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COMPOSITION OF LARGE DEVIATION PRINCIPLES AND APPLICATIONS

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0. Introduction

The purpose of this note is to present a general result on the composition of large deviation principles (Theorem 1.2) and to apply this theorem to obtain large deviations estimates for solutions of anticipating stochastic differential equations.

This problem was suggested to us by G. Benarous and we would like to thank him for his stimulating remaks.

The composition theorem is stated and proved in Section 1. Its proof is based on a recent result of P. Baldi and M. Sanz [2] about the equivalence between large deviations estimates and a continuity property. Section two contains the application of the composition result to deduce a large deviation principle for the solutions of anticipating stochastic differential equations. Two types of equations have been considered. Frist we deal with the Stratonovich stochastic differential equation studied by D. Ocone and E. Pardoux in [7]. In this case the large deviation principle we present is based on the large deviation principle for stochastic flows proved in [5], and it generalizes the large deviation results obtained previously by the authors in [5]. The second type of anticipating equation is a quasilinear equation introduced by R. Buckdahn in [3]. For this equation we generalize the results of [4].

1. Composition of large deviation principles

Let $(\Omega, \mathcal{F}, P)$ be a probability space and (E,d) be a Polish space. Consider a family $(V^\varepsilon,\ \varepsilon > 0)$ of E–valued random variables which satisfies a large deviation principle (LDP) with rate function $\lambda :\ E \to [0,+\infty]$. That means, for every open (resp. closed) subset G (resp. C) of E we have

$$\liminf_{\varepsilon\downarrow 0}\ \varepsilon\ \log\ P(V^\varepsilon \in G) \geq -\inf\ \{\lambda(f) : f \in G\}$$
$$\limsup_{\varepsilon\downarrow 0}\ \varepsilon\ \log\ P(V^\varepsilon \in C) \leq -\inf\ \{\lambda(f) : f \in C\}\,.$$

In the applications presented in the next section, V^ε will be $\sqrt{\varepsilon}\ W$ where W is a standard Brownian motion.

Suppose that (E_2, d_2) is another Polish space and $(\xi^\varepsilon,\ \varepsilon > 0)$ a family of E_2–valued random variables. Assume also that there exists a map $\xi : \{\lambda < +\infty\} \to E_2$ such that the restriction of ξ to the compact sets $\{\lambda \leq a\}$, $a \in [0, \infty)$, is continuous. The following result has been proved by P. Baldi and M. Sanz in [2, Theorem 3].

Proposition 1.1. The following properties are equivalent:

(P1) The family $\big((V^\varepsilon, \xi^\varepsilon),\ \varepsilon > 0\big)$ satisfies a LDP with rate function

$$\tilde{\lambda}(f, e) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad e = \xi(f), \\ +\infty & \text{otherwise}, \end{cases}$$

for $f \in E$, $e \in E_2$.

(P2) For every $R > 0$, $\eta > 0$ and $f \in E$ such that $\lambda(f) < \infty$ there exist $\alpha > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$

$$P\Big(d_2(\xi^\varepsilon,\ \xi(f)) \geq \eta, \quad d(V^\varepsilon, f) \leq \alpha\Big) \leq \exp\left(-\frac{R}{\varepsilon}\right).$$

Let (F, ρ) be another Polish space and let K be a compact metric space. We denote by d_1 the Euclidean distance on $\mathbb{R}^d$ and by $\tilde{d}_1$ and $\tilde{\rho}$ the distances inducing the topology of uniform convergence on $\mathcal{C}(K, \mathbb{R}^d)$ and $\mathcal{C}(K, F)$, respectively. Let $\tilde{d}$ be a distance on $\mathcal{C}(\mathbb{R}^d, F)$ inducing the topology of uniform convergence on compact subsets of $\mathbb{R}^d$.

Given any $\varepsilon > 0$ we consider random variables $X^\varepsilon : \Omega \to \mathcal{C}(K, \mathbb{R}^d)$, $Y^\varepsilon : \Omega \to \mathcal{C}(\mathbb{R}^d, F)$ and set $\tilde{X}^\varepsilon = (V^\varepsilon, X^\varepsilon)$, $\tilde{Y}^\varepsilon = (V^\varepsilon, Y^\varepsilon)$. We introduce the following assumptions, which correspond to condition (P1) of Proposition 1.1.

(H1) There exists a map $X : \{\lambda < \infty\} \to \mathcal{C}(K, \mathbb{R}^d)$ such that its restriction to the compact sets $\{\lambda \leq a\}$, $a \in [0, \infty)$, is continuous, and the family $(\tilde{X}^\varepsilon,\ \varepsilon > 0)$ satisfies a LDP with rate function

$$\tilde{\lambda}_1(f, g) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad g = X(f), \\ +\infty & \text{otherwise}, \end{cases}$$

for $f \in E$, $g \in \mathcal{C}(K, \mathbb{R}^d)$.

(H2) There exists a map $Y : \{\lambda < \infty\} \to \mathcal{C}(\mathbb{R}^d, F)$ such that its restriction to the compact sets $\{\lambda \leq a\}$, $a \in [0, \infty)$, is continuous, and the family $(\tilde{Y}^\varepsilon,\ \varepsilon > 0)$ satisfies a LDP with rate function

$$\tilde{\lambda}_2(f, h) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad h = Y(f), \\ +\infty & \text{otherwise}, \end{cases}$$

for $f \in E$, $h \in \mathcal{C}(\mathbb{R}^d, F)$.

Then the following theorem states a LDP for the composition of Y^ε and X^ε.

Theorem 1.2. Suppose that the assumptions (H1) and (H2) are satisfied. For each $\varepsilon > 0$, let $Z^\varepsilon : \Omega \to \mathcal{C}(K, F)$ be defined by $Z^\varepsilon = Y^\varepsilon \circ X^\varepsilon$. Then $(Z^\varepsilon, \varepsilon > 0)$ satisfies a LDP with rate function defined on $\mathcal{C}(K, F)$ by

$$\lambda_3(g) = \inf \{\lambda(f) : Y(f) \circ X(f) = g\}. \tag{1.1}$$

Proof: It suffices to show that the pair $(V^\varepsilon, Z^\varepsilon)$ satisfies a LDP with rate function

$$\tilde{\lambda}_3(f, g) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad g = Y(f) \circ X(f), \\ +\infty & \text{otherwise}, \end{cases}$$

where $f \in E$, $g \in \mathcal{C}(K, F)$. Note that conditions (H1) and (H2) imply that the mapping $f \longmapsto Y(f) \circ X(f)$ is continuous on the level sets $\{\lambda \le a\}$, $0 \le a < \infty$. By Proposition 1.1 this is equivalent to the continuity property (P2) for $(Z^\varepsilon,\ \varepsilon > 0)$:

(C): For every $R > 0$, $\eta > 0$ and $f \in E$ such that $\lambda(f) < \infty$ there exists $\alpha > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, if

$$A^\varepsilon = \left\{\tilde{\rho}(Y^\varepsilon \circ X^\varepsilon,\ Y(f) \circ X(f)) \ge \eta,\ d(V^\varepsilon, f) \le \alpha\right\}$$

then

$$P(A^\varepsilon) \le \exp\left(-\frac{R}{\varepsilon}\right). \tag{1.2}$$

This continuity property will be deduced from similar continuity properties for $(X^\varepsilon,\ \varepsilon > 0)$ and $(Y^\varepsilon,\ \varepsilon > 0)$, which are equivalent to our hypotheses (H1) and (H2), respectively, due again to Proposition 1.1. In order to complete the details of the proof of (C), fix $R > 0$, $\eta > 0$ and $f \in E$ such that $\lambda(f) < \infty$ and define, for any $\delta > 0$ and $\alpha > 0$

$$\begin{aligned} B^\varepsilon &= \left\{\tilde{d}_1(X^\varepsilon, X(f)) \ge \delta,\ d(V^\varepsilon, f) \le \alpha\right\} \\ C^\varepsilon &= \left\{\tilde{d}_1(X^\varepsilon, X(f)) < \delta, \quad d(V^\varepsilon, f) \le \alpha,\ \tilde{\rho}(Z^\varepsilon, Y(f) \circ X^\varepsilon) \ge \frac{\eta}{2}\right\} \\ D^\varepsilon &= \left\{\tilde{d}_1(X^\varepsilon, X(f)) < \delta,\ \tilde{\rho}(Y(f) \circ X^\varepsilon,\ Y(f) \circ X(f)) \ge \frac{\eta}{2}\right\}. \end{aligned}$$

Clearly $A^\varepsilon \subset B^\varepsilon \cup C^\varepsilon \cup D^\varepsilon$. By (H1) and Proposition 1.1 there exists $\alpha_1 > 0$, $\varepsilon_1 > 0$ such that for $0 < \alpha \le \alpha_1$, $0 < \varepsilon \le \varepsilon_1$, $P(B^\varepsilon) \le \exp\left(-\frac{R}{\varepsilon}\right)$.

Suppose $0 < \delta \leq 1$, and set $\gamma = 1 + \sup_{k \in K} |X(f)(k)|$. Then $|X^\varepsilon(\omega)| \leq \gamma$ for each $\omega \in C^\varepsilon$, and there exists $\eta' > 0$ such that

$$\begin{aligned} C^\varepsilon &\subset \left\{ d(V^\varepsilon, f) \leq \alpha,\ \sup_{|x| \leq \gamma} \rho(Y^\varepsilon(x),\ Y(f)(x)) \geq \frac{\eta}{2} \right\} \\ &\subset \left\{ d(V^\varepsilon, f) \leq \alpha,\ \tilde{d}(Y^\varepsilon,\ Y(f)) \geq \eta' \right\}. \end{aligned}$$

By (H2) and Proposition 1.1 there exists $\varepsilon_2 > 0$, $\alpha_2 > 0$ such that for $0 < \varepsilon \leq \varepsilon_2$, $0 < \alpha \leq \alpha_2$, $P(C^\varepsilon) \leq \exp\left(-\dfrac{R}{\varepsilon}\right)$. On the other hand, since $Y(f)$ is uniformly continuous on $\{x : |x| \leq \gamma\}$ we can find $\delta \in (0,1]$ such that $D^\varepsilon = \phi$. Therefore for $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$ and $\alpha \leq \min(\alpha_1, \alpha_2)$ we obtain $P(A^\varepsilon) \leq 2\exp\left(-\dfrac{R}{\varepsilon}\right)$ which completes the proof of the continuity condition (C). ∎

The following result shows that if two families of random variables are "close" and one of them satisfies a *LDP*, then the other one also satisfies a *LDP* with the same rate function.

Proposition 1.3. Let (E, d) be a Polish space, and let $(\xi^\varepsilon,\ \varepsilon > 0)$ and $(\eta^\varepsilon,\ \varepsilon > 0)$ be E–valued random variables.

Assume that:

(i) The family $(\xi^\varepsilon,\ \varepsilon > 0)$ satisfies a *LDP* with rate function $I : E \to [0, +\infty]$.

(ii) For any $\alpha > 0$

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \ \log\ P\Big(d(\xi^\varepsilon,\ \eta^\varepsilon) \geq \alpha\Big) = -\infty\,. \tag{1.3}$$

Then the family $(\eta^\varepsilon,\ \varepsilon > 0)$ also satisfies a *LDP* with rate function I.

Proof: Let G be an open subset of E. Let $g \in G$ and fix a neighborhood U of g such that

$$U_\alpha = \{x \in E : d(x, U) < \alpha\} \subset G\,.$$

Then for any $\varepsilon > 0$

$$P(\eta^\varepsilon \in G) \geq P(\xi^\varepsilon \in U) - P\Big(d(\xi^\varepsilon, \eta^\varepsilon) \geq \alpha\Big)\,.$$

Hypothesis (i) yields

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \ \log\ P(\xi^\varepsilon \in U) \geq -\inf\,\{I(f) : f \in U\} \geq -I(g)\,.$$

Therefore (ii) implies that $\liminf_{\varepsilon \downarrow 0} \varepsilon \log P(\eta^\varepsilon \in G) \geq -I(g)$, and since g is arbitrary we obtain

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log P(\eta^\varepsilon \in G) \geq -\inf \{I(g) : g \in G\}.$$

Now let F be a closed subset of E, $\alpha > 0$ and set $F_\alpha = \{x \in E : d(x, F) < \alpha\}$. Then

$$P(\eta^\varepsilon \in F) \leq P(\xi^\varepsilon \in \overline{F}_\alpha) + P\Big(d(\xi^\varepsilon, \eta^\varepsilon) \geq \alpha\Big).$$

Since $\limsup_{\varepsilon \downarrow 0} \varepsilon \log P(\xi^\varepsilon \in \overline{F}_\alpha) \leq -\inf\{I(g) : g \in \overline{F}_\alpha\}$, (ii) yields that for any $\alpha > 0$

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log P(\eta^\varepsilon \in F) \leq -\inf \{I(g) : g \in \overline{F}_\alpha\}.$$

Finally, letting $\alpha \downarrow 0$ we obtain

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log P(\eta^\varepsilon \in F) \leq -\inf \{I(g) : g \in F\}. \qquad \blacksquare$$

2. Large deviations for anticipating stochastic differential equations

In this section we will give two applications of Theorem 1.2 to the solution of anticipating stochastic differential equations. First we will consider the equations studied by Ocone and Pardoux in [7].

Let $(W_t,\ t \in [0,1])$ be a k-dimensional standard Brownian motion defined on the canonical probability space $(\Omega, \mathcal{F}, P)$. Suppose that $b, \sigma_i : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $1 \leq i \leq k$, and $m = \frac{1}{2} \sum_{i=1}^{k} \frac{\partial \sigma_i}{\partial x} \sigma_i : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are functions of class $\mathcal{C}^2$ with bounded partial derivatives up to order 2. Let $(\varphi^\varepsilon,\ \varepsilon > 0)$ denote the family of stochastic flows defined on $\mathbb{R}^d \times [0,1]$ by

$$\varphi_t^\varepsilon(x) = x + \int_0^t \sqrt{\varepsilon}\, \sigma_i \left(\varphi_s(x)\right) \circ dW_s^i + \int_0^t b\left(\varphi_s(x)\right) ds\,. \tag{2.1}$$

Here we made the convention of summation over repeated indices and the first stochastic integral is defined in the Stratonovich sense.

Define

$$\mathcal{H}_k = \Big\{f : [0,1] \longrightarrow \mathbb{R}^k : f(t) = \int_0^t \dot{f}(s)\, ds\,,$$
$$\lambda(f) := \frac{1}{2} \int_0^1 |\dot{f}_s|^2\, ds < \infty\Big\}\,,$$

and set $\lambda(f) = +\infty$ if $f \notin \mathcal{H}_k$.

Given $f \in \mathcal{H}_k$ let $h = S(f)$ denote the solution of the ordinary differential equation

$$h_t(x) = x + \int_0^t \left[\sigma\left(h_s(x)\right) \dot{f}_s + b\left(h_s(x)\right)\right] ds \ , \tag{2.2}$$

called the skeleton of $\varphi_t^1\ (x)$.

Let $E := \Omega = \mathcal{C}_0([0,1],\ \mathbb{R}^k)$ be the set of continuous functions from [0,1] into $\mathbb{R}^k$ which vanish at 0, endowed with the distance d defined by the supremum norm on [0,1], and let $\mathcal{C}([0,1] \times \mathbb{R}^d,\ \mathbb{R}^d)$ be endowed with a distance $\tilde{\rho}$ inducing the topology of uniform convergence on compact sets.

Note that by Gronwall's lemma, the restriction of S to each level set $\{\lambda \le a\}$, $a \in [0, \infty)$, is continuous.

Then, given $f \in \mathcal{H}_k$, $\eta > 0$ and $R > 0$ there exists $\alpha > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \le \varepsilon_0$

$$P\left(\tilde{\rho}(\varphi^\varepsilon,\ S(f)) \ge \eta\,,\ d(\sqrt{\varepsilon}\,W,\ f) \le \alpha\right) \le \exp\left(-\frac{R}{\varepsilon}\right). \tag{2.3}$$

This result is a kind of uniform Ventzell-Freidlin estimation and has been proved in [5, Theorem 2.1]. As a consequence, by Proposition 1.1, the pair $(\sqrt{\varepsilon}\ W,\ \varphi^\varepsilon)$ satisfies a LDP with rate function

$$I_2(f,h) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad h = S(f), \\ +\infty & \text{otherwise.} \end{cases}$$

Then we have the following result:

Proposition 2.1. Let $(X_0^\varepsilon,\ \varepsilon > 0)$ be a family of $\mathbb{R}^d$–valued random variables verifying the following condition:

(i) There exists a mapping $\zeta : \mathcal{H}_k \longrightarrow \mathbb{R}^d$ such that its restriction to the compact sets $\{\lambda \le a\}$, $a \in [0, \infty)$, is continuous, and the pair $(\sqrt{\varepsilon}\ W,\ X_0^\varepsilon)$ satisfies a LDP on $\mathcal{C}_0([0,1],\ \mathbb{R}^k) \times \mathbb{R}^d$ with rate function

$$I_1(f,g) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad g = \zeta(f), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $(\varphi^\varepsilon,\ \varepsilon > 0)$ be the stochastic flow solution of (2.1). Then $\left(Z_t^\varepsilon = \varphi_t^\varepsilon\,(X_0^\varepsilon)\,,\ t \in [0,1]\right)$ is a solution of the anticipating stochastic differential equation

$$Z_t^\varepsilon = X_0^\varepsilon + \int_0^t \sigma_i\,(Z_s^\varepsilon) \circ d\,W_s^i + \int_0^t b\,(Z_s^\varepsilon)\,ds \tag{2.4}$$

and it satisfies a LDP with rate function

$$I(g) = \inf \left\{ I(f) : S(f)\left(\zeta(f)\right) = g \right\} , \tag{2.5}$$

for any $g \in \mathcal{C}([0,1], \mathbb{R}^d)$.

Proof: It suffices to apply Theorem 1.2 in the following context. Let $K = \{0\}$, $F = \mathcal{C}([0,1], \mathbb{R}^d)$, and as above $E = \mathcal{C}_0([0,1], \mathbb{R}^k)$. Then $\mathcal{C}(K, \mathbb{R}^d) \cong \mathbb{R}^d$ and $\mathcal{C}(\mathbb{R}^d, F) \cong \mathcal{C}([0,1] \times \mathbb{R}^d, \mathbb{R}^d)$. Condition (i) implies $(H1)$ for $X = \zeta$ and $X^\varepsilon = X_0^\varepsilon$.

Moreover we have seen that $Y^\varepsilon = \varphi^\varepsilon$ and $Y = S$ satisfy condition $(H2)$. This completes the proof of the proposition. ∎

Remark 2.2. Proposition 2.1 generalizes the large deviation estimates obtained in [5]. Indeed, the condition

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log P\left(|X_0^\varepsilon - x_0| > \eta\right) = -\infty , \tag{2.6}$$

for any $\eta > 0$ and some $x_0 \in \mathbb{R}^d$, can be considered as a particular case of the assumption (i) in Proposition 2.1. More precisely, let $\zeta_0 : \mathcal{H}_k \longrightarrow \mathbb{R}^d$ be defined by $\zeta_0(f) = x_0$. Then (2.6) implies that the family $\left((\sqrt{\varepsilon}\, W, X_0^\varepsilon), \varepsilon > 0\right)$ satisfies a LDP on $\mathcal{C}_0([0,1], \mathbb{R}^k) \times \mathbb{R}^d$ with rate function

$$I_1(f,x) = \begin{cases} \lambda(f) & \text{if} \quad x = x_0 \\ +\infty & \text{if} \quad x \neq x_0 . \end{cases}$$

In fact, by Proposition 1.3 it suffices to show that the family $\left((\sqrt{\varepsilon}\, W, x_0), \varepsilon > 0\right)$ satisfies a LDP with rate function I_1 and this is straightforward.

Now we proceed to deduce large deviation estimates for a different type of anticipating stochastic differential equation. Set $K = [0,1]$ and let $(\Omega, \mathcal{F}, P)$ be the canonical probability space associated with a standard one–dimensional Brownian motion. Fix a Lipschitz function $b : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $\sigma \neq 0$. For any $\varepsilon > 0$ and $t \in [0,1]$ set

$$\psi_t^\varepsilon(\omega) = \exp\left(\sqrt{\varepsilon}\, \sigma\, \omega_t - \frac{1}{2}\, \varepsilon \sigma^2 t\right) .$$

Consider the family of transformations $A_t^\varepsilon : \Omega \longrightarrow \Omega$ defined by

$$A_t^\varepsilon(\omega)(s) = \omega_s - \sqrt{\varepsilon}\, \sigma\,(t \wedge s) ,$$

and set $T_t^\varepsilon = \left(A_t^\varepsilon\right)^{-1}$, that means

$$T_t^\varepsilon(\omega)(s) = \omega_s + \sqrt{\varepsilon}\, \sigma\,(t \wedge s) .$$

We denote by $\left(z_t^\varepsilon(\omega,x),\ t\in[0,1]\right)$ the solution of the ordinary differential equation

$$z_t^\varepsilon(\omega,x) = x + \int_0^t \left[\psi_s^\varepsilon\left(T_t^\varepsilon(\omega)\right)\right]^{-1} b\left[\psi_s^\varepsilon\left(T_t^\varepsilon(\omega)\right)\, z_s^\varepsilon(\omega,x)\right] ds\ .$$

Then if $M^\varepsilon(t,x,\omega) = z_t^\varepsilon\left(A_t^\varepsilon(\omega),\ x\right)$ we have that

$$M^\varepsilon(t,x,\omega) = x + \int_0^t \exp\left(-\sqrt{\varepsilon}\,\sigma\,\omega_s + \frac{\varepsilon}{2}\,\sigma^2 s\right) b\left[\exp\left(\sqrt{\varepsilon}\,\sigma\,\omega_s - \frac{\varepsilon}{2}\,\sigma^2 s\right) M^\varepsilon(s,x,\omega)\right] ds\,. \tag{2.7}$$

Furthermore, the stochastic flow (ζ_t^ε) solution of

$$\zeta_t^\varepsilon(x) = x + \sqrt{\varepsilon}\int_0^t \sigma\,\zeta_s^\varepsilon(x)\,dW_s + \int_0^t b\left(\zeta_s^\varepsilon(x)\right) ds \tag{2.8}$$

is given by

$$\zeta_t^\varepsilon(x) = \psi_t^\varepsilon\ M^\varepsilon(t,x)\ . \tag{2.9}$$

Given a random variable $X_0^\varepsilon \in \mathbb{D}^{1,p}$ for some $p>1$, it has been proved in [3,4] that the process $Z_t^\varepsilon = \zeta_t^\varepsilon\left[X_0^\varepsilon(A_t^\varepsilon)\right]$ has a continuous version and it is the unique solution of the anticipating quasilinear stochastic differential equation

$$Z_t^\varepsilon = X_0^\varepsilon + \sqrt{\varepsilon}\int_0^t \sigma\, Z_s^\varepsilon\ dW_s + \int_0^t b(Z_s^\varepsilon)\,ds\ , \tag{2.10}$$

where the stochastic integral is defined in the Skorohod sense. We refer the reader to [6] for the definition and main properties of the Skorohod integral and the Sobolev spaces $\mathbb{D}^{1,p}$.

We at first prove that the pair $\left((\sqrt{\varepsilon}\ W,\ \zeta^\varepsilon),\ \varepsilon>0\right)$ satisfies a LDP on $\mathcal{C}_0([0,1])\times\mathcal{C}([0,1]\times\mathbb{R})$. The very particular nature of ζ^ε, with a constant diffusion coefficient in dimension one, allows to obtain this result under milder assumptions on b than those required for the general uniform Ventzell-Freidlin estimates. As before $\tilde{\rho}$ will denote the metric on $\mathcal{C}([0,1]\times\mathbb{R})$, which induces the topology of uniform convergence on compact sets.

Let $\eta^\varepsilon(t,x,\omega)$ be the solution of the ordinary differential equation (for each fixed $\omega\in\Omega$)

$$\eta^\varepsilon(t,x,\omega) = x + \int_0^t \exp\left(-\sigma\,\omega_s + \frac{\varepsilon}{2}\,\sigma^2\,s\right) b\left[\exp\left(\sigma\,\omega_s - \frac{\varepsilon}{2}\,\sigma^2\,s\right)\eta^\varepsilon(s,x,\omega)\right] ds\,,$$

then $\eta^\varepsilon(t,x,\sqrt{\varepsilon}\,\omega) = M^\varepsilon(t,x,\omega)$. Similarly, let $\eta(t,x,\omega)$ be the solution of the differential equation

$$\eta(t,x,\omega) = x + \int_0^t \exp(-\sigma\,\omega_s)\, b\left[\exp(\sigma\,\omega_s)\,\eta(s,x,\omega)\right] ds\ .$$

Consider the mappings H^ε, $H : \mathcal{C}_0([0,1]) \longrightarrow \mathcal{C}([0,1] \times \mathbb{R})$ given by

$$H^\varepsilon(\omega)\,(t,x) = \exp\left(\sigma\,\omega_t - \frac{1}{2}\,\varepsilon\,\sigma^2\,t\right)\,\eta^\varepsilon(t,x,\omega) \tag{2.11}$$

$$H(\omega)\,(t,x) = \exp\left(\sigma\,\omega_t\right)\,\eta\,(t,x,\omega)\,. \tag{2.12}$$

Notice that for any $f \in \mathcal{H}_1 \subset \Omega$, $H(f)$ is the skeleton associated with f and the stochastic flow $(\zeta_t^\varepsilon(x))$ introduced in (2.8). That means, $H(f)$ is the solution of the ordinary differential equation

$$H(f)_t\,(x) = x + \int_0^t \Big[\sigma\,H(f)_s\,(x)\dot{f}_s + b\big(H(f)_s\,(x)\big)\Big]\,ds\,.$$

Proposition 2.2. The pair $\big((\sqrt{\varepsilon}\,W,\,\zeta^\varepsilon),\ \varepsilon > 0\big)$ satisfies a LDP on $\mathcal{C}_0([0,1])\times\mathcal{C}([0,1]\times\mathbb{R})$ with rate function

$$I_2(f,g) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad g = H(f)\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.13}$$

where

$$\lambda(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |\dot{f}_s|^2\,ds & \text{if} \quad f \in \mathcal{H}_1 \\ +\infty & \text{otherwise} \end{cases}$$

denotes the rate function of the Brownian motion.

Proof: For any $\varepsilon > 0$ set $G^\varepsilon(\omega) = \big(\omega,\,H^\varepsilon(\omega)\big)$ and $G(\omega) = \big(\omega,\,H(\omega)\big)$. Then it is not difficult to check that G and G^ε are continuous functions on $\Omega = \mathcal{C}_0([0,1])$ and $\lim_{\varepsilon\downarrow 0} G^\varepsilon = G$ uniformly on compact subsets of Ω.

Let P^ε denote the law of $\sqrt{\varepsilon}\,W$ on Ω; then $(P^\varepsilon,\ \varepsilon > 0)$ satisfies a LDP on Ω with rate function λ. Let Q^ε denote the law of $G^\varepsilon(\sqrt{\varepsilon}\,W) = \big(\sqrt{\varepsilon}\,W,\ H(\sqrt{\varepsilon}\,W)\big)$. Then (cf. [8, Theorem 2.4]) $(Q^\varepsilon,\ \varepsilon > 0)$ satisfies a LDP with rate function.

$$\begin{aligned} I_2(f,g) &= \inf\{\lambda(f') : G(f') = (f,g)\} \\ &= \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad g = H(f)\,, \\ +\infty\,, & \text{otherwise.} \end{cases} \end{aligned}$$

Furthermore, by construction $H^\varepsilon(\sqrt{\varepsilon}\,\omega) = \exp\left(\sigma\sqrt{\varepsilon}\,\omega_t - \frac{1}{2}\,\varepsilon\,\sigma^2\,t\right)\,\eta^\varepsilon\,(t,x,\sqrt{\varepsilon}\,\omega) = \psi_t^\varepsilon\,(\omega)\,M^\varepsilon(t,x,\omega) = \zeta_t^\varepsilon(x)\,(\omega)\,.$ ∎

Note that the restriction of H to the compact sets $\{\lambda \leq a\}$, $a \in [0, \infty)$, is continuous.

The second ingredient in the proof of large deviation estimates for the solution of (2.10) will be a LDP for the pair $(\sqrt{\varepsilon}\, W,\ X_0^\varepsilon(A_0^\varepsilon))$.

Proposition 2.3. Let $(X_0^\varepsilon,\ \varepsilon > 0)$ be a family of real–valued random variables verifying the following conditions:

(i) There exists a mapping $X : \mathcal{H}_1 \longrightarrow \mathbb{R}$ such that its restriction to the compact sets $\{\lambda \leq a\}$, $a \in [0, \infty)$, is continuous and the pair $\big((\sqrt{\varepsilon}\, W,\ X_0^\varepsilon),\ \varepsilon > 0\big)$ satisfies a LDP on $\mathcal{C}_0([0,1]) \times \mathbb{R}$ with rate function

$$\lambda_0(f,g) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad g = X(f), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.14}$$

(ii) For each $\varepsilon > 0$, X_0^ε belongs to $\mathbb{D}^{1,p}$, $p > 1$, and for any $M > 0$ there exists $\varepsilon_0 > 0$ such that

$$\sup_{0<\varepsilon\leq\varepsilon_0} E \int_0^1 \exp\left[M\,|D_s X_0^\varepsilon|^2\right] ds < \infty\,. \tag{2.15}$$

For any $\varepsilon > 0$ and $t \in [0,1]$, set $X_t^\varepsilon(\omega) = X_0^\varepsilon\left(A_t^\varepsilon(\omega)\right)$. Then X^ε has a version with continuous paths and $\big((\sqrt{\varepsilon}\, W, X^\varepsilon),\ \varepsilon > 0\big)$ satisfies a LDP with rate function

$$I_1(f,g) = \begin{cases} \lambda(f) & \text{if} \quad \lambda(f) < \infty \quad \text{and} \quad g_t = X(f) \quad \text{for all} \quad t \in [0,1], \\ +\infty & \text{otherwise,} \end{cases} \tag{2.16}$$

where $f \in \mathcal{C}_0([0,1])$, $g \in \mathcal{C}([0,1])$.

Proof: As it has been proved in [4, Proposition 1.3] the existence of a continuous version for (X_t^ε) follows from the formula

$$X_0^\varepsilon(A_t^\varepsilon) - X_0^\varepsilon(A_s^\varepsilon) = -\sigma\sqrt{\varepsilon}\int_s^t (D_r X_0^\varepsilon)(A_r^\varepsilon)\, dr\ ,$$

for any $s \leq t$. Condition (i) implies that $\big((\sqrt{\varepsilon}\, W,\ X_0^\varepsilon),\ \varepsilon > 0\big)$ satisfies a LDP on $\mathcal{C}_0([0,1]) \times \mathcal{C}([0,1])$ with rate function (2.16). Here we have identified X_0^ε with a constant function. Then, using Proposition 1.3 it suffices to show that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\left(\sup_{0\leq t\leq 1} |X_0^\varepsilon(A_t^\varepsilon) - X_0^\varepsilon| \geq \alpha\right) = -\infty\ , \tag{2.17}$$

for any $\alpha > 0$. This has been proved in [4, Proof of Theorem 2.1]. For the sake of completeness we give below the main steps of this proof which is based on the condition (2.15).

Given $R > 0$, let $k > 0$ be such that $P\left(\sup_{0\le t\le 1} |\sqrt{\varepsilon}\,W_t| \ge k\right) \le \exp\left(-\frac{R}{\varepsilon}\right)$. Let $B = \left\{\sup_{0\le t\le 1} |\sqrt{\varepsilon}\,W_t| \le k\right\}$. Then given M such that $\frac{M\,\alpha}{\sigma} > R$ we have

$$
\begin{aligned}
&P\left(B \cap \left\{\sup_{0\le t\le 1} |X_0^\varepsilon(A_t^\varepsilon) - X_0^\varepsilon| \ge \alpha\right\}\right) \\
\le\, & P\left(B \cap \left\{\exp\left(M \int_0^1 |(D_s X_0^\varepsilon)(A_s^\varepsilon)|^2\, ds\right) \ge \exp\left(\frac{R}{\varepsilon}\right)\right\}\right) \\
\le\, & \exp\left(-\frac{R}{\varepsilon}\right) \int_B \exp\left[M \int_0^1 |(D_s X_0^\varepsilon)(A_s^\varepsilon)|^2\, ds\right] dP \\
\le\, & \exp\left(-\frac{R}{\varepsilon}\right) \int_0^1 \int_B \exp\left(M |(D_s X_0^\varepsilon)(A_s^\varepsilon)|^2\right) dP\, ds \\
\le\, & \exp\left(-\frac{R}{\varepsilon}\right) \int_0^1 \int_{\{\sup_{0\le t\le 1} |\sqrt{\varepsilon}\,W_t| \le k'\}} \exp\left(M\, |D_s X_0^\varepsilon|^2\right) e^{-\sigma\sqrt{\varepsilon}\,W_s - \frac{\varepsilon}{2}\sigma^2 s}\, ds \\
\le\, & C' \exp\left(-\frac{R}{\varepsilon}\right),
\end{aligned}
$$

and (2.17) is proved. ∎

Now we can state the LDP for the solution of (2.10).

Proposition 2.4. Let $(X_0^\varepsilon,\ \varepsilon > 0)$ be a family of real–valued random variables verifying the assumptions (i) and (ii) of Proposition 2.3. Then the family $(Z^\varepsilon,\ \varepsilon > 0)$ of solutions of (2.10) satisfies a LDP on $\mathcal{C}([0,1])$ with rate function

$$
\tilde{I}(g) = \inf\left\{\lambda(f) : H(f)\left(t,\, X(f)\right) = g_t\,,\ t \in [0,1]\right\}, \tag{2.18}
$$

where H is defined in (2.12).

Proof: Set $d = 1$, $K = [0,1]$, $E = \mathbb{R}$, $V^\varepsilon = \sqrt{\varepsilon}\,W$ and $F = \mathcal{C}([0,1])$. We want to apply Theorem 1.2 to the random variables $X^\varepsilon = X_0^\varepsilon(A_\cdot^\varepsilon) : \Omega \longrightarrow \mathcal{C}(K, \mathbb{R})$ and $Y^\varepsilon = \zeta^\varepsilon : \Omega \longrightarrow \mathcal{C}(\mathbb{R}, F)$. By Propositions 2.2 and 2.3 the hypotheses (H1) and (H2) hold. Consequently the family of random variables $\bar{Z}^\varepsilon : \Omega \longrightarrow \mathcal{C}([0,1],\, F) \cong \mathcal{C}([0,1]^2)$ defined by $\bar{Z}^\varepsilon(s,t) = \zeta_t^\varepsilon(X_s^\varepsilon)$ satisfies a LDP with rate function

$$
\bar{I}\,(f,g) = \inf\left\{\lambda(f) : H(f)\left(t,\, X(f)\right) = g(t,s)\,,\ (s,t) \in [0,1]^2\right\}.
$$

Let $\Pi : \mathcal{C}([0,1]^2) \longrightarrow \mathcal{C}([0,1])$ be the continuous mapping defined by $\Pi(g)_t = g(t,t)$. Then $Z_t^\varepsilon = \Pi(\bar{Z}^\varepsilon)_t$ and, therefore, the family $(Z^\varepsilon,\ \varepsilon > 0)$ satisfies a LDP with the rate function (2.18). ∎

Taking into account Remark 2.2 it follows that Proposition 2.4 generalizes the large estimations results obtained in [4] for the anticipating quasilinear equation.

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