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0. Introduction.

The purpose of this paper is to analyze the integrator properties of the Skorohod integral. It has been proved in [2] that the indefinite Skorohod integral, $M_t = \delta(u \cdot \mathbf{1}_{[0,t]})$ has the property $E\{M_t - M_s \mid \mathcal{F}_{]s,t]^c}\} = 0$, for any $0 \leq s < t \leq t_0$, where $\mathcal{F}_{]s,t]^c}$ denotes the σ-field generated by the increments of a standard Brownian motion on $[0, t_0]-]s, t]$. In some sense, this property plays in the anticipating calculus the analogue role of the martingale property in the classical Itô's calculus. Therefore, it is reasonable to attempt the construction of an anticipating integral of processes with respect to M, as the nonadapted counterpart of the stochastic integral with respect to semimartingales. This paper contain several results in this direction.

In Section 2 we prove that, under suitable hypotheses on the process u, the process $M = \{M_t\,,\ 0 \leq t \leq t_0\}$ defines an L^2-stochastic measure on the σ-field of predictable sets (see [1]). Thus, the integral of predictable, bounded processes f with respect to the anticipating process M can be properly defined.

As has been pointed out in [2] in relation with the Skorohod integral, the price to be paid to remove the adaptedness of the integrand is some smoothness property precisely described with the tools of the calculus on the Wiener space. The results of Section 3 confirm this remark. Here two kinds of integrals with respect to M for anticipating processes f are presented. One of them coincides with the Skorohod integral of fu, the second one contains a correction term and is related with Stratonovich anticipating integrals.

1. Preliminaries and notation.

We denote by T the unit interval $[0,1]$. Our basic probability space $(\Omega, \mathcal{F}, P)$ will be the canonical space associated with the standard Brownian motion on T, $\{W_t\,,\ t \in T\}$.

Given $A \in \mathcal{B}(T)$, $\mathcal{F}_A$ is the σ-field generated by $W(B)$, $B \in \mathcal{B}(T)$, $B \subset A$. For $A = [0, t]$ we write $\mathcal{F}_t$ instead of $\mathcal{F}_{[0,t]}$.

Let $C_b^\infty(\mathbb{R}^n)$ be the set of all functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ which are bounded and have bounded and continuous derivatives of any order. The class $\mathcal{S}$ is defined as the set of random variables $F : \Omega \longrightarrow \mathbb{R}$ having an expression

$$F = f\,(W_{t_1}, \ldots, W_{t_n})\,, \tag{1.1}$$

where $f \in C_b^\infty(\mathbb{R}^n)$ and $t_1, \ldots, t_n \in T$. $\mathcal{S}$ is a dense subset of $L^p(\Omega)$, for any $p \in [1, \infty)$. The elements of $\mathcal{S}$ are called *smooth functionals.*

The derivative of a smooth functional F given by (1.1) is the process defined by

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,(W_{t_1}, \ldots, W_{t_n})\;\mathbf{1}_{[0,t_i]}(t)\,.$$

It can be considered as an unbounded operator defined on a dense subset of $L^2(\Omega)$ and taking values on $L^2(T \times \Omega)$. D is a closed operator, its domain is denoted by $\mathbb{D}^{1,2}$. Notice that $\mathbb{D}^{1,2}$ is the adherence of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|D\,F\|_{L^2(T\times\Omega)}\,.$$

The $n - th$ derivative of a smooth functional F is defined by iteration:

$$D^n_{\underline{t}} F = D_{t_1}\,D_{t_2}\ldots D_{t_n}\,F\,,$$

where $\underline{t} = (t_1, \ldots, t_n) \in T^n$.

By analogy with the case $n = 1$ we define the spaces $\mathbf{D}^{n,p}$, $n \geq 1$, $p > 1$, as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{n,p} = \|F\|_{L^p(\Omega)} + \sum_{i=1}^{n} \big\|\|D^i F\|_{L^2(T^i)}\big\|_{L^p(\Omega)}\,.$$

The adjoint operator of D is denoted by δ, and it is called the Skorohod integral. The operator δ is a closed and unbounded operator on $L^2(T \times \Omega)$ taking values on $L^2(\Omega)$ such that:

(i) *Dom* δ is the set of processes $u \in L^2(T \times \Omega)$ for which there exists a constant C, only depending on u, with

$$\left|E\left(\int_T D_t\,F\;u_t\;dt\right)\right| \leq C\,\|F\|_{L^2(\Omega)}\,,$$

for any $F \in \mathcal{S}$.

(ii) If $u \in Dom\,\delta$, $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the integration by parts formula:

$$E(F\,\delta(u)) = E\left(\int_T D_t F\, u_t\, dt\right),$$

for any $F \in \mathcal{S}$.

Most of the interesting properties on the Skorohod integral are proved assuming than u belongs to a class $\mathbb{L}^{1,2}$ of processes strictly contained in $Dom\,\delta$. In general, for $n \geq 1$ and $p > 1$ $\mathbb{L}^{n,p}$ is the space $L^p(T\,;\,\mathbb{D}^{n,p})$. We also denote by $\mathbb{L}^{\infty,p}$ the space $\bigcap_{n\geq 1} \mathbb{L}^{n,p}$, $p > 1$. If u belongs to $\mathbb{L}^{1,2}$, $u\,\mathbf{1}_{[0,t]}$ belongs to $Dom\,\delta$, for any $t \in T$.

We also need to introduce the set $\mathbb{L}^{1,p}_C$ of processes u in $\mathbb{L}^{1,p}$ such that there exists a version of Du satisfying the following conditions

(i) the functions $s \longmapsto D_t\, u_s$ defined on $[0,t]$ and with values on $L^p(\Omega)$ are continuous uniformly in t, and the same property holds (with a different version of Du) for these functions defined on $[t,1]$.

(ii) $\text{ess sup}_{s,t}\ E(|D_s\, u_t|^p) < \infty$.

The space $\mathbb{L}^{1,2}_C$ is connected with the definition of the Stratonovich integral.

Notice that for any process $u \in \mathbb{L}^{1,p}_C$ the limits

$$D_t^+ u_t = \lim_{\varepsilon\downarrow 0}\ D_t\, u_{t+\varepsilon}\,,$$
$$D_t^- u_t = \lim_{\varepsilon\downarrow 0}\ D_t\, u_{t-\varepsilon}\,,$$

are well defined elements of $L^p(T \times \Omega)$. In this situation we will denote by ∇ the operator $\frac{1}{2}(D^+ + D^-)$.

The reader is refered to [2] for an extensive treatement of all questions concerning the anticipating stochastic calculus.

2. Integration of bounded predictable processes with respect to a Skorohod integral.

Let $\mathcal{H}_1$ be the set of measurable processes having a representation

$$f(t,\omega) = \sum_{i=0}^{n-1} c_i \, \mathbf{1}_{A_i}(\omega) \, \mathbf{1}_{]t_i,t_{i+1}]}(t), \tag{2.1}$$

with $c_i \in \mathbf{R}$, $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$, $A_i \in \mathcal{F}_{t_i}$, $i = 0, \ldots, n-1$ and disjoint. For simplicity we will assume $|c_i| \leq 1$.

We also consider the set $\mathcal{H}_1^r$ of measurable processes

$$f(t,\omega) = \sum_{i=0}^{n-1} f_i(\omega) \, \mathbf{1}_{]t_i,t_{i+1}]}(t), \tag{2.2}$$

where f_i, $i = 0, \ldots, n-1$, are $\mathcal{F}_{t_i}$-measurable smooth functionals.

For any process $u \in \mathbb{L}^{1,2}$ we denote by M_t the Skorohod integral $\delta(u \, \mathbf{1}_{[0,t]})$, $t \geq 0$.

We define a mapping $I^M : \mathcal{H}_1 \longrightarrow L^2(\Omega, \mathcal{F}, P)$ by $I^M(f) = \sum_{i=0}^{n-1} c_i \, \mathbf{1}_{A_i}(M_{t_{i+1}} - M_{t_i})$, for f as in (2.1). $I^M(f)$ is said to be the integral of f with respect to M. The following lemma shows that $I^M(f)$ is also a Skorohod integral.

Lemma 2.1. For any $u \in \mathbf{L}^{1,2}$ and $f \in \mathcal{H}_1$, the process fu belongs to $Dom\,\delta$, and $\delta(fu) = I^M(f)$.

Proof: Assume without loss of generality that $f(t,\omega) = \mathbf{1}_A(\omega) \, \mathbf{1}_{]t_1,t_2]}(t)$, with $A \in \mathcal{F}_{t_1}$. There exists a sequence $\{\varphi^n, \, n \geq 1\}$ of $\mathcal{F}_{t_1}$-measurable smooth functionals converging to $\mathbf{1}_A$ in $L^2(\Omega)$, as n tends to infinity. Let $f^n(t,\omega) = \varphi^n(\omega) \, \mathbf{1}_{]t_1,t_2]}(t)$. The process $f^n u$ belongs to $\mathbf{L}^{1,2}$ and

$$\delta(f^n u)(\omega) = \int_{t_1}^{t_2} \varphi^n(\omega) \, u_s(u) \, dW_s = \varphi^n(\omega)\big(M_{t_2}(\omega) - M_{t_1}(\omega)\big).$$

Moreover, for any $G \in \mathcal{S}$

$$E(G \, \delta(f^n u)) = E(G \varphi^n(M_{t_2} - M_{t_1})) = E\Big(\int_T (D_t G) \, f_t^n \, u_t \, dt\Big). \tag{2.3}$$

Taking the limit as n tend to infinity in (2.3) we get

$$E(G \, \mathbf{1}_A(M_{t_2} - M_{t_1})) = E\Big(\int_{t_1}^{t_2} D_t G \, \mathbf{1}_A \, u_t \, dt\Big).$$

Consequently

$$\Big| E\Big(\int_T D_t G \, f_t \, u_t \, dt \Big) \Big| \le C \, \|G\|_{L^2(\Omega)} ,$$

with a constant C depending only on f and u.

This implies $fu \in Dom\, \delta$ and $\delta(fu) = \mathbf{1}_A (M_{t_2} - M_{t_1})$. ∎

The next result is the essential ingredient to extend the mapping I^M to the class of predictable and uniformly bounded processes.

Proposition 2.2. Let f be a process belonging to $\mathcal{H}_1$, and $u \in \mathbb{L}^{\infty,2}$. Assume that

$$\sup_n \frac{1}{n!} \, \|D^n u\|^2_{L^2(T^{n+1} \times \Omega)} < +\infty . \tag{2.4}$$

Then

$$\|I^M(f)\|_{L^2(\Omega)} \le C , \tag{2.5}$$

where C is a constant only depending on u.

Proof: Assume first that f belongs to $\mathcal{H}_1^r$. We will establish that

$$\|I^M(f)\|_{L^2(\Omega)} \le (1 + \sqrt{2}) a , \tag{2.6}$$

with

$$a^2 = \sup_n \, E\Big\{ \int_0^1 \int_0^{s_1} \cdots \int_0^{s_n} \, |f_{s_{n+1}} \, D^n_{s_n \cdots s_1} \, u_{s_{n+1}}|^2 \, d s_{n+1} \ldots d_{s_2} \, d_{s_1} \Big\} .$$

The processes fu and $f \, D^n_{\underline{s}} u$, for any fixed $\underline{s}$ and any $n \ge 1$, belong to $\mathbb{L}^{1,2}$ .Using the isometry property of the Skorohod integral (see Proposition 3.1 [3]) and the duality between the operators D and δ we get

$$\|I^M(f)\|^2_{L^2(\Omega)} = E \int_0^1 u_s^2 \, f_s^2 \, ds + 2E \int_0^1 u_s \, f_s \Big(\int_0^s f_r \, D_s u_r \, dW_r \Big) ds . \tag{2.7}$$

Indeed,

$$\begin{aligned} \|I^M(f)\|^2_{L^2(\Omega)} &= E\big(\delta(fu)\big)^2 \\ &= E \int_0^1 u_s^2 \, f_s^2 \, ds + 2E \int_0^1 \Big(\int_0^t \{ D_s(f_t u_t) \, D_t(f_s u_s) \} ds \Big) dt , \end{aligned}$$

and, on the other hand,

$$\int_0^t E\left\{D_s(f_t\, u_t) D_t(f_s u_s)\right\} ds = E\left\{f_t\, u_t \int_0^t D_t(f_s\, u_s) dW_s\right\}$$
$$= E\left\{f_t\, u_t \int_0^t f_s\, D_t\, u_s\, dW_s\right\}.$$

Therefore we obtain (2.7).

By Schwarz's inequality,

$$\|I^M(f)\|^2_{L^2(\Omega)} \leq \|u\, f\|^2_{L^2(T\times\Omega)} + 2\,\|u\, f\|_{L^2(T\times\Omega)} \left(\int_0^1 E\Big(\int_0^{s_1} f_{s_2}\, D_{s_1}\, u_{s_2}\, dW_{s_2}\Big)^2 ds_1\right)^{1/2}.$$

We can apply again the identity (2.7) replacing the process fu by $f D_{s_1} u \mathbf{1}_{[0,s_1]}$. Then

$$\|I^M(f)\|^2_{L^2(\Omega)} \leq \|u\, f\|^2_{L^2(T\times\Omega)} + 2\,\|u\, f\|_{L^2(T\times\Omega)}$$
$$\Big(E\int_0^1\int_0^{s_1} |f_{s_2}\; D_{s_1}\; u_{s_2}|^2\, ds_1\; ds_2 + 2\,(E\int_0^1\int_0^{s_1} |f_{s_2}\; D_{s_1}\; u_{s_2}|^2\, ds_1\; ds_2)^{1/2}$$
$$\cdot\,(E\int_0^1\int_0^{s_1}(\int_0^{s_2} f_{s_3}\; D^2_{s_2\, s_1}\; u_{s_3}\; dW_{s_3})^2 ds_2\; ds_1\big)^{1/2}\Big)^{1/2}.$$

Let

$$a_{n+1} = \left(E\int_0^1\int_0^{s_1}\dots\int_0^{s_n} |f_{s_{n+1}}\; D^n_{s_n\dots s_1}\; u_{s_{n+1}}|^2\; ds_{n+1}\dots ds_2\, ds_1\right)^{1/2},$$

$n \geq 0$.

Then, recursively we get

$$\|I^M(f)\|_{L^2(\Omega)} \leq \sqrt{a_1^2 + 2a_1\sqrt{a_2^2 + 2a_2\sqrt{\dots\sqrt{a_n^2 + 2a_n\, A_n}}}}\,,$$

where

$$A_n^2 = \int_0^1\int_0^{s_1}\dots\int_0^{s_{n-1}} E\Big(\int_0^{s_n} f_{s_{n+1}}\; D^n_{s_n\dots s_1}\; u_{s_{n+1}}\; dW_{s_{n+1}}\Big)^2 ds_n\dots ds_2\, ds_1\,.$$

Notice that

$$a^2 \leq \|f\|^2_\infty\; \sup_n\; \frac{1}{n!}\; \|D^n u\|^2_{L^2(T^{n+1}\times\Omega)}\,,$$

thus, hypothesis (2.4) ensures that a is finite and we can write

$$\|I^M(f)\|_{L^2(\Omega)} \leq \sqrt{a^2 + 2a\sqrt{a^2 + 2a\sqrt{\dots\sqrt{a^2 + 2a\, A_n}}}}\,. \tag{2.8}$$

Let $f : [0, \infty) \longrightarrow [0, \infty)$ be defined by $f(x) = (a^2 + 2ax)^{1/2}$. The function f is increasing and concave. Moreover $f([a,\infty)) \subset [a,\infty)$ and on $[a,\infty)$, $|f'(x)| < 1$. Hence, the rectriction of f on the set $[a,\infty)$ is a contraction and therefore there exists a unique fixed point $x_0 \in [a,\infty)$ $\big(f(x_0) = x_0\big)$. It is immediate to check that $x_0 = (1+\sqrt{2})a$. Notice that $x_0 > a$.

It follows from (2.8) that

$$\|I^M(f)\|_{L^2(\Omega)} \leq f^n(A_n), \tag{2.9}$$

for any $n \geq 1$, with the usual notation $f^n = f \circ \overset{(n}{\ldots} \circ f$.

We prove in Lemma 2.3 that hypothesis (2.4) implies $\sup_n A_n < +\infty$. Now two situations are possible:

either

(i) There exists $k \geq 1$ such that $A_k < x_0$,

or

(ii) $A_k \geq x_0$, for any $k \geq 1$.

In case (i), we have

$$\begin{aligned}
\|I^M(f)\|_{L^2(\Omega)} &\leq f^k(A_k) && \text{(by (2.9))}\\
&\leq f^k(x_0) && \text{(by the monotony of } f)\\
&= (1+\sqrt{2})\,a.
\end{aligned}$$

Assume now that condition (ii) is satisfied. We want to show that

$$\lim_{n\to\infty} f^n(A_n) = x_0. \tag{2.10}$$

Indeed,

$$\begin{aligned}
|f^n(A_n) - x_0| &= |f^n(A_n) - f(x_0)|\\
&\leq K\ |f^{n-1}(A_n) - x_0|,
\end{aligned}$$

with $K < 1$.

Recursively, we obtain

$$|f^n(A_n) - x_0| \leq K^n\ |A_n - x_0|,$$

since $\sup_n A_n < +\infty$, the convergence (2.10) follows.

Consequently

$$\|I^M(f)\|_{L^2(\Omega)} \leq \lim_{n\to\infty} f^n(A_n) = x_0 = (1+\sqrt{2})\,a,$$

and the proof of the inequality (2.6) is complete.

Assume now that $f \in \mathcal{H}_1$. There exists a sequence $\{f^m, \ m \geq 1\}$ of processes in $\mathcal{H}_1^r$, bounded by 1, such that $f^m u \longrightarrow fu$ in $L^2(T \times \Omega)$, as $m \to \infty$. From inequality (2.6) it follows that

$$\|I^M(f^m)\|^2_{L^2(\Omega)} \leq (1+\sqrt{2})^2 \sup_n E\Big\{ \int_0^1 \int_0^{s_1} \cdots \int_0^{s_n} |f^m_{s_{n+1}} D^n_{s_n \ldots s_1} u_{s_{n+1}}|^2 ds_{n+1} \ldots ds_2\, ds_1 \Big\}$$
$$\leq (1+\sqrt{2})^2 \sup_n \frac{1}{n!} \|D^n u\|^2_{L^2(T^{n+1} \times \Omega)}.$$

Using the fact that δ is closed and taking the limit, as $m \to \infty$ we obtain (2.5) with $C^2 = (1+\sqrt{2})^2 \ \sup_n \frac{1}{n!} \|D^n u\|^2_{L^2(T^{n+1}\times\Omega)}$. The proof of the proposition is now complete. ∎

Lemma 2.3. Let $f \in \mathcal{H}_1^r$ and $u \in \mathbf{L}^{\infty,2}$. Then the sequence

$$\Big\{ A_n^2 = \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} E\Big(\int_0^{s_n} f_{s_{n+1}} D^n_{s_n \ldots s_1} u_{s_{n+1}} dW_{s_{n+1}} \Big)^2 ds_n \ldots ds_2\, ds_1 , \ n \geq 0 \Big\}$$

is bounded.

Proof: By the isometry property of the Skorohod integral we have

$$\begin{aligned} A_n^2 &\leq E \int_0^1 \int_0^{s_1} \cdots \int_0^{s_n} |f_{s_{n+1}} D^n_{s_n \ldots s_1} u_{s_{n+1}}|^2 \, ds_{n+1} \ldots ds_2 \, ds_1 \\ &+ 2E \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} \int_0^{s_n} \int_0^{s_n} |f_{s_{n+1}} D^{n+1}_{r s_n \ldots s_1} u_{s_{n+1}}|^2 \, ds_{n+1} \, dr \, ds_n \ldots ds_2 \, ds_1 \\ &+ 2E \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} \int_0^{s_n} \int_0^{s_{n+1}} |D_r f_{s_{n+1}} D^n_{s_n \ldots s_1} u_{s_{n+1}}|^2 \, dr \, ds_{n+1} \, ds_n \ldots ds_2 \, ds_1 \\ &= b_1^n + 2b_2^n + 2b_3^n . \end{aligned}$$

We want to prove that $\sup_n b_i^n < \infty$, $i = 1, 2, 3$.

For the sequence $\{b_1^n , \ n \geq 0\}$ we can write

$$b_1^n \leq \|f\|^2_\infty \ \frac{1}{n!} \ \|D^n u\|^2_{L^2(T^{n+1}\times\Omega)} .$$

We also have

$$\begin{aligned} b_2^n &\leq \|f\|^2_\infty \ E \int_0^1 \Big(\int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} \int_0^{s_n} |D^{n+1}_{r\, s_n \ldots s_1} u_{s_{n+1}}|^2 \, dr \, ds_n \ldots ds_2 \, ds_1 \Big) ds_{n+1} \\ &\leq \|f\|^2_\infty \ \frac{1}{(n+1)!} \ \|D^{n+1} u\|^2_{L^2(T^{n+2}\times\Omega)} , \end{aligned}$$

and

$$b_3^n \leq \|Df\|_\infty^2 \; \frac{1}{n\,!} \; \|D^n\, u\|^2_{L^2(T^{n+1}\times\Omega)} \,.$$

Therefore, the boundedness of these sequences is ensured by hypothesis (2.4). ∎

Remark. Let $u_t = \sum_{m=0}^\infty I_m\big(f_m(\,\cdot\,;t)\big)$ be the Wiener chaos expansion of the random variable u_t. Condition (2.4) is equivalent to the following one

$$\sup_n \sum_{m=n}^\infty m!\binom{m}{n}\|f_m\|^2_{L^2(T^{m+1})} < +\infty \,.$$

We now give the main result of this section

Theorem 2.4. Let u be a process satisfying the hypotheses of Proposition 2.2, then

(a) The mapping $I^M : \mathcal{H}_1 \longrightarrow L^2(\Omega, \mathcal{F}, P)$, given by $I^M(f) = \delta(fu)$ can be extended to the set $\mathcal{P}$ of previsible and uniformly bounded processes. This extension will be still denoted by I^M.

(b) For any process f belonging to $\mathcal{P}$, the process fu also belongs to *Dom* δ and $I^M(f) = \delta(fu)$.

For $f \in \mathcal{P}$ we call $I^M(f)$ the integral of f with respect to M.

Proof: From Proposition 2.2 it follows that the set $\{I^M(f)\,,\ f \in \mathcal{H}_1\}$ is bounded in $L^2(\Omega)$.

On the other hand, if $f(t,\omega) = 1_A(\omega)\ 1_{]t_1,t_2]}(\omega)$ with $A \in \mathcal{F}_{t_1}$, the isometry property of the Skorohod integral implies that

$$\lim_{|t_2-t_1|\to 0} \ I^M(f) = 0\,,$$

in L^2. Consequently, the mapping $I^M : \mathcal{H}_1 \longrightarrow L^2(\Omega,\mathcal{F},P)$, $I^M(f) = \delta(fu)$ defines an L^2-stochastic measure on the σ-field of predictable sets (see [1]) and part (a) of the Theorem is proved.

Let $f \in \mathcal{P}$ and $\{f^n\,,\ n \geq 1\}$ a sequence of processes in $\mathcal{H}_1$ such that $f^n \longrightarrow f$ a.s. as $n \to \infty$ and in $L^p(T\times\Omega)$, $p \geq 1$.

By the results on L^p-stochastic measures we know that $I^M(f) = L^2 - \lim_{n\to\infty}\ I^M(f^n)$. On the other hand $I^M(f^n)$ equals $\delta(f^n u)$. Since δ is a closable operator on $L^2(T\times\Omega)$ taking values on $L^2(\Omega)$, we conclude that fu belongs to *Dom* δ and $\delta(fu) = I^M(f)$. This finishes the proof of the Theorem. ∎

3. Integration of non adapted processes with respect to a Skorohod integral.

In this section p will be a fixed real number, $p > 2$, $u = \{u_t\,,\ t \in T\}$ a measurable process belonging to $\mathbb{L}^{1,p}$, and $f = \{f_t\,,\ t \in T\}$ a measurable process in $\mathbb{L}^{1,q}$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. As in the preceding section M_t denotes the Skorohod integral $\delta(u\,\mathbf{1}_{[0,t]})$, for any $t \in T$. Our aim is to show how we can define an integral of f_t with respect to M_t.

We introduce the class of measurable processes $\mathcal{H}_2$ having a representation of the form

$$f(t,\omega) = \sum_{i=0}^{n-1} f_i(\omega)\ \mathbf{1}_{]t_i,t_{i+1}]}(t)\,, \tag{3.1}$$

where $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$, $f_i \in \mathbb{D}^{1,q}$ and f_i $\mathcal{F}_{]t_i,t_{i+1}]^c}$-measurable, for any $i = 0, \ldots, n-1$.

A natural definition of the integral of a process $f \in \mathcal{H}_2$ with respect to M_t is

$$I^M(f) =: \sum_{i=0}^{n-1} f_i(M_{t_{i+1}} - M_{t_i})\,, \tag{3.2}$$

whenever f is given by (3.1).

Notice that due to the measurability properties of f_i, it holds that $I^M(f) = \delta(fu)$.

The definition (3.2) gives rise to a linear mapping

$$I^M : \mathcal{H}_2 \longrightarrow L^2(\Omega)\,.$$

Moreover, the L^2-norm inequalities for the Skorohod integral imply

$$\|I^M(f)\|_{L^2(\Omega)} \leq \|f\|_{\mathbb{L}^{1,q}} \|u\|_{\mathbb{L}^{1,p}}\,. \tag{3.3}$$

Consequently I^M can be extended to a linear mapping on the closure of $\mathcal{H}_2$ with respect to the norm $\|\cdot\|_{\mathbb{L}^{1,q}}$. We will denote again by I^M this extension.

Theorem 3.1. Given $f \in \mathbb{L}^{1,q}$, there exists a sequence $\{f_n\,,\ n \geq 1\}$ of elements in $\mathcal{H}_2$ such that $f_n \to f$, as $n \to \infty$, in the $\mathbb{L}^{1,q}$ norm. Hence $I^M(f)$ is given by the L^2-limit of the sequence $\delta(f^n u)$, and coincides with $\delta(fu)$.

Proof: Denote by $\{\Pi^n\,,\ n \geq 1\}$ a sequence of partitions of T, $\Pi^n = \{0 = t_0^n < t_1^n < \ldots < t_n^n = 1\}$ with $|\Pi^n| \to 0$, as $n \to \infty$. In the sequel the superscript n will be omitted. For each $n \geq 1$ we define

$$f^n(t) = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left(\int_{t_i}^{t_{i+1}} E\left(f_s | \mathcal{F}_{]t_i,t_{i+1}]^c}\right) ds \right) \mathbf{1}_{]t_i,t_{i+1}]}(t)\,. \tag{3.4}$$

Then

$$D_r f^n(t) = \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \Big(\int_{t_i}^{t_{i+1}} E(D_r f_s \,|\, \mathcal{F}_{]t_i,t_{i+1}]^c}) ds \Big) \mathbf{1}_{]t_i,t_{i+1}]}(t)\ \mathbf{1}_{]t_i,t_{i+1}]^c}(r). \quad (3.5)$$

Notice that f^n is the conditional expectation of f in the probability space $\big(T\times\Omega\,,\ \mathcal{B}(T)\otimes\mathcal{F}\,,\ \lambda\times P\big)$ with respect to the σ-field $\mathcal{G}^n\subset\mathcal{B}(T)\otimes\mathcal{F}$ generated by $\{]t_i^n\,,\ t_{i+1}^n]\times F_i^n\,, F_i^n\in\mathcal{F}_{]t_i^n,t_{i+1}^n]^c}\,,\ 0\le i\le n-1\}$, therefore by the results on martingale convergence we get $f^n \overset{L^q(T\times\Omega)}{\longrightarrow} f$, as n tends to ∞.

A similar argument holds for the convergence of the derivatives, (see Lemma 4.2 [2]). Hence $f_n \longrightarrow f$, as $n\to\infty$ in the $\mathbb{L}^{1,q}$ norm. This convergence also implies $f_n u \longrightarrow fu$, as $n\to\infty$ in $\mathbb{L}^{1,2}$. Thus $\delta(f_n u)\longrightarrow\delta(fu)$, as $n\to\infty$ in $L^2(\Omega)$, and the proof is complete. ∎

A more natural approach to the problem of defining the integral of a process f with respect to M_t could be as follows.

Consider the class of measurable processes $\mathcal{H}_3$ having a representation of the form

$$f(t,\omega) = \sum_{i=0}^{n-1} f_i(\omega)\ \mathbf{1}_{]t_i,t_{i+1}]}(t) \quad (3.6)$$

where $0=t_0<t_1<\ldots<t_{n-1}<t_n=1$, and $f_i\in\mathbb{D}^{1,q}$.

We define $I^M(f)$ by (3.2). In this situation $I^M(f)$ is no more a Skorohod integral. In fact, we have

$$I^M(f) = \delta(fu) + \int_0^1 \Big(\sum_{i=0}^{n-1} (D_s f_i)\ \mathbf{1}_{]t_i,t_{i+1}]}(s) \Big) u_s ds.$$

As an element of $L^q(T\times\Omega)$, the process $\Big\{\sum_{i=0}^{n-1}(D_s f_i)\ \mathbf{1}_{]t_i,t_{i+1}]}(s)\ ,\ s\in T\Big\}$ coincides with $\{(\nabla f)_s\,,\ s\in T\}$, consequently

$$I^M(f) = \delta(fu) + \int_0^1 (\nabla f)_s\, u_s\, ds\ , \quad (3.7)$$

if f is given by (3.6).

The extension of I^M to a larger class than $\mathcal{H}_3$ is based on the next theorem.

Theorem 3.2. Let $f\in\mathbb{L}_C^{1,q}$ and $u\in\mathbb{L}^{1,p}$. There exists a sequence $\{f_n\,,\ n\ge1\}$ of processes in $\mathcal{H}_3$ such that

$$\delta(f^n u) + \int_0^1 (\nabla f^n)_s\, u_s\, ds \longrightarrow \delta(fu) + \int_0^1 (\nabla f)_s\, u_s\, ds\,,$$

in $L^2(\Omega)$, as $n \to \infty$.

Thus, we can define $\tilde{I}^M(f) = L^2 - \lim_{n\to\infty} I^M(f_n)$ and we have

$$\tilde{I}^M(f) = \delta(fu) + \int_0^1 (\nabla f)_s \, u_s \, ds .$$

Proof: Consider a sequence $\{\Pi^n , \ n \geq 1\}$ of partitions of T as have been described in the proof of Theorem 3.1, and define

$$f^n(t) = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left(\int_{t_i}^{t_{i+1}} f_s \, ds \right) \mathbf{1}_{]t_i, t_{i+1}]}(t). \tag{3.8}$$

Then

$$D_r f^n(t) = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left(\int_{t_i}^{t_{i+1}} D_r \, f_s \, ds \right) \mathbf{1}_{]t_i, t_{i+1}]}(t).$$

The sequence $\{f_n , \ n \geq 1\}$ belongs to the class $\mathcal{H}_3$. On the other hand $f^n \longrightarrow f$, as $n \to \infty$ in the $\mathbb{L}^{1,q}$ norm (see Lemma 4.2 [2] for the case $p = 2$). Indeed, $f^n(t)$, $\big(Df^n(t)\big)$ is the conditional expectation of f, $\big($ the H-valued random variable $Df(t)\big)$ in the probability space $\big(T \times \Omega , \ \mathcal{B}(T) \otimes \mathcal{F} , \ \lambda \times P\big)$ with respect to the σ-field generated by $]t_i^n, t_{i+1}^n] \times A$, $A \in \mathcal{F}$, and L^q-convergence theorem of martingales can be applied. Consequently $f^n u \longrightarrow fu$, as $n \to \infty$ in the $\mathbb{L}^{1,2}$ norm, and the $L^2(\Omega)$-convergence of $\delta(f^n u)$ to $\delta(fu)$ follows.

Next we establish the convergence

$$\int_0^1 (\nabla f^n)_s \, u_s ds \xrightarrow{L^2} \int_0^1 (\nabla f)_s \, u_s ds , \tag{3.9}$$

as $n \to \infty$.

To this end, notice that

$$\int_0^1 (\nabla f^n)_s \, u_s ds = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} u_s \Big(\int_{t_i}^{t_{i+1}} D_s \, f_t \; dt \Big) ds .$$

We will prove that

$$E \Big| \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \, u_s \int_s^{t_{i+1}} D_s \, f_t \, dt - \frac{1}{2} \int_0^1 (D_s^+ f_s) \, u_s \, ds \Big|^2 \tag{3.10}$$

converges to zero, as $n \to \infty$. An analogue result with D^- also holds.

The expression (3.10) can be bounded by $I_1 + I_2$, where

$$I_1 = E\Big|\sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \Big(u_s \int_s^{t_{i+1}} (D_s f_t - D_s^+ f_s)\, dt\Big)\, ds\Big|^2,$$

and

$$I_2 = E\Big|\int_0^1 u_s(D_s^+ f_s)\Big(\sum_{i=0}^{n-1} \frac{t_{i+1}-s}{t_{i+1}-t_i} \mathbf{1}_{]t_i,t_{i+1}]}(s)\Big)\, ds - \frac{1}{2}\int_0^1 (D_s^+ f_s)\, u_s\, ds\Big|^2.$$

The term I_1 is majorized by

$$C\|u\|^2_{L^p(T\times\Omega)}\big(\sup_{\substack{s<t\\|s-t|\le|\Pi^n|}} E\{|D_s f_t - D_s^+ f_s|^q\}\big)^{2/q}. \tag{3.11}$$

Indeed, using Hölder's inequality we can write

$$\begin{aligned}
I_1 &= E\Big|\int_0^1 u_s \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \mathbf{1}_{]t_i,t_{i+1}]}(s)\, \Big(\int_0^1 (D_s f_t - D_s^+ f_s)\, \mathbf{1}_{]s,t_{i+1}]}(t)\, dt\Big)\, ds\Big|^2 \\
&\le E\Big|\Big(\int_0^1 |u_s|^{p/2} ds\Big)^{2/p} \Big(\int_0^1 \Big(\sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \mathbf{1}_{]t_i,t_{i+1}]}(s)\Big(\int_0^1 (D_s f_t - D_s^+ fs) \\
&\quad \cdot \mathbf{1}_{]s,t_{i+1}]}(t)\, dt\Big)\Big)^{q/2} ds\Big)^{2/q}\Big|^2 \\
&\le \|u\|^2_{L^p(T\times\Omega)} \Big\{E\Big(\int_0^1 \Big(\sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \mathbf{1}_{]t_i,t_{i+1}]}(s)\Big(\int_0^1 (D_s f_t - D_s^+ f_s) \\
&\quad \cdot \mathbf{1}_{]s,t_{i+1}]}(t)\, dt\Big)\Big)^{q/2} ds\Big)^2\Big\}^{2/q}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&E\Big(\int_0^1 \Big(\sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \mathbf{1}_{]t_i,t_{i+1}]}(s)\Big(\int_0^1 (D_s f_t - D_s^+ f_s)\, \mathbf{1}_{]s,t_{i+1}]}(t)\, dt\Big)\Big)^{q/2} ds\Big)^2 \\
&= E\Big(\int_0^1 \sum_{i=0}^{n-1} \frac{1}{(t_{i+1}-t_i)^{q/2}} \mathbf{1}_{]t_i,t_{i+1}]}(s)\Big(\int_0^1 (D_s f_t - D_s^+ f_s)\, \mathbf{1}_{]s,t_{i+1}]}(t)\, dt\Big)^{q/2} ds\Big)^2 \\
&= E\Big(\sum_{i=0}^{n-1} \frac{1}{(t_{i+1}-t_i)^q}\Big(\int_{t_i}^{t_{i+1}} \Big(\int_0^1 (D_s f_t - D_s^+ f_s)\, \mathbf{1}_{]s,t_{i+1}]}(t)\, dt\Big)^{q/2} ds\Big)\Big)^2 \\
&\le E\Big(\sum_{i=0}^{n-1} \frac{1}{(t_{i+1}-t_i)^{q-1}} \int_{t_i}^{t_{i+1}} \Big(\int_0^1 (D_s f_t - D_s^+ f_s)\, \mathbf{1}_{]s,t_{i+1}]}(t)\, dt\Big)^q ds\Big) \\
&\le E\Big(\sum_{i=0}^{n-1} \frac{1}{(t_{i+1}-t_i)^{q-1}} \int_{t_i}^{t_{i+1}} \Big(\int_s^{t_{i+1}} |D_s f_t - D_s^+ f_s|^q\, dt\Big)(t_{i+1}-s)^{q-1}\, ds\Big) \\
&\le C \sup_{\substack{s<t\\|s-t|\le|\Pi^n|}} E\{|D_s f_t - D_s^+ f_s|^q\}.
\end{aligned}$$

Consequently $\lim_{n\to\infty} I_1 = 0$, due to the definition of $\mathbb{L}_C^{1,q}$.

We also have $\lim_{n\to\infty} I_2 = 0$. In fact the sequence

$$\left\{ \sum_{i=0}^{n-1} \frac{t_{i+1}-s}{t_{i+1}-t_i} \, 1_{]t_i,t_{i+1}]}(s) \quad , \quad n \geq 1 \right\}$$

converges to $\frac{1}{2}$ in the weak topology of $L^2(T)$, and the result follows by dominated convergence. Hence (3.9) is established and the proof of the Theorem is now complete. ∎

Remark.

The integral $\tilde{I}^M(f)$ given in Theorem 3.2 has the feature of a Stratonovich integral (compare (3.7) with equation (7.3) in Theorem 7.3 [2]).

If we replace M_t by the Stratonovich integral $N_t = \int_0^t u_s \circ dW_s$, with appropiate hypotheses on u and f, the integral $\tilde{I}^N(f)$ turns out to be the Stratonovich integral $\int_0^1 (f_s\, u_s) \circ dW_s$.

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