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## NOTE ON A SIX-VALUED EXTENSION OF THREE-VALUED LOGIC

by

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# Note on a six-valued extension of three-valued logic

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**Abstract**

In this paper we introduce a set of six logical values, which arises in the application of three-valued logics to time intervals, find its algebraic structure, and use it to define a six-valued logic. We then prove, by using algebraic properties of the class of De Morgan algebras, that this semantically defined logic can be axiomatized as Belnap's "useful" four-valued logic. Other directions of research suggested by the construction of this set of six logical values are described.

## 1. Introduction

Intelligent systems often have to make inferences within a certain time interval and with limited resources available to them. For example, consider a system that makes logical deductions based on the knowledge that it has already acquired from the external world. As additional knowledge becomes available, the system refines its knowledge base, but at any point in time it should be able to respond to a query based on its limited resources.

As an example, imagine that robbot *Robbie* has gone to the zoo and is trying to identify certain animals that it observes [14]. Let's suppose that *Robbie* is trying to verify the hypothesis that *Bozo* (one of the animals) is a carnivore. One of the rules it considers is:

If X eats meat, then X is a carnivore.

Now let's assume that *Robbie* knows that the feeding time for *Bozo* is between 10 and 11am. Therefore, it knows that it will be possible to determine whether or not *Bozo* eats meat after 10am but not before (barring any other events that may lead to finding out about *Bozo*'s eating habits). According to this knowledge, *Robbie* can

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plan its reasoning process. Either it will make a decision based on what it knows now, or given more resources, it may wait until more information becomes available.

Considering the above example, one might have the following exchanges with *Robbie*:

Q1: Does *Bozo* eat meat ?

A1: I don't know. I will know it soon (provided that the time now is some time before 10am).

A2: I don't know. I cannot know it today since they have already fed *Bozo* and I just got here (provided that the time now is some time after 11am).

Clearly, there is a difference between these two answers, in the sense that the first reflects a temporary lack of knowledge over a time interval, while the second indicates a definite lack of knowledge over the time interval. In fact, the six-valued logic introduced in [9] has been developed to capture and represent such differences, which would not be reflected if we used Kleene's three-valued logic [11] or Belnap's four-valued logic [2,3], which have a single "undefined" or "unknown" value.

The purpose of this note is to find an axiomatization of the logic introduced in [9]. To this end we first present it and then, using an algebraic argument, prove that it can be axiomatized as Belnap's well-known logic. In a final section we have included some less formalized comments on some possible directions for research suggested by the original idea of [9].

## 2. The six logical values

Consider the three-valued logic of Kleene [11], where, besides the two classical values **T** (true) and **F** (false), a third value **U** is assigned to statements whose truth values are undecidable or undefined. We are going to use these values to represent our epistemic state about the truth value of a statement at a definite time point, and we will extend them to obtain a new set of values which take into account the changes in our knowledge over a time interval.

A statement $p$ is assigned the value **T** (respectively **F**) *at a time point* when it is *known* to be true (respectively false) at that time point, while it is assigned the value **U** when it is *not known* to be either true or false at that time point. We assume that such assignment is always possible at each time point of the interval under consideration. Note, however, that here the value **U** does not merely reflect a lack of information, like in other logics, but rather a non-concluding result of the evaluation by the observer of all the available information, either from direct inspection of the facts or by other knowledge.

Let $v_t(p)$ represent the truth value of $p$ at time point $t$. We consider variations of this truth value due to changes in our information, not due to the change of the 'real' truth value itself. Therefore the assignment rules of our logic should satisfy the following conditions:

$$\text{If } v_{t1}(p) = \mathbf{T} \text{ and } t1 \leq t2, \text{ then } v_{t2}(p) = \mathbf{T}.$$

$$\text{If } v_{t1}(p) = \mathbf{F} \text{ and } t1 \leq t2, \text{ then } v_{t2}(p) = \mathbf{F}.$$

These two requirements state that if statement $p$ is decided (i.e., it is known to be true or false) at some time point, then its truth value must remain constant at any later time point. These two conditions together are called the **monotonicity** conditions. On the other hand, a statement whose truth value is not decided at a time point is assigned the value **U** at that and earlier time points, while it may become **T** or **F** at later time points (and thus to remain so until the end of the interval). However, the observer may have some additional knowledge leading him to exclude one, or both, possibilities. Thus he may arrive at a joint evaluation of the truth values of the statement over a time interval $T$. Such an evaluation can be expressed as one of the following six possibilities, which represent the six possible **patterns of change** of the truth value of a sentence over a time interval:

**Definition 1.** *Let us denote by* $M_3 = \{\mathbf{T}, \mathbf{U}, \mathbf{F}\}$ *the set of truth values a statement can have at a time point. Then the values of a statement* $p$ *over a time interval* $T$ *are the following:*

(1) *We say that* $p$ *is* **t** *in* $T$ *when* $p$ *is* **T** *over all the time points in* $T$.

(2) *We say that* $p$ *is* **f** *in* $T$ *when* $p$ *is* **F** *over all the time points in* $T$.

(3) *We say that* $p$ *is* **u** *in* $T$ *when* $p$ *is* **U** *over all the time points in* $T$.

(4) *We say that* $p$ *is* **k** *in* $T$ *when* $p$ *is* **U** *at some time points in* $T$ *and may be either* **T** *or* **F** *at other time points in* $T$.

(5) *We say that* $p$ *is* **k1** *in* $T$ *when* $p$ *is* **U** *at some time points in* $T$ *and may be* **T** *at other time points in* $T$, *but cannot be* **F** *at any time point in* $T$.

(6) *We say that* $p$ *is* **k0** *in* $T$ *when* $p$ *is* **U** *at some time points in* $T$ *and may be* **F** *at other time points in* $T$, *but cannot be* **T** *at any time point in* $T$.

*The set of these six values will be denoted by* $M_6$, *i.e.,* $M_6 = \{\mathbf{t}, \mathbf{f}, \mathbf{u}, \mathbf{k}, \mathbf{k0}, \mathbf{k1}\}$.

It is easy to see that these six cases cover all possible combinations of truth values that a statement can have over a time interval $T$, the missing case ($p$ is **T** at some points and **F** at the other ones) obviously violating the monotonicity conditions required above. We refer to the values **t** and **f** as *decided*, to the value **u** as *undecidable*, and to **k**, **k0** and **k1** as *undecided*, and among these the value **k** is sometimes called *completely undecided*. Note that we are using 'undecided' in the sense of 'not yet decided but not known to be undecidable', and thus partially opposed both to 'decided' and to 'undecidable'. To avoid confusion, in this paper we will refer to the elements of $M_3$ as the 'truth values' or 'initial values' (although calling them 'knowledge values' would perhaps be a sounder choice), and will refer to the elements of $M_6$ as just 'values', 'new values', or 'logical values', since it is upon these that we will build our logic (therefore we will use them formally as truth values, the initial ones appearing only as the motivation for the introduction and behaviour of the new ones).

As an example, let the time interval $T$ be the month of October. Now consider the following statements:

$p1$: It will rain in October.

$p2$: It will rain in December.

$p3$: It will rain on October 19.

$p4$: It will not rain in October.

It is worth noting that at no time point in interval $T$ (the month of October) can $p1$ have the truth value **F**, and that it will remain **U** unless it becomes **T** at some point. Thus, an observer having no information at all will assign to $p1$ over $T$ the value **k1**; and by similar arguments, the values he will assign to $p2$, $p3$, and $p4$ will be **u**, **k**, and **k0** respectively. However, if the observer has some information, for instance if today is October 20, say, and he knows that yesterday it did rain, then the value he will assign to statement $p3$ over $T$ will be **k1**, while the other three statements will receives the same values as in the first situation.

A warning is here in order: The use of the above examples might suggest that our logic is specially designed to deal with temporal statements. This is by no means the case. Actually, from our basic assumptions it follows that our logic cannot be applied to sentences containing temporally indefinite indices [13] like 'yesterday', 'next Sunday' and the like; such sentences change their meaning over time, and thus may violate the monotonicity conditions. As we pointed out before, for the same reason our logic cannot deal with statements whose truth value changes over time due to changes in their factual reference, like 'I am hungry' or 'My salary is $2,000 per month'.

## 3. The logical connectives

Our logic will be truth-functional with respect to the six new logical values, that is, the value of a combined statement will be the result of combining the values of its parts in the same way. Thus, the logic will be determined by the algebraic structure of $M_6$. At the same time, we want our logic to be a refinement of Kleene's three-valued logic, and since we can identifiy **t**,**u** and **f** with **T**, **U** and **F**, respectively, we are going to extend to $M_6$ the definitions of $\neg, \wedge, \vee$ on Kleene's strong three-valued logic, which follow the tables shown below.

| ∧ | F | U | T |
|---|---|---|---|
| F | F | F | F |
| U | F | U | U |
| T | F | U | T |

| ∨ | F | U | T |
|---|---|---|---|
| F | F | U | T |
| U | U | U | T |
| T | T | T | T |

| ¬ | |
|---|---|
| F | T |
| U | U |
| T | F |

Figure 1: Kleene's three-valued truth-tables.

The extension is easily made if we think of these new values as the *sets of possible truth values* that can be taken by a sentence in the interval $T$; then the following notation is implicit in Definition 1: **t** = $\{\mathbf{T}\}$, **f** = $\{\mathbf{F}\}$, **u** = $\{\mathbf{U}\}$, **k** = $\{\mathbf{T},\mathbf{F},\mathbf{U}\}$, **k1** = $\{\mathbf{T},\mathbf{U}\}$, and **k0** = $\{\mathbf{F},\mathbf{U}\}$. Now, it is clear that a combined sentence can have as values all combinations of the possible values of its components. Thus, it is enough to apply the preceding tables to the members of each set to find a new set, which will be represented by one of the six new values. Hence, to find the negation :

$$\begin{array}{rclclclcl}
\neg\mathbf{t} & = & \neg\{\mathbf{T}\} & = & \{\neg\mathbf{T}\} & = & \{\mathbf{F}\} & = & \mathbf{f} \\
\neg\mathbf{f} & = & \neg\{\mathbf{F}\} & = & \{\neg\mathbf{F}\} & = & \{\mathbf{T}\} & = & \mathbf{t} \\
\neg\mathbf{u} & = & \neg\{\mathbf{U}\} & = & \{\neg\mathbf{U}\} & = & \{\mathbf{U}\} & = & \mathbf{u} \\
\neg\mathbf{k} & = & \neg\{\mathbf{T},\mathbf{F},\mathbf{U}\} & = & \{\neg\mathbf{T},\neg\mathbf{F},\neg\mathbf{U}\} & = & \{\mathbf{F},\mathbf{T},\mathbf{U}\} & = & \mathbf{k} \\
\neg\mathbf{k1} & = & \neg\{\mathbf{T},\mathbf{U}\} & = & \{\neg\mathbf{T},\neg\mathbf{U}\} & = & \{\mathbf{F},\mathbf{U}\} & = & \mathbf{k0} \\
\neg\mathbf{k0} & = & \neg\{\mathbf{F},\mathbf{U}\} & = & \{\neg\mathbf{F},\neg\mathbf{U}\} & = & \{\mathbf{T},\mathbf{U}\} & = & \mathbf{k1}
\end{array}$$

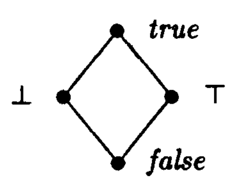
and similarly for $\wedge$ and $\vee$; we only highlight the following :

$$\begin{array}{l}
\mathbf{k}\wedge\mathbf{u} = \{\mathbf{T},\mathbf{F},\mathbf{U}\}\wedge\{\mathbf{U}\} = \{\mathbf{T}\wedge\mathbf{U},\,\mathbf{F}\wedge\mathbf{U},\,\mathbf{U}\wedge\mathbf{U},\} = \{\mathbf{U},\mathbf{F},\mathbf{U}\} = \mathbf{k0} \\
\mathbf{k}\vee\mathbf{u} = \{\mathbf{T},\mathbf{F},\mathbf{U}\}\vee\{\mathbf{U}\} = \{\mathbf{T}\vee\mathbf{U},\,\mathbf{F}\vee\mathbf{U},\,\mathbf{U}\vee\mathbf{U},\} = \{\mathbf{T},\mathbf{U},\mathbf{U}\} = \mathbf{k1}
\end{array}$$

From the above it is clear that if we had simply added the "undecided" value **k** to $M_3$ in order to capture the differences mentioned in Section 1, then the two values **k0** and **k1** would have been generated to obtain a set of values closed under these connectives.

With the operations $\wedge$ and $\vee$ the set $M_6$ has the structure of the finite (hence complete) distributive lattice shown in Figure 2. We refer to the partial order defined by this lattice as the *truth order* and denote it by $\leq_t$, or, if no confusion is likely to arise, simply by $\leq$. This ordering relation expresses in a natural way the relationships between the *degrees of truth* implicit in the six values: It is natural to say that the degree of truth of a sentence which is evaluated to be completely undecided (**k**) over $T$ is greater than the degree of truth of one which we know cannot be true at any point in $T$ (**k0**) and is lower than one which we know cannot be false at any point in $T$ (**k1**), while cannot be compared to the one of a sentence which is known to be undecidable (**u**).

If we consider the operation $\neg$, we can easily check that it is an idempotent dual automorphism of the lattice structure of $M_6$, that is, it satisfies $x = \neg\neg x$ and the so-called De Morgan laws $\neg(x \wedge y) = \neg x \vee \neg y$, and $\neg(x \vee y) = \neg x \wedge \neg y$. Moreover, if we take into account the existence of a maximum element for $\leq_t$, **t**, then we conclude that $M_6$ has the algebraic structure of a *De Morgan algebra*, see [1, chapter XI]. The proof of the main result of Section 4 will be based on the following properties of De Morgan algebras (other useful universal algebraic concepts and results can be



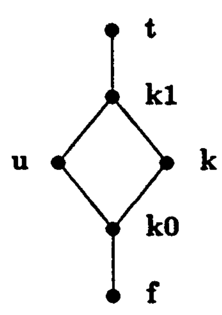




Figure 2: The lattices $M_4$ (left) and $M_6$ (right) under the truth order $\leq_t$.

found, for instance, in chapters I,II and XI of [1]). Recall that a *variety*, or *equational class*, is a class of algebras of the same similarity type definable by a set of equations; the variety generated by an algebra is defined to be the least variety containing the generator. Then we have:

**Theorem 2.** *The variety generated by the algebra $M_6$ is the class of all De Morgan algebras.*

**Proof:** As we have just said, $M_6$ is itself a De Morgan algebra, therefore the variety generated by $M_6$ will be a subvariety of the class of all De Morgan algebras. According to Theorem XI.3.1 of [1], the only proper such subvarieties are those of Boolean algebras and of Kleene algebras. But $M_6$ is certainly not a Boolean algebra, neither is it a Kleene algebra, since it does not satisfy the inequality $x \wedge \neg x \leq y \vee \neg y$ (put for instance $x = \mathbf{k}$ and $y = \mathbf{u}$). Thus $M_6$ does not belong to any of the proper subvarieties, therefore the subvariety it generates must be the whole class of De Morgan algebras. □

The class of De Morgan algebras is normally considered to be generated by the four-element algebra $M_4$ shown in Figure 2 (this algebra is called $M_2$ in [1, p. 214ff], **4** or **L4** in [3], and $\mathcal{FOUR}$ in [5]). Thus, both $M_4$ and $M_6$ generate the same equational class. As a consequence we have:

**Corollary 3.** *Any equation true in $M_4$ is true in $M_6$ and conversely.*

**Proof:** From its definition it follows that the variety generated by an algebra is exactly the class of all algebras which satisfy all the equations true in the generator. The result then follows directly from Theorem 2 and the above observations. □

## 4. The six-valued logic

In this Section we are going to use $M_6$ to define a logic, that is, a relation of consequence between formulas of a specified formal language, and to prove that it can be axiomatized in the same form as Belnap's four-valued logic, that is, as relations of consequence these two logics will be the same.

Let us consider *Form* the set of sentential formulas built over some set of sentential variables *Var* together with the logical connectives $\wedge$ (binary: conjunction, 'and'), $\vee$ (binary: disjunction, 'or') and $\neg$ (unary: negation, 'not'), and with two constant symbols $\top$ (the truth constant, 'true') and $\perp$ (the falsum constant, 'false'). Thus *Form* is the absolutely free algebra of similarity type (2,2,1,0,0) over the set of generators *Var*. Its operations are denoted by the same symbols used in $M_6$ and in any other algebra of the same similarity type. If we want to use $M_6$ as the set of logical values that sentences can have, then we must begin with the following:

**Definition 4.** *We call* **valuation** *any assignment of values of the set $M_6$ to the sentential variables of Var, that is, any mapping $v : Var \longrightarrow M_6$. These mappings are extended to the whole set Form with the usual formulas*

$$v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$$

$$
\begin{aligned}
v(\varphi \vee \psi) &= v(\varphi) \vee v(\psi) \\
v(\neg\varphi) &= \neg v(\varphi) \\
v(\top) &= \mathbf{t} \\
v(\bot) &= \mathbf{f}.
\end{aligned}
$$

*Equivalently, due to the freeness of the algebra Form, we can just say that valuations are the homomorphisms from the algebra Form to the algebra $M_6$.*

The **basic idea** in the definition of our logic will be that a sentence $\psi$ follows from another one $\varphi$, as far as this semantics is concerned, whenever it gives a better approximation to truth in every situation, that is, whenever it has a greater degree of truth in every assignment. Here is where the truth-order of $M_6$ enters the picture: we want a relation $\models_6$ of semantical entailment such that, in symbols, $\varphi \models_6 \psi$ whenever $v(\varphi) \leq_t v(\psi)$ for all valuations $v$. Note that we are not using the algebra $M_6$ as a logical matrix in the customary way (see [15] for instance) taking $\{\mathbf{t}\}$ as the 'designated' subset, that is, we do not say that $\psi$ follows from $\varphi$ just when $v(\varphi) = \mathbf{t}$ implies $v(\psi) = \mathbf{t}$ for all valuations $v$. This would be too strong, and the resulting logic would recognize as valid many inferences which are not so in terms of degrees of truth; instead we use the finer structure of order in $M_6$. This idea can be extended to arbitrary sets of premises by means of two conventions:

1) That the joint value of a finite set of sentences is just the value of their conjunction (thus giving the conjunction symbol $\wedge$ the content of its intuitive interpretation 'and').
2) That the relation of logical consequence corresponding to $\models_6$ must be *finitary*, that is, the consequences of an infinite set are just the consequences of its finite subsets.

If we adopt these two conventions we arrive naturally at the following:

**Definition 5.** *Let $\Sigma \subseteq$ Form and $\psi \in$ Form. We then say that $\Sigma \models_6 \psi$ if and only if there are $\varphi_1, \ldots, \varphi_n \in \Sigma$ (for some $n \geq 1$) such that for any valuation $v$, $v(\varphi_1) \wedge \ldots \wedge v(\varphi_n) \leq v(\psi)$. Moreover, for any $\varphi, \psi \in$ Form, we put $\varphi \mathrel{\models\!\!\!\dashv}_6 \psi$ if and only if $\varphi \models_6 \psi$ and $\psi \models_6 \varphi$, that is, if and only if for any valuation $v$, $v(\varphi) = v(\psi)$.*

The symbol '$\models$' indicates that we are dealing with an entailment relation defined by semantical methods, and the subscript '6' indicates that the set of values is in fact $M_6$. Now we are going to see that this logic can represent, in some sense, the equations which are true in this algebra. Note that, since the similarity type of $M_6$ is the same as the one of *Form*, the equations interpretable in this algebra can be written in the form $\varphi \approx \psi$, where $\varphi$ and $\psi$ are sentences of *Form*; we just have to *think* that the sentential variables now represent arbitrary elements of $M_6$ instead of arbitrary sentences of the language (of course, this also holds for any algebra of the same similarity type). We then have at once:

**Proposition 6.** *An equation $\varphi \approx \psi$ is true in $M_6$ if and only if $\varphi \mathrel{\models\!\!\!\dashv}_6 \psi$.*

**Proof:** An equation is true in an algebra if and only if its two sides receive the same interpretation when the variables are substituted by elements of that algebra, in any form. Formally expressed, this is to say that for any valuation $v$ from *Form* into $M_6$, we have that $v(\varphi) = v(\psi)$, and by definition 5 this is equivalent to $\varphi \mathrel{\vDash\!\!\mid}_6 \psi$. □

Now we are ready to compare the logic $\models_6$ with Belnap's. Let us denote the latter by $\models_4$; it is defined (see [3,5]) in exactly the same way as $\models_6$ with $M_4$ in the place of $M_6$. Similarly, we also have $\mathrel{\vDash\!\!\mid}_4$, and it holds that the equation $\varphi \approx \psi$ is true in $M_4$ if and only if $\varphi \mathrel{\vDash\!\!\mid}_4 \psi$. (More details on the connections between $\models_4$ and the class of De Morgan algebras can be found in [6,7,8]) Then:

**Theorem 7.** $\models_6 \;=\; \models_4$.

**Proof:** Because of the two conventions we adopted in defining both logics, to prove that they are equal it is enough to prove that they are equal on single formulas, that is, to prove that for any $\varphi, \psi \in$ *Form*, $\varphi \models_6 \psi$ if and only if $\varphi \models_4 \psi$. But both algebras are lattices, and thus the ordering relation can be expressed equationally, namely $a \leq b \;\Leftrightarrow\; a = a \wedge b$, for any $a, b$ in $M_4$ and also in $M_6$. Hence, the two logics satisfy $\varphi \models_i \psi$ if and only if $\varphi \mathrel{\vDash\!\!\mid}_i \varphi \wedge \psi$, for $i = 4, 6$. But from Proposition 6 and Corollary 3 we know that $\mathrel{\vDash\!\!\mid}_6 = \mathrel{\vDash\!\!\mid}_4$, therefore we conclude that $\models_6 = \models_4$. □

**Corollary 8 (Completeness).** *The following set of rules is adequate and complete for the logic* $\models_6$*:*

$$\textit{(Axiom)} \quad \varphi \vdash \varphi$$

$$\textit{(Weakening)} \quad \frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi} \qquad\qquad \textit{(Cut)} \quad \frac{\Gamma \vdash \varphi \qquad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}$$

$$(\wedge \vdash) \quad \frac{\Gamma, \varphi, \psi \vdash \xi}{\Gamma, \varphi \wedge \psi \vdash \xi} \qquad\qquad (\vdash \wedge) \quad \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}$$

$$(\vee \vdash) \quad \frac{\Gamma, \varphi \vdash \xi \qquad \Gamma, \psi \vdash \xi}{\Gamma, \varphi \vee \psi \vdash \xi} \qquad\qquad (\vdash \vee) \quad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$(\neg) \quad \frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}$$

$$(\neg\neg \vdash) \quad \frac{\Gamma, \varphi \vdash \psi}{\Gamma, \neg\neg\varphi \vdash \psi} \qquad\qquad (\vdash \neg\neg) \quad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \neg\neg\varphi}$$

*where* $\varphi, \psi, \xi$ *are arbitrary elements of Form*, $\Gamma$ *is a non-empty and finite subset of Form, and the rules have been expressed in a standard Gentzen-style formalism. Note that the rules of Exchange and Contraction are implicit in this formalism since we use subsets rather than sequences of formulas.*

**Proof:** See [7, Theorem 2] and also [3, pp. 15,16]. A Hilbert-style formalism will be presented in [8]. □

Having reached the goal of this section, we would like to point out that the equality of the two logics $\models_6$ and $\models_4$ does not imply that the two sets of logical values are identical in any sense, or that either set can be interpreted in terms of the other. This equality is a formal result, which as a consequence, for instance, tells us that we can obtain a syntactic formulation of $\models_6$ by looking at one of $\models_4$; but from the point of view of applications, it may happen that each situation requires the use of only one of the sets of logical values. For instance, it is clear that in our six-valued model we do not take into account the possibility of contradictory informations coming from independent sources, which is the basis for [2,3] and a host of subsequent works, and conversely, these works only refer to values at time points, not over intervals, which was the basis for [9] and related papers.

## 5. Other directions of research

In this section we merely indicate some additional features of our logic, which may lead to different lines of research.

### The knowledge-order, and weak interlaced bilattices

We have so far considered the set $M_6$ with some structure, namely the one given by the ordering relation $\leq_t$ known as *truth order*, which arises from the original ordering of $M_3$ understood as a 'degree of truth' order. However, the six elements of $M_6$ can also be naturally ordered based on the completeness of the knowledge they represent. In the AI literature, this partial order has been referred to as the ***knowledge order*** [10] and denoted by $\leq_k$. In a knowledge order, we say that a valuation $v$ tells us more about a sentence $\psi$ than about a sentence $\varphi$ when $v(\varphi) \leq_k v(\psi)$. This partial ordering naturally exists in $M_6$ in the sense that the knowledge or information content of a sentence with value **t**, **u** or **f** is higher than that concerning a sentence with undecided value (**k**, **k1** or **k0**); and similarly the degree of knowledge of a sentence with completely undecided value (**k**) is lower than that concerning one with value **k1** or **k0**. It is clear that the three decided values are incomparable with each other in this partial ordering, and also the two not completely undecided. Figure 3 is a Hasse diagram displaying the knowledge ordering $\leq_k$ in $M_6$; note that if we read this diagram from left to right then it also represents the truth order $\leq_t$ in $M_6$.

If we think of the members of $M_6$ as subsets of $M_3$, then the knowledge order can be simply described as the inverse of the inclusion relationship: the greater a subset is, the less information it contains. This is so because each new value is just the set of *possible* values that a sentence can take over the time interval. This is in contrast to

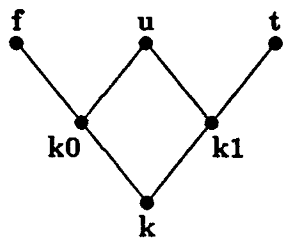



Figure 3: The set $M_6$ ordered under the knowledge order $\leq_k$

the parallel definition of the knowledge order in Belnap's logic developed in Section 2 of [5], where each new value represents the set of values that a sentence actually (and simultaneously) has: in such interpretation, the more values a sentence has, the more information we have about it.

It is clear from Figure 3 that under the knowledge order $\leq_k$ the set $M_6$ is not a lattice: a meet exists for any subset, and thus any subset having a common upper bound has a join; but in general joins need not exist. Actually it can be checked that this is an example of what Fitting calls a *complete semilattice* (see definition 3.7 of [5]): A partially ordered set that is closed under arbitrary meets and under joins of directed subsets, which is a stronger property. (A subset $D$ is directed if for all $x, y \in D$ there is a $z \in D$ such that $x \leq z$ and $y \leq z$).

The existence of these two partial orders together with the good relationships holding between them seem to be closely related to the notion of *bilattice*, introduced by Ginsberg (see [10] for a survey) and further refined by Fitting [5] under the name of *interlaced bilattice*. By slightly modifying Fitting's definition we can introduce the notion of **weak interlaced bilattice**: A set $W$ together with two partial orderings $\leq_t$ and $\leq_k$ such that: (1) $\leq_t$ gives $W$ the structure of a complete lattice, (2) $\leq_k$ gives $W$ the structure of a complete semilattice, (3) the meet and join operations for $\leq_t$ are monotone with respect to $\leq_k$, and (4) the meet and (whenever it exists) the join for $\leq_k$ are monotone with respect to $\leq_t$. (See Fitting's comments, after his Definition 3.1, on how to interpret the "interlacing" conditions (3) and (4) when they involve infinite operations.) Since in some applications (see [4] for instance) the completeness of both orders is deleted in the definition of (interlaced) bilattice, we can also delete it here.

It can be easily verified that $M_6$ with the truth order and the knowledge order introduced in this paper is a weak interlaced bilattice, either with or without the completeness requirement. In the second part of this section we will see that this is a very common structure. On the other hand, Ginsberg and Fitting (among others) have developed a powerful machinery to take advantage of the functionality and versatility of the notion of (interlaced) bilattice, opening its application to many areas of Computer Science. It should now be possible to extend this machinery to weak interlaced bilattices and provide a setting for extensions to logic programming to deal with truth values of sentences over time intervals rather than time points.

## The intervals construction, and weak bilattices

The reader might have observed that the truth order $\leq_t$ on $M_6$ was introduced in Section 3 in a rather ad hoc way, only as a by-product of the connectives $\wedge, \vee$. However, the elements of $M_6$ are not just some of the subsets of $M_3$, but they are exactly all the (non-empty) *intervals* of this set when ordered like in Kleene's logic: $\mathbf{F} \leq \mathbf{U} \leq \mathbf{T}$. It so happens that this is only a particular case of the general construction outlined below, where the definitions of the two orderings and their lattice operations turn out to have a direct and natural form.

Let us start with a 'basic' set of truth values $L$ with an ordering relation $\leq$ which gives it the structure of a complete lattice, whose meet we denote by $\wedge$ and whose join we denote by $\vee$, and with minimum $\mathbf{F}$ and maximum $\mathbf{T}$. Then we consider the set of all (non-empty) **intervals** of $L$ :

$$I(L) = \{ [a,b] : a,b \in L \text{ with } a \leq b \}$$

where we put $[a,b] = \{x \in L : a \leq x \leq b\}$. Then we can define a truth order and a knowledge order over intervals in a fairly natural way: An interval is truth-greater than another when it is 'closer to truth **T**' in the following sense:

$$[a,b] \leq_t [c,d] \quad \text{iff} \quad a \leq c \text{ and } b \leq d$$

while, as we have already explained, an interval contains more information when it is smaller:

$$[a,b] \leq_k [c,d] \;\; \text{iff} \;\; a \leq c \text{ and } d \leq b \text{ , that is, } [c,d] \subseteq [a,b]$$

It can be easily checked that these two relations are partial orders in $I(L)$ and that this set becomes a weak interlaced bilattice in the sense stated above. The minimum of the truth order is $\mathbf{f} = [\mathbf{F},\mathbf{F}] = \{\mathbf{F}\}$ and its maximum is $\mathbf{t} = [\mathbf{T},\mathbf{T}] = \{\mathbf{T}\}$ while the minimum for the knowledge order is $\mathbf{k} = [\mathbf{T},\mathbf{F}] = L$. There is no maximum for $\leq_k$ but each point $p \in L$ gives a maximal element of $I(L)$, namely $[p,p] = \{p\}$. The meet and join operations for the truth order are respectively $[a,b] \wedge [c,d] = [a \wedge c, b \wedge d]$ and $[a,b] \vee [c,d] = [a \vee c, b \vee d]$, the meet for the knowledge order is $[a,b] \otimes [c,d] = [a \wedge c, b \vee d]$, and the join for the knowledge order exists if and only if the intervals are not disjoint, and then it is $[a,b] \oplus [c,d] = [a \vee c, b \wedge d]$. It is interesting to remark the formal similarity between the definitions found in this construction and the ones in the so-called 'world-based bilattices' of [10, Section 4.5].

We can then see that $M_6 = I(M_3)$ in its whole structure, that is, both its two orders and their lattice operations are instances of this general construction. But there is more, the negation can also be treated in a similar way: Suppose that $L$ has a negation $\neg$ giving it the structure of a De Morgan algebra. Then we can define the following operation in $I(L)$: $\neg[a,b] = [\neg b, \neg a]$. It can be checked that this satisfies, for arbitrary $I_1, I_2 \in I(L)$, that $I_1 \leq_t I_2$ implies $\neg I_2 \leq_t I_1$, that $I_1 \leq_k I_2$ implies $\neg I_1 \leq_k I_2$, and that $\neg\neg I_1 = I_1$. Thus, this is a weak interlaced bilattice *with negation* in the sense of [5]. Moreover this negation satisfies the De Morgan laws with respect to $\wedge$ and $\vee$, that is, $I(L)$ is a De Morgan algebra; but it also satisfies $\neg(I_1 \otimes I_2) = \neg I_1 \otimes \neg I_2$, and $\neg(I_1 \oplus I_2) = \neg I_1 \oplus \neg I_2$ whenever the joins exist. Therefore $I(L)$ satisfies all the requirements stated in Definition 4.1 of [10] to be a bilattice, except the upwards completeness of $\leq_k$. This suggests to us the introduction of a notion of **weak bilattice** which would bear to the notion of bilattice the same relationship that the notion of weak interlaced bilattice to the one of interlaced bilattice.

The above constructions show us that if $L$ is any complete De Morgan algebra then $I(L)$ is both a weak interlaced bilattice with negation and a weak bilattice. The development and study of these constructions and of the general concept of weak bilattice, as well as the study of the class of De Morgan algebras of intervals, together with the develpment of applications generalizing those of bilattices, are some of the directions of research that show up naturally in this situation. Note that the consideration of the intervals of any set of truth values is a natural issue in many kinds of applications dealing with incomplete information, aproximate reasoning, etc.

The initial set $L$ will in most cases be the real interval $[0,1]$, like in probabilistic logic [12], or some subset of it closed under negation (i.e., symmetrical with respect to $\frac{1}{2}$), like in fuzzy logic [16], where other lattices are also used as sets of values. See [10] for more references.

### Extension to other multi-valued logics

Up to now we have extended to an arbitrary set of truth values the mathematical construction which led us from Kleene's three-valued logic to our six-valued logic. But it is obvious that this does not constitute an extension of the original idea of [9]. In fact, the previous constructions did not take the time factor into account. Doing so while starting from a many-valued logic presents more difficulties, and as now we do not see any clear and general solution.

Actually, there were two elements in our original idea: First, we considered an 'undecided' value **U** representing failure to assign any of the accepted truth values to a sentence. Second, we considered the variations of the assignments of truth values (including **U**) due to changes in our information. Many systems of many-valued logic have dealt with the first issue by treating **U** as the empty subset; and it is indeed possible to do so in our construction $L \mapsto I(L)$ thus obtaining a set $I^*(L) = I(L) \cup \{\emptyset\}$ which turns out to be just a bilattice in the ordinary sense (the presence of $\emptyset$ preventing $\oplus$ from being monotone with respect to $\leq_t$). The resulting mathematical theory is beautiful, but this again does not help us in our purpose of taking the passage of time into account.

In principle the scheme could be as follows: We start with an initial set of values $L$ (which may well be $I(V)$, or $I^*(V)$, for some other more basic set $V$) understood as a structure of degrees of truth (for instance, a complete De Morgan algebra). We assume that at each time point, a sentence can be assigned one of these values, or else we assign to it a new value **U**, thus obtaining a second set $L^* = L \cup \{\mathbf{U}\}$. We should then select the order in $L^*$, which could compare **U** to the other truth values, and state a refined monotonicity condition which would determine the family of subsets of $L^*$ that a sentence can take over a time interval.

But it seems there is no universal recipe on how to perform these two choices. They may depend on the interpretation of the initial truth values $L$. For instance, if they are degrees of truth in the sense of fuzzy logic, or probabilities, then assigning one of these values is a sort of 'final evaluation' and should not be changed by the acquisition of more information; this would result in a very restrictive monotonicity condition. On the other hand, if they represent degrees or certainty, or degrees of belief for (or against) a sentence, or approximations to the 'real' truth value (like an interval), then a sentence can increase (or decrease) its value as soon as more information is available, and thus the monotonicity condition would not be so restrictive. And the same can be said of the problem of ordering **U** with respect to $L$, that is, of deciding about the degree of truth of the undecided: It seems always reasonable to have $\mathbf{F} \leq \mathbf{U} \leq \mathbf{T}$, but nothing more seems to be reasonably general.

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