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INTERPOLATION THEOREMS OF SOME WEIGHTED QUASI-BANACH SPACES

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Interpolation theorems of some weighted quasi-Banach spaces

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Abstract. We give several results concerning weighted Hardy's inequalities for the case $q < 1$, and using some techniques based upon a reiteration theorem, we study the weighted version of several interpolation theorems for quasi-Banach spaces.

§1. Introduction. In a series of papers (see [16], [17] and [5]) several authors have considered the interpolation of some quasi-Banach spaces with respect to the complex method of interpolation, and more generally for the case of families indexed by the boundary of the unit disk, (see [4]). The idea to identify the intermediate spaces obtained is to get a reiteration formula that goes back to the work of [9]. In [10] a generalization of this formula is given to the case of weighted Banach spaces and as a consequence, the interpolation spaces, for the weighted Lorentz spaces, are found in the Banach case. We take up now the interpolation of these spaces in the quasi-Banach case for which we will improve the results of [16], by using the techniques of [10] for a certain class of weights.

In §2 we prove some results related to the weighted Hardy's inequalities. In section §3 we show an interpolation theorem for families of log-subharmonic operators. In §4 we complete the interpolation results with a partial reiteration formula. Finally in §5 we obtain as an application, the results mentioned above, concerning the weighted Lorentz spaces, and some others for the weighted Hardy spaces.

In this paper, we denote by $L^p(w)$ the space of functions f satisfying $fw \in L^p$, and C is a constant which may be different from line to line, but irrelevant to the conclusion otherwise.

§2. Weighted inequalities. We will consider a class of weights satisfying Hardy's inequalties for $0 < q \leq \infty$, restricted to non-increasing functions. These weights were characterized in [1] and turn out to be precisely the weights w for which the

Hardy-Littlewood maximal operator M, is bounded on $\Lambda^q(w)$, the weighted Lorentz space. The weights for which the inequality holds for all functions (i.e., boundedness of the Hardy operator S, on weighted L^q), were characterized in [12]. A first result shows that for the case $0 < q < 1$ there exists a big difference when restricted to non-increasing functions, since the only weight satisfying the inequality for all functions is the trivial weight, (we give a stronger result by considering the weak-L^q version).

THEOREM 2.1. *Let S be the Hardy operator*

$$Sf(t) = \frac{1}{t}\int_0^t f(s)\,ds,$$

and suppose that w is a non-negative locally integrable function on $(0,\infty)$. If $0 < q < 1$, w is continuous on an interval I and satisfies

$$\int_{\{s:\, Sf(s)>t\}} w(s)\,ds \le \frac{C}{t^q}\int_0^\infty |f(t)|^q w(t)\,dt,$$

for some $C > 0$, all $t > 0$ and all $f = \chi_{I'}$ (I' subinterval of I), then $w \equiv 0$ on I.

PROOF: Suppose $\infty > w(t_0) \neq 0$ and let $\varepsilon > 0$ be such that $w \neq 0$ on the interval $I' = (t_0 - \varepsilon, t_0 + \varepsilon) \subset I$. Let $t > 0$ and let $f = \chi_{I'}$. Then it is very easy to show that

$$\{s : Sf(s) > t\} = \begin{cases} \emptyset & \text{if } t > 2\varepsilon/(t_0+\varepsilon) \\ \left(\dfrac{t_0-\varepsilon}{1-t}, \dfrac{2\varepsilon}{t}\right) & \text{if } 0 < t \le 2\varepsilon/(t_0+\varepsilon). \end{cases}$$

Hence,

$$\int_{(t_0-\varepsilon)/(1-t)}^{2\varepsilon/t} w(s)\,ds \le \frac{C}{t^q}\int_{t_0-\varepsilon}^{t_0+\varepsilon} w(s)\,ds.$$

With $t = \varepsilon/(t_0+\varepsilon)$ we get

$$\frac{\varepsilon^{q-1}}{(t_0+\varepsilon)^q}\int_{(t_0^2-\varepsilon^2)/t_0}^{2(t_0+\varepsilon)} w(s)\,ds \le \frac{C}{2\varepsilon}\int_{t_0-\varepsilon}^{t_0+\varepsilon} w(s)\,ds.$$

If we now let ε go to infinity, we get that the left hand side tends to infinity, (since $0 < q < 1$), and the right hand side tends to $Cw(t_0)$, giving a contradiction. ∎

In view of the results of [1] and [12], it is now natural to consider what happens with the boundedness of S on $\Lambda^q(w)$, $0 < q \leq \infty$. We show that this is equivalent to the boundedness of M on the same space and, as a consequence we can give a way to construct weights under these conditions, in terms of some function parameters, (see [10] for some related results). Recall that a function $f \in \Lambda^q(w)$ if

$$\|f\|_{\Lambda^q(w)} = \left(\int_0^\infty (f^*(t)w(t))^q\, dt\right)^{1/q} < \infty,$$

where f^* is the non-increasing rearrangement of f.

PROPOSITION 2.2. *Suppose w is a weight on $\mathbf{R}^+$ and $0 < q \leq \infty$. Then S is a bounded operator on $\Lambda^q(w)$ if and only if M is bounded on $\Lambda^q(w)$.*

PROOF: If $S\colon \Lambda^q(w) \longrightarrow \Lambda^q(w)$ boundedly, then

$$\int_0^\infty ((Sf)^*(t)w(t))^q\, dt \leq C \int_0^\infty (f^*(t)w(t))^q\, dt. \tag{1}$$

We want to show (see [1]) that if f is a non-increasing positive function, then

$$\int_0^\infty (Sf(t)w(t))^q\, dt \leq C \int_0^\infty (f(t)w(t))^q\, dt, \tag{2}$$

which is trivial by (1) and the fact that $f^* = f$ and $(Sf)^* = Sf$.

Conversely if (2) holds then, since, for all $t > 0$,

$$\int_0^t f \leq \int_0^t f^*$$

we have that $(Sf)^*(t) \leq S(f^*)(t)$ and hence,

$$\begin{aligned}\int_0^\infty ((Sf)^*(t)w(t))^q\, dt &\leq \int_0^\infty (S(f^*)(t)w(t))^q\, dt \\ &\leq C \int_0^\infty (f^*(t)w(t))^q\, dt,\end{aligned}$$

which is (1). ∎

DEFINITION 2.3. (See [8]) *Suppose $\varphi : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$, $\varphi \in C^1$. We say that φ is a function parameter ($\varphi \in B_\Psi$) if*

$$0 < \alpha_\varphi = \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \leq \sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)} = \beta_\varphi < 1.$$

This family of function parameters enjoys a great deal of properties, (see e.g. [14]) useful to obtain weighted inequalities and interpolation theorems for both real and complex methods, as we will see later.

COROLLARY 2.4. *Suppose $\varphi \in B_\Psi$, $0 < q \leq \infty$ and let $w(t) = t^{1-1/q}/\varphi(t)$. Then S is a bounded operator on $\Lambda^q(w)$.*

PROOF: By the previous proposition and the characterization of [1], we know that it suffices to show that for every $r > 0$,

$$\int_r^\infty \left(\frac{w(x)}{x}\right)^q dx \leq \frac{C}{r^q} \int_0^r (w(x))^q\, dx.$$

If we denote by $\overline{\varphi}(t) = \sup_{s>0} \dfrac{\varphi(st)}{\varphi(s)}$ and using properies (4), (12) and (13) in [8] we obtain,

$$\int_r^\infty \left(\frac{w(x)}{x}\right)^q dx = \int_r^\infty \left(\frac{1}{\varphi(x)}\right)^q \frac{dx}{x} \leq \left(\frac{1}{\varphi(r)}\right)^q \int_0^1 (\overline{\varphi}(t))^q \frac{dt}{t}$$

$$\leq \frac{q}{r^q}\left(\int_0^r \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t}\right)\left(\int_0^1 (\overline{\varphi}(t))^q \frac{dt}{t}\right) = C_{q,\varphi}\frac{1}{r^q}\int_0^r (w(t))^q\, dt. \blacksquare$$

REMARK: If $1 \leq q$, then much more can be said about w. In fact, in [10] it was proved that w satisfies Hardy's inequalities for all functions, and not only restricted to non-increasing ones, as the corollary shows.

§3. An interpolation theorem for families of log-subharmonic operators.

The complex method of interpolation for families of quasi-Banach spaces, as given in [16] and [17], follows the same ideas as the Banach case in [4]. Let us give the main definitions involved in order to fix the notation used in the sequel.

Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. To simplify notation we shall write $\theta \in \mathbf{T}$ instead of $e^{i\theta} \in \mathbf{T}$. Let $\{B(\theta)\}_{\theta\in\mathbf{T}}$ be a family of quasi-Banach spaces, and denote by

$C(\theta)$ the constants in the quasi-triangle inequality. We say that this family is an interpolation family (of quasi-Banach spaces) if each $B(\theta)$ is continuously embedded in a Hausdorff topological space U, the function $\theta \longrightarrow \|b\|_{B(\theta)}$ is measurable for each $b \in \cap_{\theta \in \mathbf{T}} B(\theta)$, and $\log C(\theta) \in L^1(\mathbf{T})$. Set

$$\mathcal{B} = \left\{ b \in \cap_{\theta \in \mathbf{T}} B(\theta) : \int_0^{2\pi} \log^+ \|b\|_{B(\theta)} \, d\theta < \infty \right\}.$$

$\mathcal{B}$ is called the log-intersection space of the given family and U is called the containing space.

Let us denote by $\mathcal{G} = (B(\cdot), \mathbf{T})$ the space of all $\mathcal{B}$-valued analytic functions of the form $g(z) = \sum_{j=1}^{m} \mathcal{X}_j(z) b_j$ for which $\|g\|_\infty = \sup_\theta \|g(\theta)\|_{B(\theta)} < \infty$, where $\mathcal{X}_j \in N^+$ and $b_j \in \mathcal{B}$, $j = 1, ..., m$. (N^+ denotes the positive Nevalinna class for the unit disc $D = \{z \in \mathbf{C} : |z| \leq 1\}$.) For every $b \in \mathcal{B}$, and $z \in D$ we define

$$|a|_z = \inf \left\{ \|g\|_\infty : \; g \in \mathcal{G}, \; g(z) = b \right\}.$$

If N_z denotes the set of elements of $\mathcal{B}$ such that $|a|_z = 0$, the completion $B(z)$ of $(\mathcal{B}/N_z, |\cdot|_z)$ will be called the interpolation space at z of the family $\{B(\theta)\}_{\theta \in \mathbf{T}}$. It can be shown that this is a quasi-Banach space with quasi-triangle constant

$$C(z) = \exp\left(\int_{\mathbf{T}} \log C(\theta) P_z(\theta) \, d\theta \right),$$

where $P_z(\theta)$ is the Poisson kernel.

In order to prove our next result, we need the following definition.

DEFINITION 3.1. *Let $\mathcal{N}$ be the set of all measurable complex-valued functions on some measure space (X, μ) and let $\{B(\theta)\}_{\theta \in \mathbf{T}}$ be an interpolation family of quasi-Banach spaces with containing space U. A family of operators $M_z : U \longrightarrow \mathcal{N}$, $z \in D$, is said to be log-subharmonic if for all $f \in \mathcal{G}(B(\cdot), \mathbf{T})$ and* a.e. *$x \in X$,*

$$\log |M_z f(z)(x)|$$

is subharmonic in D.

THEOREM 3.2. *Let $\{M_z\}_{z\in D}$ be a log-subharmonic family of operators as in Definition 3.1 and such that $M_z f(z)(\cdot)$ is locally integrable in x. Suppose that for all $a \in \mathcal{B}$ we have*

$$\|M_\theta a\|_{L^{p(\theta)}(w_\theta)} \le \eta(\theta)|a|_\theta, \tag{3}$$

where $0 < p(\theta) \le \infty$, $\dfrac{1}{p(\cdot)} \in L^1(\mathbf{T})$ *and* $\log\eta \in L^1(\mathbf{T})$. *Then, for all $a \in \mathcal{B}$ and $z \in D$,*

$$\|M_z a\|_{L^{p(z)}(w_z)} \le \eta(z)|a|_z,$$

where $\dfrac{1}{p(z)} = \displaystyle\int_{\mathbf{T}} \frac{1}{p(\theta)} P_z(\theta)\,d\theta$ *and* $\eta(z) = \exp\left(\displaystyle\int_{\mathbf{T}} (\log\eta(\theta))P_z(\theta)\,d\theta\right)$.

PROOF: Let $a \in \mathcal{B}$; there exists $f(z) = \sum_{j=1}^{n} \varphi_j(z)a_j \in \mathcal{G}(B(\cdot),\mathbf{T})$ such that $f(z_0) = a$ and

$$\|f\|_\infty \le |a|_{z_0}(1+\varepsilon), \tag{4}$$

$z_0 \in D$, $\varepsilon > 0$ fixed.

We claim the following:

The function $z \longrightarrow \log\|M_z f(z)(\cdot)\|_{L^{p(z)}}$ is subharmonic in D.

The claim gives the result since, by (3) and (4),

$$\begin{aligned}
\|M_{z_0} a\|_{L^{p(z_0)}} &= \|M_{z_0} f(z_0)\|_{L^{p(z_0)}}\\
&\le \exp\left(\int_{\mathbf{T}} (\log\|M_\theta f(\theta)\|_{L^{p(\theta)}})P_{z_0}(\theta)\,d\theta\right)\\
&\le \exp\left(\int_{\mathbf{T}} (\log(\eta(\theta)|f(\theta)|_\theta))P_{z_0}(\theta)\,d\theta\right)\\
&\le \|f\|_\infty \eta(z_0) \le |a|_{z_0}(1+\varepsilon)\eta(z_0).
\end{aligned}$$

Let us now give the proof of the claim. Let $\rho > 0$ be such that $\overline{B_\rho(z_0)} \subset D$ and $r > 0$ such that $0 < r < p(z)$ for all $z \in \overline{B_\rho(z_0)}$. This is posible since $p(z)$ is strictely positive on D. It is enough to show

$$\log\|M_{z_0} f(z_0)(\cdot)\|_{L^{p(z_0)}} \le \frac{1}{2\pi}\int_{\mathbf{T}} \log\|M_{z_0+\rho e^{i\theta}} f(z_0+\rho e^{i\theta})(\cdot)\|_{L^{p(z_0+\rho e^{i\theta})}}\,d\theta$$

for any such $\rho > 0$. Define $l(z) = 1 - \dfrac{r}{p(z)}$ and let g be a simple and positive function on X of the form $\sum_{j=1}^{N} \alpha_j \chi_{E_j}$, with $\alpha_j > 0$ and E_j pairwise disjoint sets of finite measure. Then $g(x)^{l(z)}$ is a log-subharmonic function in the disc for every fixed x. Consider

$$\begin{aligned} I(z) &= \int_X g(x)^{l(z)} \left|M_z f(z)(x)\right|^r dx \\ &= \sum_{j=1}^{N} \alpha_j^{l(z)} \int_{E_j} \left|M_z f(z)(x)\right|^r dx \\ &= \sum_{j=1}^{N} \beta_j(z), \end{aligned}$$

which is well defined since $M_z f(z)(\cdot)$ is locally integrable in x.

We want to prove that $I(z)$ is log-subharmonic and therefore it is enough to show that each $\beta_j(z)$ is log-subharmonic. Since,

$$\log \beta_j(z) = l(z) \log \alpha_j + \log \left(\int_{E_j} \left|M_z f(z)(x)\right|^r dx \right),$$

it remains to show that

$$\delta_j(z) = \log \left(\int_{E_j} \left|M_z f(z)(x)\right|^r dx \right)$$

is subharmonic.

Since, by hypothesis, $\left|M_z f(z)(x)\right|$ is log-subharmonic, $\left|M f(z)(x)\right|^r$ is also log-subharmonic, and so, for any complex number α,

$$\begin{aligned} \left|e^{\alpha z_0}\right| e^{\delta_j(z_0)} &= \left|e^{\alpha z_0}\right| \int_{E_j} \left|M_{z_0} f(z_0)(x)\right|^r dx \\ &\leq \int_{E_j} \left(\frac{1}{2\pi} \int_{\mathbf{T}} \left|e^{\alpha(z_0+\rho e^{i\theta})}\right| \left|M_{z_0+\rho e^{i\theta}} f(z_0+\rho e^{i\theta})(x)\right|^r d\theta \right) dx \\ &= \frac{1}{2\pi} \int_{\mathbf{T}} \left|e^{\alpha(z_0+\rho e^{i\theta})}\right| e^{\delta_j(z_0+\rho e^{i\theta})} dx. \end{aligned}$$

This shows that $\log e^{\delta_j(z)} = \delta_j(z)$ is subharmonic, by Radö's criteria. Thus, if $\|g\|_{L^1} = 1$ and using Hölder's inequality with exponents $q = p(z_0 + \rho e^{i\theta})/r > 1$ and

$q' = 1/l(z_0 + \rho e^{i\theta})$ we have,

$$\begin{aligned}
&\log\left(\int_X g(x)^{l(z_0)} \left|M_{z_0} f(z_0)(x)\right|^r dx\right) \\
&= \log I(z_0) \leq \frac{1}{2\pi}\int_{\mathbf{T}} \log I(z_0 + \rho e^{i\theta})\, d\theta \\
&= \frac{1}{2\pi}\int_{\mathbf{T}} \log\left(\int_X g(x)^{l(z_0+\rho e^{i\theta})} \left|M_{z_0+\rho e^{i\theta}} f(z_0+\rho e^{i\theta})(x)\right|^r dx\right) d\theta \\
&\leq \frac{1}{2\pi}\int_{\mathbf{T}} \log\left(\|g\|_{L^1}^{l(z_0+\rho e^{i\theta})} \|M_{z_0+\rho e^{i\theta}} f(z_0+\rho e^{i\theta})(\cdot)\|^r_{L^{p(z_0+\rho e^{i\theta})}}\right) d\theta \\
&= \frac{r}{2\pi}\int_{\mathbf{T}} \log \|M_{z_0+\rho e^{i\theta}} f(z_0+\rho e^{i\theta})(\cdot)\|_{L^{p(z_0+\rho e^{i\theta})}}\, d\theta.
\end{aligned}$$

Therefore, taking the supremum over all such functions g, we have

$$\begin{aligned}
&\log \|M_{z_0} f(z_0)(\cdot)\|^r_{L^{p(z_0)}} = r \log \|M_{z_0} f(z_0)\|_{L^{p(z_0)}} \\
&\leq \frac{r}{2\pi}\int_{\mathbf{T}} \log \|M_{z_0+\rho e^{i\theta}} f(z_0+\rho e^{i\theta})(\cdot)\|_{L^{p(z_0+\rho e^{i\theta})}}\, d\theta,
\end{aligned}$$

which proves the claim and hence the theorem. ∎

REMARK: This theorem improves Theorem 2.3 of **[16]** for analytic families of operators M_z such that $\log\left|M_z f(z)(x)\right|$ is subharmonic, which generalizes the well-known result of Stein and Weiss, (see **[15]**).

Theorem 3.2 gives us a theorem for operators with values in weighted L^p spaces, which is stated after the following definition.

DEFINITION 3.3. (See **[16]**)

Let $\mathcal{M}$ be the set of measurable complex-valued functions on some measure space (Y, ν). An operator M mapping $\mathcal{M}$ into the class $\mathcal{N}^+$ of non-negative-valued measurable functions on some other measure space (X, μ) is said to be of maximal type, provided it satisfies:

(a) $M(\lambda a) = |\lambda| Ma,\qquad$ *for all* $\lambda \in \mathbf{C}$, $a \in \mathcal{M}$.

(b) $M(a) = M(|a|),\qquad a \in \mathcal{M}$.

(c) $M(a)(x) \leq M(b)(x),\qquad$ *if* $|a(y)| \leq |b(y)|$.

(d) $M\left(\int_{\mathbf{T}} f(\cdot, \theta)\, d\theta\right)(x) \leq \int_{\mathbf{T}} M(f(\cdot, \theta))(x)\, d\theta$.

THEOREM 3.4. *Let $\{B(\theta)\}_{\theta\in\mathbf{T}}$ be an interpolation family of quasi-Banach spaces, $\{w_\theta\}$ a family of weights in $\mathcal{N}^+$ and $M : U \longrightarrow \mathcal{N}^+$ an operator which can be expressed as the composition $M_0 \circ L$ of a linear operator L and of a maximal type operator M_0. Suppose that for all $a \in \mathcal{B}$ we have*

$$\|Ma\|_{L^{p(\theta)}(w_\theta)} \leq \eta(\theta)|a|_\theta,$$

where $0 < p(\theta) \leq \infty$, $\frac{1}{p(\cdot)} \in L^1(\mathbf{T})$ and $\log\eta \in L^1(\mathbf{T})$. Then, for all $a \in \mathcal{B}$

$$\|Ma\|_{L^{p(z)}(w_z)} \leq \eta(z)|a|_z,$$

where $\frac{1}{p(z)} = \int_{\mathbf{T}} \frac{1}{p(\theta)} P_z(\theta)\,d\theta, \qquad \eta(z) = \exp\left(\int_{\mathbf{T}} (\log\eta(\theta)) P_z(\theta)\,d\theta\right)$ *and*

$$w_z(x) = \exp\left(\int_{\mathbf{T}} (\log w_\theta(x)) P_z(\theta)\,d\theta\right).$$

PROOF: Let $M_z(u)(x) = M(u)(x)w_z(x)$, $u \in U$. It is enough to observe that $\{M_z\}_{z\in D}$ is a log-subharmonic family of operators, since for all $f \in \mathcal{G}(B(\cdot),\mathbf{T})$

$$\log\left|M_z f(z)(x)\right| = \log M_z f(z)(x) + \log w_z(x),$$

and the first factor is subharmonic (by Proposition 2.1 of [16]) and the second is a harmonic function. The result now follows by applying Theorem 3.2. ∎

§4. Reiteration.

As we have mentioned in the introduction, our goal is to find a partial reiteration theorem for the interpolation of some weighted quasi-Banach spaces. These spaces will be constructed by means of the real interpolation method with a function parameter, that we now review.

Let A_0, A_1 be two quasi-Banach spaces. We say that A_0 and A_1 are compatible if there is a Hausdorff topological vector space $\mathcal{U}$ such that A_0 and A_1 are subspaces of $\mathcal{U}$. We then can form their sum and their intersection. We set $\overline{A} = (A_0, A_1)$ and $\Sigma(\overline{A}) = A_0 + A_1$ and define the Peetre K-functional by

$$K(t,a) = K(t,a;\overline{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \qquad \text{all } t > 0,\ a \in \Sigma(\overline{A}).$$

We also need to introduce the $\tilde{J}$-functional. Set $\Delta(\overline{A}) = A_0 \cap A_1$, and for each $a \in \Delta(\overline{A})$ and $t > 0$, define

$$\tilde{J}(t,a) = \tilde{J}(t,a;\overline{A}) = \max(|a|_0, t|a|_1),$$

where $|a|_0 = \lim_{t\to\infty} K(t,a)$ and $|a|_1 = \lim_{t\to 0} K(t,a)/t$ are the so called Gagliardo completion norms, (see [3]).

Given a couple $\overline{A} = (A_0, A_1)$ of compatible quasi-Banach spaces, a function parameter $\varphi \in B_\Psi$ and $0 < q \le \infty$ we define

$$\overline{A}_{\varphi,q,K} = (A_0, A_1)_{\varphi,q,K} = \left\{ a \in \Sigma(\overline{A}) : \|a\|^q_{\varphi,q,K} = \int_0^\infty \left(\frac{1}{\varphi(t)} K(t,a) \right)^q \frac{dt}{t} < \infty \right\}.$$

This is the continuous version of the norm. There exists an analogous discrete norm, (see Proposition 2.6 of [10]). For the $\tilde{J}$-functional, we give directly this discrete definition:

$$\overline{A}_{\varphi,q,\tilde{J}} = \left\{ a \in \Sigma(\overline{A}) : \ a = \sum_n a_n \right\},$$

where the sum converges in $\Sigma(\overline{A})$ and $a_n \in \Delta(\overline{A})$. In this case, we have

$$\|a\|_{\overline{A}_{\varphi,q,\tilde{J}}} = \inf_{\{a_n\}_n} \left(\sum_n \left(\frac{1}{\varphi(2^n)} \tilde{J}(2^n, a_n) \right)^q \right)^{1/q}.$$

It can be proved that these two definitions give rise to an equivalent quasi-norm on the intermediate space $\overline{A}_{\varphi,q}$. We will only show the inequality needed for our main result.

LEMMA 4.1. *Set $\rho = \log 2 / \log(2\max(C_0, C_1))$, where C_i is the quasi-triangle constant for A_i, and suppose $a \in \Delta(\overline{A})$. Then*

$$\|a\|_{\varphi,q;K} \le K_{\varphi,q} \|a\|_{\varphi,q;\tilde{J}},$$

where

$$K_{\varphi,q} = (\overline{\varphi}(2))^2 2^{1/\rho} (\log 2)^{1/q - 1/\rho} \left(\int_0^\infty \left(\min(1, 1/t) \overline{\varphi}(t) \right)^\rho \frac{dt}{t} \right)^{1/\rho}. \tag{5}$$

PROOF: Suppose $a \in \Delta(\overline{A})$ and $a = \sum_\nu a_\nu$, with $a_\nu \in \Delta(\overline{A})$. It is known that for $\rho = \log 2/\log(2\max(C_0, C_1))$ we have

$$K(t,a) \le 2^{1/\rho}\Big(\sum_\nu (K(t,a_\nu)^\rho\Big)^{1/\rho},$$

(see [3]). Using now the fact that $K(t,a_\nu) \le \min(1,t/s)\tilde{J}(s,a_\nu)$ and a change of variables (with $t = 2^\mu$), we obtain

$$K(2^\mu,a) \le 2^{1/\rho}\Big(\sum_\nu (\min(1,2^\nu)\tilde{J}(2^{\mu-\nu},a_{\mu-\nu}))^\rho\Big)^{1/\rho},$$

and hence, using the discrete characterization for $\|\cdot\|_{\varphi,q;K}$, (see Proposition 2.6 of [10]),

$$\|a\|_{\varphi,q;K} \le \overline{\varphi}(2)(\log 2)^{1/q}\Big(\sum_\mu \Big(\frac{1}{\varphi(2^\mu)}K(2^\mu,a)\Big)^q\Big)^{1/q}$$

$$\le \overline{\varphi}(2)(\log 2)^{1/q}2^{1/\rho}\Big(\sum_\nu \min(1,2^{\nu\rho})\overline{\varphi}(2^{-\nu\rho})\Big)^{1/\rho}\Big(\sum_\mu \Big(\frac{1}{\varphi(2^\mu)}\tilde{J}(2^\mu,a_\mu)\Big)^q\Big)^{1/q}.$$

But, by a monotonicity argument,

$$\Big(\sum_\nu \min(1,2^{\nu\rho})\overline{\varphi}(2^{-\nu\rho})\Big)^{1/\rho} \le \frac{\overline{\varphi}(2)}{(\log 2)^{1/\rho}}\Big(\int_0^\infty (\min(1,1/t)\overline{\varphi}(t))^\rho\frac{dt}{t}\Big)^{1/\rho},$$

and hence,

$$\|a\|_{\varphi,q;K} \le (\overline{\varphi}(2))^2 2^{1/\rho}(\log 2)^{1/q-1/\rho}\Big(\int_0^\infty (\min(1,1/t)\overline{\varphi}(t))^\rho\frac{dt}{t}\Big)^{1/\rho}\|a\|_{\varphi,q;\tilde{J}},$$

and the integral is finite by Proposition 1.1 of [8]. ∎

DEFINITION 4.2. *Let $\overline{B} = (B_0, B_1)$ be a pair of compatible quasi-Banach spaces with quasi-triangle constants C_0 and C_1 respectively. We say that a pair of measurable functions $F : \mathbf{T}\times\mathbf{R}^+ \longrightarrow \mathbf{R}^+$ and $q : \mathbf{T} \longrightarrow (0,\infty]$ satisfies condition S if for $\rho(\theta) = \min(\rho, q(\theta))$, ($\rho$ as above), we have*

(i) $F_\theta(t) = F(\theta,t) \in B_\Psi$, *for every $\theta \in \mathbf{T}$.*

(ii) $\int_{\mathbf{T}} \log \overline{F_\theta}(\max(2, C_1/C_0))\, d\theta < \infty.$

(iii) $\int_{\mathbf{T}} \frac{1}{q(\theta)}\, d\theta < \infty.$

(iv) $\int_{\mathbf{T}} \log^+ \left(\int_0^\infty (\overline{F_\theta}(t) \min(1, 1/t))^{q(\theta)} \frac{dt}{t} \right)^{1/q(\theta)} d\theta < \infty.$

(v) $\int_{\mathbf{T}} \left| \log \left(\int_0^\infty (\overline{F_\theta}(t) \min(1, 1/t))^{\rho(\theta)} \frac{dt}{t} \right)^{1/\rho(\theta)} \right| d\theta < \infty.$

LEMMA 4.3. *If B_0, B_1 are quasi-Banach spaces with constants C_0 and C_1, $\varphi \in B_\Psi$ and $0 < q \leq \infty$, then $(B_0, B_1)_{\varphi,q}$ is a quasi-Banach space with constant*

$$C_{\varphi,q} = C_0 \overline{\varphi}(C_1/C_0) \max(1, 2^{1/q-1}). \tag{6}$$

PROOF: Suppose $a, b \in (B_0, B_1)_{\varphi,q}$. Since

$$K(t, a+b) \leq C_0 \big(K(C_1 t/C_0, a) + K(C_1 t/C_0, b)\big),$$

using the properties that φ satisfies, (see for example Proposition 1.1 of [8]), we have,

$$\begin{aligned}
\|a+b\|_{\varphi,q} &= \left(\int_0^\infty \left(\frac{1}{\varphi(t)} K(t, a+b) \right)^q \frac{dt}{t} \right)^{1/q} \\
&\leq \left(\int_0^\infty \left(\frac{1}{\varphi(t)} C_0 (K(C_1 t/C_0, a) + K(C_1 t/C_0, b)) \right)^q \frac{dt}{t} \right)^{1/q} \\
&\leq C_0 \max(1, 2^{1/q-1}) \Bigg(\left(\int_0^\infty \left(\frac{1}{\varphi(t)} K(C_1 t/C_0, a) \right)^q \frac{dt}{t} \right)^{1/q} + \\
&\qquad + \left(\int_0^\infty \left(\frac{1}{\varphi(t)} K(C_1 t/C_0, b) \right)^q \frac{dt}{t} \right)^{1/q} \Bigg) \\
&= C_0 \max(1, 2^{1/q-1}) \Bigg(\left(\int_0^\infty \left(\frac{1}{\varphi(C_0 t/C_1)} K(t, a) \right)^q \frac{dt}{t} \right)^{1/q} + \\
&\qquad + \left(\int_0^\infty \left(\frac{1}{\varphi(C_0 t/C_1)} K(t, b) \right)^q \frac{dt}{t} \right)^{1/q} \Bigg) \\
&\leq C_0 \max(1, 2^{1/q-1}) \overline{\varphi}(C_1/C_0) (\|a\|_{\varphi,q} + \|b\|_{\varphi,q}) \cdot \blacksquare
\end{aligned}$$

LEMMA 4.4. *Let (B_0, B_1) be a pair of comparable quasi-Banach spaces and F and q satisfying condition S. Set $B(\theta) = (B_0, B_1)_{F_\theta, q(\theta)}$. Let $\mathcal{B}$ be the log-intersection space of the family $\{B(\theta)\}_{\theta \in \mathbf{T}}$. Then*

$$B_0 \cap B_1 \subset \mathcal{B}.$$

PROOF: Let $b \in B_0 \cap B_1$. Then,

$$\begin{aligned}
\|b\|_{F_\theta, q(\theta)} &= \left(\int_0^\infty \left(\frac{1}{F_\theta(t)} K(t,b) \right)^{q(\theta)} \frac{dt}{t} \right)^{1/q(\theta)} \\
&\leq \|b\|_{B_0 \cap B_1} \left(\int_0^\infty \left(\frac{1}{F_\theta(t)} \min(1,t) \right)^{q(\theta)} \frac{dt}{t} \right)^{1/q(\theta)} \\
&\leq \|b\|_{B_0 \cap B_1} \left(\int_0^\infty \left(\overline{F_\theta}(t) \min(1, 1/t) \right)^{q(\theta)} \frac{dt}{t} \right)^{1/q(\theta)}.
\end{aligned}$$

The desired result now follows from (iv) of condition S, for the case $q(\theta) < \infty$. If $q(\theta) = \infty$, using the estimate (see for example Proposition 1.1.9 of [**14**]),

$$F_\theta(t) \geq \min(t^{\alpha_{F_\theta}}, t^{\beta_{F_\theta}}),$$

we obtain

$$\begin{aligned}
\|b\|_{F_\theta, q(\theta)} &= \sup_{t>0} \frac{K(t,b)}{F_\theta(t)} \leq \|b\|_{B_0 \cap B_1} \sup_{t>0} \frac{\min(1,t)}{F_\theta(t)} \\
&\leq \|b\|_{B_0 \cap B_1} \max\left(\sup_{t \leq 1} t^{-\alpha_{F_\theta}+1}, \sup_{t \geq 1} t^{-\beta_{F_\theta}} \right) = \|b\|_{B_0 \cap B_1}. \blacksquare
\end{aligned}$$

THEOREM 4.5. *Let (B_0, B_1) be a pair of comparable quasi-Banach spaces and F and q satisfying condition S. Set $B(\theta) = (B_0, B_1)_{F_\theta, q(\theta)}$. Then $\{B(\theta)\}_{\theta \in \mathbf{T}}$ is an interpolation family of quasi-Banach spaces and*

$$(B_0, B_1)_{F_z, q(z)} \subset [B(\theta)]_z, \tag{7}$$

where $F_z(t) = \exp\left(\int_{\mathbf{T}} \log F(\theta, t) P_z(\theta)\, d\theta \right)$ *and* $\frac{1}{q(z)} = \int_{\mathbf{T}} \frac{1}{q(\theta)} P_z(\theta)\, d\theta$.

PROOF: The fact that $F_z \in B_\Psi$, $z \in D$, needed to define the space $(B_0, B_1)_{F_z, q(z)}$, is proved as in [**10**].

Our next step is to show that $\{B(\theta)\}_{\theta\in\mathbf{T}}$ is an interpolation family of quasi-Banach spaces in the sense of [16]. First $B(\theta)\subset B_0+B_1$ and

$$\|a\|_{B_0+B_1}\le k(\theta)\|a\|_{B(\theta)},$$

where

$$k(\theta)=\left(\int_0^\infty \left((\overline{F_\theta}(s))^{-1}\min(1,s)\right)^{q(\theta)}\frac{ds}{s}\right)^{-1/q(\theta)},$$

(see Proposition 2.4 of [10]). Secondly, if $C(\theta)=C_{F_\theta,q(\theta)}$ as in (6), the fact that $\log C(\theta)\in L^1(\mathbf{T})$ is a consequence of condition S.

We are going to show that if $b\in B_0\cap B_1$

$$|b|_z\le C(z)\|b\|_{F_z,q(z)},$$

where F_z and $q(z)$ are as above.

By Proposition 3.19 of [16], we know that there exists a representation of b of the form $b=\sum_{|n|\le N}b_n$, with $b_n\in B_0\cap B_1$ such that

$$\tilde{J}(2^n,b_n)\le 4\max(C_0,C_1)K(2^n,b).$$

Fix $t>0$ and set $G_\theta(t)=tF_\theta'(t)/F_\theta(t)$. Let $\widetilde{G}(\xi,t)$, $\xi\in D$ be the harmonic conjugate of $G(\cdot,t)$ normalized by $\widetilde{G}(z,t)=0$. Similarly, let $(1/q)^\sim$ be the harmonic conjugate of $1/q$ such that $(1/q)^\sim(z)=0$. Set $W(\xi,t)=G(\xi,t)+i\widetilde{G}(\xi,t)$, and

$$\frac{1}{s(\xi)}=\frac{1}{q(\xi)}+i\left(\frac{1}{q}\right)^\sim(\xi),\qquad \xi\in D.$$

Let $H(\xi,t)$ be so that $W(\xi,t)=\dfrac{tH'(\xi,t)}{H(\xi,t)}$; that is

$$H(\xi,t)=\exp\left(\int_1^t\frac{W(\xi,s)}{s}\,ds\right).$$

Define

$$B_n(\xi)=\frac{H(\xi,2^n)}{F_z(2^n)}\left(\frac{\tilde{J}(2^n,b_n)}{F_z(2^n)}\right)^{q(z)/s(\xi)-1}\frac{K(z)}{K(\xi)}\frac{L(z)}{L(\xi)}\|b\|_{F_z,q(z)}^{1-q(z)/q(\xi)}$$

where

$$K(\xi)=\exp\left(\int_{\mathbf{T}}\log K(\theta)H_\xi(\theta)\,d\theta\right)$$

and $K(\theta)=K_{F_\theta,q(\theta)}$ is the constant in (5), and

$$L(\xi)=\exp\left(\int_{\mathbf{T}}\log L(\theta)H_\xi(\theta)\,d\theta\right)$$

where

$$L(\theta)=4\max(C_0,C_1)^{q(z)/q(\theta)}\left(\frac{\overline{F_\theta}(2)}{\log 2}\right)^{q(z)/q(\theta)}.$$

$K(\xi)$ and $L(\xi)$ are well defined by condition S. To show that $B_n\in N^+(D)$ we observe that it can be written as a quotient of two functions in H^∞, so that the one at the bottom has no singular part, (see [7]):

$$|H(\xi,2^n)|=\exp\left(\int_1^{2^n}\frac{W(\xi,2^n)}{s}\,ds\right)\le 2^n,$$

since $W(\xi,s)\le 1$ and for the other factors one observes that they are exponentials of either bounded functions or functions whose reciprocals are bounded (and therefore have no singular part).

Define

$$g(\xi)=\sum_{|n|\le N}b_nB_n(\xi).$$

Observe that $g(z)=b$. Also,

$$\|g(\theta)\|_{F_\theta,q(\theta);K}\le K(\theta)\|g(\theta)\|_{F_\theta,q(\theta);\tilde{J}}$$

$$\le K(\theta)\Bigg(\sum_{|n|\le N}\left(F_\theta(2^n)^{-1}\tilde{J}(2^n,b_nB_n(\theta))\right)^{q(\theta)}\Bigg)^{1/q(\theta)}$$

$$=K(z)\Bigg(\sum_{|n|\le N}\left(F_\theta(2^n)^{-1}\left|\frac{H(\theta,2^n)}{F_z(2^n)}\left(\frac{\tilde{J}(2^n,b_n)}{F_z(2^n)}\right)^{q(z)/s(\theta)-1}\right|\tilde{J}(2^n,b_n)\right)^{q(\theta)}\Bigg)^{1/q(\theta)}\times$$

$$\times\frac{L(z)}{L(\theta)}\|b\|_{F_z,q(z)}^{1-q(z)/q(\theta)}$$

$$=K(z)\frac{L(z)}{L(\theta)}\Big(\sum_{|n|\le N}(F_z(2^n)^{-1}\tilde{J}(2^n,b_n))^{q(z)}\Big)^{1/q(\theta)}\|b\|_{F_z,q(z)}^{1-q(z)/q(\theta)}$$

$$\le K(z)\frac{L(z)}{L(\theta)}(4\max(C_0,C_1))^{q(z)/q(\theta)}\Big(\sum_{|n|\le N}(F_z(2^n)^{-1}K(2^n,b))^{q(z)}\Big)^{1/q(\theta)}\times$$

$$\times\|b\|_{F_z,q(z)}^{1-q(z)/q(\theta)}$$

$$\le K(z)\frac{L(z)}{L(\theta)}(4\max(C_0,C_1))^{q(z)/q(\theta)}\left(\frac{\overline{F_\theta}(2)}{\log 2}\right)^{q(z)/q(\theta)}\|b\|_{F_z,q(z);K}^{q(z)/q(\theta)+1-q(z)/q(\theta)}$$

$$=K(z)L(z)\|b\|_{F_z,q(z);K}<\infty.$$

Using the density of $B_0\cap B_1$ in $(B_0,B_1)_{F_z,q(z)}$, we conclude the proof of (7). ∎

REMARK: We do not know if the equality $(B_0,B_1)_{F_z,q(z)}=[B(\theta)]_z$ holds for the case of quasi-Banach spaces, which we know is true for Banach spaces, (see **[10]**).

§5. Applications to some weighted quasi-Banach spaces.

The results given in the previous sections provide us with a very useful tool to obtain, for example, the intermediate spaces for some weighted spaces, namely the weighted Lorentz spaces, for which there has been lately a great deal of interest (see **[1]**, **[10]**, **[13]**). For our second example, we will give a partial inclusion and will make some comments about the full answer.

PROPOSITION 5.1. *Suppose* $w:\mathbf{T}\times\mathbf{R}^+\longrightarrow\mathbf{R}^+$ *and* $q:\mathbf{T}\longrightarrow(0,\infty]$ *are two measurable functions satisfying that* w_θ *is a weight for the Hardy operator on* $\Lambda^{q(\theta)}(w_\theta)$, *(see* Proposition 2.2*),* $1/q\in L^1(\mathbf{T})$ *and*

$$\log C^+(w_\theta,q(\theta))\in L^1(\mathbf{T}),$$

where $C^+(w_\theta,q(\theta))$ *is the constant for the boundedness of* S *on* $\Lambda^{q(\theta)}(w_\theta)$*. Then, if* $\Lambda^{q(\theta)}(w_\theta)$ *is an interpolation family of quasi-Banach spaces,*

$$\left[\Lambda^{q(\theta)}(w_\theta)\right]_z\subset\Lambda^{q(z)}(w_z),$$

where $1/q(z)$ *is the harmonic extension of* $1/q(\theta)$ *and*

$$w_z(t)=\exp\left(\int_{\mathbf{T}}\log(w_\theta(t))P_z(\theta)\,d\theta\right).$$

PROOF: Let $Mf(t) = f^{**}(t) = S(f^*)(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$. M is an operator of maximal type, since by the equality, (see [2])

$$Mf(t) = \sup_{|E|=t} \frac{1}{t}\int_E f(s)\,ds$$

it is easy to verify conditions (a), (b), (c) and (d) of Definition 3.3. Moreover,

$$M : \Lambda^{q(\theta)}(w_\theta) \longrightarrow L^{q(\theta)}(w_\theta),$$

since,

$$\|Mf\|_{L^{q(\theta)}(w_\theta)} = \left(\int_0^\infty (S(f^*)(t)w_\theta(t))^{q(\theta)}dt\right)^{1/q(\theta)}$$
$$\leq C^+(w_\theta, q(\theta))\left(\int_0^\infty (f^*(t)w_\theta(t))^{q(\theta)}dt\right)^{1/q(\theta)} = C^+(w_\theta, q(\theta))\|f\|_{\Lambda^{q(\theta)}(w_\theta)}.$$

We can now apply Theorem 3.4 to conclude

$$\|Mf\|_{L^{q(z)}(w_z)} \leq C(w_z, q(z))\|f\|_{[\Lambda^{q(\theta)}(w_\theta)]_z},$$

where,

$$C(w_z, q(z)) = \exp\left(\int_{\mathbf{T}} \log(C^+(w_\theta, q(\theta)))P_z(\theta)\,d\theta\right).$$

Since $Mf = f^{**}$, and $f^* \leq f^{**}$, the above inequality implies,

$$\|f\|_{\Lambda^{q(z)}(w_z)} = \|f^*\|_{L^{q(z)}(w_z)} \leq \|f^{**}\|_{L^{q(z)}(w_z)}$$
$$= \|Mf\|_{L^{q(z)}(w_z)} \leq C(w_z, q(z))\|f\|_{[\Lambda^{q(\theta)}(w_\theta)]_z},$$

which, by a density argument, proves the desired result. ∎

To prove the converse, we will need to recall the following result (see [11]):

LEMMA 5.2. *If $\varphi \in B_\Psi$, $0 < q \leq \infty$ and we set $w(t) = t^{1-1/q}/\varphi(t)$, then $(L^1, L^\infty)_{\varphi,q} = \Lambda^q(w)$, with equivalent quasi-norms.*

THEOREM 5.3. *Let $\overline{B} = (B_0, B_1)$ be a pair of compatible quasi-Banach spaces with quasi-triangle constants C_0 and C_1 respectively, and suppose that the functions $F : \mathbf{T} \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ and $q : \mathbf{T} \longrightarrow (0, \infty]$ satisfy condition S. Set*

$$w(\theta, t) = t^{1-1/q(\theta)}/F_\theta(t).$$

Then, $\{\Lambda^{q(\theta)}(w(\theta,\cdot))\}_{\theta\in\mathbf{T}}$ *is an interpolation family of quasi-Banach spaces and*

$$[\Lambda^{q(\theta)}(w(\theta,\cdot))]_z = \Lambda^{q(z)}(w(z,\cdot)),$$

with equivalent quasi-norms, where

$$\frac{1}{q(z)} = \int_{\mathbf{T}} \frac{1}{q(\theta)} P_z(\theta)\, d\theta$$

and

$$w(z,s) = \exp\left(\int_{\mathbf{T}} (\log w(\theta,s)) P_z(\theta)\, d\theta\right).$$

PROOF: By Lemma 5.2

$$\Lambda^{q(\theta)}(w(\theta,\cdot)) = (L^1, L^\infty)_{\varphi(\theta),q(\theta)}.$$

By Theorem 4.5, $\{\Lambda^{q(\theta)}(w(\theta,\cdot))\}_{\theta\in\mathbf{T}}$ is an interpolation family, and

$$\Lambda^{q(z)}(w(z,\cdot)) = (L^1, L^\infty)_{\varphi_z,q(z)} \subset [\Lambda^{q(\theta)}(w(\theta,\cdot))]_z.$$

The other inclusion is a trivial consequence of the fact that by Corollary 2.4 and condition S, we are under the hypothesis of Proposition 5.1. ∎

Theorem 3.4 can also be applied to other situations in which the spaces are definided in terms of maximal functions, namely the H^p spaces over very general domains (homogeneous type, product domains,...). Our next theorem gives results for weighted H^p spaces. We say that $f \in H^p(w)$, $0 < p \le \infty$, if for $\varphi \in \mathcal{S}$, with $\int \varphi \neq 0$,

$$Mf(x) = \sup_{|x-y|<t} |f * \varphi_t(y)| \in L^p(w^{1/p}),$$

and we denote $\|f\|_{H^p(w)} = \|Mf\|_{L^p(w^{1/p})}$.

THEOREM 5.4. *Let* $0 < p(\theta) \le \infty$ *and* $w : \mathbf{T}\times\mathbf{R}^+ \longrightarrow \mathbf{R}^+$ *be two measurable functions, with* $1/p \in L^1(\theta)$. *If* $\{H^{p(\theta)}(w_\theta)\}_{\theta\in\mathbf{T}}$ *is an interpolation family of quasi-Banach spaces, then*

$$\left[H^{p(\theta)}(w_\theta)\right]_z \subset H^{p(z)}(w_z),$$

where $1/p(z)$ *is the harmonic extension of* $1/p(\theta)$ *and*

$$\frac{\log w_z(x)}{p(z)} = \int_{\mathbf{T}} \frac{\log w_\theta(x)}{p(\theta)} P_z(\theta)\, d\theta.$$

PROOF: It is clear that $Mf(x) = \sup_{|x-y|<t} |f * \varphi_t(y)|$ is an operator of maximal type associated to $H^{p(\theta)}(w_\theta)$. Moreover, $\|Mf\|_{L^{p(\theta)}(w_\theta^{1/p(\theta)})} = \|f\|_{H^{p(\theta)}(w_\theta)}$, so that by Theorem 3.4,

$$\|f\|_{H^{p(z)}(w_z)} = \|Mf\|_{L^{p(z)}(w_z^{1/p(z)})} \leq \|f\|_{[H^{p(\theta)}(w_\theta)]_z}$$

and a density argument will finish the proof. ∎

REMARK: Equality of spaces on the previous result is still an open question. For the case where the weight is constant, an affirmative answer was found in [16].

References

[1] M. A. Ariño and B. Muckenhoupt, *Maximal Functions on Classical Lorentz Spaces and Hardy's Inequalities with Weights for Nonincreasing Functions*, Preprint.

[2] C. Bennett and R. Sharpley, "Interpolation of Operators," Academic Press, 1988.

[3] J. Bergh and J. Löfström, "Interpolation spaces. An introduction," Springer-Verlag, 1976.

[4] R. Coifman. M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss, *A theory of complex interpolation for families of Banach spaces*, Adv. in Math. **43** (1982), 203-229.

[5] M. Cwikel, M. Milman and Y. Sagher, *Complex Interpolation of Some Quasi-Banach spaces*, Jour. of Funct. Anal. **65** (1986), 339-347.

[6] J. García-Cuerva and J. L. Rubio de Francia, "Weighted Norm Inequalities and Related Topics," Mathematical Studies, vol.116, North-Holland, 1985.

[7] J. B. Garnett, "Bounded Analytic Functions," Academic Press, 1981.

[8] J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand **42** (1978), 289-305.

[9] E. Hernández, *A relation between two interpolation methods*, Proc. Conf. Lund, Lecture Notes in Mathematics **1070** (1983), 81-91.

[10] E. Hernández and J. Soria, *Spaces of Lorentz Type and Complex Interpolation*, To appear in Arkiv för Matematik.

[11] C. Merucci, *Interpolation Réelle: Compacité, Dualité, Réitération et Espaces de Lorentz*, Manuscript (1980).

[12] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Mathematica **T. XLIV** (1972), 31-38.

[13] E. Sawyer, *Boundedness of Classical Operators on Classical Lorentz Spaces*, Studia Mathematica **T. XCVI** (1990), 145-158.

[14] J. Soria, *Tent spaces based on weighted Lorentz spaces. Carleson measures*, Ph.D. Thesis, Washington University (1990).

[15] E. M. Stein and G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton University Press, 1971.

[16] A. Tabacco, *Some techniques for the characterization of intermediate spaces*, Preprint.

[17] —, *Complex interpolation for families of quasi-Banach spaces*, Ind. U. Jour. **37** (1988), 1-21.

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