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WEDGE CANCELLATION AND GENUS

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§1. INTRODUCTION

Let P be the set of homotopy types of base–pointed finite CW–complexes. The wedge operation $X \vee Y$ (union with base points identified) gives P a structure of commutative monoid. It is known that P is not a cancellation monoid; that is, there are examples of spaces X, Y, Z such that

$$X \vee Z \simeq Y \vee Z\ , \qquad \text{but} \qquad X \not\simeq Y\ .$$

This phenomenon was first observed in the stable category by Freyd; see [3] and [4]. Freyd studied its relationship with the genus of the spaces (in the stable sense) and proved the following.

Theorem. In the stable homotopy category of finite CW–complexes, the following conditions are equivalent:

(i) There is a space Z such that $X \vee Z \simeq Y \vee Z$.
(ii) $X \vee B \simeq Y \vee B$, where B is the wedge of spheres with the same Betti numbers as X.
(iii) X and Y are of the same (stable) genus.

The same sort of results as Freyd obtained are expected to hold in the unstable case; however the situation is more complicated here. In [6], Hilton gave examples of spaces X, Y such that for a certain sphere S,

$$\begin{aligned} &X \vee S \simeq Y \vee S & \qquad & X \not\simeq Y \\ &2X \simeq 2Y & & 2X \not\simeq X \vee Y\ . \end{aligned}$$

In these examples, X and Y are CW–complexes with three cells in dimensions $0, n$ and m, $m-1 > n \geq 1$, and with attaching maps $\alpha, \beta \in \pi_{m-1}(S^n)$ of finite order. For this kind of spaces E.A. Molnar [9] proved the following.

Theorem. Let α and β be elements of finite order in $\pi_{m-1}(S^n)$, $m-1 > n \geq 1$, and denote by C_α and C_β their mapping cones. Then the following conditions are equivalent:

(i) C_α and C_β are of the same genus.
(ii) $C_\alpha \vee S^n \vee S^m \simeq C_\beta \vee S^n \vee S^m$.
(iii) There is a wedge of spheres T such that $C_\alpha \vee T \simeq C_\beta \vee T$.

If, in addition, n and the orders of α and β are odd, these conditions are also equivalent to

(iv) $C_\alpha \vee S^n \simeq C_\beta \vee S^n$.
(v) $C_\alpha \vee S^m \simeq C_\beta \vee S^m$.

Actually, Hilton and Roitberg had already proved in [6] that (i) always implies (v) and that, if α is a suspension element, (i) also implies (iv).

In this paper we shall study the case of certain CW–complexes with a finite number of cells in dimension n and one cell in dimension m. Our spaces will be mapping cones C_α and C_β of elements α and β in $\pi_{m-1}(\bigvee^k S^n)$, of finite order. By the Hilton–Milnor Theorem,

$$\pi_{m-1}(\vee^k S^n) \cong \bigoplus_{i=1}^{k} \pi_{m-1}(S^n) \oplus \bigoplus_{j} \pi_{m-1}(S^{n_j}) ,$$

where the S^{n_j} are spheres of dimension greater than $2(n-1)$. The direct summands $\pi_{m-1}(S^{n_j})$ are embedded in $\pi_{m-1}(\bigvee^k S^n)$ by composition with certain Whitehead products, so the suspension elements α in $\pi_{m-1}(\bigvee^k S^n)$ belong to the subgroup $\bigoplus_{i=1}^{k} \pi_{m-1}(S^n)$ each component being itself a suspension element in $\pi_{m-1}(S^n)$. Our main result is the following.

Theorem. Let α and β be elements of finite order in $\pi_{m-1}(\bigvee^k S^n)$, $m-1 > n > 2$. If α is a suspension element then the following conditions are equivalent:

(i) $C_\alpha \sim C_\beta$.
(ii) $C_\alpha \vee \bigvee^k S^n \simeq C_\beta \vee \bigvee^k S^n$.
(iii) There is a wedge of spheres T such that $C_\alpha \vee T \simeq C_\beta \vee T$.

We shall prove this in §3. The proof uses a criterion for C_α and C_β to be of the same genus, that will be proved in §2. We also give an example that shows that, in general, (i) does not imply $C_\alpha \vee S^m \simeq C_\beta \vee S^m$.

In this paper all the spaces are finite CW–complexes with base point. A basic reference for p–localization and genus of these spaces is [8].

We shall always suppose $m-1 > n > 2$.

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§2. A CRITERION FOR $C_\alpha \sim C_\beta$ TO HOLD

Suppose

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & X & \longrightarrow & C_\alpha \\ \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\delta} \\ B & \xrightarrow{\beta} & Y & \longrightarrow & C_\beta \end{array}$$

is a commutative diagram such that φ, δ are homotopy equivalences and A is a Moore–space $K'(G, n)$. If G is free, or if $n \geq 2$ and Y is 2–connected, then there is a homotopy equivalence $\psi : A \to B$ completing the diagram, see [5]. In particular, if $\alpha, \beta \in \pi_{m-1}(\bigvee^k S^n)$, $m - 1 > n > 2$, then a homotopy equivalence between the p–localizations $C_{\alpha(p)}$ and $C_{\beta(p)}$ arises from a homotopy commutative diagram

$$\begin{array}{ccc} S^{m-1}_{(p)} & \xrightarrow{\alpha(p)} & \bigvee^k S^n_{(p)} \\ {\scriptstyle\overline{\psi}}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ S^{m-1}_{(p)} & \xrightarrow{\beta(p)} & \bigvee^k S^n_{(p)} \end{array},$$

where $\overline{\psi}$ and $\overline{\varphi}$ are homotopy equivalences. Thus, $\overline{\psi}$ is a unit in $[S^{m-1}_{(p)}, S^{m-1}_{(p)}] \cong \mathbf{Z}_{(p)}$, the p–localization of the ring $\mathbf{Z}$. Similarly, $[\bigvee^k S^n_{(p)}, \bigvee^k S^n_{(p)}]$ is isomorphic to the ring of $k \times k$ matrices over $\mathbf{Z}_{(p)}$, and $\overline{\varphi}$ is a homotopy equivalence if and only if its determinant is a unit in $\mathbf{Z}_{(p)}$. In fact, when α is a suspension element, we can assume that $\overline{\psi}$, and the elements in the matrix of $\overline{\varphi}$, are integers. For, if l is the least common multiple of the denominators that appear in $\overline{\psi}$ and in the entries of the matrix of $\overline{\varphi}$, then l is coprime to p. Now, take maps $\psi' : S^{m-1}_{(p)} \to S^{m-1}_{(p)}$ and $\varphi' : \bigvee^k S^n_{(p)} \to \bigvee^k S^n_{(p)}$ with matrices l and lI, respectively, where I is the $k \times k$ identity matrix. Clearly $\varphi' \circ \alpha(p) \simeq \alpha(p) \circ \psi'$ and the homotopy equivalences $\varphi = \overline{\varphi} \circ \varphi'$ and $\psi = \overline{\psi} \circ \psi'$ can replace $\overline{\varphi}$ and $\overline{\psi}$ in the above diagram.

In our arguments, we will often use the same symbol to denote a self–map between a wedge of spheres and its matrix.

Proposition 1. Let $\alpha, \beta \in \pi_{m-1}(\bigvee^k S^n)$ be of finite order with α a suspension element. Then $C_{\alpha(p)} \simeq C_{\beta(p)}$ if and only if there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{m-1} & \xrightarrow{\alpha} & \bigvee^k S^n \\ \tilde{\psi}\downarrow & & \downarrow\tilde{\varphi} \\ S^{m-1} & \xrightarrow{\beta} & \bigvee^k S^n \end{array},$$

where $\tilde{\psi}$ and $\tilde{\varphi}$ have integer matrices with determinants coprime to p.

Proof. If a prime q does not divide the order $|\alpha|$ of α, then $\alpha(q) = 0$; otherwise, $|\alpha(q)|$ is a power of q. So we can find an integer s coprime to p and divisible by $|\alpha(q)|$ and $|\beta(q)|$ for all primes $q \neq p$. Now, if $C_{\alpha(p)} \simeq C_{\beta(p)}$, take ψ and φ as above and define $\tilde{\psi} = s\psi$, $\tilde{\varphi} = s\varphi$. It is easy to see that $(\tilde{\varphi} \circ \alpha)(q) = 0 = \beta(q) \circ \tilde{\psi} = (\beta \circ \tilde{\psi})(q)$ for any prime $q \neq p$ and $(\tilde{\varphi} \circ \alpha)(p) = (\beta \circ \tilde{\psi})(p)$. This implies $\tilde{\varphi} \circ \alpha = \beta \circ \tilde{\psi}$. #

The following criterion — that generalises theorem 1.9. in [9] — provides us with an useful tool for the study of further results.

Theorem 2. Let $\alpha, \beta \in \pi_{m-1}(\bigvee^k S^n)$ be elements of finite order with α a suspension element. Then, $C_\alpha \sim C_\beta$ if and only if there is a map $\varphi : \bigvee^k S^n \to \bigvee^k S^n$ such that $\beta = \varphi \circ \alpha$ and $\det \varphi$ is coprime to $|\alpha|$.

This follows from the next three lemmas.

Lemma 3. Let β be an element of finite order in $\pi_{m-1}(\bigvee^k S^n)$. If, for each prime p, $\overline{\psi}_p$ is an integer coprime to p, then there is a map $\psi : S^{m-1} \to S^{m-1}$ of degree coprime to $|\beta|$, such that for every prime p,

$$(\beta \circ \psi)(p) = \overline{\psi}_p \beta(p)$$

Proof. By the Chinese Remainder Theorem we can find $\psi \in \mathbb{Z}$ such that $\psi \equiv \overline{\psi}_p \mod |\beta(p)|$ for every p dividing $|\beta|$. Let us use ψ to denote also the map $S^{m-1} \to S^{m-1}$ of degree ψ. Then $(\beta \circ \psi)(p) = \psi\beta(p) = \overline{\psi}_p \beta(p)$ for every p.

In order to prove that $(\psi, |\beta|) = 1$, assume that q is a prime number dividing ψ and $|\beta|$. Since $\psi \equiv \overline{\psi}_q \mod |\beta(q)|$, q divides $\overline{\psi}_q$ which contradicts the hypothesis. #

Now, let α be a suspension element in $\pi_{m-1}(\bigvee^k S^n)$. As we observed in the introduction,

$$\alpha = (\alpha^1, \ldots, \alpha^k) \in \bigoplus_{i=1}^{k} \pi_{m-1}(S^n) \subset \pi_{m-1}(\bigvee^k S^n),$$

each α^j being a suspension. α determines a one–column matrix, with elements in $\pi_{m-1}(S^n)$, that we will also denote by α. Then the composite of α and a map $\varphi : \bigvee^k S^n \to \bigvee^k S^n$ is obtained by matrix multiplication; see [7] Lemma 3.

Lemma 4. Let $\alpha \in \pi_{m-1}(\bigvee^k S^n)$ be a suspension element of finite order. If, for each prime p, $\overline{\varphi}_p$ is an integer matrix with determinant coprime to p, then there is a map $\varphi : \bigvee^k S^n \to \bigvee^k S^n$ such that the determinant of its matrix is coprime to $|\alpha|$, and for every prime p,

$$(\varphi \circ \alpha)(p) = \overline{\varphi}_p \alpha(p) .$$

Here $\overline{\varphi}_p \alpha(p)$ denotes matrix multiplication.

Proof. Let $\overline{\varphi}_p = (c_{pij})$. By the Chinese Remainder Theorem we can find integers c_{ij} such that

$$\begin{array}{lll} c_{ij} \equiv c_{pij} & \bmod |\alpha^i(p)| & \text{if } p \text{ divides } |\alpha^i| , \\ c_{ij} \equiv c_{pij} & \bmod p & \text{if } p \text{ divides } |\alpha| \text{ but does not divide } |\alpha^i| . \end{array}$$

Denote by $\varphi : \bigvee^k S^n \to \bigvee^k S^n$ the map associated with the matrix (c_{ij}). Then for every prime p we have

$$\begin{aligned} (\varphi \circ \alpha)(p) = \varphi\alpha(p) &= \left(\sum_i c_{ij}\alpha^i(p) \right) = \\ &= \left(\sum_i c_{pij}\alpha^i(p) \right) = \overline{\varphi}_p \alpha(p) . \end{aligned}$$

In order to prove that $(\det \varphi, |\alpha|) = 1$, suppose that a prime q divides $\det \varphi$ and $|\alpha|$. Clearly, $\det \varphi \equiv \det \overline{\varphi}_q \ (mod\, q)$, so q must divide $\det \overline{\varphi}_q$, which contradicts the hypotheses. #

Lemma 5. Let γ be an element of finite order in $\pi_{m-1}(\bigvee^k S^n)$. If $\varphi \circ \gamma = 0$ and $\det \varphi \neq 0$, then

(i) there is a map φ' with scalar matrix $q'I$ such that $\varphi' \circ \gamma = 0$,
(ii) q' divides $\det \varphi$,
(iii) q' and $|\gamma|$ have the same prime divisors.

Proof. Take $\overline{\varphi}$ such that $\overline{\varphi}\varphi = \det \varphi\, I$. Then $\det \varphi\, I \circ \gamma = \overline{\varphi} \circ \varphi \circ \gamma = 0$.

Let $\det \varphi = q' \cdot q''$, where $(q'', |\gamma|) = 1$ and if p divides q', then p divides $|\gamma|$. For each prime p dividing $|\gamma|$, the map $q''I : \pi_n(\bigvee^k S^n_{(p)}) \to \pi_n(\bigvee^k S^n_{(p)})$

is an isomorphism, since $(p, q'') = 1$. But $\gamma(p) \neq 0$, so $q''I \circ \gamma(p) \neq 0$ and, therefore,

$$q''I \circ \gamma \neq 0\,.$$

Observe that, in general, for $\alpha \in \pi_{m-1}(\bigvee^k S^n)$ of finite order, if $q''I \circ \alpha = 0$ and $(q'', |\alpha|) = 1$, then $\alpha = 0$. In particular, for $\alpha = q'I \circ \gamma$, $q''I \circ (q'I \circ \gamma) = \det \varphi I \circ \gamma = 0$. Moreover, $|\gamma|(q'I \circ \gamma) = q'I \circ |\gamma|\gamma = 0$ implies that $|q'I \circ \gamma|$ divides $|\gamma|$ so that $|q'I \circ \gamma|$ is coprime to q''. Thus $q'I \circ \gamma = 0$.

To prove (iii) observe that, by the definition of q', if a prime p divides q', p also divides $|\gamma|$. Now, if p divides $|\gamma|$ but p does not divide q', then $\gamma(p) \neq 0$ and $q'I : \pi_n(\bigvee^k S^n_{(p)}) \to \pi_n(\bigvee^k S^n_{(p)})$ is an isomorphism. Therefore $q'I \circ \gamma(p) \neq 0$, and $q'I \circ \gamma \neq 0$ which is not true.#

Proof of Theorem 2. Suppose $C_\alpha \sim C_\beta$. Then, for every prime p there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{m-1}_{(p)} & \xrightarrow{\alpha(p)} & \bigvee^k S^n_{(p)} \\ {\scriptstyle \overline{\psi}_p}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}_p} \\ S^{m-1}_{(p)} & \xrightarrow{\beta(p)} & \bigvee^k S^n_{(p)} \end{array}$$

as in Proposition 1. Take maps ψ and φ as in Lemmas 3 and 4. Then $(\beta \circ \psi)(p) = (\varphi \circ \alpha)(p)$ for every prime p, so that $\beta \circ \psi = \varphi \circ \alpha$, $(\psi, |\beta|) = 1$ and $(\det \varphi, |\alpha|) = 1$. But, clearly, $|\beta(p)| = |\alpha(p)|$ for every prime p, so that $|\alpha| = |\beta|$.

Now take integers r, s such that $r\psi + s|\beta| = 1$. Then $\beta = (r\varphi) \circ \alpha$ and $det(r\varphi) = r^k det\,\varphi$ is coprime to $|\alpha| = |\beta|$.

Conversely assume there is a map φ such that $\beta = \varphi \circ \alpha$, with $(\det \varphi, |\alpha|) = 1$. We first prove that $|\alpha| = |\beta|$. On one hand, $|\alpha|\beta = |\alpha|(\varphi \circ \alpha) = \varphi \circ |\alpha|\,\alpha = 0$, so $|\beta|$ divides $|\alpha|$. On the other hand, $0 = |\beta|\,\beta = |\beta|(\varphi \circ \alpha) = \varphi \circ |\beta|\,\alpha$ and, by Lemma 5, there is an integer q' such that $q'I \circ |\beta|\,\alpha = 0$ and p divides q' if and only if p divides $|\,|\beta|\,\alpha|$ which, in turn, divides $|\alpha|$. Hence, if $|\beta|\,\alpha \neq 0$, $(q', |\alpha|) \neq 1$ and $(\det \varphi, |\alpha|) \neq 1$ against the hypothesis. So $|\beta|\,\alpha = 0$ and $|\alpha|$ divides $|\beta|$. Therefore, $|\alpha| = |\beta|$.

Now the conclusion follows easily from Proposition 1. #

§3. The main theorem

Theorem 6. Let α and β be elements of finite order in $\pi_{m-1}(\bigvee^k S^n)$, $m - 1 > n > 1$. If α is a suspension element then the following conditions are equivalent:

(i) $C_\alpha \sim C_\beta$.

(ii) $C_\alpha \vee \bigvee^k S^n \simeq C_\beta \vee \bigvee^k S^n$.
(iii) There is a wedge of spheres T such that $C_\alpha \vee T \simeq C_\beta \vee T$.

Proof. (i) $\Rightarrow$ (ii). Assume that $C_\alpha \sim C_\beta$. Theorem 2 tells us that there is a map φ such that $\beta = \varphi \circ \alpha$ and $(\det \varphi, |\alpha|) = 1$. Choose r, s such that $r \det \varphi - s|\alpha| = 1$, and a matrix $\overline{\varphi}$ such that $\overline{\varphi}\varphi = \det \varphi I$. The diagram

$$\begin{array}{ccccc} S^{m-1} & \xrightarrow{(\alpha,0)} & \bigvee^k S^n \vee \bigvee^k S^n & \longrightarrow & C_\alpha \vee \bigvee^k S^n \\ {\scriptstyle 1}\downarrow & & {\scriptstyle \Phi}\downarrow & & \\ S^{m-1} & \xrightarrow{(\beta,0)} & \bigvee^k S^n \vee \bigvee^k S^n & \longrightarrow & C_\beta \vee \bigvee^k S^n \end{array}$$

where $\Phi = \begin{pmatrix} \varphi & sI \\ |\alpha| I & r\overline{\varphi} \end{pmatrix}$, is homotopy commutative. Moreover

$$\Phi = \begin{pmatrix} \varphi & 0 \\ |\alpha| I & I \end{pmatrix} \begin{pmatrix} I & s(\det \varphi)^{-1}\overline{\varphi} \\ 0 & (\det \varphi)^{-1}\overline{\varphi} \end{pmatrix}$$

so that Φ is unimodular and we obtain a homotopy equivalence $C_\alpha \vee \bigvee^k S^n \simeq C_\beta \vee \bigvee^k S^n$. This proves (ii).

Obviously (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). Suppose that $C_\alpha \vee T \simeq C_\beta \vee T$. We may assume that T has no spheres of dimension less than n, for if $T = T' \vee \bigvee S^{k_i}$ with $k_i < n$ and T' is a wedge of spheres of dimension $\geq n$, then we may suppose that the given homotopy equivalence takes $\bigvee S^{k_i}$ into $\bigvee S^{k_i}$ and, hence, that it induces a homotopy equivalence $C_\alpha \vee T' \to C_\beta \vee T'$. We may also assume that T has no spheres of dimension greater than m, by restricting to the m-skeletons. Finally, we may assume that the number of n-spheres in T is k times the number of m spheres, by adding as many spheres as necessary.

Now, consider the diagram

$$\begin{array}{ccccc} S^{m-1} \vee \ldots \vee S^{m-1} & \xrightarrow{\alpha \vee 0} & \bigvee^k S^n \vee \ldots \vee \bigvee^k S^n \vee T'' & \longrightarrow & C_\alpha \vee T \\ {\scriptstyle \psi}\downarrow & & {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \simeq} \\ S^{m-1} \vee \ldots \vee S^{m-1} & \xrightarrow{\beta \vee 0} & \bigvee^k S^n \vee \ldots \vee \bigvee^k S^n \vee T'' & \longrightarrow & C_\beta \vee T, \end{array}$$

where T'' is the wedge of spheres in T of dimension strictly between n and m. We may assume Φ cellular, so that ψ and Φ are represented by unimodular matrices of the form

$$\psi = \begin{pmatrix} b_{11} & \ldots & b_{1r} \\ \vdots & & \\ b_{r1} & \ldots & b_{rr} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \overline{\varphi} & A \\ 0 & B \end{pmatrix},$$

and $\overline{\varphi} = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1r} \\ \vdots & & \\ \varphi_{r1} & \cdots & \varphi_{rr} \end{pmatrix}$ is also unimodular; here each φ_{ij} is a $k \times k$–matrix. From the commutativity of the left square in the diagram we get

$$\beta \circ b_{11} = \varphi_{11} \circ \alpha \,, \quad \beta \circ b_{1j} = 0 \,, \quad \varphi_{j1} \circ \alpha = 0 \,, \quad \text{for} \quad j = 2, \dots, r \,.$$

In particular, $0 = \beta \circ b_{1j} = b_{1j}\beta$ implies that $|\beta|$ divides b_{1j}, $2 \le j \le r$. Hence $(b_{11}, |\beta|) = 1$.

On the other hand, for each prime p, we can find matrices A_j over $\mathbb{Z}(p)$ such that the determinant of $\varphi_{11} + \sum_j A_j \varphi_{j1}$ is a unit in $\mathbb{Z}(p)$; see [1], Lemma 6.4. Now, we can always choose an integer N coprime to p and such that the matrices NA_j have integer entries. Write

$$\Theta = N\varphi_{11} + \Sigma_j N A_j \varphi_{j1} \quad \text{and} \quad b'_{11} = N b_{11} \,.$$

Then we have

$$\Theta \circ \alpha = N\varphi_{11} \circ \alpha = \beta \circ b'_{11}$$

Thus $\Theta \circ |\beta|\alpha = |\beta|(\Theta \circ \alpha) = |\beta| b'_{11}\beta = 0$ and, by Lemma 5, if a prime divides the order of $|\beta|\alpha$ it also divides $\det \Theta$. So we have

$$(|\,|\beta|\alpha\,|, p) = 1 \text{ for every prime } p \Rightarrow |\beta|\alpha = 0 \Rightarrow |\alpha| \text{divides} |\beta| \,.$$

But

$$|\alpha| b_{11}\beta = |\alpha|(\beta \circ b_{11}) = |\alpha|(\varphi_{11} \circ \alpha) = \varphi_{11} \circ |\alpha|\alpha = 0$$
$$\Rightarrow |\beta| = |b_{11}\beta| \text{ divides } |\alpha| \,.$$

Therefore $|\alpha| = |\beta|$. Now, if $p \nmid |\alpha|$, $\alpha(p) = 0 = \beta(p)$ and $C_{\alpha(p)} \simeq C_{\beta(p)}$; if $p \,|\, |\alpha|$, $C_{\alpha(p)} \simeq C_{\beta(p)}$ follows from $\Theta \circ \alpha = \beta \circ b_{11}$, since both $\det \Theta$ and b_{11} are coprime to p. This proves $C_\alpha \sim C_\beta$. #

Theorem 6 shows, in particular, that $C_\alpha \vee S^m \simeq C_\beta \vee S^m$ implies $C_\alpha \sim C_\beta$. The converse was proved for $k = 1$ by Hilton and Roitberg in [6], but it does not hold when $k \ge 2$, as the following shows.

Example. Consider $\alpha \in (\alpha^1, \alpha^2) \in \pi_{m-1}(S^n) + \pi_{m-1}(S^n) \subset \pi_{m-1}(S^n \vee S^n)$, where α^1 and α^2 are elements of order 5 such that the subgroups they generate are disjoint. Take $\beta = (-\alpha^2, 2\alpha^1)$. By Theorem 2, $C_\alpha \sim C_\beta$. Now, suppose $C_\alpha \vee S^m \simeq C_\beta \vee S^m$. Then there is a homotopy commutative diagram

$$\begin{array}{ccccc}
S^{m-1} \vee S^{m-1} & \xrightarrow{<\alpha,0>} & \bigvee^2 S^n & \longrightarrow & C_\alpha \vee S^m \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \big\downarrow \simeq & & \Phi \big\downarrow \simeq & & \big\downarrow \simeq \\
S^{m-1} \vee S^{m-1} & \xrightarrow{<\beta,0>} & \bigvee^2 S^n & \longrightarrow & C_\beta \vee S^m \,.
\end{array}$$

Thus $\Phi \circ \alpha = a\beta = a\left(\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right) \circ \alpha\right)$ so all the entries of the matrix $\Phi - a\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right)$ are divisible by 5 and $\det \Phi \equiv 2a^2$, modulo 5. Hence $\det \Phi \neq \pm 1$, so Φ is not a homotopy equivalence. Thus $C_\alpha \vee S^m \not\simeq C_\beta \vee S^m$.

Our final theorem simply says that wedge cancellation by means of k n–spheres (or one m–sphere) holds exactly in the same cases as wedge cancellation of more than k n–spheres (or several m–spheres).

Theorem 7. Suppose $\alpha, \beta \in \pi_{m-1}(\bigvee^k S^n)$ of finite order.

(i) $C_\alpha \vee S^m \vee \ldots \vee S^m \simeq C_\beta \vee S^m \vee \ldots \vee S^m$ if and only if $C_\alpha \vee S^m \simeq C_\beta \vee S^m$.

(ii) If α is a suspension element and $l \geq k$ then $C_\alpha \vee \bigvee^l S^n \simeq C_\alpha \vee \bigvee^l S^n$ if and only if $C_\alpha \vee \bigvee^k S^n \simeq C_\beta \vee \bigvee^k S^n$.

Proof. (i) Consider the homotopy commutative diagram

$$\begin{array}{ccccc}
S^{m-1} \vee \ldots \vee S^{m-1} & \xrightarrow{<\alpha,0,\ldots,0>} & \bigvee^k S^n & \longrightarrow & C_\alpha \vee S^m \vee \ldots \vee S^m \\
\psi \big\downarrow & & \varphi \big\downarrow & & \big\downarrow \simeq \\
S^{m-1} \vee \ldots \vee S^{m-1} & \xrightarrow{<\beta,0,\ldots,0>} & \bigvee^k S^n & \longrightarrow & C_\beta \vee S^m \vee \ldots \vee S^m
\end{array}$$

where $\psi = (b_{ij})$ and φ are unimodular. The commutativity of the left square implies that $\beta \circ b_{11} = \varphi \circ \alpha$ and $\beta \circ b_{i1} = 0$ for $2 \leq i \leq r$. In particular, $|\beta| \mid b_{i1}, 2 \leq i \leq r$. Hence $(b_{11}, |\beta|) = 1$.

Take integers x and y, such that $b_{11}y - |\beta| x = 1$, and consider

$$\begin{array}{ccccc}
S^{m-1} \vee S^{m-1} & \xrightarrow{<\alpha,0>} & \bigvee^k S^n & \longrightarrow & C_\alpha \vee S^m \\
\left(\begin{smallmatrix} b_{11} & |\beta| \\ x & y \end{smallmatrix}\right) \big\downarrow & & \big\downarrow \varphi & & \\
S^{m-1} \vee S^{m-1} & \xrightarrow{<\beta,0>} & \bigvee^k S^n & \longrightarrow & C_\beta \vee S^m
\end{array}$$

Using matrix multiplication one easily checks that the left square is homotopy commutative, so the vertical maps induce a homotopy equivalence between the mapping cones: $C_\alpha \vee S^m \simeq C_\beta \vee S^m$.

(ii) is an obvious consequence of Theorem 6. #

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