are bounded, and $t_1, \ldots, t_n$ are in I . The elements of $\mathcal{S}$ are called smooth functionals and form a dense subspace of $L^2(\Omega)$.

We recall that the derivative of a smooth functional F of the form (1.3) is the stochastic process $\{D_tF,\, t \in [0,1]\}$ given by

$$D_tF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(W(t_1), \ldots, W(t_n)\right) 1_{[0,t_i]}(t).$$

This defines an unbounded closable operator on $L^2(\Omega)$ with values on $L^2(I \times \Omega)$. Then we define the space $\mathbb{D}^{1,2}$ as the domain of D, that means, $\mathbb{D}^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|DF\|_{L^2(I\times\Omega)}.$$

Denote by δ the adjoint operator of D, and by $Dom\,\delta$ its domain. As we have mentioned in the introduction, δ coincides with the Skorohod integral, and we will write $\delta(u) = \int_I u_t dW_t$, for $u \in Dom\,\delta$.

More generally, we can define the N-th derivative of $F \in \mathcal{S}$ by $D^N_{t_1\dots t_N} F = D_{t_1} D_{t_2} \dots D_{t_N} F$, and for any $p > 1$ the space $\mathbb{D}^{N,p}$ is the completion of $\mathcal{S}$ with respect to the norm $\|F\|_{N,p} = \|F\|_{L^p(\Omega)} + \sum_{k=1}^N \| \, \|D^k F\|_{L^2(I^k)} \|_{L^p(\Omega)}$.

The following result will we needed in section 3.

Proposition 1.2. Let $F \in \mathbb{D}^{1,2}$. Then for all $0 \le s \le t \le 1$

$$F = E(F|\mathcal{F}_{[s,t]^c}) + \int_s^t E\left(D_r F|\mathcal{F}_{[r,t]^c}\right) dW_r \,. \tag{1.4}$$

We refer the reader to [9] (Proposition A.1) for the proof of this fact, which is an extension of a well-known result on the representation of Wiener functionals.

2. Sufficient condition for the Doob–Meyer decomposition of an $\mathcal{S}$–quasimartingale.

Consider the simple example where $I = \mathbb{N}$, the set of all natural numbers, and let $X = \{X_n, \, n \ge 0\}$ be an integrable process such that

$$E\left\{X_m - X_n \,|\, \mathcal{F}_{(n,m]^c}\right\} \ge 0 \,, \text{ for any } \quad 0 \le n < m \,, \tag{2.1}$$

and $X_0 = 0$.

Define $A_n = \sum_{i=0}^{n-1} E\left\{X_{i+1} - X_i \,|\, \mathcal{F}_{\{i+1\}^c}\right\}$, for $n > 0$, $A_0 = 0$ and $M_n = X_n - A_n$. Then $\{A_n, \, n \ge 0\}$ is an integrable, positive and increasing process, while $\{M_n, \, n \ge 0\}$ is an $\mathcal{S}$–martingale.

The decomposition $X_n = M_n + A_n$ is the analogue of the Doob–Meyer representation of a discrete submartingale.

The purpose of this section is to extend this decomposition to continuous time $\mathcal{S}$–quasimartingales. Along this section $X = \{X_t, \, t \in [0,1]\}$ is assumed to be an integrable process.

Let Δ^n be the dyadic partition of [0,1] of order n. For every $t \in [0,1]$ we define

$$\Delta^n(X)_t = \sum_{i=0}^{2^n-1} \left| E\left\{ X_{t\wedge(i+1)2^{-n}} - X_{t\wedge i2^{-n}} \middle| \mathcal{F}_{(t\wedge i2^{-n}, \, t\wedge(i+1)2^{-n}]^c} \right\} \right| ,$$

$$\Delta^n(X)_t^+ = \sum_{i=0}^{2^n-1} \left(E\left\{ X_{t\wedge(i+1)2^{-n}} - X_{t\wedge i2^{-n}} \middle| \mathcal{F}_{(t\wedge i2^{-n}, \, t\wedge(i+1)2^{-n}]^c} \right\} \right)^+ ,$$

and $\Delta^n(X)_t^-$ in the analogous way.

We consider the set of random variables $I_X(t) = \{\Delta^n(X)_t\,,\, n \geq 1\}$. We also consider the sets $I_X^+(t)$ and $I_X^-(t)$ defined as $I_X(t)$ with $\Delta^n(X)_t$ replaced by $\Delta^n(X)_t^+$ and by $\Delta^n(X)_t^-$, respectively.

To simplify the notations, when $t = 1$ we will write $\Delta^n(X), I_X, I_X^+, I_X^-$ instead of $\Delta^n(X)_1, I_X(1), I_X^+(1), I_X^-(1)$, respectively. Notice that if $n \leq m$, $E\{\Delta^n(X)\} \leq E\{\Delta^m(X)\}$. Thus, if X is continuous in L^1 and $\sup_n E\{\Delta^n(X)\} < +\infty$, then X is an S–quasimartingale.

Let $Q(X)_t = \sup_n E\{\Delta^n(X)_t\}$. If X is an L^1–continuous S–quasimartingale it is not difficult to prove that the function $t \to Q(X)_t$, defined on [0,1], is continuous. The same property holds for $Q(X)_t^+ = \sup_n E\{\Delta^n(X)_t^+\}$ and $Q(X)_t^- = \sup_n E\{\Delta^n(X)_t^-\}$ as well.

Consider the following hypothesis:

(H_1) The set I_X is uniformly integrable.

We can state the main result of this section.

Theorem 2.1. Let X be an L^1–continuous S–quasimartingale. Assume that hypothesis (H_1) holds. Then, there exists a decomposition

$$X = M + A\,, \tag{2.2}$$

where M is an S–martingale and A is a process with paths of bounded variation. Moreover, $A_0 = 0$, A is L^1–continuous and has a.s. right continuous paths. If X is an S–submartingale (that means, $E\{X_t - X_s|\mathcal{F}_{(s,t]^c}\} \geq 0$, for any $s \leq t$), then A is increasing.

Proof. Let $\bar{\mathbb{Q}} = \{r_n, n \geq 1\}$ be the set of all rational dyadic numbers contained in [0,1]. We first show the existence of a subsequence $\{n^{(n)}, n \geq 1\} \subset \mathbb{N}$, and integrable random variables $\{A_{r_n}^+, n \geq 1\}$, $\{A_{r_n}^-, n \geq 1\}$, such that

$$\Delta^{n^{(n)}}(X)_{r_i}^+ \to A_{r_i}^+,\ \Delta^{n^{(n)}}(X)_{r_i}^- \to A_{r_i}^-\,, \tag{2.3}$$

respectively, in the weak topology $\sigma(L^1, L^\infty)$, as $n \to \infty$, for any $i \geq 1$.

Indeed, the sets $I_X^+(r_1)$ and $I_X^-(r_1)$ are uniformly integrable. Thus there exists a subsequence $\{\Delta^{n^{(1)}}, n \geq 1\}$ of $\{\Delta^n, n \geq 1\}$, and integrable random variables $A_{r_1}^+$, $A_{r_1}^-$ such that

$$\Delta^{n^{(1)}}(X)_{r_1}^+ \overset{\sigma(L^1,L^\infty)}{\longrightarrow} A_{r_1}^+\,,$$

$$\Delta^{n^{(1)}}(X)_{r_1}^- \overset{\sigma(L^1,L^\infty)}{\longrightarrow} A_{r_1}^-\,,$$

as $n \to \infty$.

By this way we can construct recursively a subsequence $\{\Delta^{n^{(k+1)}}, n \geq 1\}$ of $\{\Delta^{n^{(k)}}, n \geq 1\}$, and integrable random variables $A^+_{r_{k+1}}$, $A^-_{r_{k+1}}$ $k \geq 1$ such that

$$\Delta^{n^{(k+1)}}(X)^+_{r_{k+1}} \overset{\sigma(L^1,L^\infty)}{\longrightarrow} A^+_{r_{k+1}},$$

and

$$\Delta^{n^{(k+1)}}(X)^-_{r_{k+1}} \overset{\sigma(L^1,L^\infty)}{\longrightarrow} A^-_{r_{k+1}},$$

as $n \to \infty$.

The diagonal sequence $\{\Delta^{n^{(n)}}, n \geq 1\}$, and the integrable random variables $\{A^+_{r_n}, A^-_{r_n}, n \geq 1\}$ satisfy (2.3). From now on we will write $a(n) = n^{(n)}$. The processes $\{A^+_{r_n}, n \geq 1\}$ and $\{A^-_{r_n}, n \geq 1\}$ obtained by this procedure are increasing. Indeed, assume that $r_i < r_j$. Then, for some $n_0 > 0$ we will have $r_i = k2^{-n_0}$, $r_j = \ell 2^{-n_0}$, $k \neq \ell$. Then, for any $n \geq n_0$

$$\Delta^n(X)^+_{r_j} - \Delta^n(X)^+_{r_i} = \sum_{h=2^{n-n_0}k}^{2^{n-n_0}\ell-1} \left(E\left\{X_{(h+1)2^{-n}} - X_{h2^{-n}} \,|\, \mathcal{F}_{\left(h2^{-n},\,(h+1)2^{-n}\right]^c}\right\}\right)^+ \geq 0.$$

Hence $A^+_{r_j} - A^+_{r_i} \geq 0$, because it is the weak limit of a positive sequence. The same arguments apply to $\{A^-_{r_n}, n \geq 1\}$. We set $A_{r_n} = A^+_{r_n} - A^-_{r_n}$, $r_n \in \bar{\mathbb{Q}}$.
The continuity of $Q(X)^+_t$ and $Q(X)^-_t$ entails that $\{A_{r_n}, n \geq 1\}$ is uniformly continuous in L^1. In fact, assuming that $r_i < r_j$, we have

$$\begin{aligned} E\,|A_{r_j} - A_{r_i}| &\leq E\,(A^+_{r_j} - A^+_{r_i}) + E\,(A^-_{r_j} - A^-_{r_i}) \\ &= \left(Q(X)^+_{r_j} - Q(X)^+_{r_i}\right) + \left(Q(X)^-_{r_j} - Q(X)^-_{r_i}\right) \to 0, \end{aligned}$$

as $(r_j - r_i) \to 0$.

The process $\{M_{r_n} = X_{r_n} - A_{r_n}, n \geq 1\}$ is an S–martingale. To prove this fact take $r_i < r_j$ and a bounded random variable ξ which is $\mathcal{F}_{(r_i,r_j]^c}$–measurable. Using the same notations as before it follows that:

$$E\{\xi(A_{r_j} - A_{r_i})\} = \lim_{n\to\infty} E\Big\{\xi \sum_{h=2^{a(n)-n_0}k}^{2^{a(n)-n_0}\ell-1} E\{X_{(h+1)2^{-n}} - X_{h2^{-n}} \,|\mathcal{F}_{(h2^{-n},(h+1)2^{-n}]^c}\}\Big\}.$$

Consequently

$$\begin{aligned} E\left\{\xi(A_{r_j} - A_{r_i})\right\} &= \lim_{n\to\infty} E\left\{\xi \sum_{h=2^{a(n)-n_0}k}^{2^{a(n)-n_0}\ell-1} E\left\{X_{(h+1)2^{-n}} - X_{h2^{-n}} \,|\, \mathcal{F}_{(r_i,r_j]^c}\right\}\right\} \\ &= E\left\{\xi E\left\{X_{\ell 2^{-n_0}} - X_{k2^{-n_0}} \,|\, \mathcal{F}_{(r_i,r_j]^c}\right\}\right\} \\ &= E\left\{\xi(X_{r_j} - X_{r_i}\right\}, \end{aligned}$$

and the desired result follows.

Finally, for any $t \in [0,1] \cap \bar{\mathbb{Q}}^c$ we define $A_t^+ = \lim_{\substack{r \in \mathbb{Q} \\ r \downarrow t}} A_r^+$, $A_t^- = \lim_{\substack{r \in \mathbb{Q} \\ r \downarrow t}} A_r^-$, and $A_t = A_t^+ - A_t^-$. Notice that the process $\{A_t,\, t \in [0,1]\}$ has bounded variation paths and is continuous in L^1.

Moreover $\{M_t = X_t - A_t,\, t \in [0,1]\}$ is an S–martingale. Indeed, let $s,t \in [0,1]$, $s < t$ and consider sequences of rational dyadic numbers $\{s_n,\, n \geq 1\}$, $\{t_n,\, n \geq 1\}$ such that $s_n < t_n$ for any $n \geq 1$, $s_n \downarrow s$ and $t_n \uparrow t$ as $n \to \infty$. Then

$$E\left\{X_t - X_s \,|\, \mathcal{F}_{(s,t]^c}\right\} = L^1 - \lim_{n\to\infty} E\left\{X_{t_n} - X_{s_n} \,|\, \mathcal{F}_{(s,t]^c}\right\},$$

by the L^1–continuity of X. On the other hand

$$\begin{aligned} E\left\{X_{t_n} - X_{s_n} \,|\, \mathcal{F}_{(s,t]^c}\right\} &= E\left\{E\left\{X_{t_n} - X_{s_n} \,|\, \mathcal{F}_{(s_n,t_n]^c}\right\} \,|\, \mathcal{F}_{(s,t]^c}\right\} \\ &= E\left\{A_{t_n} - A_{s_n} \,|\, \mathcal{F}_{(s,t]^c}\right\} \xrightarrow{L^1} E\left\{A_t - A_s \,|\, \mathcal{F}_{(s,t]^c}\right\}. \end{aligned}$$

This finishes the proof of the theorem. □

Remark. Assume that we are dealing with the Brownian case, and that the process X in Theorem 2.1 is adapted. Then, so are A and M. Furthermore, in this case M is a martingale.

3. Necessary and sufficient conditions for the Doob–Meyer decomposition of an S–quasimartingale

In this section we will study the Brownian case. This is our fundamental assumption. In the first part we prove that, if the measure induced by the process A constructed in Theorem 2.1 is absolutely continuous, then the hypotheses (H_1) is also necessary for the Doob–Meyer decomposition of an L^1–continuous S–quasimartingale. In the second part we consider the set of L^1–continuous S–quasimartingales which are S–submartingales. We introduce a new hypotheses (H_3) and prove that it is a necessary and sufficient condition for the Doob–Meyer decomposition to hold. As a by–product of our results we will obtain the uniqueness of the representation in this particular case. In the last part we will study a family of S–quasimartingales derived by transformation of Skorohod integrals.

Assume that X is a process such that $X = M + A$ where M is an S–martingale and A an integrable process of bounded variation, with $A_0 = 0$. With the notation of the previous section we have

$$I_X = \left\{\Delta^n(A) = \sum_{i=0}^{2^n-1} \left| E\left\{A_{(i+1)2^{-n}} - A_{i2^{-n}} \,|\, \mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right\} \right|,\, n \geq 1\right\}. \qquad (3.1)$$

The process $A = \{A_t,\, t \in [0,1]\}$ induces a signed measure on the product space $[0,1] \times \Omega$ defined by $\lambda_A((s,t] \times F) = E\big(1_F(A_t - A_s)\big)$, $F \in \mathcal{F}$. Let us introduce the following condition:

(H_2) There exists a deterministic (positive) measure μ on $\mathcal{B}([0,1])$, and a measurable process $\{a_s(\omega), (s,\omega) \in [0,1] \times \Omega\}$ such that

$$\lambda_A(G) = E \int_G a_s(\omega)\mu(ds), \quad \text{for any} \quad G \in \mathcal{B}([0,1]) \otimes \mathcal{F}.$$

We can now establish the following result:

Proposition 3.1. Let X be an S–quasimartingale. Assume that $X = M + A$, where M is an S–martingale and A an integrable process whose paths are right continuous and with bounded variation, and $A_0 = 0$. Furthermore, assume that (H_2) is satisfied. Then (H_1) holds.

Proof. Set $\mathcal{F}_t = \mathcal{F}_{[0,t]}$ and $\mathcal{F}^t = \mathcal{F}_{[t,1]}$. We notice that $\mathcal{F}^t$ is the σ–algebra generated by the increments $\{W_u - W_1,\, t \le u \le 1\}$. Let $W^u = W_{1-u} - W_1$. Then $\{W^u,\, 0 \le u \le 1\}$ is also a Brownian motion, and $\mathcal{F}^t = \sigma\{W^u,\, 0 \le u \le 1-t\}$. Given $s_0, t_0 \in [0,1]$ with $s_0 + t_0 = 1$, we have that $\{W_s,\, 0 \le s \le s_0\}$ and $\{W^s,\, 0 \le s \le t_0\}$ are independent. Consequently, the two–parameter filtration $\{\mathcal{F}_{(s,t]^c} = \mathcal{F}_s \vee \mathcal{F}^t,\, 0 \le s < t \le 1\}$ satisfy the usual conditions (F1) to (F4) of Cairoli and Walsh [2].

Let ξ be a measurable and bounded random variable. Define

$$Z_n(t,\omega) = \sum_{i=0}^{2^n-1} E\left\{\xi \,|\, \mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right\} 1_{(i2^{-n},(i+1)2^{-n}]}(t).$$

Then, by the results on convergence of two–parameter martingale sequences, we have that, for any fixed $t \in [0,1]$, $Z_n(t,\omega) \to \xi$ a.s. as n tends to ∞.

The process A can be decomposed as the difference of two increasing integrable processes $A = A^{(1)} - A^{(2)}$. Clearly $\Delta^n(A) \le \Delta^n(A^{(1)}) + \Delta^n(A^{(2)})$. So, in order to establish the uniform integrability of I_X we can assume without loss of generality that A is increasing.

We want to prove that for any $\xi \in L^\infty$ we have

$$E\{\xi\Delta^n(A)\} \to E\{\xi A_1\}, \quad \text{as } n \to \infty \tag{3.2}$$

We have

$$\begin{aligned} E\{\xi\Delta^n(A)\} &= E\Big\{\sum_{i=0}^{2^n-1} \big(A_{(i+1)2^{-n}} - A_{i2^{-n}}\big) E\left\{\xi \,|\, \mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right\}\Big\} \\ &= E\int_0^1 Z_n(t,\omega)dA_t = \int_{[0,1]\times\Omega} Z_n(t,\omega)d\lambda_A(t,\omega). \end{aligned} \tag{3.3}$$

Let $G = \{(t,\omega) \in [0,1] \times \Omega : \lim_{n\to\infty} Z_n(t,\omega) \neq \xi(\omega)\}$. Then $\lambda_A(G) = 0$.
Indeed, by Fubini's theorem and using the fact that the sections $G_t = \{\omega : (t,\omega) \in G\}$ have probability zero for all t, we have

$$\lambda_A(G) = E\int_0^1 1_G(t,\omega)a_t(\omega)\mu(dt) = \int_0^1 \Big(\int_\Omega 1_G(t,\omega)a_t(\omega)dP\Big)\mu(dt) = 0.$$

Hence, by dominated convergence

$$\lim_{n\to\infty}\int_{[0,1]\times\Omega} Z_n(t,\omega)d\lambda_A(t,\omega) = \int_{[0,1]\times\Omega}\xi(t,\omega)d\lambda_A(t,\omega) = E\{\xi A_1\}.$$

Then, by (3.3) we obtain the desired convergence. $\square$

In order to introduce a new hypothesis we give the following definition.

Definition 3.2. A set $\mathcal{H}$ of random variables is said to be *weakly uniformly integrable* if $\mathcal{H}$ is weakly sequentially compact for the weak topology $\sigma(L^1, L^\infty \cap \mathbb{D}^{1,2})$. That means , for every sequence of elements in $\mathcal{H}$ one can extract a further subsequence which converges in the weak topology $\sigma(L^1, L^\infty \cap \mathbb{D}^{1,2})$.
We can now state condition (H_3) as follows.
(H_3) The set $I_X(t) = \{\Delta^n(X)_t,\ n \geq 1\}$ is weakly uniformly integrable for any $t \in \bar{\mathbb{Q}}$, where $\bar{\mathbb{Q}}$ denotes the set of the rational dyadic numbers in [0,1].
Notice that (H_3) is weaker than (H_1).

The remaining of this section is devoted to prove the following result.

Theorem 3.3. (1) Let X be an L^1–continuous S–submartingale. Assume that (H_3) is satisfied. Then $X = M + A$, where M is an S–martingale and A is an increasing process. Moreover $A_0 = 0$, A is L^1–continuous and has a.s. right continuous paths, and this decomposition is unique.
Conversely,
(2) If $X = M + A$, with M an S–martingale and A an increasing, integrable process with $A_0 = 0$, then, X is a S–submartingale and (H_3) is satisfied.

The proof of this theorem is based on several lemmas.

Lemma 3.4. Let ξ be any random variable belonging to $\mathbb{D}^{1,2}$ and $\{A_t,\ t \in [0,1]\}$ an integrable, increasing process with $A_0 = 0$. As before, we set

$$\Delta^n(A) = \sum_{i=0}^{2^n-1} E\left\{A_{(i+1)2^{-n}} - A_{i2^{-n}} \,|\, \mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right\}.$$

Then, for any $k > 0$,

$$\sup_{\substack{\|\xi\|_\infty \le k \\ \|D\xi\|^2_{L^2([0,1]\times\Omega)} \le \varepsilon}} \sup_n \left| E\left\{\Delta^n(A)\xi - A_1\xi\right\}\right| \to 0\,,$$

as ε tends to zero.

Proof. By the definition of $\Delta^n(A)$ and the properties of the conditional expectation

$$\left|E\left\{\Delta^n(A)\xi - A_1\xi\right\}\right| = \left|E\left\{\sum_{i=0}^{2^n-1}\left(A_{(i+1)2^{-n}} - A_{i2^{-n}}\right)\left(\xi - E|\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right)\right\}\right| \tag{3.4}$$

Set $Y_n = \sum_{i=0}^{2^n-1}\left(A_{(i+1)2^{-n}} - A_{i2^{-n}}\right)\left(\xi - E\left(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right)\right.$. Fix $\delta > 0$ and $M > 0$; we have the following inequalities:

$$\left|\int_{\{A_1 > M, \max_i |\xi - E(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c})| > \delta\}} Y_n dP\right| \le 2\|\xi\|_\infty \int_{\{A_1 > M\}} A_1 dP\,. \tag{3.5}$$

$$\begin{aligned}
&\left|\int_{\{A_1 \le M, \max_i |\xi - E\left(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right)| > \delta\}} Y_n dP\right| \\
&\le 2\|\xi\|_\infty M\, P\left\{\max_i |\xi - E(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c})| > \delta\right\} \\
&\le 2\|\xi\|_\infty M \sum_{i=1}^{2^n-1} P\left\{|\xi - E\left(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right)| > \delta\right\} \\
&\le 2\|\xi\|_\infty M\,\delta^{-2} \sum_{i=1}^{2^n-1} E|\xi - E\left(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right)|^2 \\
&= 2\|\xi\|_\infty M\,\delta^{-2} \sum_{i=1}^{2^n-1} E\left|\int_{i2^{-n}}^{(i+1)2^{-n}} E\left(D_r\xi|\mathcal{F}_{(r,(i+1)2^{-n}]^c}\right) dW_r\right|^2, \qquad (see (1.4)) \\
&= 2\|\xi\|_\infty M\,\delta^{-2} \sum_{i=1}^{2^{-1}} E\int_{i2^{-n}}^{(i+1)2^{-n}} |E\left(D_r\xi|\mathcal{F}_{(r,(i+1)2^{-n}]^c}\right)|^2 dr \\
&\le 2\|\xi\|_\infty M\,\delta^{-2} \sum_{i=1}^{2^n-1} \int_{i2^{-n}}^{(i+1)2^{-n}} E\,|D_r\xi|^2 dr = 2\|\xi\|_\infty M\delta^{-2}\int_0^1 E|D_r\xi|^2 dr \\
&= 2\|\xi\|_\infty M\delta^{-2}\|D\xi\|^2_{L^2([0,1]\times\Omega)}\,.
\end{aligned} \tag{3.6}$$

$$\left|\int_{\{\max_i |\xi - E\left(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}\right)| \le \delta\}} Y_n dP\right| \le \delta E(A_1)\,. \tag{3.7}$$

Since $A_1 \in L^1(\Omega)$ given $\eta > 0$ we can choose $M > 0$ such that $\int_{\{A_1>M\}} A_1 dP \le \frac{\eta}{6k}$. Hence the supremum when $\|\xi\|_\infty \le k$ of the right hand side of (3.5) can be bounded by $\frac{\eta}{3}$. By the same reason, by choosing $\delta = \frac{\eta}{3E(A_1)}$, the right hand side of (3.7) can be majorized by $\frac{\eta}{3}$.

Finally taking $\varepsilon_0 = \frac{\eta^3}{3} \frac{1}{18kM(EA_1)^2}$, the supremum when $\|\xi\|_\infty \le k$ and $\|D\xi\|^2_{L^2((0,1]\times\Omega)} \le \varepsilon$ with $\varepsilon \le \varepsilon_0$ of the right side of (3.6) is also bounded by $\frac{\eta}{3}$. Thus, we have the result. $\square$

Lemma 3.5. Let A be a process satisfying the same hypotheses as in the preceding lemma. For any random variable $\xi \in L^\infty \cap \mathbb{D}^{1,2}$ such that $\int_0^1 E|D_r\xi|^4 dr < +\infty$, it holds that

$$\lim_{n\to\infty} E\{\Delta^n(A)\xi\} = E\{A_1\xi\}.$$

Proof. Let $\xi \in L^\infty \cap \mathbb{D}^{1,2}$. For any $\delta > 0$ we have

$$P\{\max_{0\le i\le 2^n-1} |\xi - E(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c})| > \delta\} \le \sum_{i=0}^{2^n-1} P\{|\xi - E(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c})| > \delta\}$$

$$\le \frac{1}{\delta^4}\sum_{i=0}^{2^n-1} E\{|\xi - E(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c}|^4\}$$

$$= \frac{1}{\delta^4}\sum_{i=0}^{2^n-1} E\{|\int_{i2^n}^{(i+1)2^n)} E(D_r\xi|\mathcal{F}_{(r,(i+1)2^{-n}]^c})dW_r|^4\} \qquad \text{(by Proposition 1.2)}$$

$$\le \frac{C}{\delta^4}\sum_{i=0}^{2^n-1} E\Big(\int_{i2^{-n}}^{(i+1)2^{-n}} |E(D_r\xi|\mathcal{F}_{(r,(i+1)2^{-n}]^c})|^2 dr\Big)^2$$

$$\le \frac{C}{\delta^4}\sum_{i=0}^{2^n-1} \frac{1}{2^n}\int_{i2^{-n}}^{(i+1)2^{-n}} E|D_r\xi|^4 dr$$

$$\le \frac{C}{\delta^4}\frac{1}{2^n}\int_0^1 E|D_r\xi|^4 dr.$$

Consequently, if $\xi \in L^\infty \cap \mathbb{D}^{1,2}$, the sequence

$$\{\max_{0\le i\le 2^n-1} |\xi - E(\xi|\mathcal{F}_{(i2^{-n},(i+1)2^{-n}]^c})|,\ n \ge 1\}$$

converges in probability to zero, as n tends to infinity. Using this fact and the same arguments as in the proof of Lemma 3.4 we get the desired result. $\square$

Remark. The conclusions of Lemmas 3.4 and 3.5 also hold if we replace $\Delta^n(A)$ by $\Delta^n(A)_t = \sum_{i=0}^{2^n-1} E\{A_{t\wedge(i+1)2^{-n}} - A_{t\wedge i2^{-n}}|\mathcal{F}_{(t\wedge(i2^{-n},t\wedge(i+1)2^{-n}]^c}\}$ and A_1 by A_t, $t \in [0,1]$, respectively.

Proposition 3.6. Let A be a process satisfying the hypotheses of Lemma 3.4. For any random variable $\xi \in L^\infty \cap \mathbb{D}^{1,2}$ we have

$$\lim_{n\to\infty} E\left\{\Delta^n(A)_t\, \xi\right\} = E\{A_t\, \xi\},$$

for any $t \in [0,1]$.

Proof. Given $\xi \in L^\infty \cap \mathbb{D}^{1,2}$, and $\varepsilon > 0$, let η_ε be a smooth functional such that $\|\eta_\varepsilon\|_\infty \le k$, for some constant k, $E\int_0^1 |D_r\eta_\varepsilon|^4 dr < \infty$, and $\|D\xi - D\eta_\varepsilon\|_{L^2([0,1]\times\Omega)} < \varepsilon$.

The functional η_ε can be obtained using the following argument. First consider a sequence of smooth functionals $\{\xi_n,\, n \ge 1\}$ such that ξ_n converges to ξ in $\mathbb{D}^{1,2}$ and a.s., when $n \to \infty$. Let $a < b$ be real numbers such that $a < b$ and $\xi(\omega) \in (a,b)$ a.s. consider a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(\xi_n)$ converges to ξ in $\mathbb{D}^{1,2}$, as n tends to ∞. The sequence $\{\varphi(\xi_n),\, n \ge 1\}$ is bounded and satisfies $E\int_0^1 |D_r\varphi(\xi_n)|^4 dr < +\infty$, for any $n \ge 1$.

By Lemma 3.5 we know that $\lim_{n\to\infty} E\left\{\Delta^n(A)_t\eta_\varepsilon\right\} = E\{A_t\eta_\varepsilon\}$. Then

$$\begin{aligned}&\overline{\lim_{n\to\infty}} \left|E\left\{\Delta^n(A)_t\xi - A_t\xi\right\}\right| \le \overline{\lim_{n\to\infty}} \left|E\left\{\Delta^n(A)_t(\xi-\eta_\varepsilon) - A_t(\xi-\eta_\varepsilon)\right\}\right| \\ &+ \lim_{n\to\infty} \left|E\left\{\Delta^n(A)_t\eta_\varepsilon - A_t\eta_\varepsilon\right\}\right|,\end{aligned}$$

and the right hand side of this inequality is zero, due to the preceding remark and Lemma 3.4. □

Lemma 3.7. Let A be a Borel subset of [0,1] and $X \in L^1(\Omega, \mathcal{F}_A, P)$. If $E(X\eta) = 0$ for all $\eta \in \mathbb{D}^{1,2} \cap L^\infty(\Omega, \mathcal{F}_A, P)$, then $X = 0$.

Proof. It is immediate because we can take $\eta = \varphi(W(B_1), \dots, W(B_n))$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $B_i \subset A$. □

Proof. of Theorem 3.3.

(1) Assume first that X is an L^1–continuous S–submartingale satisfying (H_3). The construction of the increasing process A follows by the same arguments than in the proof of Theorem 2.1 but replacing the convergence in $\sigma(L^1, L^\infty)$ by the convergence in $\sigma(L^1, L^\infty \cap \mathbb{D}^{1,2})$. This process A verifies $E\left\{\xi[(X_{r_j} - X_{r_i}) - (A_{r_j} - A_{r_i})]\right\} = 0$, for any $r_i, r_j \in \bar{\mathbb{Q}}$, $r_i < r_j$, and for any $\mathcal{F}_{(r_i,r_j]^c}$–measurable random variable $\xi \in L^\infty \cap \mathbb{D}^{1,2}$. Thus

$$E\left\{\xi\, E\{[(X_{r_j} - X_{r_i}) - (A_{r_j} - A_{r_i})]|\mathcal{F}_{(r_i,r_j]^c}\}\right\} = 0.$$

By Lemma 3.7 it follows that

$$E\left\{X_{r_j} - X_{r_i}|\mathcal{F}_{(r_j,r_i]^c}\right\} = E\left\{A_{r_j} - A_{r_i}|\mathcal{F}_{(r_i,r_j]^c}\right\},$$

thus $M_{r_n} = X_{r_n} - A_{r_n}$, $r_n \in \bar{\mathbf{Q}}$ is an S–martingale, and we continue as in the proof of Theorem 2.1.

Let us now establish the *uniqueness* of the Doob–Meyer decomposition. Assume that X is an S–submartingale such that $X = M_1 + A_1 = M_2 + A_2$, where M_i, are S–martingales and A_i right continuous, integrable processes with increasing paths, and $A_i(0) = 0$, $i = 1, 2$. Then for any $t \in [0,1]$, $\Delta^n(A_1)_t = \Delta^n(A_2)_t$. Hence, by Proposition 3.6, for any $t \in [0,1]$

$$E\left\{(A_1(t) - A_2(t))\xi\right\} = 0,$$

for any $\xi \in L^\infty \cap \mathbb{D}^{1,2}$. Thus, by Lemma 3.7, $A_1(t) = A_2(t)$, a.s for a fixed $t \in [0,1]$. The right continuity of A_i, $i = 1,2$, implies $A_1(t) = A_2(t)$, for any $t \in [0,1]$, a.s., proving uniqueness.

(2) Assume that $X = M + A$, where M is an S–martingale and A an increasing, integrable process with $A_0 = 0$. It is clear that X is an S–submartingale. On the other hand $\Delta^n(X)_t = \Delta^n(A)_t$, and property ($\mathrm{H}_3$) follows from Proposition 3.6.

The proof of the theorem is now complete. $\square$

Remark. The method used in the proof of Theorem 3.3 cannot be adapted to the case of an S–quasimartingale. So we do not have a sufficient condition (similar to condition (H_3)) for the Doob–Meyer decomposition of an S–quasimartingale. A necessary condition is provided by the results of Section 2. The uniqueness of the decomposition in the Brownian case can be deduced by the quadratic variation properties of the Skorohod integral (see [9] and [10, pag 105]). In Corollary 2.4 of [7] these is also a uniqueness property under different conditions.

Following [9], we set $\mathbb{L}^{N,p} = L^p([0,1]; \mathbb{D}^{N,p})$ for all $N \geq 1$ and $p \geq 1$. It is known that if $u \in \mathbb{L}^{2,2}$ then $u \cdot 1_{[0,t]} \in \operatorname{Dom}\delta$ for any $0 \leq t \leq 1$.

The next proposition gives an example of S–quasimartingale and its Doob–Meyer decomposition.

Proposition 3.8. Let $u \in \mathbb{L}^{2,2}$, $X_t = \delta(u \cdot 1_{[0,t]})$, $0 \leq t \leq 1$, and $f : \mathbf{R} \to \mathbf{R}$ a $\mathcal{C}^2$ function with an uniformly continuous and bounded second derivative. Then, $f(X_t) = M_t + A_t$, where

(i) $M = \{M_t,\ 0 \leq t \leq 1\}$ is an S–martingale,

(ii) $A = \left\{A_t = \frac{1}{2}\int_0^t f''(X_s)u_s^2 ds + \int_0^t f''(X_s)\left(\int_0^s D_s u_r dW_r\right)u_s ds,\ 0 \leq t \leq 1\right\}$

is a continuous, bounded variation process and $E\int_0^1 |dA_t| < +\infty$.

Consequently $\{f(X_t),\ 0 \leq t \leq 1\}$ is an S–martingale.

Proof. It is clear that the process A given in (ii) is continuous and of bounded variation.

The property $E\int_0^1 |dA_t| < +\infty$ follows from Schwarz's inequality, the isometry of the Skorohod integral and the fact that $u \in \mathbb{L}^{2,2}$. So, it only remains to prove that $M = \{M_t = f(X_t) - A_t,\ 0 \le t \le 1\}$ is an S–martingale. Notice that the properties of f and A ensure the integrability of M.

In order to show that M_t is an S–martingale we will use the techniques developed in [9] for the proof of the extended Itô formula.

Let $t, t' \in [0,1]$, $t < t'$, and $\pi^n = \{t = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = t'\}$ be a refining sequence of partitions of $[t,t']$ whose mesh tends to zero. We will write t_i instead of $t_{i,n}$, for the sake of simplicity. Let F be a $\mathcal{F}_{(t,t']^c}$–measurable and bounded element of $\mathcal{S}$, then

$$
\begin{aligned}
&E\left\{(f(X_{t'}) - f(X_t))\,F\right\}\\
&= E\Big\{\Big(\sum_{i=0}^{n-1}\big(f'(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \frac{1}{2}f''(\bar{X}_i)(X_{t_{i+1}} - X_{t_i})^2\big)\Big)\,F\Big\},
\end{aligned}
$$

where $\bar{X}_i$ denotes a random point between X_{t_i} and $X_{t_{i+1}}$. Define

$$
C_1^n = E\Big\{\big(\sum_{i=0}^{n-1} f'(X_{t_i})(X_{t_{i+1}} - X_{t_i})\big)\,F\Big\},
$$

and

$$
C_2^n = E\Big\{\big(\sum_{i=0}^{n-1} f''(\bar{X}_i)(X_{t_{i+1}} - X_{t_i})^2\big)\,F\Big\}.
$$

Our aim is to prove that

$$
\lim_{n\to\infty} C_1^n = E\Big\{\Big(\int_t^{t'} f''(X_s)\big(\int_0^s D_s u_r dW_r\big)u_s ds\Big)\,F\Big\} \tag{3.8}
$$

and

$$
\lim_{n\to\infty} C_2^n = E\Big\{\big(\int_t^{t'} f''(X_s)u_s^2 ds\big)\,F\Big\}. \tag{3.9}
$$

Since F is $\mathcal{F}_{(t,t']^c}$–measurable, and using the duality between D and δ, we have

$$
\begin{aligned}
C_1^n &= E\Big\{\sum_{i=0}^{n-1} F\,f'(X_{t_i})\big(\int_{t_i}^{t_{i+1}} u_s dW_s\big)\Big\}\\
&= E\Big\{F\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f''(X_{t_i})\big(\int_0^{t_i} D_s u_r dW_r\big)u_s ds\Big\}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
E\Big\{\Big|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}&[f''(X_{t_i})(\int_0^{t_i} D_s u_r dW_r) - f''(X_s)(\int_0^s D_s u_r dW_r)]u_s ds\Big|\Big\} \\
&\le E\Big\{\Big|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f''(X_{t_i})(\int_{t_i}^s D_s u_r DW_r)u_s ds\Big|\Big\} \\
&+ E\Big\{\Big|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} [f''(X_{t_i}) - f''(X_s)](\int_0^s D_s u_r dW_r)u_s ds\Big|\Big\} \\
&\le \|f''\|_\infty \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} E\Big(\Big|\int_{t_i}^s D_s u_r dW_r\Big||u_s|\Big)ds + \beta_n ,
\end{aligned}
$$

where $\beta_n = E\Big\{\Big|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[f''(X_{t_i}) - f''(X_s)](\int_0^s D_s u_r dW_r)u_s ds\Big|\Big\}$.
Using Schwarz's inequality and the isometry of the Skorohod integral, we can majorize the first term in this last inequality by

$$
\begin{aligned}
\|f''\|_\infty\Big(E\int_0^1 u_s^2 ds\Big)^{1/2}\Big(E\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^s |D_s u_r|^2 dr ds \\
+ E\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_0^1\int_{t_i}^s |D_\theta D_s u_r|^2 dr d\theta ds\Big)^{1/2},
\end{aligned}
$$

which tends to zero as n tends to infinity, since $u \in \mathbb{L}^{2,2}$.

In order to estimate the term β_n we proceed as follows. Fix $\varepsilon > 0$ and let $v \in L^4([0,1]\times\Omega)$ such that $\|u - v\|_{L^2([0,1]\times\Omega)} < \varepsilon$. Then we have

$$
\begin{aligned}
\beta_n \le &\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} E\Big\{|f''(X_{t_i}) - f''(X_s)|\,\Big|\int_0^s D_s u_r dW_r\Big||u_s - v_s|\Big\}\,ds \\
&+ \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} E\Big\{|f''(X_{t_i}) - f''(X_s)|\,\Big|\int_0^s D_s u_r dW_r\Big||v_s|\Big\}\,ds \\
&\le 2\|f''\|_\infty\,\{\int_0^1 E(|\int_0^s D_s u_r dW_r|^2)\,ds\}^{1/2}\,\varepsilon \qquad (3.10) \\
&+ \Big\{\int_0^1 E\,(|\int_0^s D_s u_r dW_r|^2)\,ds\Big\}^{1/2}\|v\|_{L^4([0,1]\times\Omega)} \sup_{\substack{|s-r|\le|\pi^n| \\ s,r\in(t,t']}} \Big(E(|f''(X_r) - f''(X_s)|^4)\Big)^{1/4}.
\end{aligned}
$$

The second summand in (3.10) converges to zero as n tends to infinity because $\{f''(X_t),\, 0 \le t \le 1\}$ is continuous in L^p for all p. The proof of (3.8) is now complete.

Let us now prove (3.9). Suppose that $n \geq m$, and for any $i = 1, \dots n$ denote by $t_i^{(m)}$ the point of the partition π^m which is closer to t_i^n from the left. Then we have

$$
\begin{aligned}
&\Big|\sum_{i=0}^{n-1} f''(\bar{X}_i)(X_{t_{i+1}} - X_{t_i})^2 - \int_0^t f''(X_s)u_s^2 ds\Big| \\
&\leq D_1 + D_2 + D_3.
\end{aligned}
$$

where

$$
D_1 = \Big|\sum_{i=0}^{n-1} [f''(\bar{X}_i) - f''(X_{t_i^{(m)}})](X_{t_{i+1}} - X_{t_i})^2\Big|,
$$

$$
D_2 = \Big|\sum_{j=0}^{m-1} \int_{t_j^m}^{t_{j+1}^m} f''(X_{t_j^m})u_s^2 ds - \int_0^t f''(X_s)u_s^2 ds\Big|,
$$

and

$$
D_3 = \Big|\sum_{i=0}^{n-1} f''(X_{t_i^{(m)}})(X_{t_{i+1}} - X_{t_i})^2 - \sum_{j=0}^{m-1} \int_{t_j^m}^{t_{j+1}^m} f''(X_{t_j^m})u_s^2 ds\Big|.
$$

Consider the set $\mathcal{S}_H$ of step processes of the form $v_t = \sum_{i=0}^{n-1} \mathbf{1}_{]s_i, s_{i+1}]}(t)\, F_i$, where $F_i \in \mathcal{S}$. The class $\mathcal{S}_H$ is dense in $\mathbb{L}^{1,2}$. Consequently, given $\varepsilon > 0$, we can find a process $v \in \mathcal{S}_H$ such that $\|u - v\|_{\mathbb{L}^{1,2}} < \varepsilon$. Set $Y_t = \int_0^t v_s dW_s$. Then, we have,

$$
\begin{aligned}
E(D_1) &\leq E\Big(\Big|\sum_{i=0}^{n-1} [f''(\bar{X}_i) - f''(X_{t_i^{(m)}})]\, \{(X_{t_{i+1}} - X_{t_i})^2 - (Y_{t_{i+1}} - Y_{t_i})^2\Big|\Big) \\
&+ E\Big(\Big|\sum_{i=0}^{n-1} [f''(\bar{X}_i) - f''(X_{t_i^{(m)}})]\, (Y_{t_{i_1}} - Y_{t_i})^2\Big|\Big) \\
&\leq 2\|f''\|_\infty \|u - v\|_{\mathbb{L}^{1,2}} \|u + v\|_{\mathbb{L}^{1,2}} + \gamma_n
\end{aligned}
$$

where $\quad \gamma_n = E\Big(\big|\sum_{i=0}^{n-1} [f''(\bar{X}_i) - f''(X_{t_i^{(m)}})](Y_{t_{i+1}} - Y_{t_i})^2\big|\Big)$.

Hence it suffices to show that $\lim_n \gamma_n = 0$. We can replace $f''(\bar{X}_i)$ by $\int_0^1 2(1-\lambda) f''(X_{t_i} + \lambda(X_{t_{i+1}} - X_{t_i})\, d\lambda$. Set $\wedge_i(\lambda) = \big|f''\big(X_{t_i} + \lambda(X_{t_{i+1}} - X_{t_i}\big) - f''(X_{t_i}^{(m)})\big|$. Then we have

$$
\begin{aligned}
\gamma_n &\leq E \sum_{i=0}^{n-1} \int_0^1 2(1-\lambda) \wedge_i(\lambda)(Y_{t_{i+1}} - Y_{t_i})^2\, d\lambda \\
&\leq \delta \sum_{i=0}^{n-1} E\big[(Y_{t_{i+1}} - Y_{t_i})^2\big] \\
&+ \sum_{i=0}^{n-1} \int_0^1 2(1-\lambda) E\big[\wedge_i(\lambda) \mathbf{1}_{\{\wedge_i(\lambda) > \delta\}} (Y_{t_{i+1}} - Y_{t_i})^2\big]\, d\lambda \\
&\leq \delta \|v\|_{\mathbb{L}^{1,2}} \qquad\qquad (3.11) \\
&+ 2\|f''\|_\infty \sum_{i=0}^{n-1} \int_0^1 2(1-\lambda) E\big[\mathbf{1}_{\{|X_{t_i} - X_{t_i^{(m)}} + \lambda(X_{t_{i+1}} - X_{t_i})| > \eta\}} \cdot (Y_{t_{i+1}} - Y_{t_i})^2\big]\, d\lambda,
\end{aligned}
$$

where for any $\delta > 0$, the number $\eta > 0$ is such that if $|x-y| < \eta$ then $|f''(x)-f''(y)| < \delta$. The second summand in (3.11) is bounded by

$$
\begin{aligned}
&\frac{2}{\eta}\|f''\|_\infty \sum_{i=0}^{n-1} \int_0^1 2(1-\lambda) E\Big(|X_{t_i} - X_{t_i^{(m)}} + \lambda(X_{t_{i+1}} - X_{t_i})|\,|Y_{t_{i+1}} - Y_{t_i}|^2\Big)\, d\lambda \\
&\leq \frac{2\|f''\|_\infty}{\eta} \sup_{\lambda\in[0,1]} \sum_{i=0}^{n-1} E\Big(\int_{t_i}^{t_{i+1}} v_s D_s\big[(Y_{t_{i+1}} - Y_{t_i})|X_{t_i} - X_{t_i^{(m)}} + \lambda(X_{t_{i+1}} - X_{t_i})|\big]\, ds\Big) \\
&\leq \frac{2\|f''\|_\infty}{\eta} \sup_{\lambda\in[0,1]} \sum_{i=0}^{n-1} E\Big(\int_{t_i}^{t_{i+1}} (v_s^2 + |v_s \int_{t_i}^{t_{i+1}} D_s v_r dW_r|)\big|X_{t_i} - X_{t_i^{(m)}} + \lambda(X_{t_{i+1}} - X_{t_i})\big|\, ds \\
&+ \int_{t_i}^{t_{i+1}} |v_s||Y_{t_{i+1}} - Y_{t_i}|\,(\lambda|u_s| + |\int_{t_i^{(m)}}^{t_i} D_s u_r dW_r| + |\int_{t_i}^{t_{i+1}} D_s u_r dW_r|)\, ds\Big) \\
&\leq \frac{2\|f''\|_\infty}{\eta}\Big(\sup_{\substack{|r-s|\leq|\pi^m|\\ r,s\in(t,t']}} 2E(|X_s - X_r|) \Big[\|v\|_\infty^2 + \|v\|_\infty \sup_{\substack{\alpha,\beta\\ s\in[\alpha,\beta]}} \Big(E(|\int_\alpha^\beta D_s v_r dW_r|^2)\Big)^{1/2}\Big] \\
&+ \Big[E\big(\sup_{\substack{|r-s|\leq|\pi^m|\\ r,s\in(t,t']}} (|Y_t - Y_s|^2)\big)\Big]^{1/2} \|v\|_\infty \Big[E\big(\int_0^1 |u_s|^2 ds + \\
&+ E\sum_{i=0}^{n-1} \int_0^1 \Big(|\int_{t_i^{(m)}}^{t_i} D_s u_r dW_r|^2 + |\int_{t_i}^{t_{i+1}} D_s u_r dW_r|^2\Big)\, ds\Big],
\end{aligned}
$$

and this expression converges to zero as $m \to \infty$. The arguments used before to show that $\lim_n \beta_n = 0$ imply $\lim_{m\to\infty} D_2 = 0$.

Finally,

$$
D_3 = \Big|\sum_{j=0}^{m-1} f''(X_{t_j^m})\Big\{ \sum_{k:t_k^n\in[t_j^{(m)},t_{j+1}^{(m)})} (X_{t_{k+1}^n} - X_{t_k^n})^2 - \int_{t_j^m}^{t_{j+1}^m} u_s^2 ds\Big\}\Big|
$$

tends to zero in L^2, as n tends to infinity, for each m fixed, as follows from Theorem 5.4 of [9]. This completes the proof of (3.9), and the proposition is established. □

Remark. Notice that the assumption of Proposition 3.8 are not strong enough to ensure the continuity of the processes X_t and M_t (see [9]). In fact, a sufficient condition for the continuity of X_t is $u \in \mathbb{L}^{2,p}$ with $p > 4$, and in this case the proof of Proposition 3.8 could be shortened.

Actually, in this case we can apply the Itô formula (see [9]) and get $M_t = \int_0^t f'(X_s)u_s dW_s$ with $f'(X_s)u_s$ locally in $\mathbb{L}^{1,2}$.

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