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THE SQUARE PEG PROBLEM

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Abstract

The Square Peg Problem, also known as Toeplitz' Conjecture, is an unsolved problem in the mathematical areas of geometry and topology which states the following: every Jordan curve in the plane inscribes a square.

Although it seems like an innocent statement, many authors throughout the last century have tried, but failed, to solve it. It is proved to be true with certain "smoothness conditions" applied on the curve, but the general case is still an open problem. We intend to give a general historical view of the known approaches and, more specifically, focus on an important result that allowed the Square Peg Problem to be true for a great sort of curves: Walter Stromquist's theorem.

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Chapter 1

Introduction

The Square Peg Problem, also known as the Inscribed Square Problem or Toeplitz' Conjecture, is a still open problem which numerous authors have tried to solve for the last hundred years. It is usually attributed to the German mathematician Otto Toeplitz. The problem asks whether every Jordan curve in the plane has four points which form a square:

Conjecture 1.1 (Square Peg Problem)**.** *Every continuous simple closed curve in the plane* $\gamma : S^1 \longrightarrow \mathbb{R}^2$ *contains four points that are the vertices of a square.*

A continuous simple closed curve in the plane is also called a Jordan curve, and it is the same as a continuous injective map from the unit circle into de plane. In other words, a topological embedding $S^1 \hookrightarrow \mathbb{R}^2$.
Notice that Conjecture 1.1 does not require the square to lie fully inside the curve. We can see an example in Figure 1.1.

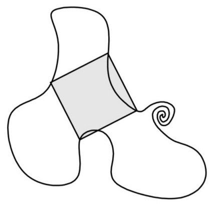

Figure 1.1: Example of the Square Peg Problem

The problem has been solved for curves that are "smooth enough", but remains open for the continuous general case. Several authors found positive results by varying the smoothness condition, which has become weaker over the years. Most of these studies have in common the claim that there is an odd number of inscribed squares in such a smooth curve, since proving this fact is a guarantee of the existence of a square.

But, why is the general case still unsolved? One may think that it is natural to approach the problem by approximating a given continuous curve by a sequence of smooth curves, which, each of them, is known to inscribe a square (using one of the existing smooth results). By converging the sequence we would get, therefore, an inscribed square in the original continuous curve. In other words, given a Jordan curve γ approximated by a sequence γ_n of certain smooth curves, then, since any γ_n inscribes a square Q_n, there is a converging subsequence $(Q_{n_k})_k$ (by compactness) whose limit is an inscribed square for the curve γ. However, it could happen that the approximated inscribed squares degenerate to a point in the limit. Even if the given curve γ is smooth except for a single singular point, all squares in the approximating curves could concentrate into that singular point in the limit. This problem is the reason why it has not been possible to remove the smoothness condition in the known positive results of the problem so far.

The aim of this text is to show some of the proofs obtained throughout these years. We focus on a particular result that had a great impact on the problem's history: Walter Stromquist's theorem. He proved the Square Peg Problem for curves that are "locally monotone", a smoothness condition that is satisfied for curves that are convex, polygon or, under certain restrictions, piecewise of class C^1 (in Section 3.4 we give the accurate definition as well as the mentioned proof). We also provide an abstract of the proof of the first historical result (satisfied for "smooth enough" convex curves, found in Section 3.2.2) and a survey through most of the solutions found during the last hundred of years.

In order to see the previous matter, we have divided the work into the following three parts:

In Chapter 2, we recall the definition of Jordan curve, state the Jordan curve theorem and give a short proof of it.

We start Chapter 3 by giving a deeper historical review of the so far obtained results. We complement this part by showing some easy results as a motivation of the problem and then proceed to overview two of the historical proofs which meant an important progress of the Square Peg Problem.
At the end of the chapter, we provide some topological tools required to follow Stromquist's proof. These are from simplicial and singular homology. We conclude Chapter 3 by showing Walter Stromquist's proof, which is our main object of study.

We have added an extra chapter, Chapter 4, to remark the furthest result of the Square Peg Problem, provided by the mathematician Terence Tao. It is a quick review of his work, since its difficulty escapes from the degree's level.

Chapter 2

Jordan Curve Theorem

The Square Peg Problem is stated for a Jordan curve, as we have seen in the first chapter. There is a classical theorem in topology which asserts an essential fact about Jordan curves: the Jordan curve theorem. This theorem was originally conjectured by Camile Jordan in 1892.
In this chapter, we will introduce the Jordan curve theorem as well as a short proof of it, using the theorem of the Brouwer fixed point. This ingenious proof belongs to Ryuji Maehara, who wrote it in [6]. But, before we start, let us give the definition of a Jordan curve:

Definition 2.1. *A Jordan curve is a continuous simple closed curve in* $\mathbb{R}^2$*. That is, the image of an injective continuous map of the unit circle into the plane:*

$$\gamma : S^1 \longrightarrow \mathbb{R}^2.$$

Thus, a Jordan curve is the homeomorphic image of a circle.
We will also be talking about "arcs". We denote by **arc** the homeomorphic image of a closed interval $[a, b]$, where $a < b$.

Now we are ready to introduce the chapter's main theorem:

Theorem 2.2 (Jordan Curve Theorem)**.** *The complement in the plane* $\mathbb{R}^2$ *of a Jordan curve* J *consists of two components, each of which has* J *as its boundary.*

In other words, the Jordan curve theorem states something geometrically visual: a simple closed curve always divides the plane into two parts, the outer and the inner part. Let us note two facts concerning the components of $\mathbb{R}^2 - J$, where J is the Jordan curve:

1. $\mathbb{R}^2 - J$ has exactly one unbounded component.

2. Each component of $\mathbb{R}^2 - J$ is path-connected and open.

The assertion (1) follows from the boundedness of J, and (2) from the local path-connectedness of $\mathbb{R}^2$ and the closedness of J.

In order to prove the Jordan curve theorem, we will first introduce two theorems: the Brouwer fixed point and Tietze extension theorem. The Brouwer fixed point theorem will be used during the main proof, as we mentioned before, while Tietze extension theorem will be applied to prove our first lemma (Lemma 2.5, found below).

Theorem 2.3 (Brouwer Fixed Point Theorem)**.** *Every continuous map from a disk into itself has a fixed point.*

Theorem 2.4 (Tietze Extension Theorem)**.** *Let X be a normal space and A a closed subspace of X.*

1. *Any continuous map from A to the closed interval $[a, b]$ of $\mathbb{R}$ can be extended to a map from all X to $[a, b]$.*

2. *Any continuous map from A to $\mathbb{R}$ can be extended to a continuous map from all X to $\mathbb{R}$.*

We will now introduce and prove two lemmas which will be needed during Jordan curve theorem's proof. Once we have proved them, we will finally proceed with the chapter's main proof.

Lemma 2.5. *If $\mathbb{R}^2 - J$ is not connected, then each component has J as its boundary.*

Proof. By assumption, $\mathbb{R}^2 - J$ has at least two components. Let U be an arbitrary component. Since any other component W is disjoint from U and is open, we have that W contains no point of the closure $\overline{U}$ and, hence, no point of the boundary $\overline{U} \cap U^c$ of U. Thus, $\overline{U} \cap U^c \subset J$.

Now we suppose that $\overline{U} \cap U^c \neq J$. We will show that this leads to a contradiction.

Since $\overline{U} \cap U^c \neq J$, there exists an arc $A \subset J$ such that $\overline{U} \cap U^c \subset A$. By (1), we know that $\mathbb{R}^2 - J$ has at least one bounded component. Let o be a point in a bounded component. If U itself is bounded, we choose o in U.

Let D be a large disk with center o such that its interior contains J. Then the boundary of D, which we will note by S, is contained in the unbounded component of $\mathbb{R}^2 - J$. Since A is an arc, we know it is homeomorphic to the interval $[0, 1]$. By the *Tietze Extension Theorem*, the identity map $A \longrightarrow A$ has a continuous extension $r : D \longrightarrow A$. This is possible since D is a normal space and A is a closed interval of $\mathbb{R}$.

We define a map $q : D \longrightarrow D - \{o\}$, according as if U is bounded or not. If U is bounded we define it by

$$q(z) = \begin{cases} r(z) & for\ z \in \overline{U}, \\ z & for\ z \in U^c, \end{cases}$$

Otherwise by

$$q(z) = \begin{cases} z & for\ z \in \overline{U}, \\ r(z) & for\ z \in U^c. \end{cases}$$

By $\overline{U} \cap U^c \subset A$, the intersection of the two closed sets $\overline{U}$ and U^c lies on A, on which r is the identity map. Thus, q is well defined and continuous. Note that $q(z) = z$ if $z \in S$.

Now let $p : D - \{o\} \longrightarrow S$ be the natural projection and let $t : S \longrightarrow S$ be the antipodal map. Then the composition $t \circ p \circ q : D \longrightarrow S \subset D$ has no fixed point. This contradicts the Brouwer fixed point theorem and, therefore, we prove the lemma. □

The previous proof implicity contains a proof that no arc separates $\mathbb{R}^2$. This is often a lemma to the Jordan curve theorem.

Before getting into the next lemma, let us denote the rectangular set $\{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ in the plane $\mathbb{R}^2$ as $E(a, b; c, d)$, where $a < b$ and $c < d$.

Lemma 2.6. *Let $h(t) = (h_1(t), h_2(t))$ and $v(t) = (v_1(t), v_2(t))$, where $-1 \leq t \leq 1$, be continuous paths in $E(a, b; c, d)$ satisfying*

$$h_1(-1) = a,\ h_1(1) = b,\ v_2(-1) = c,\ v_2(1) = d.$$

Then the two paths meet, i.e., $h(s) = v(t)$ for some s,t in $[-1, 1]$.

In other words, when $t = -1$, $h(-1) = (a, h_2(-1))$ and $v(-1) = (v_1(-1), c)$. Whereas when $t = 1$, we have $h(1) = (b, h_2(1))$ and $v(1) = (v_1(1), d)$. This means that $h(t)$ will trace its path horizontally from a to b, while $v(t)$ will do it vertically from c to d. Then, both will meet at some point.

Proof. Suppose $h(s) \neq v(t)$ for all s, t. Let $N(s, t)$ denote the maximum-norm of $h(s)-v(t)$

$$N(s, t) = max\{|h_1(s) - v_1(t)|,\ |h_2(s) - v_2(t)|\}.$$

Then $N(s, t) \neq 0$ for all s, t. We define a continuous map F from $E(-1, 1; -1, 1)$ into itself by

$$F(s, t) = \left(\frac{v_1(t) - h_1(s)}{N(s, t)}, \frac{h_2(s) - v_2(t)}{N(s, t)} \right).$$

Notice that if $N(s, t)$ takes the value $|h_1(s) - v_1(t)|$, then

$$F(s, t) = \left(\pm 1, \frac{h_2(s) - v_2(t)}{|h_1(s) - v_1(t)|} \right).$$

While if $N(s, t) = |h_2(s) - v_2(t)|$, then

$$F(s, t) = \left(\frac{v_1(t) - h_1(s)}{|h_2(s) - v_2(t)|}, \pm 1 \right).$$

Therefore, the image of F is in the boundary of $E(-1, 1; -1, 1)$ and, in fact, the map F goes from $E(-1, 1; -1, 1)$ into itself, as we previously said.

We need to see that F has no fixed point, so it contradicts the Brouwer fixed point theorem.
Assume $F(s_0, t_0) = (s_0, t_0)$. By the above remark, we have $|s_0| = 1$ or $|t_0| = 1$. Suppose,

for example, $s_0 = -1$. Since $h_1(-1) = a$, $v_1(t_0) - a \geq 0$. Also, $N(-1, t_0) \geq 0$. Then, the first coordinate of $F(-1, t_0)$, $(v_1(t_0) - h_1(-1)/N(-1, t_0))$, is nonnegative. Hence, it can not equal s_0 $(= -1)$. Similarly, the other cases of $|s_0| = 1$ or $|t_0| = 1$ can not occur. This implies that $F(s, t)$ has no fixed point. But since $E(-1, 1; -1, 1)$ is homeomorphic to a disk, this contradicts the Brouwer fixed point theorem. □

Now that we have seen both lemmas, we can prove the Jordan curve theorem.

Proof (Jordan curve theorem). By Lemma 2.5, we only need to show that $\mathbb{R}^2 - J$ has one and only one bounded component. This is because by assertion (1) $\mathbb{R}^2 - J$ has exactly one unbounded component. So if we see, that besides this one, $\mathbb{R}^2 - J$ has one bounded component, Lemma 2.5 will imply that both components have J as their boundary, which is exactly what Jordan curve theorem says.

The proof will be divided into three parts: establishing the notation and defining a point z_0 in $\mathbb{R}^2 - J$, proving that the component U containing z_0 is bounded and proving that there is no bounded component other than U.

Since J is compact, there exist points a, b in J such that the distance $\|a - b\|$ is the largest. We may assume that $a = (-1, 0)$ and $b = (1, 0)$. Then the rectangular set $E(-1, 1; -2, 2)$ contains J, and its boundary, that we will denote by Γ, meets J at exactly two points a and b, as shown in Figure 2.1.

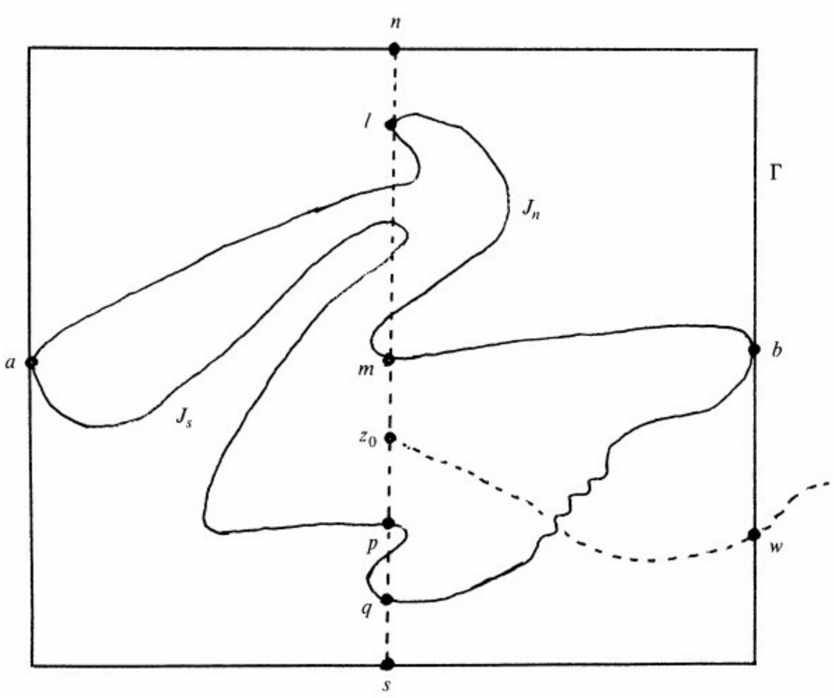

Figure 2.1: Construction of $E(-1, 1; -2, 2)$ containing J

Let n be the middle point of the top side of $E(-1, 1; -2, 2)$, $n = (0, 2)$, and s the middle point of the bottom side, $s = (0, -2)$. The segment $\overline{ns}$ meets J by Lemma 2.6, where we take $h(t)$ as J and $v(t)$ as $\overline{ns}$. Let l be the y-maximal point (that means the point

$(0, y)$ with maximal y) in $J \cap \overline{ns}$. Points a and b divide J into two arcs. We denote the one containing l by J_n, and the other by J_s. Let m be the y-minimal point in $J_n \cap \overline{ns}$. Possibly $l = m$. Then the segment $\overline{ms}$ meets J_s; otherwise the path $\overline{nl} + \widehat{\mathrm{lm}} + \overline{ms}$ (where $\widehat{\mathrm{lm}}$ denotes the subarc of J_n with end points l and m) could not meet J_s, contradicting Lemma 2.6. Let p and q denote the y-maximal point and the y-minimal point in $J_s \cap \overline{ms}$, respectively. Finally, let z_0 be the middle point of the segment $\overline{mp}$.

We have defined our point z_0, which we have ensured to be in $\mathbb{R}^2 - J$ via the previous construction and, more specifically, in the supposed bounded component of $\mathbb{R}^2 - J$. Now we will show that U, the component of $\mathbb{R}^2 - J$ that contains z_0, is, in fact, bounded. To prove so, let us suppose U is unbounded. Since U is path-connected, there exists a path α in U from z_0 to a point outside $E(-1, 1; -2, 2)$. Let w be the first point at which α meets the boundary Γ of $E(-1, 1; -2, 2)$. We denote α_w the part of α from z_0 to w. If w is on the lower half of Γ, we can find a path $\widehat{\mathrm{ws}}$ in Γ from w to s which contains neither a nor b. Now consider the path $\overline{nl} + \widehat{\mathrm{lm}} + \overline{mz_0} + \alpha_w + \widehat{\mathrm{ws}}$. This path does not meet J_s, by hypothesis, contradicting Lemma 2.6. Similarly, if w is on the upper half of Γ, the path $\overline{sz_0} + \alpha_w + \widehat{\mathrm{wn}}$ fails to meet J_n, where $\widehat{\mathrm{wn}}$ is the shortest path in Γ from w to n. The contradiction shows that U is a bounded component.

We have proven the existence of the bounded component. Let us see its uniqueness. Suppose that there exists another bounded component W ($W \neq U$) of $\mathbb{R}^2 - J$. Clearly, $W \subset E(-1, 1; -2, 2)$. We denote by β the path $\overline{nl} + \widehat{\mathrm{lm}} + \overline{mp} + \widehat{\mathrm{pq}} + \overline{qs}$, where $\widehat{\mathrm{pq}}$ is the subarc of J_s, from p to q. Notice that β has no point of W. This is because the path $\overline{mp}$ belongs to U, since it meets z_0, and we have supposed that $W \neq U$. Also, the subarcs $\widehat{\mathrm{lm}}$ and $\widehat{\mathrm{pq}}$ can not be in W because $\widehat{\mathrm{lm}} \subset J_n$ and $\widehat{\mathrm{pq}} \subset J_s$. Since a and b are not on β, there are circular neighborhoods V_a, V_b of a, b, respectively, such that each of them contains no point of β. By Lemma 2.5, a and b are in the closure $\overline{W}$. Hence, there exist $a_1 \in W \cap V_a$ and $b_1 \in W \cap V_b$.
Since W is path-connected, there exists a path from a_1 to b_1. Let $\widehat{a_1 b_1}$ be this path in W. Then, the path $\overline{aa_1} + \widehat{a_1 b_1} + \overline{b_1 b}$ fails to meet β, because we have shown that β has no point of W. But by Lemma 2.6 we know that the path intersects β. This is a contradiction and, therefore, this completes our proof. $\square$

Chapter 3

Square Peg Problem

In this chapter we will focus on the history of the Square Peg Problem. We will start by introducing the most important studies with their chronological order and overviewing some of these proofs. More precisely, we will see in detail the first proof provided by Arnold Emch, so we acknowledge the initial attempts of the problem. At the end of the chapter we will study Walter Stromquist's proof and see how he managed to get one of the weakest smoothness conditions.

3.1 History of the problem

Most part of the historical chronology found in this section has been extracted from the survey [7] made by Benjamin Matschke.

The Square Peg Problem first appeared in 1911. Otto Toeplitz gave a talk whose second part had the title "On some problems in topology", in which he mentioned the problem and claimed to have the solution only for convex curves. However, it seems that Toeplitz never published the proof.

In 1913, Arnold Emch proved it for "smooth enough" convex curves [1], which was the first major result. Two years later, Emch published in [2] a further proof that required a weaker smoothness condition and in which he wrote that he did not know that Toeplitz and his students discovered the problem independently two years earlier. Despite this fact, the conjecture is usually attributed to Toeplitz. Emch presented a third paper in 1916 [3], where he proved the Square Peg Problem for curves that are piecewise analytic with a finite number of inflexions and other singularities. We will show his first proof in the next section (Section 3.2.2).

In 1929, Schnirelmann proved the problem for a class of curves that is slightly larger than C^2: curves with piecewise continuous curvature. This paper was published in a Russian publication, but an extended version, which corrects some minor errors, was published posthumously in 1944 [13]. Schnirelmann noted that for a generic curve, the parity of the number of inscribed squares must be invariant as the curve is deformed. The proof uses a local lemma on existence of inscribed square for closed curves. Heinrich Guggenheimer studied Schnirelmann's proof and claimed that it still contained errors. He corrected several technical points and concluded that the curve needs to have a bounded variation so the proof works.

In 1961, Jerrard [5] examined the case of real-analytic curves (curves whose coordinate functions are each real-analytic) and showed that each such curve admits an odd number of inscribed squares.

In 1989, Walter Stromquist [14] proved one of the furthest results of the Square Peg Problem. He introduced the smoothness condition "locally monotone" which is satisfied when the curve is convex, polygon or, under certain restrictions, piecewise of class C^1. We will study this proof in detail at the end of this chapter (Section 3.4).

Other results are due to Hebbert, when γ is a quadrilateral, Nielsen and Wright for curves that are symmetric across a line or about a point (see Section 3.2.1), Pak [12] for piecewise linear curves, Cantarella, Denne, McCleary for C^1 curves and Matschke [8] for an open dense class of curves (curves that did not contain small trapezoids of a certain form), as well as curves that were contained in annuli in which the ratio between the outer and the inner radius is at most $1 + \sqrt{2}$.

The latest result is due to Terence Tao [15] in 2017, where he proves the Square Peg Problem if the curve is the union of two Lipschitz graphs that agree at the endpoints, and whose Lipschitz constants are strictly less than one. The condition he provides goes a little bit further than the condition of locally monotone. We will overview Tao's paper in the next chapter (Chapter 4).

3.2 Several results

3.2.1 Witty and easy results

Before we get into the important dense results obtained by some of the previously mentioned authors, we would like to offer some quick and clever proofs as a motivation of the problem.

Square inscribed in a function

Suppose we form a simple closed curve by adjoining the segment of the x-axis from $x = 0$ to $x = 1$ to the graph of a continuous function

$$f : [0, 1] \longrightarrow \mathbb{R}$$

such that $f(0) = f(1) = 0$ and $f(x) > 0$, for $0 < x < 1$, as seen in Figure 3.1. Let us see that this curve inscribes a square.

Proof. We will only need the extreme value theorem and the intermediate value theorem in order to proof it.

First, by the extreme value theorem, the function f assumes its maximum value m at some x_1. In other words, $f(x_1) = m$, where $f(x) \leq m$ for any value of x.

Now, we define a function $g : [0, 1] \longrightarrow \mathbb{R}$ such that

$$g(x) = x + f(x).$$

Then, we have

$$g(0) = 0$$
$$g(1) = 1$$

Using the intermediate value theorem, there is some x_2 between 0 and 1 so that

$$g(x_2) = x_1.$$

Now, we define another function h like

$$h(x) = f(x) - f(g(x)).$$

To assure that this definition makes sense, we will consider $f(x) = 0$ for all values of x outside $[0, 1]$. Then, $h(x_1) = f(x_1) - f(g(x_1)) = m - f(g(x_1))$ which is positive or zero, since m is the maximum value of the function f. But we also have $h(x_2) = f(x_2) - f(g(x_2)) = f(x_2) - f(x_1) = f(x_2) - m$, which is negative or zero.
By the intermediate value theorem, there is a value x_0 between x_1 and x_2 such that $h(x_0) = 0$, which is the same as

$$f(x_0) = f(x_0 + f(x_0)).$$

Therefore, the points $x = x_0$ and $x = x_0 + f(x_0)$ on the x-axis are the base corners of the inscribed square, as shown in Figure 3.1. □

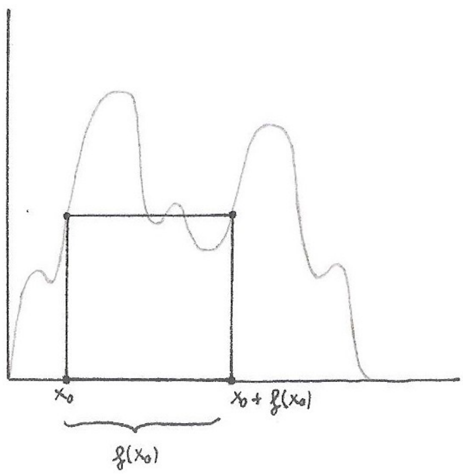

Figure 3.1: Square inscribed in a function

Square inscribed in symmetric curve

In this case, we are going to see the theorem proved by Mark J. Nielsen and S.E. Wright in [11], where the curve is a Jordan curve symmetric about a point.

Theorem 3.1. *Every simple closed curve that is symmetric about the origin has an inscribed square.*

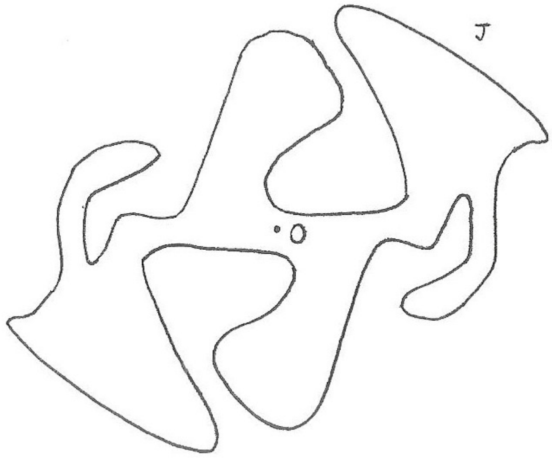

Figure 3.2: Jordan curve symmetric about O

Proof. Let J be a Jordan curve symmetric about the origin O. This means that for each point P on J, the point $-P$ (whose coordinates are negatives of the coordinates of P) is also on J. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the function that rotates each point in the plane 90 degrees about the origin.

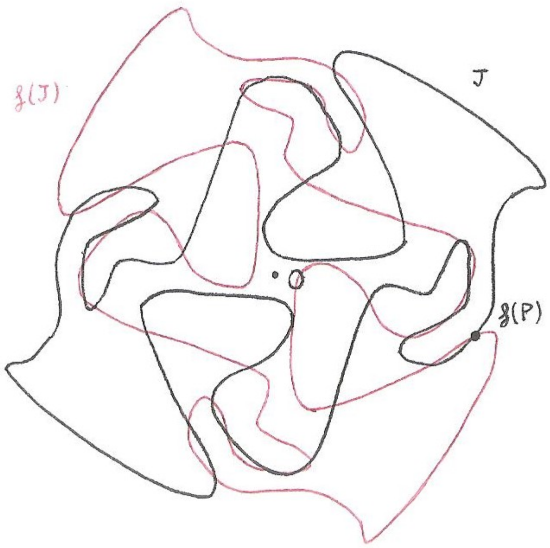

Figure 3.3: J and its rotated image $f(J)$

We can see in Figure 3.3 that J and its rotated image $f(J)$ intersect each other. If we suppose they do, they must share a point in common. Then, if we choose a point $f(P)$ on $f(J)$ that is also on J, the points $P, f(P), -P, -f(P)$ form the vertices of a square (see Figure 3.4). And, since all four points are on J, then we have that J inscribes a square.

All that remains now is to see that J and $f(J)$ intersect. To show this, consider P_{near} and P_{far} to be the points on the curve J that are at minimum and maximum distance from the origin respectively. Then, $f(P_{near})$ is as close to O as any point of J, since f

does not change any distance, and $f(P_{far})$ is as far from O as any point of J. Thus, we have the cases of $f(P_{near})$ being inside or on J and $f(P_{far})$ being outside or on J.
If either of them is on J, we are done, since J and $f(J)$ would intersect on that point. Otherwise, if $f(P_{near})$ is inside J and $f(P_{far})$ is outside J, then $f(J)$ must connect both points, and hence it must cross J somewhere (because of the Jordan curve theorem). □

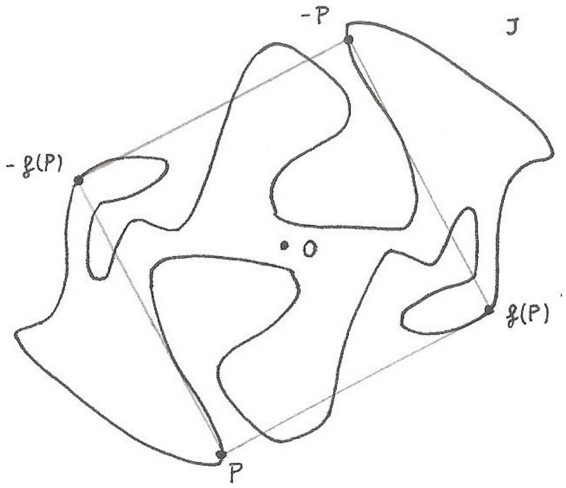

Figure 3.4: $P, f(P), -P, -f(P)$ form the vertices of a square

3.2.2 Arnold Emch

Arnold Emch provided the first result of the Square Peg Problem for convex curves. Here we will give an overview of his paper *Some Properties of Closed Convex Curves in a Plane*, noting the theorems and previous conditions he states before proving the main theorem: *It is always possible to inscribe a square in an oval.*
Emch denotes a "closed convex curve" with the term "oval", as we will define below. The aim of his work, then, is to prove that *at least* one square is inscribed in a convex closed curve.

Let us first introduce his definition of "oval". He considers a *convex domain* as the regular definition of convex domain (a finite, closed domain which contains with any two points also the entire segment between them), indicating that through every point of the boundary of such a domain, there is at least one straight line so that all points of the domain which are not on the line lie only on one side of the line. This line is denoted with the term *supporting line*. Then, an *oval* encloses such a domain, including the oval itself. Emch defines it parametrically by two distinct continuous single-valued periodic functions

$$x = \phi(t), \; y = \psi(t),$$

where $t \in \mathbb{R}$ and both have period w.

Now, he places several restrictions on these functions. First, he considers its derivatives $\phi'(t), \psi'(t)$ w-periodic and continuous for all definite values of t. He excludes singular points by assuming that both derivatives $\phi'(t), \psi'(t)$ do not vanish simultaneously for any values of t and he excludes any straight portions of the boundary by including in the definition of an oval that for no parts of the period-interval, the functions $\phi(t)$ and $\psi(t)$ remain constant or depend linearly upon t.

He also assumes that at every point of the oval there exists a definite tangent and, if the point of tangency varies continuously, the direction of the tangent varies continuously. Also, for $t = 0$ or $t = w$, none of the functions $\phi(t), \psi(t), \phi'(t), \psi'(t)$ vanish and we have $\phi(0) = \phi(w),\ \phi'(0) = \phi'(w),\ \psi(0) = \psi(w),\ \psi'(0) = \psi'(w)$.

Now that we have defined all the conditions on the oval imposed by Emch, we will introduce some theorems he states first in order to prove the main result afterwards.

Theorem 3.2. *There are always two and only two tangents to an oval parallel to any given direction.*

This is true since if there were three distinct parallel tangents t_1, t_2, t_3, with t_2 between the other two, then there would be points belonging to the domain of both sides of t_2, which can not occur (according to the property of a supporting line).

Emch denotes by "reentrant quadrangle" a quadrangle $A_1A_2A_3A_4$ in which one of the vertices lies within the boundary of the triangle formed by the remaining three, as seen in Figure 3.5, where the mentioned vertex corresponds to A_4. This is shown by getting in contradiction with the assertion that all points outside of a segment A_iA_j are excluded from the domain. Then, we have:

Theorem 3.3. *No reentrant quadrangle can be inscribed in an oval.*

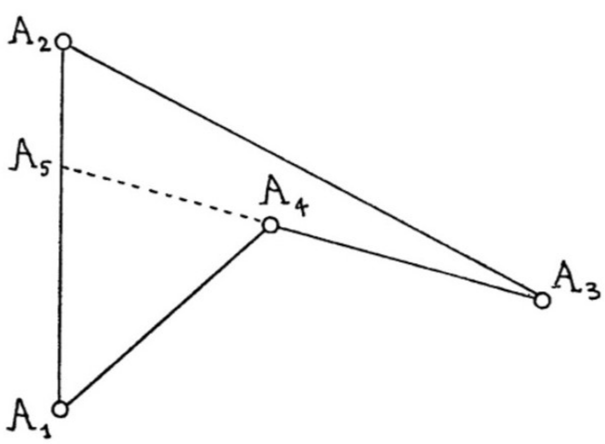

Figure 3.5: Reentrant quadrangle $A_1A_2A_3A_4$.

Theorem 3.4. *Two distinct rhombs with corresponding parallel sides or parallel axes can never be inscribed in the same oval.*

This is proved by using the fact that wherever the two rhombs are placed, four of their vertices will always form a reentrant quadrangle. Thus, no oval can pass through them.

Theorem 3.5. *Let a and b be two distinct real numbers, and let* $\lambda = \phi_1(\theta), \mu = \psi_1(\theta)$ *be two uniform continuous real functions of a parameter* θ*, subject to the only condition that two distinct values* α *and* β *of the parameter* θ *exist, so that*

$$\phi_1(\alpha) = \psi_1(\beta) = a,$$
$$\phi_1(\beta) = \psi_1(\alpha) = b;$$

then there exists at least one value of θ*, say* $\theta = \gamma$*, for which* $\phi_1(\gamma) = \psi_1(\gamma)$*.*

This follows from the fact that $\phi_1(\theta) - \psi_1(\theta)$ is continuous and, hence, takes every value between $a-b$ and $b-a$. That is, there is at least one value of θ, γ, such that $\phi_1(\gamma) = \psi_1(\gamma)$.

Using all the previous results, Emch proves the main theorem:

Theorem 3.6. *It is always possible to inscribe a square in an oval.*

Proof. Emch uses a geometric construction to prove the existence of a rhombus inside an oval, seen in Figure 3.8. He proceeds as follows:
Having our curve placed in the plane, we assume any point O, outside the curve, and we draw any line l_α through this point. Now, we draw the chords of the oval parallel to l_α, determining their respective midpoints, and we add the tangents parallel to l_α as well as their points of tangency, denoted by S_α and T_α. If we connect all the midpoints, we obtain a continuous curve C_α from S_α to T_α. In his later publications, Emch will denote this locus of midpoints by *median*. This construction is seen in Figure 3.6.

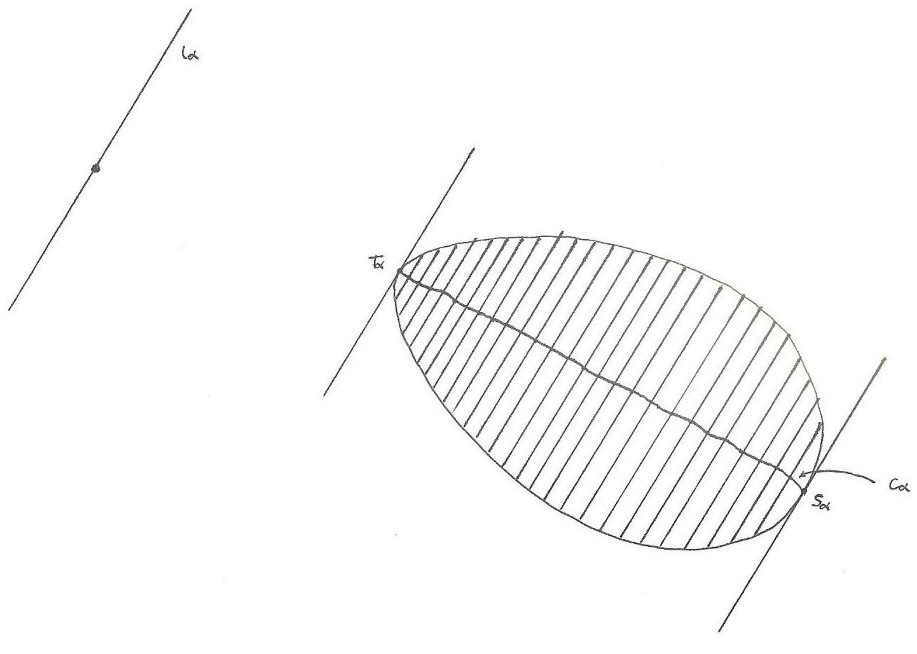

Figure 3.6: Chords of the oval parallel to l_α and the curve C_α through their midpoints.

Next, we draw through O a line l_β perpendicular to l_α, and we repeat the same construction with respect to this line. Thus, we obtain a continuous curve C_β extending from S_β to T_β, as well as the two tangents of the oval parallel to l_β.

The two tangents parallel to l_α and the two parallel to l_β form a rectangle. Also, the curves C_α and C_β necessarily intersect within the domain of the oval. In fact, there is always *only one* real point of intersection between C_α and C_β, that we will denote as $P_{\alpha\beta}$. This is true, since if there were two points of intersection, then there would exist two rhombs with parallel axes inscribed in the oval, in contradiction with Theorem 3.4. We can see the construction of the chords parallel to l_β and the curve C_β in Figure 3.7, where the old construction is on a slightly softer colour and the newer one is on a stronger one.

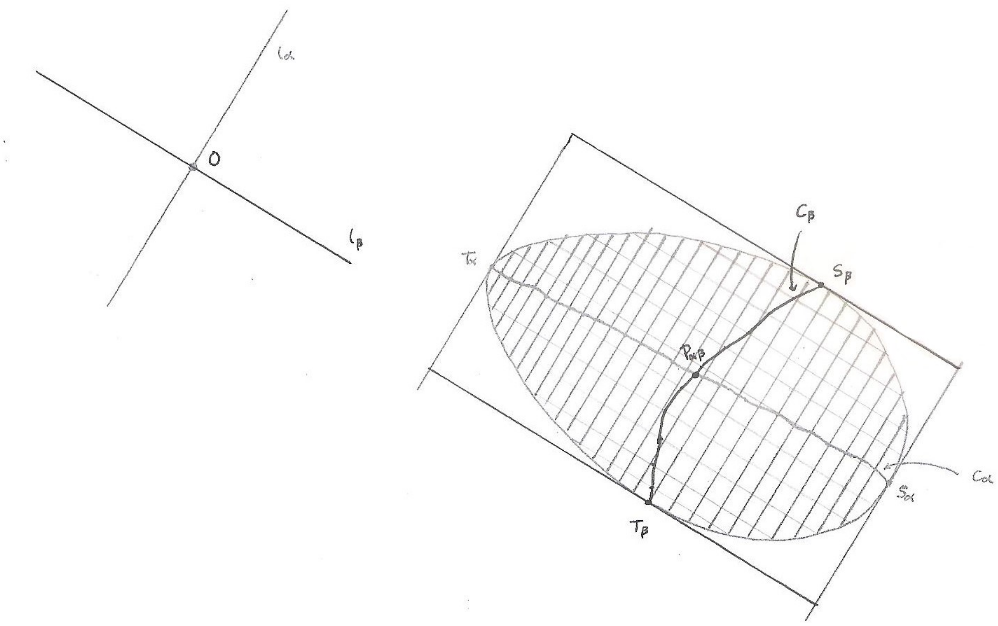

Figure 3.7: Chords of the oval parallel to l_β and the curve C_β through their midpoints.

If we draw the lines parallel to l_α and l_β that pass through $P_{\alpha\beta}$, we see that they intersect the oval at four points A, B, A', B', which form a rhombus $ABA'B'$. Thus, with every pair of orthogonal rays l_α and l_β through O is associated one definite rhombus inscribed in the oval, and this same rhombus is evidently obtained when l_α and l_β are interchanged.

Now that we have shown that a rhombus is inscribed in the curve, the final step is to prove that this rhombus becomes a square.

Emch notes that if we turn a line l_ξ through O continuously from l_α to l_β, then its orthogonal ray l_η will turn in the same sense from l_β to l_α. The corresponding curves C_ξ and C_η also change continuously, since their extremities S_ξ, S_η, T_ξ and T_η on the oval change continuously. Therefore, their point of intersection $P_{\xi\eta}$ describes a continuous curve and hence the corresponding rhombus, denoted by $XYX'Y'$, changes continuously. Then, the axes $\lambda = XX'$ and $\mu = YY'$ of this rhombus may be expressed as uniform and continuous functions of a parameter θ associated with the direction of l_ξ, within the interval between l_α and l_β (including these limits). This θ may be chosen as the positive angle between l_ξ and the positive part of the axis X.

Now, if we designate the diagonals of the original rhombus by a and b, the parameters associated with l_α and l_β by α and β, and the axes of the rhombus by

$$\lambda = \phi(\theta), \quad \mu = \psi(\theta)$$

within the interval $\alpha \leq \theta \leq \beta$, where λ and μ are the uniform and continuous functions of θ, then we have

$$a = \phi(\alpha), \quad b = \psi(\alpha).$$

If now we turn the line l_ξ from l_α to l_β, the rhombus $XYX'Y'$ changes from $ABA'B'$ to $BA'B'A$, so that in the second position we have

$$\lambda = \phi(\beta) = b, \quad \mu = \psi(\beta) = a.$$

That is, the diagonals are interchanged in the positions $\theta = \alpha$ and $\theta = \beta$. Thus, the situation is exactly as stated in Theorem 3.5. There exists, therefore, at least one direction l_γ for which $\phi(\gamma) = \psi(\gamma)$ $(\lambda = \mu)$, i.e., where the rhombus becomes a square. In Figure 3.8 we have an overview of the whole construction. □

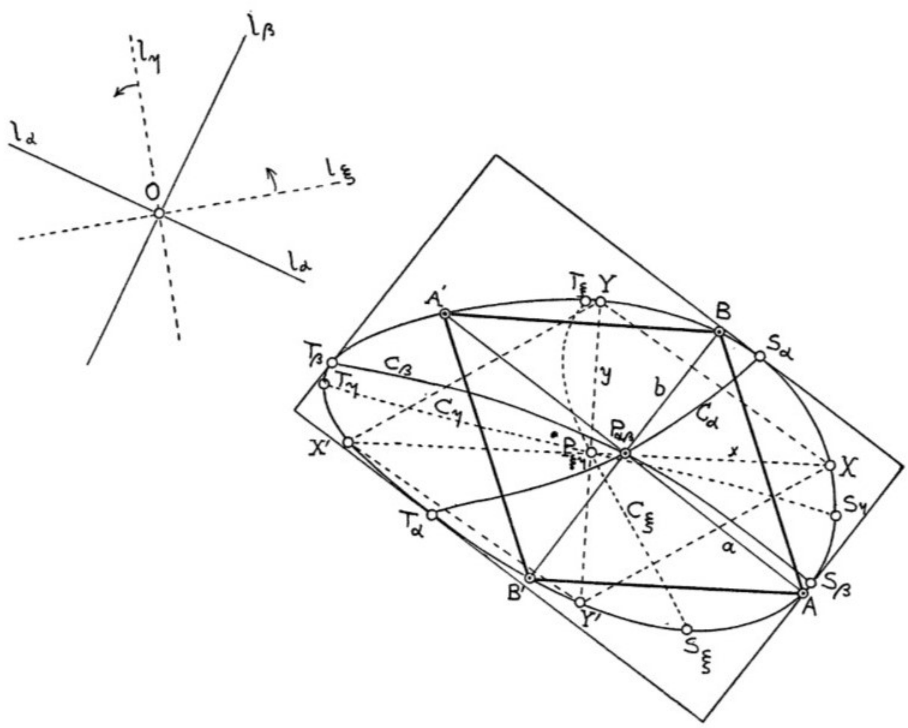

Figure 3.8: Rhombs in the oval

Notice that Emch gives strong conditions on the curve, such as being differential and excluding straight portions of the curve. This, as we mentioned, was the first approach (or at least the first one known) of the Square Peg Problem. When we introduce Stromquist's proof at the end of this chapter, we will see that his results are much stronger than this one.

Let us briefly summarize the idea of the above proof using the notation Emch uses in his later publications. He denotes by set of *medians* the set of all midpoints of the secants (chords) of the curve parallel to a given direction. He shows that all medians of a closed convex analytic curve are continuous and analytic curves. With each pair of directions σ and τ there are associated two medians M_σ and M_τ, which always intersect in one and only one point. Moreover, he sees that with every pair (σ, τ) there is a rhomb associated whose diagonals are parallel to σ and τ and whose vertices lie on the curve. Finally, he concludes that to every closed convex analytic curve without rectilinear segments, at least one square may be inscribed.

Besides this paper, Emch published two further proofs in [2] and [3], as we mentioned before. In his second paper, *On the Medians of a Closed Convex Polygon*, he proves that at least one square may be inscribed in any convex rectilinear polygon, and generally in any closed convex curve formed by a finite number of ordinary analytic arcs. In his third paper, *On some properties of the Medians of Closed Continuous Curves formed by Analytic Arcs*, he generalises his result for any closed continuous curve composed of a finite number of analytic arcs with a finite number of inflexions and other singularities.

3.2.3 Lev Schnirelmann

In 1929, Lev G. Schnirelmann offered a solution for curves with piecewise continuous curvature.

The following summary's idea is obtained from [7], since we were not able to access to Schnirelmann's official publication.

Schnirelmann's idea is to describe the set of inscribed squares as a preimage. For instance, in the following way:
Let $\gamma : S^1 \longrightarrow \mathbb{R}^2$ be the given curve. The space $(S^1)^4$ parameterizes quadrilaterals that are inscribed in γ. We construct a so-called *test-map*

$$f_\gamma : (S^1)^4 \longrightarrow \mathbb{R}^6,$$

which sends a 4-tuple (x_1, x_2, x_3, x_4) of points on the circle to the mutual distances between $\gamma(x_1), \ldots, \gamma(x_4) \in \mathbb{R}^2$, i.e., to $(||\gamma(x_1)-\gamma(x_2)||, ||\gamma(x_2)-\gamma(x_3)||, ||\gamma(x_3)-\gamma(x_4)||, ||\gamma(x_4)-\gamma(x_1)||, ||\gamma(x_1)-\gamma(x_3)||, ||\gamma(x_2)-\gamma(x_4)||)$.

Let V be the 2-dimensional linear subspace of $\mathbb{R}^6$ that corresponds to the points where all four edges are of equal length and the two diagonals are of equal length. The preimage $f_\gamma^{-1}(V)$ is parameterizing the set of inscribed squares. Within this set, there are degenerate components. That is, points where $x_1 = x_2 = x_3 = x_4$ and, more generally, 4-tuples where $x_1 = x_3$ and $x_2 = x_4$.

Schnirelmann claims that an ellipse inscribes exactly one square up to symmetry. Using some smooth isotopy, he deforms the ellipse into the given curve γ via other curves γ_t, where $t \in [0, 1]$. By smoothness, these inscribed squares do not come close to the degenerate quadrilaterals during the deformation. Thus, they do not degenerate to a point. Therefore, the nondegenerate part of all preimages $f_{\gamma_t}^{-1}(V)$ forms a 1-manifold that connects the solution sets for γ and the ellipse. Using this result, he sees that the parities of the number of inscribed squares on γ and on the ellipse coincide, leading to the conclusion that any smooth curve inscribes generically an odd number of squares.

3.3 Previous tools

In this section we will give some previous needed theory in order to follow the proof provided by Walter Stromquist. The following homology theory is extracted from [10] and [4]. We only intend to give a basic and general idea of it, in order to understand the tools required in the next section. This is why, we will state a summarized scheme.

Our object of study will be simplicial and singular homology. Simplicial homology depends on an associated topological space which allows a triangulation (we will work with *simplicial complexes*, see Definition 3.10), this is why it makes it easy to compute. Singular homology, on the other hand, is better adapted to theory than computation, which gives it much advantage over singular homology. It is defined for all topological spaces and depends only on the topology, not on any triangulation.
We will begin with the properties of simplicial homology, followed by the ones for sigular homology. At the end, we will see that both agree for spaces that can be triangulated.

In this section, $\mathbb{R}^N$ will denote the affine euclidian space of dimension N with the topology fixed with the euclidian distance.
Let $v_0, v_1, \ldots, v_n$ be points of $\mathbb{R}^N$. These are affinely independent if the vectors $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent. Equivalently, these points are affinely independent if there is no affine subvariety of $\mathbb{R}^N$ of dimension less than n where the points lie. In this case, every point of the generated affine subvariety allows a unique expression represented by $x = \lambda_0 v_0 + \ldots + \lambda_n v_n$, where $\lambda_i \in \mathbb{R}$ verify $\sum_{i=0}^{n} \lambda_i = 1$

Definition 3.7 (Simplex). *Let $N > 0$ and $v_0, v_1, \ldots, v_n$, with $n \geq 0$, $n+1$ affinely independent points of $\mathbb{R}^N$. A n-dimensional simplex of vertices $v_0, \ldots, v_n$ is a subset $\Delta(v_0, \ldots, v_n)$ of $\mathbb{R}^N$ defined by*

$$\Delta(v_0, \ldots, v_n) = \{x \in \mathbb{R}^N \mid x = \sum_{i=0}^{n} \lambda_i v_i,\ \sum_{i=0}^{n} \lambda_i = 1\ , \lambda_i \geq 0, i = 0, \ldots, n\}.$$

When referring to a simplex of dimension n we will use the expression *n-simplex*. Thus, 0-simplices are points, 1-simplices are segments, 2-simplices are triangles, etc., as shown in Figure 3.9.

Lemma 3.8. *1. Every simplex is a compact, connected, locally path-connected and contractible space.*
2. Any two n-dimensional simplices are homeomorphic.

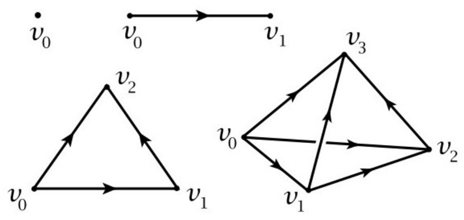

Figure 3.9: Example of 0-simplex, 1-simplex, 2-simplex and 3-simplex

Definition 3.9. *Let $\Delta(v_0, \ldots, v_n)$ be an n-simplex and $k \geq 0$. We call k-dimensional faces of $\Delta(v_0, \ldots, v_n)$ the simplices*

$$\Delta(v_{i_0}, v_{i_1}, \ldots, v_{i_k}),$$

where $0 \leq i_0 < \ldots < i_k \leq n$.

If we delete one of the $n+1$ vertices of an n-simplex $\Delta(v_0, \ldots, v_n)$, then the remaining n vertices span an $(n-1)$-simplex which is a face of $\Delta(v_0, \ldots, v_n)$. We adopt the following convention: The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the largest simplex.

Definition 3.10. *A simplicial complex is a finite set of simplices of $\mathbb{R}^N$, $K = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, such that*

1. *If σ_i is a simplex of K, then all the faces of σ_i belong to K.*
2. *If σ_i and σ_j are simplices of K, then $\sigma_i \cap \sigma_j = \varnothing$ or $\sigma_i \cap \sigma_j$ is a face of σ_i and σ_j.*

Definition 3.11. *Given a simplicial complex K, the simplicial polyhedron associated to K, and denoted by $|K|$, is the subspace of $\mathbb{R}^N$ defined by the reunion of all the simplices of K:*

$$|K| = \cup_{\sigma_i \in K} \sigma_i.$$

The simplicial polyhedrons are topological spaces. This fact will allow us to relate both homologies (the simplicial and the singular one) on the polyhedrons.
If K is a simplicial complex, the simplices $\sigma_i \in K$ are the *faces* of K and K is a *triangulation* of $|K|$.

Definition 3.12. *Let K be an ordered simplicial complex. For every $p \geq 0$, it is called group of p-dimensional chains of K, and is denoted by $C_p(K)$, the free abelian group generated by the set of p-dimensional ordered faces. In other words, the group of p-dimensional chains is*

$$C_p(K) := \oplus \mathbb{Z}[\sigma],$$

where the direct sum extents to the set of p-dimensional ordered faces.

If $p > dimK$, then $C_p(K) = 0$.

The p-dimensional chains can be written as finite formal sums $\sum_i k_i \sigma_i$, where $k_i \in \mathbb{Z}$ are coefficients. Such a sum can be thought of as a finite collection (or "chain") of p-simplices in K with integer multiplicities.

For instance, in the case of a 2-simplex $[v_0, v_1, v_2]$, it has associated the ordered simplicial complex formed by the 2-dimensional face $[v_0, v_1, v_2]$, the 1-dimensional faces $[v_0, v_1], [v_0, v_2], [v_1, v_2]$ and the 0-dimensional faces $[v_0], [v_1], [v_2]$. Then, we would have:

$$\begin{aligned} C_0(\Delta) &= \mathbb{Z}[v_0] \oplus \mathbb{Z}[v_1] \oplus \mathbb{Z}[v_2] \cong \mathbb{Z}^3, \\ C_1(\Delta) &= \mathbb{Z}[v_0, v_1] \oplus \mathbb{Z}[v_0, v_2] \oplus \mathbb{Z}[v_1, v_2] \cong \mathbb{Z}^3, \\ C_2(\Delta) &= \mathbb{Z}[v_0, v_1, v_2] \cong \mathbb{Z}. \end{aligned}$$

As we can see in Figure 3.10, the boundary of a n-simplex $[v_0, \ldots, v_n]$ consists of the various $(n-1)$-dimensional simplices $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ (where the hat over $\widehat{v_i}$ indicates that this vertex is removed from the sequence $v_0, \ldots, v_n$). In terms of chains, we would like to say that the boundary of $[v_0, \ldots, v_n]$ is the $(n-1)$-chain formed by the sum of the faces $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$. However, it turns out to be better to insert certain signs and let the boundary of $[v_0, \ldots, v_n]$ be $\sum_i (-1)^i [v_0, \ldots, \widehat{v_i}, \ldots, v_n]$. This is because the signs will take the orientation into account, so that all faces of the simplex are oriented. The "boundary operator", which we will define below, is the algebraic version of this geometrical concept.

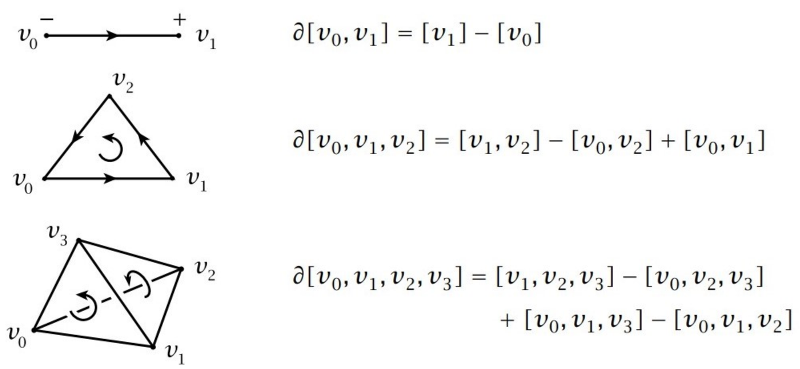

Figure 3.10: Examples of the boundary operator

Definition 3.13. *Let K be an ordered simplicial complex. For every $p \geq 1$, the boundary operator $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$ is the morphism of abelian groups defined by*

$$\partial_p[v_{i_0}, \ldots, v_{i_p}] = \sum_{k=0}^{p} (-1)^k [v_{i_0}, \ldots, \widehat{v_{i_k}}, \ldots, v_{i_p}],$$

where $[v_{i_0}, \ldots, \widehat{v_{i_k}}, \ldots, v_{i_p}]$ denotes the ordered $(p-1)$-simplex obtained when we delete the vertex v_{i_k}. For $p = 0$, we define $\partial_0 = 0$.

As we can see in Figure 3.10, for the case of a 1-simplex $[v_0, v_1]$ we would have $\partial_1[v_0, v_1] = [v_1] - [v_0]$. Also, for the case of the 2-simplex $[v_0, v_1, v_2]$, we would have

$$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

This, in fact, gives us the simplex ordered.

In the following proposition we introduce an important property of the boundary operator. This fact will be applied in the demonstration of Stromquist.

Proposition 3.14. *Let K be an ordered simplicial complex. The composition*

$$C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$$

is null for every $p \geq 1$, i.e., $\partial^2 = 0$.

We are now ready to define the simplicial homology group, which is our main object of study in this section.

Definition 3.15. *Let K be a simplicial complex. For every $p \geq 0$, the p^{th} simplicial homology group of K is defined as the quotient group*

$$H_p(K) := Z_p(K)/B_p(K),$$

where

$$Z_p(K) := ker\ (\partial_p : C_p(K) \longrightarrow C_{p-1}(K))$$

is the group of p-dimensional cycles of K and

$$B_p(K) := im\ (\partial_{p+1} : C_{p+1}(K) \longrightarrow C_p(K))$$

is the group of p-dimensional boundaries.

Notice that $H_p(K) = 0$ if $p > dimK$.

The simplicial homology is only applied to one class of topological spaces: the polyhedrons. However, the singular homology is defined for every topological space, is functorial regarding continuous maps and matches with the simplicial homology on polyhedrons. Everything described above can be applied equally to singular simplices.

Let Δ^p denote a p-simplex, where $p \geq 0$ is an integer.

Definition 3.16. *Let X be a topological space. A singular p-simplex of X is a continuous map $\sigma : \Delta^p \longrightarrow X$.*

Definition 3.17. *Let X be a topological space. The group of singular p-dimensional chains of X, denoted by $S_p(X)$, is the free abelian group generated by the singular p-simplices of X:*

$$S_p(X) = \{\sum_{i=1}^{r} \lambda_i \sigma_i;\ \lambda_i \in \mathbb{Z},\ \sigma_i : \Delta^p \longrightarrow X \ continuous\}.$$

The boundary operator is introduced as $\partial_p : S_p(X) \longrightarrow S_{p-1}(X)$ and it is defined by

$$\partial_p(\sigma) = \sum_{i=0}^{p} (-1)^i \sigma \circ \delta_i,$$

where $\delta_i : \Delta^{p-1} \longrightarrow \Delta^p$, $0 \leq i \leq p$, is the continuous map defined by

$$\delta_i(x_0, x_1, \ldots, x_{p-1}) = (x_0, x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{p-1}).$$

So we can define the singular homology group by

$$H_p(X) = Ker\ \partial_p / Im\ \partial_{p+1}.$$

Proposition 3.18. *Let X, Y be topological spaces and $f : X \longrightarrow Y$ a continuous map. Then f induces a morphism of complexes*

$$S_*(f) : S_*(X) \longrightarrow S_*(Y)$$

such that

$$S_p(f)(\sigma) = f \circ \sigma,$$

for every singular p-simplex σ of X.

Now that we have translated all the simplicial properties to the singular field, it is time to introduce the functoriality of the singular homology, where **Top** will represent the category of topological spaces and **Ab** the category of abelian groups:

Proposition 3.19. *Let X, Y be topological spaces and $f : X \longrightarrow Y$ a continuous map. Then, the morphism $S_*(f)$ induces a group morphism*

$$H_*(f) : H_*(X) \longrightarrow H_*(Y)$$

such that, if $[c]$ is a cycle of X, then $H_*(f)([c]) = [S_*(f)(c)]$.
If $f = id_X$, then

$$H_*(id_X) = id_{H_*(X)},$$

and, if $g : Y \longrightarrow Z$ is another continuous map, then

$$H_*(g \circ f) = H_*(g) \circ H_*(f).$$

Thus, H_* defines a functor

$$H_* : \textbf{Top} \longrightarrow \textbf{Ab}_*.$$

From this functoriality it immediately follows the topological invariance of the singular homology:

Theorem 3.20 (Topological invariance)**.** *Let X, Y be topological spaces and $f : X \longrightarrow Y$ a homeomorphism. Then, the morphism*

$$H_*(f) : H_*(X) \longrightarrow H_*(Y)$$

is an isomorphism.

We also have that singular homology is invariant under homotopy. Recall that, given two topological spaces X and Y, they are of the same homotopy type if there exists $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ continuous such that the compositions $f \circ g$ and $g \circ f$ are homotopic to the corresponding identities: $f \circ g \simeq id_Y, g \circ f \simeq id_X$.

Theorem 3.21 (Homotopic invariance)**.** *Let X, Y be topological spaces and $f, g : X \longrightarrow Y$ continuous maps. Then, if f is homotopic to g,*

$$H_*(f) = H_*(g) : H_*(X) \longrightarrow H_*(Y).$$

As an immediate consequence of this result, we get:

Corollary 3.22. *Let X and Y be topological spaces of the same type of homotopy. Then, the groups of singular homology of X and Y are isomorphic, i.e., $H_*(X) \cong H_*(Y)$.*

We have seen the basic properties of simplicial and singular homology required for Stromquist's proof. We need a final step to relate both homologies. We show that these properties allow us to compare the simplicial homology of a polyhedron to its singular homology.

Let K be an ordered simplicial complex and $V = \{v_1, v_2, \ldots, v_r\}$ its vertices. Let $C_*(K)$ be the complex of simplicial chains of K. The polyhedron $|K|$ associated to the complex K is a topological space and, hence, we can consider the complex of singular chains of $|K|$: $S_*(|K|)$.

If s is an ordered p-dimensional simplicial face of K, $s = [v_{i_0}, \ldots, v_{i_p}]$, s defines a singular simplex p-dimensional $\sigma = (v_{i_0}, \ldots, v_{i_p})$. Then, we have the map

$$\begin{aligned} \nu : C_*(K) &\longrightarrow S_*(|K|) \\ s &\longmapsto \sigma \end{aligned}$$

Lemma 3.23. *The morphism ν is a morphism of complexes, natural in K.*

Therefore, ν induces a morphism in the respective homologies

$$H_*(\nu) : H_*(C_*(K)) \longrightarrow H_*(S_*(|K|)),$$

that is, we have the morphism

$$H_*(\nu) : H_*(K) \longrightarrow H_*(|K|)$$

from the simplicial homology of the simplicial complex K to the singular homology of the topologic space $|K|$.

3.4 Walter Stromquist's proof

In this section we present Walter Stromquist's result, which is titled *Inscribed squares and square-like quadrilaterals in closed curves.* We will provide some extra annotations to his original proof.

In this proof, we use the term *smooth* to refer to a curve which has a continuous turning tangent, i.e., which is C^1. We will show that for every smooth curve in $\mathbb{R}^n$, there is a quadrilateral with equal sides and equal diagonals whose vertices lie on the curve. In the case of a smooth plane curve, the quadrilateral corresponds to a square.

We will give a weaker smoothness condition which will still satisfy the existence of an inscribed square for curves that are convex, polygon or piecewise of class C^1 (with certain restrictions that we will state later).

Definitions and necessary tools Let us begin with some initial definitions. A **simple closed curve** is a continuous function $w : \mathbb{R} \longrightarrow \mathbb{R}^n$ which satisfies $w(x) = w(y)$ if, and only if, $x - y$ is an integer, where w is determined by its values in $[0, 1]$. This can also be seen as an injective continuous map $S^1 \longrightarrow \mathbb{R}^n$ where, if $n = 2$, it is a Jordan curve. Henceforth, "curve" will mean simple closed curve.

The curve w is **smooth** if it has a continuous non-vanishing derivative. Any curve with a continuously turning tangent can be parameterized so as to be smooth in this sense.

A quadrilateral is **inscribed** in the curve w if all of its vertices lie on w. In the case of $\mathbb{R}^n$, a quadrilateral may be inscribed even if its sides do not lie in the interior of the curve.

We shall introduce some required tools before getting into the first lemma. We will work with a simplex Q, which will represent the set of quadrilaterals inscribed in w. That is, all possible geometric elements inscribed in w that could be the square we want. We will also work with four subsets Q_1, Q_2, Q_3, Q_4 that will cover our Q and whose intersection will correspond to a set of inscribed rhombuses (we will see the definition of *rhombus* at the end of this section). This set is going to be our main object of study, as we will see later.

Denote by Q the set

$$Q = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1\}.$$

This is a 4-simplex with vertices

$$\begin{aligned} v_0 &= (1, 1, 1, 1), \\ v_1 &= (0, 1, 1, 1), \\ v_2 &= (0, 0, 1, 1), \\ v_3 &= (0, 0, 0, 1), \\ v_4 &= (0, 0, 0, 0), \end{aligned}$$

and faces $F_0, \ldots, F_4$ numbered so that each F_i is opposite v_i. Each face F_i is generated by all the vertices stated above except for v_i. For instance, the face F_0 is formed by v_1, v_2, v_3, v_4.
We will use the notation $\mathbf{x}$ to represent an element (x_1, x_2, x_3, x_4) of Q.

In Figure 3.11 we illustrate the simplex Q. Since it is a 4-dimensional object, it is difficult to represent. Let us note some details to clarify the form of our Q. Each face F_i is a tetrahedron, since it is a 3-simplex. However, notice that in Figure 3.11 we have not drawn F_0 and F_4 as tetrahedrons. Also, the face $F_0 \cap F_4$ is a 2-simplex, which means that it is actually a triangle. But in our illustration we have represented it as a line segment (the right edge of the figure).

Given a curve w, we associate with each point $\mathbf{x} \in Q$ an inscribed quadrilateral with vertices $w(x_1), w(x_2), w(x_3), w(x_4)$. That is, we identify the points of Q with the corresponding geometric figures on w and refer to points of Q as quadrilaterals. Thus, Q represents the set of quadrilaterals inscribed in w.

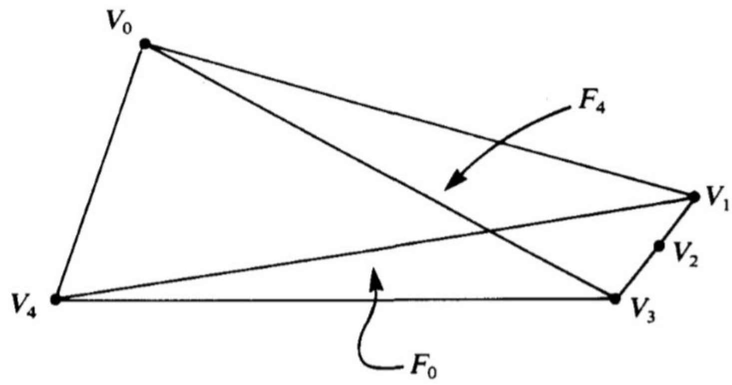

Figure 3.11: The simplex Q of quadrilaterals

The vertices of each quadrilateral have the same cyclic order in the quadrilateral as in w. Some of these quadrilaterals are degenerate (have one or more sides of zero length), and some of the degenerate quadrilaterals are one-point quadrilaterals (all four sides have zero length). That is, the possibility $x_1 = x_2 = x_3 = x_4$ is not excluded.

Let us mention an important fact about the faces F_0 and F_4. You can think of F_0 as quadrilaterals with the vertex $w(0)$ fixed, while F_4 is the face where quadrilaterals have vertex $w(1)$ fixed. Thus, both faces are morally the same since $w(0) = w(1)$ (by definition of simple closed curve). In other words, if we define $h : F_0 \longrightarrow F_4$ by

$$h(0, x, y, z) = (x, y, z, 1),$$

then $h(x)$ represents the same quadrilateral as $\mathbf{x}$ but with the vertices numbered differently. Hence, the points of F_4 represent the same quadrilaterals as the points of F_0.

Now, for each $i = 1, 2, 3, 4$, we define

$$s_i(\mathbf{x}) = \parallel w(x_{i+1}) - w(x_i) \parallel$$

as the **length of the i-th side** of the quadrilateral corresponding to $\mathbf{x}$. For the case $i = 4$, x_{i+1} becomes x_1. Each s_i is a continuous function on Q.

For each i, we define Q_i to be the closure of the set

$$\{\mathbf{x} \in Q^0 \mid s_i(\mathbf{x}) = \max_j s_j(\mathbf{x})\}.$$

Here Q^0 denotes the interior of Q. Q_i is the set of quadrilaterals whose i-th side is their longest side, except that a quadrilateral in ∂Q is included in Q_i only if it is the limit of quadrilaterals in Q^0 with the same property.
Each Q_i is still closed and we still have $\bigcup_i Q_i = Q$. The purpose of this device is to prevent one-point quadrilaterals from being elements of every Q_i, as we will need afterwards. But the device works perfectly only if the curve is sufficiently smooth. We will see this smoothness requirement later.

Now we will introduce a lemma that requires w to be smooth. We are going to prove its stronger version later (see Lemma 3.30), as well as specify the smoothness condition that we will apply.

We will work as it follows: first we will prove the Square Peg Problem for curves that satisfy the following Lemma 3.24, i.e., which are smooth. Once we have this proof, we will introduce a new smoothness condition called "Condition A" that will still guarantee this Lemma 3.24 and, hence, the proof of the Square Peg Problem will still be implied. Therefore, the following lemma is a tool to be able to prove the Square Peg Problem later for this so called "Condition A". Let us first introduce it and then explain its meaning.

Lemma 3.24. *If w is a smooth curve, then each one-point quadrilateral is contained in only one set Q_i. In particular, $v_i \in Q_i$ for $i = 1, 2, 3$ and $\boldsymbol{x} \in Q_4$ for each $\boldsymbol{x}$ on the edge connecting v_0 and v_4.*

In order to understand better the above lemma, we will explain it more precisely. The vertices v_0, v_1, v_2, v_3, v_4 are one-point quadrilaterals. This is because $w(0) = w(1)$, by definition of w, and hence for each v_i we have $s_1 = s_2 = s_3 = s_4 = 0$, where $i = 0, 1, 2, 3, 4$. Then, we will have that $v_1 \in Q_1, v_2 \in Q_2, v_3 \in Q_3$, whereas v_0, v_4 and the whole edge between them will be in Q_4 (this fact will be proven in the stronger version of the lemma). So, why do we consider the case of the edge connecting v_0 and v_4 differently?

Since $v_0 = (1, 1, 1, 1)$ and $v_4 = (0, 0, 0, 0)$, all points placed on this edge are represented by $(\lambda, \lambda, \lambda, \lambda)$, where $\lambda \in [0, 1]$. Therefore, all their associated quadrilaterals are one-point quadrilaterals. This does not occur on the other edges of Q. For instance, the points on the edge connecting $v_3 = (0, 0, 0, 1)$ and v_4 are represented by the quadruple $(0, 0, 0, \lambda)$. These correspond to quadrilaterals with only one side with no zero length (except for the limit cases where $\lambda = 0$ or $\lambda = 1$). It works similarly for the edges between the other vertices.

Therefore, all points settled on the edges joining our five vertices (excluding the vertices themselves) represent quadrilaterals with three sides of zero length, except for the case of the edge linking v_0 and v_4, where they are associated to one-point quadrilaterals.

Furthermore, we can take a look at the points on the faces of Q. The points situated on the 2-simplices are associated to a quadrilateral with one side of zero length. For example, the points in $F_0 \cap F_4$ are represented by $(0, \mu, \lambda, 1)$, where $\mu, \lambda \in [0, 1]$, and since our curve w is closed ($w(0) = w(1)$), two of its vertices are associated implying that its quadrilateral is degenerate (has one side of zero length).

Also, the points found on faces F_1, F_2 and F_3 are, as well, degenerate quadrilaterals with one of their sides with length equal to zero. But, points situated in the interior of F_0 and F_4 might be non degenerate quadrilaterals. In the case of F_0, its points are represented by $(0, \lambda, \mu, \gamma)$ and for F_4 we have $(\lambda, \mu, \gamma, 1)$, where $\lambda, \mu, \gamma \in [0, 1]$. Both of them with one of its vertices fixed ($w(0)$ and $w(1)$ respectively). Therefore, their points are non degenerate unless $\lambda = \mu$, $\lambda = \gamma$ or $\mu = \gamma$.

We intend to avoid all of the degenerate cases, so we can proof the existence of a quadrilateral with equal sides and equal diagonals. This is why we will work with faces F_0 and F_4, where their points are generically non degenerate quadrilaterals.

For $i = 1, 2, 3$ notice that Q_i includes v_i but does not intersect the opposite face F_i (where $s_i(x) = 0$). For instance, $v_1 \in Q_1$, but the points of F_1 are represented by $(\lambda, \lambda, \mu, \gamma)$, $\lambda, \mu, \gamma \in [0, 1]$. Then $s_1 = \|w(\lambda) - w(\lambda)\| = 0$, which means that s_1 is not the largest side of any associated quadrilateral in F_1. The case of $i = 4$ is different: Q_4 includes the entire segment from v_0 to v_4, and avoids the opposite 2-simplex $F_0 \cap F_4$.

The smoothness of w is only required for this lemma. Henceforth, we assume that w is any curve for which the result of Lemma 3.24 is valid.

Now let $R = \bigcap_i Q_i$. Since we defined Q_i as the set of quadrilaterals whose i-th side is their longest side and R as the intersection of all Q_i, then R corresponds to the set of those quadrilaterals which have all four sides with the same value. We call a point $\mathbf{x} \in R$ a *rhombus*. Thus, a rhombus corresponds to an inscribed quadrilateral whose sides are equal and nonzero (we made sure that each one-point quadrilateral can only be in **one** Q_i). A *square-like quadrilateral* is a rhombus which satisfies

$$d_{13}(\mathbf{x}) = d_{24}(\mathbf{x})$$

where

$$d_{13}(\mathbf{x}) = \|w(x_3) - w(x_1)\|$$

and

$$d_{24}(\mathbf{x}) = \|w(x_4) - w(x_2)\|.$$

In other words, a square-like quadrilateral is an inscribed quadrilateral with equal sides and equal diagonals. In $\mathbb{R}^2$, a square-like quadrilateral is an inscribed square, but we use the term *rhombus* to include equilateral quadrilaterals which do not lie in a plane.

A **thin** rhombus is one which satisfies $d_{13} \geq d_{24}$, and a **fat** rhombus is one such that $d_{13} \leq d_{24}$. A rhombus which is both thin and fat is a **square-like quadrilateral**.

Denote by R_{THIN} and R_{FAT} the subsets of R consisting of thin and fat rhombuses respectively. Then, $R = R_{THIN} \cup R_{FAT}$. Note that if $\mathbf{x} \in F_0$ is a rhombus, then so is $h(\mathbf{x})$. But if $\mathbf{x}$ is thin, then $h(\mathbf{x})$ is fat and vice versa.

Before continuing, let us make a quick overview of the general strategy. So far we have seen the definitions and tools required to set the problem we would like to solve. Our strategy will consist on studying the set of rhombuses in Q, and especially the rhombuses on F_0 and F_4. We will show that there must be, in a sense, an odd number of rhombuses in F_0. Notice that if we proved that the number of rhombuses is even, we would not discard the case of a zero number of rhombuses. Therefore, we need to see that the number is odd.

Moreover, if the number of thin rhombuses on F_0 is even, for instance, then the number of fat rhombuses on this face must be odd. The correspondence based on the function h shows that these parities must be reversed on F_4. But we will also show that the parities can not be reversed - in effect because the bottom face can be lifted continuously up through Q onto the top face - unless R_{THIN} and R_{FAT} intersect.

To prove all of this we will need some machinery from homology theory.

The degree of a set of rhombuses In this part of the section we will define the concept of the *degree* of a subset K of a simplex, given a cover of the simplex by closed sets. In the next part, our K will be a set of rhombuses, and the degree will be our way of counting the rhombuses modulo 2. Additionally, all homology groups will be simplicial homology groups with coefficients in $\mathbb{Z}_2$.

Let A be an n-simplex. We will use the notations v_i and F_i for the vertices and faces of A respectively. A **cover of closed vertex neighborhoods** in A, or simply a *cover*, is a family of closed subsets $A_0, \dots, A_n$ of A such that $v_i \in A_i$ but $F_i \cap A_i = \emptyset$ for each i, and such that $\bigcup_i A_i = A$. This cover is denoted as $\{A_i\}$. We are going to show that $\bigcap A_i$ is nonempty and that, in a certain well defined sense, it is odd.

Let $\{A_i\}$ be a cover. Let K be any subset of $\bigcap A_i$ which is both open and closed relative to $\bigcap A_i$. We include the possibilities $K = \emptyset$ and $K = \bigcap A_i$. In other words, we take K as a connected component of $\bigcap A_i$ and we want to define the degree of the cover $\{A_i\}$ around our K. This degree is an element of $\mathbb{Z}_2$; in effect it tells whether we should consider K to be even or odd.

A **reversing map** for the cover $\{A_i\}$ is a function

$$f : \left(A - \bigcap A_i\right) \longrightarrow \partial A$$

which maps each set A_i into the opposite face F_i; that is, $f(A_i) \subseteq F_i$ for each i. We see this fact more precisely once we define our f. Let us show that a reversing map always exists. For each i, let $d(x, A_i)$ denote the distance from x to A_i and define f by

$$f(x) = \sum_i \frac{d(x, A_i)}{\sum_j d(x, A_j)} v_i.$$

The sum $\sum_j d(x, A_j)$ normalises each coefficient, since it is the sum of all distances from $\mathbf{x}$ to each A_i, for $i = 0, \dots, n$. Notice that we can not have the case where all distances have zero value, since no $\mathbf{x}$ is in the intersection $\bigcap A_i$.
Given a k, if $\mathbf{x}$ belongs to A_k, then

$$d(\mathbf{x}, A_k) = 0.$$

Therefore we would get a lineal combination with all vertices except the vertex v_k, which implies that $f(\mathbf{x})$ is in F_k (face which does not have v_k). Thus, each $\mathbf{x}$ is sent to the opposite face of the subset where it belongs.

Normalising the coefficients ensures that the image is restricted to that specific boundary; it gives $f(\mathbf{x})$ as a convex combination of the vertices v_i so that $f(\mathbf{x}) \in A$.

Now let $L = \bigcap A_i - K$. Triangulate A finely enough that no simplex of the triangulation touches both K and L. Let Γ be the n-chain consisting of the n-simplices which touch K. We triangulated A such that only one face of each n-simplex touches K. A face of an n-simplex is an $(n-1)$-simplex. Then, the boundary of Γ represents an homology class $\gamma \in H_{n-1}(A - \bigcap A_i)$. The $(n-1)$-dimension is due to the faces of the n-simplices touching K; and it is an homology of $A - \bigcap A_i$ because the n-simplices only touch K with one of their faces. Therefore, the boundary of Γ consists of the n-simplices (contained in A) without the faces that do not touch K (which are in the set $\bigcap A_i$ since $\bigcap A_i = L \cup K$). We denote this class by γ_K. It is the unique homology class that surrounds all of K but none of L.

The **degree** of $\{A_i\}$ around K is defined as the image $f_*(\gamma_K)$ in $H_{n-1}(\partial A)$, where f is any reversing map for $\{A_i\}$. The group $H_{n-1}(\partial A)$ can be identified with $\mathbb{Z}_2$, and we will regard the degree to be an element of $\mathbb{Z}_2$. Its value is independent of the choices made in the above construction. We will denote it by $deg_{\{A_i\}}K$ or, when the cover is identified by context, by $degK$.

In particular, suppose B is any simplex and $f : A - \bigcap A_i \longrightarrow \partial B$ is any function which maps each A_i into a different face of B. Then the degree of K could just as well have been defined as $f_*(\gamma_K)$ in $H_{n-1}(\partial B)$, since there is an isomorphism $g : B \longrightarrow A$ which makes gf a reversing map, and $(gf)_*(\gamma_K) = f_*(\gamma_K)$ (both being regarded as elements of $\mathbb{Z}_2$). In this case we say that f is "isomorphic to a reversing map".

We can collect some facts about these degrees. If $K = \emptyset$, then $\gamma_K = 0$ (the zero element of $H_{n-1}(A - \bigcap A_i)$). If K is the disjoint union of open and closed subsets K_1 and K_2, then $\gamma_K = \gamma_{K_1} + \gamma_{K_2}$. Therefore we have the following lemmas:

Lemma 3.25. *$deg\ \emptyset = 0$.*

Lemma 3.26. *If $K = K_1 \cup K_2$, then $degK = degK_1 + degK_2$.*

Lemma 3.27. *$deg(\bigcap A_i) = 1$.*

In other words, Lemma 3.27 tells us that the degree of $\bigcap A_i$ is odd, as we wanted to show. Let us see its proof.

Proof (Lemma 6). Let S^{n-1} denote the $(n-1)$-sphere, and let $g : S^{n-1} \longrightarrow S^{n-1}$ be any continuous map without fixed points. We see that a map that has no fixed points is homotopic to the antipodal map of S^{n-1}:

Let $a : S^{n-1} \longrightarrow S^{n-1}$ be the antipodal map of the sphere. Since g has no fixed points, we have that $g(x) \neq x$, for $x \in S^{n-1}$. We denote by O the center of the sphere. We know that the segment $[a(x), x]$, for $x \in S^{n-1}$, passes through the center O. This implies that the segment $[a(x), g(x)]$ does not pass through O (using that $g(x) \neq x$). Thus, $tg(x) + (1-t)a(x)$ does not pass through O either, for $t \in [0,1]$, implying that it cannot be equal to zero. Taking into account this result, we can define an homotopy

$$\begin{aligned} H : I \times S^{n-1} &\longrightarrow S^{n-1} \\ (t, x) &\longmapsto \frac{tg(x) + (1-t)a(x)}{||tg(x) + (1-t)a(x||}. \end{aligned}$$

This is continuous and, therefore, well-defined.

Then we have that g is homotopic to the antipodal map, as we wanted to show. This implies that g and a generate the same map in homology; that is, $g_* = a_*$. But a is an homeomorphism of the sphere; and so $g_* : H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(S^{n-1})$ is the identity map. Now, since A is an n-simplex, ∂A is homeomorphic to S^{n-1} and any reversing map f restricts to a function $\partial A \longrightarrow \partial A$ without fixed points, so f induces the identity map on $H_{n-1}(\partial A) = \mathbb{Z}_2$. Looking at ∂A itself as a representative of the homology class that generates $H_{n-1}(\partial A)$, we can conclude that $f_*(\partial A) = 1$. But ∂A surrounds $\bigcap A_i$, and so represents $\gamma_{\cap A_i}$. Therefore $f_*(\gamma_{\cap A_i}) = 1$, which proves the lemma. □

The main result We are now ready to return to the problem of searching for a square-like quadrilateral in the simplex Q.

In Lemma 3.24 we saw that if w is smooth, then each one-point quadrilateral is in a different Q_i. Here we are going to prove the Square Peg Problem for the case of w being smooth, and hence for the curves that satisfy Lemma 3.24.

Theorem 3.28. *If w is a smooth curve, then w admits an inscribed quadrilateral with equal sides and equal diagonals.*

Proof. Let Q, R and their subsets be as before. We will suppose that there is no square-like quadrilateral in Q. Then, R can be written as a disjoint union $R = R_{THIN} \sqcup R_{FAT}$. The face F_0 is a simplex, and it has a cover $\{F_0 \cap Q_i\}$ of closed vertex neighborhoods for $i = 1, 2, 3, 4$ (each Q_i includes v_i, this is why we can build this cover with the family of closed subsets $F_0 \cap Q_1, F_0 \cap Q_2, F_0 \cap Q_3$ and $F_0 \cap Q_4$). Since $R = \bigcap_i Q_i$, the intersection of the above sets in the cover is

$$F_0 \cap R = (F_0 \cap R_{THIN}) \sqcup (F_0 \cap R_{FAT}).$$

From Lemma 3.26 and Lemma 3.27 we have

$$deg(F_0 \cap R_{THIN}) + deg(F_0 \cap R_{FAT}) = deg(F_0 \cap R) = 1,$$

from which we deduce that

$$deg(F_0 \cap R_{THIN}) \neq deg(F_0 \cap R_{FAT}). \tag{3.1}$$

We will obtain a contradiction by showing that each side of (3.1) is equal to $deg(F_4 \cap R_{THIN})$, which is measured in the simplex F_4. Notice that this idea is reasonable, since $deg(F_0 \cap R_{FAT})$ and $deg(F_4 \cap R_{THIN})$ should be equal because the quadrilaterals on both faces are the same but with the vertices numbered differently and $deg(F_0 \cap R_{THIN}) = deg(F_4 \cap R_{THIN})$ should also occur since the degree of R_{THIN} must change continuously from one face to the other (as we will prove afterwards).

The map $h : F_0 \longrightarrow F_4$ is not only an homeomorphism of the faces, it is also an isomorphism of the covers $\{F_0 \cap Q_i\}$ and $\{F_4 \cap Q_i\}$. As we have mentioned, h will change $F_0 \cap R_{FAT}$ into $F_4 \cap R_{THIN}$ because they are the same quadrilaterals but with the order of the vertices changed. Which means that the longest diagonal of one rhombus (d_{24} for the fat rhombus) will become the shortest one, while the originally shortest diagonal will play the role of the new longest diagonal (d_{13}). Thus,

$$deg_{\{F_0 \cap Q_i\}}(F_0 \cap R_{FAT}) = deg_{\{F_4 \cap Q_i\}}(h(F_0 \cap R_{FAT})) = deg_{\{F_4 \cap Q_i\}}(F_4 \cap R_{THIN}).$$

That takes care of the right side of (3.1).

Let us prove the left side of (3.1) by showing that $deg(F_0 \cap R_{THIN}) = deg(F_4 \cap R_{THIN})$. We claim that $degR_{THIN}$ is the same whether it is measured in F_0 or in F_4. Intuitively, this is true because the degree of R_{THIN} must change continuously (i.e., remain constant) as we progress smoothly up through slices of Q from the bottom face to the top face of the simplex. Making this arguments precisely takes some work.
Construct $f : (Q - R) \longrightarrow \partial F_0$ as follows:

$$d(\mathbf{x}, Q_i) = \text{distance from } \mathbf{x} \text{ to } Q_i;$$

$$f(\mathbf{x}) = \sum_{i=1}^{4} \frac{d(\mathbf{x}, Q_i)}{\sum_j d(\mathbf{x}, Q_j)} v_i.$$

Then f maps each Q_i into the face $F_0 \cap F_i$ of F_0. Therefore f restricted to F_0 is a reversing map for the cover $\{F_0 \cap Q_i\}$, and also f restricted to F_4 is isomorphic to a reversing map for the cover $\{F_4 \cap Q_i\}$.

Triangulate Q finely enough that no simplex touches both R_{THIN} and R_{FAT}, and let Δ be the 4-chain consisting of simplices of the triangulation which touch R_{THIN}. Now $\partial\Delta$ is a 3-chain in $(Q - R)$. Let Γ be the 3-chain consisting of those simplices in $\partial\Delta$ which are not contained in F_0 or F_4. In the simplest cases, Γ can be thought of as a tube surrounding R_{THIN}, and with its ends abutting F_0 and F_4, as in the figure shown below. More generally, Γ may consist of many tubes and more complicated shapes.

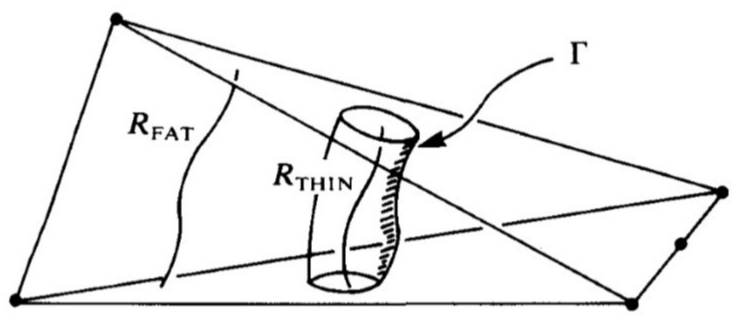

Figure 3.12: Simplest case of Γ

Now the boundary $\partial\Gamma$ is a 2-chain which must represent the zero element of $H_2(Q - R)$. This is because $\partial\Gamma$ is the boundary of the boundary, and by homology this is zero. Therefore $f_*(\partial\Gamma) = 0 \in H_2(F_0)$. But $\partial\Gamma$ contains two components: one surrounds $(R_{THIN} \cap F_0)$ in F_0, and one surrounds $(R_{THIN} \cap F_4)$ in F_4. We have:

$$0 = f_*(\partial\Gamma) = f_*(\partial\Gamma \cap F_0) + f_*(\partial\Gamma \cap F_4);$$

so

$$f_*(\partial\Gamma \cap F_0) = f_*(\partial\Gamma \cap F_4) \tag{3.2}$$

But the left side of (3.2) measures the degree of $R_{THIN} \cap F_0$, and the right side measures the degree of $R_{THIN} \cap F_4$. Therefore, these two degrees are equal, completing the promised contradiction and establishing the theorem. □

The smoothness requirement. So far we have proved the Square Peg Problem for curves that are smooth which satisfy Lemma 3.24. In this part we give a weaker hypothesis: Condition A. It is sufficient for Lemma 3.24. and, therefore, for the existence of an inscribed square or square-like quadrilateral. Smooth curves satisfy it, and so do polygons with only obtuse angles (corners where the curve changes direction by less than 90°). We will talk about *chords*. A **chord** is a line segment joining two points of w.

Definition 3.29. *A curve w satisfies Condition A if each point $w(y)$ of the curve has a neighborhood $U(y)$ in $\mathbb{R}^n$ such that no two chords in $U(y)$ are perpendicular.*

This definition is purely geometric: it depends only on the image $\operatorname{Im} w$ and not on the parameterization. An equivalent definition, intuitively, is that each point of the curve has a neighborhood (in the curve) in which any two chords (oriented in the direction of the curve) differ in direction by less than 90°. More precisely, we have the following: if w satisfies Condition A, then each $y \in \mathbb{R}$ has a neighborhood $(y - \mu, y + \mu)$ such that, if $x_1, x_2, x_3, x_4 \in (y - \mu, y + \mu)$ with $x_1 < x_2$ and $x_3 < x_4$, then

$$(w(x_2) - w(x_1)) \cdot (w(x_4) - w(x_3)) > 0.$$

The periodicity of w and the compactness of $[0, 1]$ insure that μ can be chosen independently of y.

As we have mentioned, this condition is satisfied for smooth curves and polygons with all their angles being obtuse:
Any point $w(y)$ situated on a smooth curve w will always have a sufficiently small neighborhood $U(y)$ in which all the chords will form an angle $> 90°$. In Figure 3.13 we can see that, although it may seem that the chords will be perpendicular in an specific neighborhood, we can choose an even smaller neighborhood where no two chords will differ in direction by less than 90°.

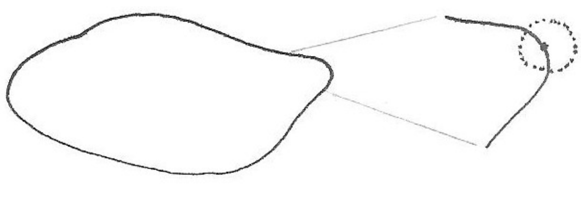

Figure 3.13: Smooth curves satisfy Condition A

For the case of polygons, we ask them to have all their angles obtuse. In Figure 3.14 we have an example of a polygon with an angle of $< 90°$. In this case, although we choose a small neighborhood in this corner, we will always have at least two chords that form the same angle of the polygon's angle. And, since this is $< 90°$, we will not have Condition A satisfied. On the other hand, if our polygon had all their angles obtuse, there would always be a neighborhood where the chords form, at least, the same angle as the respective obtuse angle. Thus, Condition A would be satisfied.

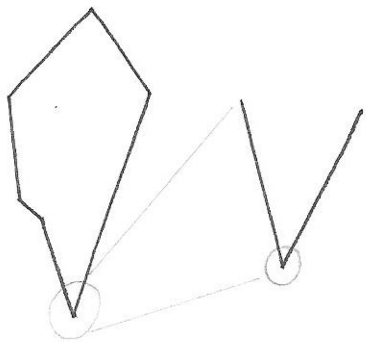

Figure 3.14: Polygons with a non obtuse angle do not satisfy Condition A

Now, let us introduce Lemma 3.24 for curves that satisfy Condition A. If we prove that curves with this weaker smoothness condition satisfy 3.24, then this will imply that they admit an inscribed square, as shown in Theorem 3.28.

Lemma 3.30. *If w satisfies Condition A, then each one-point quadrilateral in Q is contained in exactly one set Q_i. In particular, $v_i \in Q_i$ for $i = 1, 2, 3$ and $\boldsymbol{y} \in Q_4$ for each $\boldsymbol{y}$ on the edge connecting v_0 and v_4.*

Proof. Let $\mathbf{y} = (y, y, y, y)$ be a one-point quadrilateral on the edge from v_0 to v_4. We shall show that $\mathbf{y}$ has a neighborhood in Q such that, for any $\mathbf{x}$ in the neighborhood which is also in Q^0, the fourth side of $\mathbf{x}$ is the unique longest side. This will imply that $\mathbf{y}$ is in Q_4, but not in any other Q_i.

Let μ be as above. The required neighborhood of $\mathbf{y}$ consists of those elements $\mathbf{x} \in Q$ whose coordinates $x_1, \ldots, x_4$ are in $(y-\mu, y+\mu)$. Let $\mathbf{x}$ be an element of this neighborhood which is also in Q^0. Therefore we have $0 < x_1 < \ldots < x_4 < 1$. Let z_1, z_2, z_3, z_4 be vectors in $\mathbb{R}^n$ representing the sides of the quadrilateral $\mathbf{x}$. We have $z_1 = w(x_{i+1}) - w(x_i)$ for $i = 1, 2, 3$ and $z_4 = w(x_4) - w(x_1)$. We will show that z_4 is the longest of these sides. Let us see, for instance, that $z_4 > z_2$. We have

$$z_4 = z_1 + z_2 + z_3,$$

so

$$z_4 \cdot z_2 = z_1 \cdot z_2 + z_2 \cdot z_2 + z_3 \cdot z_2,$$

and since all of these dot products are positive, in particular we have

$$z_4 \cdot z_2 > z_2 \cdot z_2.$$

But if z_4 has a larger component in the direction of z_2 than z_2 itself does, then the fourth side must be strictly longer that the second side. Similarly, the fourth side is longer than the first and the third sides. Since the fourth side is the unique longest side, $\mathbf{x}$ is in Q_4 and no other Q_i's. Then, if it is true for every $\mathbf{x} \in Q^0$ sufficiently near y, it is also true for $\mathbf{y}$ itself.

This proof, taken literally, works for v_0 and v_4. The cases of v_1, v_2 and v_3 require more delicacy in their statement, but are essentially similar. □

Lemma 3.30 implies the following improvement of Theorem 3.28:

Theorem 3.31. *If w satisfies Condition A, then w admits an inscribed quadrilateral with equal sides and equal diagonals.*

Locally monotone curves in $\mathbb{R}^2$. In this part we define a much less restrictive smoothness condition: *local monotonicity.* We will see that this condition still guarantees the existence of an inscribed square. Smooth curves, convex curves polygons, and most piecewise C^1 curves satisfy this condition. This is, therefore, the strongest result provided by Stromquist.

Let us begin by defining this smoothness condition and all the previous definitions required to understand it.

Definition 3.32. *A **segment** of a curve w corresponding to an interval (a, b) is the restriction of the function to that interval. That is: $w_{|(a,b)}$.*

We call $(b - a)$ the length of the segment. This length is measured in parameter space and not in $\mathbb{R}^2$.

Definition 3.33. *The segment is **monotone in the direction u** (where u is a non-zero vector in $\mathbb{R}^2$) if the dot product $w(x) \cdot u$ is a strictly increasing function of x, for $x \in (a, b)$.*

We can think of this as the orthogonal projection of $w(x)$ onto u. Notice that if u was a unit vector, we would have $w(x) \cdot u = \|w(x)\| \cdot \|u\| = \cos\alpha$, where α is the angle formed by $w(x)$ and u.

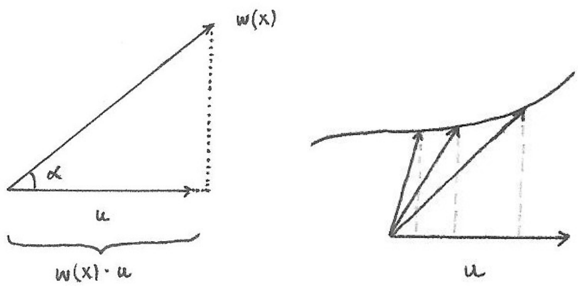

Figure 3.15: Segment monotone in the direction of u

No square inscribed in w (at least, not one with its vertices in the same cyclic order in the square as in the curve) can be inscribed in a monotone segment of w. If the vertices of the square were all placed on a monotone segment, it would imply that the square itself is monotone to a given direction, but this is not true.

Definition 3.34. *The curve is* ***locally monotone*** *if, for every $y \in \mathbb{R}$, there is an interval $(y-\mu, y+\mu)$ and a direction $u(y)$ such that $w_{|(y-\mu,y+\mu)}$ is monotone in the direction of u(y).*

If w has this property, then the periodicity of w and the compactness of $[0,1]$ allow us to choose the number μ to be a constant, independent of y. In this case, every segment of w with length at most 2μ is monotone in some direction, and we say that w is *locally monotone with constant* μ.

Since locally monotonicity is a geometric condition, we offer an equivalent definition which is more visual and, hence, easier to understand:

Definition 3.35 (Equivalent definition of 2.12)**.** *A curve w is* ***locally monotone*** *if for every point $w(y)$ of the curve, there are a neighborhood $U(y)$ in $\mathbb{R}^2$ and a direction $n(y)$ such that no chord of the curve is contained in $U(y)$ and parallel to $n(y)$. The direction $n(y)$ is normal to $u(y)$ and can be thought of as a kind of a normal vector (even if the curve is not differentiable).*

We will motivate this definition with the examples in Figure 3.16 and Figure 3.17. Let p be a point of the curve w. We consider the neighborhood of p in $\mathbb{R}^2$ to be the open ball $B(p,\epsilon)$, with center p and radius $\epsilon > 0$. To check whether w is locally monotone, we need to see that no chord of w contained in $B(p,\epsilon)$ is parallel to $n(p)$.

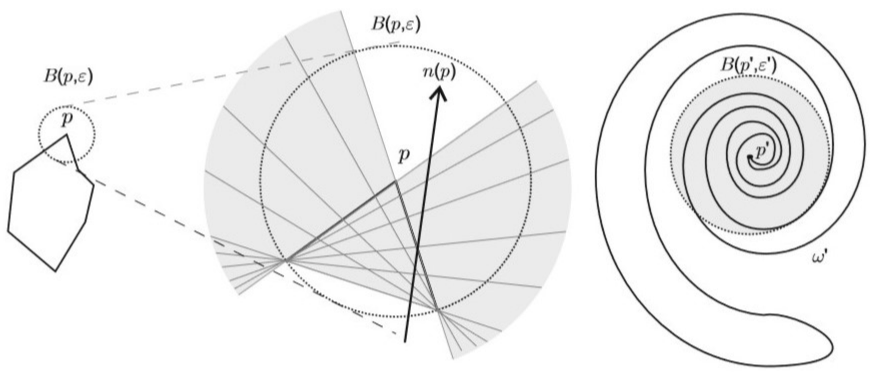

Figure 3.16: Locally monotone curves

In Figure 3.16 we have two curves that are locally monotone. Let us start with the figure on the left side. The shaded region represents the union of lines containing a chord in $w \cap B(p,\epsilon)$. In this case, we have two regions separating the plane. Then, any vector crossing the shaded region may play the role of $n(p)$ (as seen in the image).

As long as w is smooth in the neighborhood of p, this union of lines is close to a straight line and finding $n(p)$ is simple. But if w has sudden changes in $B(p,\epsilon)$, as shown in the figure on the right side, then this shaded region covers a greater area. In our example, since the union of lines containing a chord in $w' \cap B(p',\epsilon')$ covers all the plane, there does not exist any possible assignment of $n(p')$. Then, to see in this case that w' is locally monotone we may use a smaller value of ϵ', so we can choose a $n(p')$.

In Figure 3.17 we have an example of a non locally monotone curve. We have replaced the spiral found in Figure 3.16 with another spiral which has infinitely many turns, but preserving the Jordan curve. The resulting curve will not be locally monotone.

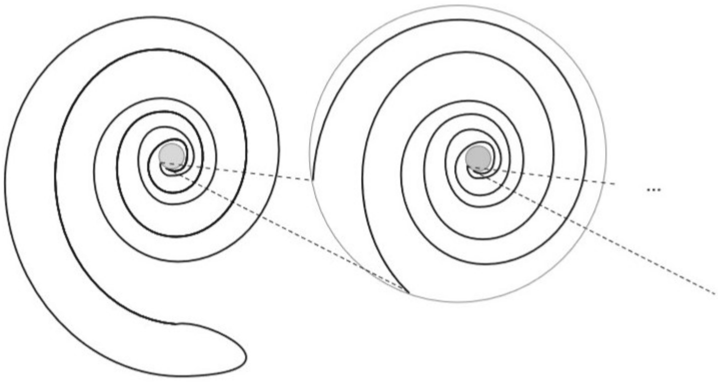

Figure 3.17: Non-locally monotone curve

As we have mentioned before, smooth curves, convex curves polygons and most piecewise C^1 curves are locally monotone curves.

Let us start by seeing that smooth curves are locally monotone. Note that these are C^1 curves. Using Definition 3.35, we could take $n(y)$ to be the normal vector on each neighborhood $U(y)$. No chord contained in $U(y)$ would be parallel to this vector. We could also use Definition 3.34 by taking $u(y) = w'(y)$. In this case, for every $y \in \mathbb{R}$ there would be an interval where w would be monotone in the direction of $w'(y)$.

The case of convex curves is even simpler. To see that convex curves are locally monotone, take $n(y)$ to be a vector from $w(y)$ toward any interior point.

Let us make precise which piecewise C^1 curves are locally monotone. We say that a curve is piecewise C^1 if there exist numbers $x_0, \ldots, x_k$ satisfying $0 = x_0 < \ldots < x_k = 1$ such that w has a continuous non-vanishing derivative w' on each interval $[x_{i-1}, x_i]$, including the one-sided derivatives at the endpoints. We denote the one-sided derivatives at x_i by $w'_-(x_i)$ and $w'_+(x_i)$. The curve has a **cusp** at x_i if these two vectors point in opposite directions; that is, if $w'_+(x_i)$ is a negative multiple of $w'_-(x_i)$. Then it can be shown that a piecewise C^1 curve without cusps is locally monotone. For the direction $u(x_i)$ take any convex combination of $w'_-(x_i)$ and $w'_+(x_i)$.
Geometrically, we could see why a piecewise C^1 curve *with cusps* is non locally monotone using the same tools we used in the above examples. In this case, the shaded region would cover the whole plane, incapacitating the choice of $n(p)$.

We are now ready to extend the main result to locally monotone curves in R^2.

Theorem 3.36. *If w is a locally monotone curve in R^2, then w admits an inscribed square.*

Proof. Assume that w is locally monotone with constant μ.

Our strategy in this proof will be to approximate w using smooth curves w_ϵ, each of which contains an inscribed square by Theorem 3.28. As $\epsilon \to 0$, a subsequence of these inscribed squares converge to a square inscribed in w. The difficulty is in proving that the limiting square does not have size zero, i.e., is non degenerate. To show this, we will first show that each w_ϵ is itself locally monotone with constant at least $\frac{1}{2}\mu$. This will allow us to establish a lower bound on the size of the square we find in w_ϵ.

For this proof, we define the size $|\mathbf{x}|$ of an element $\mathbf{x} = (x_1, x_2, x_3, x_4) \in Q$ to be the smallest of these four numbers: $(x_4 - x_1), ((1+x_3) - x_4), ((1+x_2) - x_3)$ or $((1+x_1) - x_2)$. Therefore, $|\mathbf{x}|$ is the same as the length of the smallest segment of the curve that can contain all of the points $w(x_1), w(x_2), w(x_3), w(x_4)$. Note that $|\mathbf{x}|$ is measured in parameter space, and it is not directly related to the lengths of the sides of the corresponding quadrilateral or any other measurement in $\mathbb{R}^2$. Nevertheless, the only quadrilaterals with size zero are the one-point quadrilaterals and, since $|\mathbf{x}|$ is continuous on Q, any sequence of quadrilaterals whose sizes have a positive lower bound can not converge to a one-point quadrilateral.

Now, let $\epsilon > 0$ and choose $\delta > 0$ such that $|x - y| < \delta$ implies that $\|w(x) - w(y)\| < \epsilon$. In any case, choose $\delta < \frac{1}{2}\mu$. Define $w_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}^2$ by

$$w_\epsilon(x) = \frac{1}{\delta}\int_{t=0}^{\delta} w(x+t)dt.$$

Then, w_ϵ satisfies $\|w_\epsilon(x) - w(x)\| < \epsilon$, since

$$\|w_\epsilon(x) - w(x)\| = \|\frac{1}{\delta}\int_{t=0}^{\delta} w(x+t)dt - \frac{1}{\delta}\int_{t=0}^{\delta} w(x)dt\| \leq \frac{1}{\delta}\int_{t=0}^{\delta} \|w(x+t) - w(x)dt\| < \epsilon$$

for all x, and w_ϵ has a continuous non-vanishing derivative given by

$$w'_\epsilon(x) = \frac{1}{\delta}(w(x+\delta) - w(x)).$$

For sufficiently small ϵ, using the local monotonicity of w, we see that w_ϵ is actually a smooth curve. We may, therefore, apply Theorem 3.28 to find a square S_ϵ inscribed in w_ϵ, whose vertices are in the same cyclic order in the square as in the curve.

We will show that w_ϵ is locally monotone with constant $\frac{1}{2}\mu$. Let $y \in \mathbb{R}$ and let $u(y)$ be chosen such that $w_{|(y-\mu, y+\mu)}$ is monotone in the direction $u(y)$. Let x_1, x_2 be contained in $(y - \frac{1}{2}\mu, y + \frac{1}{2}\mu)$, with $x_1 < x_2$. Then

$$(w_\epsilon(x_2) - w_\epsilon(x_1)) \cdot u(y) = \frac{1}{\delta}\int_{t=0}^{\delta} (w(x_2+t) - w(x_1+t))dt) \cdot u(y) > 0,$$

because we have chosen w to be monotone in the direction $u(y)$. That is, the monotonicity of w forces the integrand to be strictly positive. Therefore, the chord from $w_\epsilon(x_1)$ to $w_\epsilon(x_2)$ has a positive component in the direction of $u(y)$, as required.

It follows that the inscribed square S_ϵ can not have size less than μ, or its vertices would be contained in some interval of length μ, in which w_ϵ would have to be monotone (but we have seen that a square can not be inscribed in a monotone segment of the curve). We repeat this construction for a sequence of values of ϵ approaching zero. Some subsequence of the squares S_ϵ will converge to a square S, which must have size at least μ and must be inscribed in w. This completes the proof of Theorem 3.36. □

Chapter 4

Latest study

After a whole century's history, the Square Peg Problem remains open for its full generality. The latest work is due to Terence Tao, who published in 2017 *An integration approach to the Toeplitz Square Peg Problem* [15].

Terence Chi-Shen Tao, borned in Australia in 1975, has been professor of mathematics at the University of California, Los Angeles (UCLA) since 1999. He started to learn calculus when he was 7, as well as beginning high school at that same age. By 9 he was already very good at university-level calculus. He earned his Ph.D. from Princeton University when he was 20, and UCLA promoted him to full professor at age 24.

Tao was awarded the prestigious Fields Medal in 2006 "for his contributions to partial differential equations, combinatorics, harmonic analysis and additive number theory". Many mathematicians, like John Garnett, professor and former chair of mathematics at UCLA, think that Tao could be the best mathematician in the world right now.

Tao himself tried to solve the Square Peg Problem, as we have mentioned. Here, we intend to give a general idea of his new approach to the problem and explain in what differs from the other previous results.

In his paper, Tao proposes to modify the homological approach to the problem, by focusing not only on the parity of intersections between some geometric objects associated to the curve, but also on bounding certain integrals associated to these curves. He still needs to establish a certain regularity (i.e., smoothness condition) on the curves, as in previous results from other authors, in order to initially define these integrals. Thus, there is still an inability to remove smoothness conditions. But, the integrals enjoy more stability properties under limits than intersection numbers. This is why it may offer a way to establish more cases of the Square Peg Problem.

Tao gives a positive result, which appears to be new:

For any $I \subset \mathbb{R}$ and any function $f : I \longrightarrow \mathbb{R}$, we define the *graph* $\mathbf{Graph}_f : I \longrightarrow \mathbb{R}^2$ to be the function $\mathbf{Graph}_f(t) := (t, f(t))$, so in particular, $\mathbf{Graph}_f(I) \subset \mathbb{R}^2$ is the set

$$\mathbf{Graph}_f(I) := \{(t, f(t)) : t \in I\}.$$

Such a function f is said to be *C-Lipschitz* for a given $C > 0$ if $|f(s) - f(t)| \leq C|s - t|$ for all $s, t \in I$.

This C is actually giving us the approximate maximum slope the curve can have. Since

$$\frac{|f(s) - f(t)|}{|s - t|} \leq C,$$

we would have that there is an angle α such that

$$\tan \alpha = \frac{|f(s) - f(t)|}{|s - t|}$$

which is smaller that the value of C.

Theorem 4.1 (The case of small Lipschitz constant)**.** *Let $[t_0, t_1]$ be an interval, and let $f, g : [t_0, t_1] \longrightarrow \mathbb{R}$ be $(1 - \epsilon)$-Lipschitz functions for some $\epsilon > 0$. Suppose also that $f(t_0) = g(t_0)$, $f(t_1) = g(t_1)$, and $f(t) < g(t)$ for all $t_0 < t < t_1$. Then, the set*

$$\boldsymbol{Graph}_f([t_0, t_1]) \cup \boldsymbol{Graph}_g([t_0, t_1])$$

inscribes a square.

In other words, the Square Peg Problem holds for curves that traverse two Lipschitz graphs, as long as the Lipschitz constants are strictly less than one. Tao notes that the condition of having Lipschitz constant less than one is superficially similar to the property of being locally monotone, as Stromquist considered in his result. However, due to a potentially unbounded amount of oscillations at the endpoints $\mathbf{Graph}_f(t_0) = \mathbf{Graph}_g(t_0)$ and $\mathbf{Graph}_f(t_1) = \mathbf{Graph}_g(t_1)$, the sets in Theorem 4.1 are not necessarily locally monotone at the endpoints. Thus, the results in Stromquist's proof do not directly imply Theorem 4.1. Similarly for the other existing positive results on the Square Peg Problem. Therefore, by proving this fact, he achieves a slightly further case of the Square Peg Problem regarding the so far seen previous results.

Tao explains that without the hypothesis of small Lipschitz constant, the Jordan curve theorem would no longer be available, and he does not know how to adapt the argument to prove the Square Peg Problem in full generality.

Conclusions

Although the Square Peg Problem looks like an innocent conjecture, it has been an open problem for more than a century. We have shown several results throughout this text, some with weaker smoothness conditions than others.

Stromquist made an incredible step by proving it for *locally monotone* curves, since it includes a great variety of curves, like the smooth ones. However, it seems like we are still not able to remove a certain smoothness condition, incapacitating, therefore, the proof for its full generality.

At present, Terence Tao is the mathematician who has gone further on this topic. But he has needed too complex machinery in order to only sightly weaken Stromquist's result.

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