Universitat de Barcelona

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GRAU DE MATEMÀTIQUES

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Univalent functions. The Bieberbach conjecture

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Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 19 de juny de 2019

Abstract

In this work, we will study the theory holomorphic and univalent functions in proper simply connected domains of $\mathbb{C}$; in particular on the case where the domain is the unit disk. We will expose the most important results of the theory, and focus especially on one of its major problems: the Bierberbach conjecture (BC), stated in 1916 by Ludwig Bieberbach, and proved in 1984 by Louis de Branges, which claims:

Bieberbach's Conjecture. *The coefficients of each analytic and univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk, with $f(0) = 0$ and $f'(0) = 1$ satisfy:*

$$|a_n| \leq n, \quad \textit{for} \quad n = 2, 3, \cdots$$

Strict inequality holds for every n unless f is a rotation of the Koebe function.

2010 Mathematics Subject Classification. 30C20, 30C50, 30C55, 30C75

Acknowledgments

First of all, I would like to thank my tutor, Dr. Xavier Massaneda, for his weekly dedication, his good disposition and his great help throughout the realization of the project.

Also, I would like to make special mention to my family, in particular, my mother, my father and my brother, who during these four years have given me their trust, their encouragement, and above all, their unconditional support and affection, essential to have been able to get where I am now. Thank you.

Per aspera ad astra

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Chapter 1

Introduction

Given an open subset $U \subset \mathbb{C}$, an *univalent* function $f : U \to \mathbb{C}$ is an holomorphic function that is also injective. The property of univalence, much stronger in the complex case than in the real, led to the development of a theory of univalent functions, born around the turn of the past century, and still active field of research.

In this work, we will study the theory of holomorphic and univalent functions, mostly in the unit disk. We will show its most relevant results, and analyze one of the main problems of the theory: the Bieberbach conjecture (BC).

But let's start at the beginning. The first question we can ask ourselves is why considering the unit disk is sufficient to generalize many of the results to simply connected domains.

Consider the function $f : \mathbb{D} \to \mathbb{C}$. We see that f is a conformal mapping from the unit disk onto its image $f(\mathbb{D})$. Thus, in particular, we have that the image is also a simply connected domain, and not all $\mathbb{C}$, since otherwise, by the Liouville theorem, we would have that the inverse function f^{-1} would be constant.

Now, the Riemann mapping theorem in its inverse version tells us that, for every proper simply connected domain $U \subset \mathbb{C}$, there exists a univalent function that maps conformally $\mathbb{D}$ to U. In addition, this mapping becomes unique if we impose the conditions: $f(\zeta) = 0$ and $f'(\zeta) > 0$, for any $\zeta \in \mathbb{D}$.

Thus, we observe that the theorem provides a one-to-one correspondence between open simply connected proper subsets of the complex plane and univalent functions $f : \mathbb{D} \to \mathbb{C}$, with the previous conditions.

In our case, we will focus on the following normalization: $f(0) = 0$ and $f'(0) = 1$. Observe that, with these conditions, we can simplify the functions by eliminating irrelevant constants, so it is easier to work with them. These analytic and univalent functions that fulfill with the standardization, form what we call the class S:

$$S = \{f : \mathbb{D} \to \mathbb{C} \quad \text{analytic and univalent,} \quad \text{with} \quad f(0) = 0; \quad f'(0) = 1\}.$$

This class has a key property: it is compact, that is, closed and locally bounded. Proof of this are the theorems of distortion and growth, which control, as their name indicate, the distortion of any function of the class S and its derivative, by sharp bounds.

It must be said that any univalent function in $\mathbb{D}$ can be transformed so that it belongs to class S, and vice versa. For example, if g is a univalent function in $\mathbb{D}$, then $f(z) = \frac{g(z)-g(0)}{g'(0)} \in S$.

Thus, it is for this, and for the good properties of the class S that we have been citing, that many of the results related to the univalent functions theory, are proven for this particular class.

The most classic example of a function of the class S is the so-called Koebe function:

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} nz^n.$$

If we look at its second expression, we see that it is the Cayley transform $z \mapsto \frac{1+z}{1-z}$ squared, and normalized so that it belongs to S. It maps $\mathbb{D}$ to $\mathbb{C} \setminus (\infty, \frac{1}{4}]$. This function has a very important role since it is extremal in many results about univalent functions.

Among all of the results that we will expose, we will focus on one of the main problems of the theory of univalent functions: Bierberbach's conjecture, which is at heart, an assertion about extremality of the Koebe function. The BC claims:

Bieberbach's Conjecture. *The coefficients of each function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, satisfy:*

$$|a_n| \leq n, \quad \textit{for} \quad n = 2, 3, \cdots$$

Strict inequality holds for every n unless f is a rotation of the Koebe function.

This was formulated by Ludwig Bieberbach in 1916, after proving that $|a_2| \leq 2$, with equality if only if the function f was the Koebe function or one of its rotations.

The Bieberbach conjecture, although it's easy to state, has stood as a challenge to many mathematicians for decades, who tried to solve it, unsuccessfully, but who developed different methods that became part of the subject. It wasn't until 1984, 68 years after it was enunciated, that the mathematician Louis de Branges came up with a lengthy, complicated but correct proof of the conjecture.

De Branges' contribution was celebrated at an international symposium held at Purdue in March 1985, where many new problems and directions for research were proposed.

1.1 Structure of the work

In order to introduce progressively the most important concepts until arriving at the BC, I have structured the work in the following way:

The **Chapter 2**, Preliminars, is a compilation of basic results of complex analysis. We start by defining holomorphic function, and its relation with real differentiability. We also introduce the concept of conformal mapping, laying emphasis in Möbius transformations: we see that these transformations from the disk onto itself are exactly the authomorphisms of the disk. Finally, we state two theorems that will be very useful for the following results: the Green theorem, and the most important, the Riemann Mapping theorem, basic piece in the theory of univalent functions.

After this, in **Chapter 3**, we go on to expose some classic results on univalent functions. After defining the class S, we see some of its most used properties, such as the rotation or the square root transformation property. From the class S, we define the class Σ, of analytic and univalent functions g in $\Delta = \{z : |z| > 1\}$, with $g(\infty) = \infty$. Once this is shown, we go on to the first result: the Area theorem, proved by Gronwall in 1914. This result led to the proof of the Bieberbach theorem (1916), a fundamental result, considered the starting point from which the BC arises. The theorem claims that if $f \in S$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function. The proof of it illustrates standard ideas and techniques of univalent function theory, and shows how the Koebe function arise as extrema.

However, its importance lies not only in being the start of the BC, but also in the applications and consequences that have resulted from it. As a first application, we highlight the Koebe One-Quarter theorem, which affirms that the range of each function $f \in S$ contains the disk $D(0, \frac{1}{4})$. Other relevant consequences are the Distortion theorem and the Growth theorem, that give and upper and lower bound of $|f'(z)|$ and $|f(z)|$ respectively, for $f \in S$.

Once seen with detail all these results, in **Chapter 4** we focus on the main problem of the matter: The Bieberbach conjecture and its proof. As an evidence of its difficulty, we first show that with the basic tools of analysis we cannot attain a better bound that $|a_n| \leq \dfrac{n^2e^2}{4}$, which doesn't even get the order of growth of the coefficients right.

Although the general proof didn't come until 1984, since its formulation, many mathematicians have tried to prove particular cases of the theorem. We can generally group these cases in three types: proof for a particular n, proof of results of the form $|a_n| \leq Cn$, for some constant C, and proof of the BC for some subclasses of S.

Results of the first type came slowly, and they has been proved only up to $n = 6$. Among them, we emphasize the case $n = 3$, proved in 1923 by K.Loewner, since some of the tools used in the general proof, as the Loewner differential equation, that we will explain later, came from this demonstration.

With regard to the results of the second type, we highlight the Littlewood theorem, stated in 1925, that affirms that the coefficients of a function $f \in S$ satisfy $|a_n| \leq en$, for $n \leq 2$. It was the first result proved for all $n \geq 1$ that show that the Bierberbach conjecture had the correct order of magnitude. From this, sharper bounds were found, at the sacrifice of simplicity.

Finally, we show the proof of the BC for two subclasses of S: for the subclass of starlike functions, proved by R. Nevalinna in 1920, and for the subclass of functions with real coefficients, proved by J. Dieudonné and W. Rogosinki in 1931. Stronger results have been obtained after by more powerful methods. For example, Schiffer used a variational method to show that $\Re\{a_n\} \leq n$ for all functions $f \in S$, whose coefficients $a_2, \cdots, a_{n-1}$ are real. However, we will not dwell on this, as it is a very broad subject, and we want to focus on proof the BC itself.

To close the work, in **Chapter 5** we try to give a general idea of the proof of the Bieberbach conjecture done by L. de Branges in 1984. We don't go into it in depth, because as we have mentioned before, it is quite complicated, and some of the details are beyond my scope.

As a first remark, we shall say that De Branges did not directly prove the conjecture, but a result that implied it. The main point of the proof was to consider the functions $f \in S$ as the initial point of a family of analytic and univalent functions $\{f_t\}_{t\geq 0}$ that satisfy the Loewner differential equation:

$$\partial_t f_t(z) = z f_t'(z) p_t(z), \tag{1.1}$$

where $p_t(z)$ is a certain holomorphic function from $\mathbb{D}$ onto the right half of the plane, with $p_t(0) = 1$. This family of functions is called a Loewner chain and has the form:

$$f_t(z) = f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad 0 \leq t < +\infty.$$

Thus, using this Loewner chain, and with the help of the Loewner equation, de Branges proved the Milin conjecture, an inequality related to the logarithmic coefficients of a function $f \in S$. It was proved that this result implied the Robertson conjecture, an inequality related to the coefficients of the odd functions in S. And this one, in turn, implied the Bieberbach conjecture.

Chapter 2

Preliminaries

In this section we will recall some basic results of complex analysis, which will be used in further chapters.

2.1 Holomorphic functions and conformal mappings

Definition 2.1. *A **holomorphic function** f is a complex-valued function of complex variable that is complex differentiable (or analytic) in a neighbourdhood of every point of its domain.*

Recall that f is complex differentiable if for every $z_0 \in \mathbb{C}$, the function f has a derivative at z_0 defined by:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

.

The relation between differentiability in the real sense and in the complex can be expressed by this result:

Proposition 2.2. *Let Ω be a subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ a function, and $z_0 = x_0 + \imath y_0 \in \mathbb{C}$. Then, the following statements are equivalent:*

a) f is differentiable in $\mathbb{R}^2$ at (x_0, y_0), with $Df(x_0, y_0) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

b) f is holomorphic at z_0 with $f'(z_0) = \alpha + \imath\beta = \mu \neq 0$.

Moreover, if any of these is fulfilled , then the function complies with the Cauchy Riemann equations:

$$\begin{cases} u_x = v_y \\ u_y = -v_x, \end{cases} \tag{2.1}$$

where $f = u + \imath v$, *and* $u = \Re f$, $v = \Im f$.

A complex function $w = f(z)$ may be viewed geometrically as a mapping from the z-plane to a region in the w-plane, defining $z = x + iy$, and $w = u(x, y) + iv(x, y)$.

One remarkable property mappings can have is conformality, which means locally preservation of orientation and angles.

Definition 2.3. *Let* Ω *be an open subset of* $\mathbb{C}$. *A function* $f : \Omega \to \mathbb{C}$ *is called* ***conformal*** *at a point* $z_0 \in \Omega$ *if it preserves angles between curves through* z_0, *as well as preserving orientation.*

In other words: Let γ_1, γ_2 be differentiable curves in Ω with $\gamma_1(0) = \gamma_2(0) = z_0$ and consider the image curves by f, $\Gamma_1(t) = f(\gamma_1(t))$, and $\Gamma_2(t) = f(\gamma_2(t))$. Then f is conformal at z_0 if the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$ is the same as the angle between $\Gamma_1'(0)$ and $\Gamma_2'(0)$.

The conformal property may be described, as well, in terms of the Jacobian derivative matrix of a coordinate transformation. If the Jacobian matrix of the transformation is everywhere a scalar times an orientation-preserving rotation matrix, then the transformation is conformal.

Thus, for a function to be conformal, it must fulfil these conditions.

Proposition 2.4. *Let* Ω *be a open subset of* $\mathbb{C}$, *and let* $z_0 \in \Omega$. *A function* $f : \Omega \to \mathbb{C}$ *is conformal at* z_0 *if and only if* f *is holomorphic at* z_0 *and* $f'(z_0) \neq 0$.

Proof. The first implication can be proved by using Proposition 2.2, and the fact that the Cauchy-Riemann equations are satisfied if $Df : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-lineal. On the other hand, the converse comes directly from the definition of conformality, considering two arbitrary differentiable curves, and their images. $\square$

2.2 Möbius tranformations. Authomorphisms of $\mathbb{D}$

A well-known example of conformal mappings are the **Möbius transformations**, that is, the linear fractional transformations:

$$f(z) = \frac{az + b}{cz + d} \quad , \quad ad - bc \neq 0$$

where a, b, c, d $\in \mathbb{C}$.

These functions provide a conformal mapping from the extended plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto itself. $\hat{\mathbb{C}}$ is also called the Riemann sphere, as it can be identified, by stereographic projection, with the surface of a sphere.

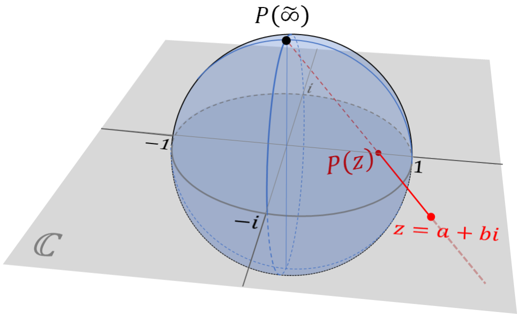

Figure 2.1: Riemann sphere

We should pay special attention to the **Möbius transformations from the unit disk onto itself**, as they will appear constantly during the work.

We will see next that this subgroup of Möbius transformations coincides with the **automorphisms of the disk** $\mathbb{D}$.

Definition 2.5. *A function $\varphi : \mathbb{D} \to \mathbb{D}$ is an **automorphism** of $\mathbb{D}$ if it is holomorphic and bijective. We denote by Aut($\mathbb{D}$) the group of such automorphisms.*

Before that, we recall a basic result on holomorphic functions from the disk onto itself.

Lemma 2.6. (Schwarz's Lemma) *Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be holomorphic, with f(0) = 0. Then,*

(1) $|f(z)| \leq |z|$, for all $z \in \mathbb{D}$.

(2) $|f'(0)| \leq 1$.

Moreover, if $|f(z_0)| = |z_0|$ for some non-zero $z_0 \in \mathbb{D}$ or $|f'(0)| = 1$, then $f(z) = \lambda z$, with $|\lambda| = 1$.

Proof. Define the function:

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Observe that g is continuous in $\mathbb{D}$. As it is holomorphic at $\mathbb{D} \setminus \{0\}$, by Morera's theorem, we deduce that g is holomorphic in $\mathbb{D}$.

Consider now the closed disk $\overline{D(0,r)}$, with radius $r < 1$. By the maximum modulus principle for $z \in \overline{D(0,r)}$,

$$|g(z)| \leq \max_{|z|=r} |g(z)| = \frac{\max\limits_{|z|=r} |f(z)|}{r} \leq \frac{1}{r}.$$

Letting $r \to 1$, we obtain $|g(z)| \leq 1$, that is, $|f(z)| \leq |z|$, for all $z \in \mathbb{D}$. Moreover, from the definition of $g(z)$, it follows that $|g(0)| = |f'(0)| \leq 1$.

Now, suppose equality occurs for either (1) or (2). Then, $|g(z_0)| = 1$ for some $z_0 \in \mathbb{D}$. By the maximum modulus principle, $g(z) = \lambda$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, which means that $f(z) = \lambda z$, $|\lambda| = 1$. □

Thus, once having seen this lemma, we state the following result:

Proposition 2.7.

$$Aut(\mathbb{D}) = \left\{ \frac{a-z}{1-\bar{a}z} e^{\imath\theta}, \quad \theta \in [0, 2\pi), \quad a \in \mathbb{D} \right\}.$$

Proof. Let $\varphi_a(z) = \dfrac{a-z}{1-\bar{a}z} e^{\imath\theta}$. Notice that it is a Möbius transformation. Let's see that it goes from $\mathbb{D}$ onto itself and that it is bijective.

It can be verified by direct computation that

$$1 - \left| \frac{a-z}{1-\bar{a}z} \right|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2}, \tag{2.2}$$

for any $a, z \in \mathbb{C}$, $\bar{a}z \neq 1$.

Thus, observe that if $|z| = 1$, then the right part of (2.2) is zero, and $|\varphi_a(z)| = 1$, which means that $\varphi_a(z)$ preserves the unit circle. On the other hand, if $|z| < 1$, then we get that $|\varphi_a(z)| < 1$, i.e, $\varphi_a(z)$ maps $\mathbb{D}$ onto itself.

We clearly see that this function is holomorphic. Moreover, the injectivity is direct, by being a Möbius transformation. Therefore, we conclude that φ_a is an automorphism of $\mathbb{D}$.

Now, let $\phi : \mathbb{D} \to \mathbb{D}$ be an authomorphism, and let $a \in \mathbb{D}$ be such that $\phi(a) = 0$. Consider, as well, the Möbius transformation:

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad \text{which exchanges } a \text{ and } 0,$$

and the composition

$$h = \phi \circ \varphi_a : \mathbb{D} \to \mathbb{D}.$$

We observe that h is holomorphic in $\mathbb{D}$, and $h(0) = (\phi \circ \varphi_a)(0) = \phi(\varphi_a(0)) = \phi(a) = 0$. So, by the Scharwz lemma:

$$|h(z)| \leq |z|, \quad z \in \mathbb{D}.$$

The same argument applied to h^{-1} yields:

$$|h^{-1}(w)| \leq |w|, \quad w \in \mathbb{D}.$$

For $w = h(z)$, this is:

$$|z| \leq |h(z)|, \quad z \in \mathbb{D}.$$

All combined, we have $|z| \leq |h(z)| \leq |z|$, of what we extract that $|h(z)| = |z|$ and, by the maximum modulus principle, $h(z) = (\phi \circ \varphi_a)(z) = \lambda z$, $\lambda = e^{\imath\theta}$, $\theta \in [0, 2\pi)$.

Writing $u = \varphi_a(z)$, we obtain:

$$\phi(u) = e^{\imath\theta}\varphi_a^{-1}(u)$$

which is a Möbius tranformation.

It remains to be seen that all the Möbius transformations φ from $\mathbb{D}$ to $\mathbb{D}$ are of the form

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}e^{\imath\theta}, \quad \text{for some} \quad a \in \mathbb{D}.$$

We know that if $\varphi : \mathbb{D} \to \mathbb{D}$ is a Möbius transformation, there exists a point $a \in \mathbb{D}$ such that $\varphi(a) = 0$. Then, the symmetric point of a with respect to the unit disk goes to the symmetric point of 0, which is ∞. Considering this, we see that φ has the form:

$$\varphi(z) = c\frac{z - a}{z - \frac{1}{\bar{a}}} = c\frac{a - z}{1 - \bar{a}z}, \quad \text{for some constant} \quad c \in \mathbb{C}.$$

Finally, since for $|z| = 1$, we have that $|\varphi(z)| = 1$, we deduce that $|c| = 1$, so $c = e^{\imath\theta}$. □

Now, we show some relevant results, which will be necessary for the next proofs.

We start with Green's theorem, in its complex version.

Theorem 2.8. (Complex version of Green's Theorem). *Let γ be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{C}$, and let R be the region bounded by γ. If P and Q are C^1-functions of $(z, \bar{z})$ defined in an open region containing R, then:*

$$\int_\gamma \left(Pdz + Qd\bar{z}\right) = \iint_R \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial \bar{z}}\right) dz\, d\bar{z}.$$

Proof. Let: $z = x + iy$, $\bar{z} = x - iy$, so that $dz = dx + idy$, $d\bar{z} = dx - idy$.

Then:

$$\int_\gamma \left(Pdz + Qd\bar{z}\right) = \int_\gamma P(dx + \imath dy) + Q(dx - \imath dy) = \int_\gamma (P + Q)dx + \imath(P - Q)dy.$$

By Green's theorem,

$$\int_\gamma (P + Q)dx + \imath(P - Q)dy = \iint_R \left(\frac{\partial \imath(P - Q)}{\partial x} - \frac{\partial(P + Q)}{\partial y}\right) dx\, dy$$

$$= \iint_R \left(\left(\imath \frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} \right) + \left(-\imath \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y} \right) \right) dx\, dy.$$

On the other hand, by calculation, we get that:

$$dxdy = d\left(\frac{z+\bar{z}}{2}\right) d\left(\frac{z-\bar{z}}{2\imath}\right) = \frac{1}{4\imath}\left(-dzd\bar{z} + d\bar{z}dz\right)$$
$$= \frac{1}{2\imath} d\bar{z}dz = \frac{\imath}{2} dzd\bar{z}.$$

Moreover, we can rewrite the expressions inside the integral as:

$$\imath \frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} = \imath \left(\frac{\partial P}{\partial x} + \imath \frac{\partial P}{\partial y} \right) = 2\imath \frac{\partial P}{\partial \bar{z}}$$

$$-\imath \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y} = -\imath \left(\frac{\partial Q}{\partial x} - \imath \frac{\partial Q}{\partial y} \right) = -2\imath \frac{\partial Q}{\partial z}$$

Thus, we finally obtain:

$$\int_\gamma \left(Pdz + Qd\bar{z}\right) = \iint_R \left(\left(\imath \frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} \right) + \left(-\imath \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y} \right) \right) dx\, dy$$
$$= \iint_R \left(2\imath \frac{\partial P}{\partial \bar{z}} - 2\imath \frac{\partial Q}{\partial z} \right) \frac{\imath}{2}\, dz\, d\bar{z} = \iint_R \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial \bar{z}} \right) dz\, d\bar{z}.$$

□

To end this chapter, we state the Riemann Mapping Theorem (1851), a very important result from which from we will depart to develop the univalent function theory.

Theorem 2.9. (Riemann Mapping Theorem). *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then, there is a unique function f which maps D conformally onto the unit disk and has the properties $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

Chapter 3

Univalent functions theory

In this chapter, we will introduce the univalent functions, the type of functions to which the Bieberbach Conjecture applies, and show some relevant related results.

Definition 3.1. *An holomorphic function f is called* ***univalent*** *in a domain $D \subset \mathbb{C}$ if it is injective.*

The function f is locally univalent at a point $z_0 \in D$ if it is univalent in some neighbourhood of z_0.

For analytic functions, the local univalence is equivalent to the condition $f'(z_0) \neq 0$, as a univalent mapping is a conformal homeomorphism.

3.1 The class S

By the Riemman Mapping theorem (Theorem 2.9), we know that to study the univalent functions in simply connected domains, it is enough to look at this functions at the unit disk. Moreover, they are unique if they satisfy $f(\zeta) = 0$ and $f'(\zeta) > 0$, for any $\zeta \in \mathbb{D}$.

We shall pay special attention to the **class** S of univalent and analytic functions $f : \mathbb{D} \to \mathbb{C}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. A function $f \in S$ has a Taylor series expansion of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + ..., \quad |z| < 1$$

This normalization is made to simplify the functions in S by eliminating irrelevant constants. If f is any univalent analytic function on $\mathbb{D}$, then $g(z) = \dfrac{f(z) - f(0)}{f'(0)}$ is in S.

Geometrically speaking, studying g rather than f corresponds to first translating the image domain by the vector $f(0)$, dilating by the factor $|f'(0)|$, and rotating through the angle $\arg(f'(0))$, which is reversible.

The class S is preserved under a number of elementary transformations, as:

1. *Conjugation*: If $f \in S$, and $g(z) = \overline{f(\bar{z})} = z + \bar{a}_2 z^2 + \bar{a}_3 z^3 + ...$, then $g \in S$.

2. *Rotation*: If $f \in S$, and $g(z) = e^{-\imath\theta} f(e^{\imath\theta} z)$, then $g \in S$.

3. *Disk automorphism*: If $f : \mathbb{D} \to \mathbb{D}$ is univalent, and
$$g(z) = \frac{f\left(\dfrac{z+a}{1+\bar{a}z}\right) - f(a)}{(1-|a|^2)f'(a)}, \quad |a| < 1$$
then $g \in S$.

4. *Square-root transformation*: If $f \in S$, and $g(z) = \sqrt{f(z^2)}$, then $g \in S$.

   As this one cannot be checked so directly, let's prove it:

   Let $f(z^2) = z^2(1 + a_2 z^2 + a_3 z^4 + \cdots)$.
   First, we observe that, since $f(z^2) = 0$ is and only if $z = 0$, we can define a single-valued branch of the square root, so that:
$$g(z) = \sqrt{f(z^2)}$$
   This is equivalent to write: $g(z)^2 = f(z^2)$. Notice that as $f(z^2)$ doesn't have coefficients of odd degree, $g(z)$ cannot have coefficients with even degree. Therefore, g is an odd function, so $g(z) = -g(z)$.
   Let's check the univalence: if $g(z_1) = g(z_2)$, then $f(z_1^2) = f(z_2^2)$. As f is univalent, $z_1^2 = z_2^2$ and $z_1 = \pm z_2$. But, if $z_1 = -z_2$, we would have that $g(z_1) = g(z_2) = -g(z_1)$, which implies that $g(z_1) = 0$, and $z_1 = 0$. In either case, we obtain that $z_1 = z_2$, so g is univalent. Finally, we see that $g(0) = 0$ and $g'(0) = 1$, so $g \in S$.

The leading example of a function of class S is the **Koebe function**:
$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + ...$$

This function will play a significant role in the theory of univalent functions, as $k(z)$ is the extremal function for multiple properties and results for functions in S. It can be written this way as well:
$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$
which helps us to describe in more detail the mapping. We see that $k(z)$ is a composition of the following mappings:

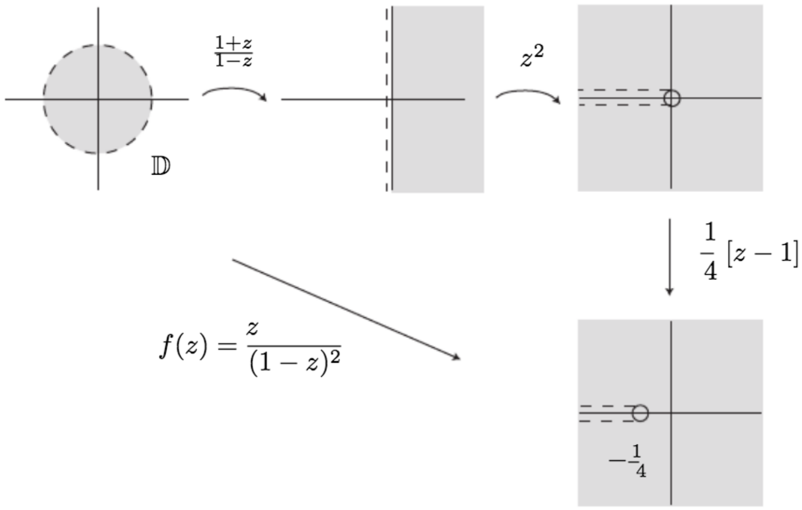

Figure 3.1: The image of the unit disk under the Koebe function

- $t(z) = \dfrac{1+z}{1-z}$, which maps $\mathbb{D}$ to the positive half plane $\{\Re w > 0\}$.
- $s(z) = t(z)^2$, which maps $\{\Re w > 0\}$ to the entire plane, minus the nonpositive real axis $\mathbb{C} \setminus (-\infty, 0]$.
- $w(z) = \dfrac{1}{4}[s(z) - 1]$, which is a translation of factor 1 to the left, and then, a dilation with factor $\frac{1}{4}$.

Thus, the Koebe function maps D onto $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

A little more generally, for any $\theta \in [0, 2\pi)$ we can consider the rotations of the Koebe function:

$$k_\theta(z) = \frac{z}{(1 - e^{\imath\theta} z)^2},$$

that maps $\mathbb{D}$ to the complex numbers with the half-line $\{-re^{\imath\imath\theta} : r \geq \frac{1}{4}\}$ removed. Thus, we see that they also play the role of extremal functions.

Ludwig Bieberbach conjectured that $k(z)$ was at the other extreme from the identity function, that is, among all functions in S, the Koebe function had the largest possible coefficients.

The fact that we can affirm there exists a bound for each coefficient of a function f in S , and that it exists a function whose coefficients attain that bound, is due to the fact that the class S is a compact subspace of the space of all analytic functions of $\mathbb{D}$.

Let's explain the compactness of S in a little more depth.

The property that S is closed is consequence of the Hurwitz theorem.

Proposition 3.2. *Let $\{f_n\}$, $f_n : \mathbb{D} \to \mathbb{C}$ be a sequence of functions in S. If $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function f, then $f \in S$.*

Proof. First of all, we see that if $\{f_n\} \in S$, then f also fulfills the normalization, as $\lim_{n\to\infty} f_n(0) = 0 = f(0)$ and $\lim_{n\to\infty} f_n'(0) = 1 = f'(0)$. So, it remains to be seen that f is univalent.

Let $z_0 \in \mathbb{D}$ be an arbitrary point. Let's see that $f(z) \neq f(z_0)$ for any other $z \in \mathbb{D}$, $z \neq z_0$. Let's consider the domain: $\mathbb{D} \setminus \{z_0\}$ We define the functions:

$$g_n(z) = f_n(z) - f_n(z_0)$$

Notice that, in this domain, g_n have no zeros. Moreover, $g_n(z)$ converges to $g(z) = f(z) - f(z_0)$. We know, by another consequence of the Hurwitz's theorem, that either $g(z) \neq 0$ for any $z \in \mathbb{D}\setminus\{z_0\}$, which means that $f(z) \neq f(z_0)$, for all $z \in \mathbb{D}$, $z \neq z_0$; or $g \equiv 0$, which implies that f is constant, impossible since $f'(0) = 1$. Therefore, we conclude that f is univalent, so $f \in S$. □

Finally, it remains to be seen that S is bounded. This is a consequence of the Growth theorem, a result followed from Bieberbach's theorem (see page 23).

Closely related to S is the **class** Σ of functions:

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + ...$$

analytic and univalent in $\Delta = \{z : |z| > 1\}$, except for a simple pole at infinity $g(\infty) = \infty$, with residue 1.

Let $E = \mathbb{C} \setminus g(\Delta)$ denote the complement of the image domain of a function $g \in \Sigma$. Then, g maps Δ onto the complement of the compact connected set E.

Sometimes, it is convenient to distinguish some subclasses of Σ, as:

- Σ' = $\{g \in \Sigma : g(z) \neq 0 \text{ in } \Delta\}$. There is a one-to-one correspondence between S and Σ': for each function $f \in S$, $f(\frac{1}{z})^{-1} \in \Sigma'$. This transformation is called inversion.
- Σ_0 = $\{g \in \Sigma : b_0 = 0\}$.
- $\tilde{\Sigma}$ = $\{g \in \Sigma : |E| = 0\}$. This functions are called full-mappings.

3.2 The Bieberbach theorem. Consequences

A significant theorem of univalent function theory is the Bierberbach theorem, first proposed in 1916 by the German mathematician Ludwig Bierberbach. In it, he estimates the second coefficient a_2 of a function f of class S. This theorem is the one that gave rise to Bieberbach's conjecture, result that caused a considerable development in this area, due in many occasions, to the attempts to reach it.

Theorem 3.3. (Bieberbach's Theorem). *If $f \in S$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.*

To prove the theorem, we need first another fundamental theorem, the Area theorem, discovered by Gronwall in 1914.

Theorem 3.4. (Area Theorem) *If $g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n} \in \Sigma$, then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

with equality if and only if $g \in \tilde{\Sigma}$.

Proof. Let $E = \mathbb{C} \setminus g(\Delta)$, and let C_r be the image under f of the circle $|z| = r$.
We want to compute the area of E in terms of the coefficients. As E can be quite irregular, let's approximate it from outside by domains $E_r = \mathbb{C} \setminus \{g(z); |z| > r\}$. Since g is univalent, C_r is a smooth simple closed curve which encloses a domain E_r. Taking $w = x + iy$, we have:

$$A(E_r) = \iint_{E_r} dx\,dy = \frac{1}{2\imath} \iint_{E_r} d\bar{w}\,dw,$$

where $dxdy = \frac{1}{2\imath} d\bar{w}dw$.
Now, using Green's theorem (Theorem 2.8), taking $P = \bar{w}$ and $Q = 0$, we obtain:

$$A(E_r) = \frac{1}{2i} \int_{C_r} \bar{w}\,dw = \frac{1}{2i} \int_{|z|=r} \overline{g(z)} g'(z)\,dz = \frac{1}{2} \int_0^{2\pi} re^{\imath\theta} g'(re^{\imath\theta}) \overline{g(re^{\imath\theta})}\,d\theta.$$

Writing g and g' as their Taylor series, we get:

$$\begin{aligned}
A(E_r) &= \frac{1}{2} \int_0^{2\pi} \left(re^{\imath\theta} - \sum_{n=1}^{\infty} nb_n r^{-n} e^{-\imath n\theta} \right) \left(re^{-\imath\theta} + \sum_{m=0}^{\infty} m\overline{b_m} r^{-m} e^{\imath m\theta} \right) d\theta \\
&= \pi r^2 + \frac{1}{2} \int_0^{2\pi} \left(\sum_{m=0}^{\infty} m\overline{b_m} r^{-(m-1)} e^{\imath(m+1)\theta} \right) d\theta - \frac{1}{2} \int_0^{2\pi} \left(\sum_{n=1}^{\infty} nb_n r^{-(n-1)} e^{-\imath(n+1)\theta} \right) d\theta \\
&\quad - \frac{1}{2} \int_0^{2\pi} \left(\sum_{n=1}^{\infty} nb_n r^{-n} e^{-\imath n\theta} \sum_{m=0}^{\infty} m\overline{b_m} r^{-m} e^{\imath m\theta} \right) d\theta \\
&= \pi r^2 + \frac{1}{2} \sum_{m=0}^{\infty} m\overline{b_m} r^{-(m-1)} \int_0^{2\pi} e^{\imath(m+1)\theta}\,d\theta - \frac{1}{2} \sum_{n=1}^{\infty} nb_n r^{-(n-1)} \int_0^{2\pi} e^{-\imath(n+1)\theta}\,d\theta \\
&\quad - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} nb_n m\overline{b_m} \int_0^{2\pi} e^{\imath(m-n)\theta}\,d\theta.
\end{aligned}$$

Since

$$\int_0^{2\pi} e^{\imath k\theta} d\theta = \begin{cases} 2\pi & \text{if } k = 0 \\ 0 & \text{if } k \neq 0, \end{cases} \tag{3.1}$$

we obtain that:

$$A(E_r) = \pi r^2 + 0 + 0 - \pi \sum_{n=1}^{\infty} n|bn|^2 r^{2n} = \pi\left(r^2 - \sum_{n=1}^{\infty} n|bn|^2 r^{2n}\right).$$

As $A(E_r) \geq 0$, we deduce that, for every $m \geq 1$:

$$\sum_{n=1}^{m} n|bn|^2 r^{2n} \leq r^2.$$

Finally, letting $r \to 1$ and $m \to \infty$, we obtain:

$$\sum_{n=1}^{\infty} n|bn|^2 \leq 1.$$

$\square$

This theorem gives us an immediate corollary, useful to prove the Bieberbach theorem:

Corollary 3.5. *If $g \in \Sigma$, then $|b_1| \leq 1$, with equality if and only if g has the form:*

$$g(z) = z + b_0 + \frac{b_1}{z}, \quad |b_1| = 1.$$

Proof. If $g \in \Sigma$, by the Area Theorem, we have:

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

It follows that, for every $n \geq 1$, $n|b_n|^2 \leq 1$. In particular, for $n = 1$, $|b_1|^2 \leq 1$, so $|b_1| \leq 1$.

On the other hand, if $|b_1| = 1$, then necessarily $b_n = 0$ for all $n > 1$. Therefore, g has the form:

$$g(z) = z + b_0 + \frac{b_1}{z}, \quad \text{with} \quad |b_1| = 1.$$

The converse is direct by definition of g. $\square$

Shown this results, let's prove the Bieberbach Theorem.

Proof. Given $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we construct the following auxiliary functions:

$$g(z) = \sqrt{f(z^2)}$$
$$h(z) = \frac{1}{g(\frac{1}{z})}.$$

The first function $g(z)$ is a square root transformation of f. It has been proved that $g \in S$ (Property 4). Let's write its coefficients in terms of the coefficients of f:

Let $g(z) = z + b_3 z^3 + b_5 z^5 + \cdots$. We know that $g(z)^2 = f(z^2)$. Grouping the coefficients by the power of z, we obtain:

$$\begin{aligned} n = 4: &\quad a_2 = 2b_3 \to b_3 = \frac{a_3}{2} \\ n = 6: &\quad a_3 = 2b_5 + b_3^2 \to b_5 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{4}\right) \\ &\dots \end{aligned}$$

Thus, we re-write $g(z)$ as:

$$g(z) = z + \frac{a_3}{2} z^3 + \frac{1}{2}\left(a_3 - \frac{a_2^2}{4}\right) z^5 + \cdots$$

Now, let's see when is $h(z)$ univalent and how is its Laurent series.

We compute: $g\left(\frac{1}{z}\right) = \frac{1}{z} + \frac{b_3}{z^3} + \frac{b_5}{z^5} + \cdots$. Observe that, as $g \in S$, $g(\frac{1}{z})$ is analytic and univalent for $|z| > 1$. Consequently, $h(z) = \frac{1}{g(\frac{1}{z})}$ is univalent and analytic in Δ and, moreover, $h(\infty) = \infty$. Therefore, $h \in \Sigma$, and has the form:

$$h(z) = z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

Let's compute the coefficients in terms of the a_n.

Since $h(z) = \frac{1}{g(\frac{1}{z})}$, we have $g(\frac{1}{z})h(z) = 1$. Writing Taylor's development of the functions,

$$1 = \left(\frac{1}{z} + \frac{b_3}{z^3} + \frac{b_5}{z^5} + \cdots\right)\left(z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right).$$

Now, multiplying and grouping the coefficients by the power of z, and equating to zero, we get:

$$\begin{aligned} n = -1: &\quad c_0 = 0 \\ n = -2: &\quad c_1 + b_3 = 0 \to c_1 = -b_3 = -\frac{a_2}{2} \\ n = -3: &\quad c_2 + b_3 c_0 = 0 \to c_2 = 0 \\ &\dots \end{aligned}$$

Finally, we obtain:

$$h(z) = z - \frac{a_2}{2z} + \cdots$$

As $h \in \Sigma$, by Corollary 3.5, we have that $|\frac{a_2}{2}| \leq 1$, that is $|a_2| \leq 2$, with equality if and only if

$$h(z) = z + \frac{b}{z}, \quad |b| = 1.$$

It only remains to be seen that this is equivalent to f being a rotation of the Koebe function.
Notice that :

$$h(z) = \frac{1}{g(\frac{1}{z})} \quad \text{is equivalent to} \quad g(z) = \frac{1}{h(\frac{1}{z})}, \quad z \in \mathbb{D}.$$

So, we have:

$$h(z) = z + \frac{b}{z} \quad \text{if and only if} \quad h(\frac{1}{z}) = \frac{1}{z} + bz.$$

Thus,

$$g(z) = \frac{1}{\frac{1}{z} + bz}, \quad \text{that is} \quad g(z) = \frac{z}{1 + bz^2}.$$

As $g(z) = \sqrt{f(z^2)}$, then:

$$g(z)^2 = \frac{z^2}{(1 + bz^2)^2} = f(z^2).$$

With the change of variable $t = z^2$, we obtain:

$$f(t) = \frac{t}{(1 + bt)^2}, \quad |b| = 1, \quad t \in \mathbb{D}$$

which is a rotation of the Koebe function. □

3.2.1 The Koebe One-Quarter theorem

As a first application of Bieberbach's theorem, we have the Koebe One-Quarter Theorem, which afirms that the range of each function $f \in S$ contains some disk $|z| < r$ centered at the origin. Already in 1907, Koebe conjectured that $r \leq \frac{1}{4}$, with the maximum attained by the Koebe function. Later in 1916, Bieberbach proved the theorem, showing that this constant cannot be improved.

Theorem 3.6. (Koebe's One-Quarter Theorem). *The range of every function f of class S contains the disk $D(0, \frac{1}{4})$.*

Proof. Let $w \in \mathbb{C}$ a point, let $f \in S$ be a function that omits the value w, and consider the function:

$$g(z) = \frac{wf(z)}{w - f(z)}.$$

Let's see that g is analytic and univalent in $\mathbb{D}$. We can write:

$$g(z) = (h \circ f)(z), \quad \text{where} \quad h(z) = \frac{wz}{w - z}.$$

So, as f is univalent and analytic in $\mathbb{D}$, it only needs to be checked the univalence of h. Suppose $h(z_1) = h(z_2)$. Then,

$$\frac{wz_1}{w - z_1} = \frac{wz_2}{w - z_2}.$$

This implies

$$wz_1 - z_1 z_2 = z_2 w - z_1 z_2 \quad \text{hence,} \quad z_1 = z_2.$$

On the other hand, h is analytic for every $z \neq w$. Therefore, g is analytic and univalent in $\mathbb{D}$, for being composition of analyitic and univalent functions in $\mathbb{D}$. We now compute the derivatives of g, in order to write its development at the origin:

$$\begin{aligned}
g(z) &= \frac{wf(z)}{w - f(z)} && \to g(0) = 0 \\
g'(z) &= \frac{w^2 f'(z)}{(w - f(z))^2} && \to g'(0) = 1 \\
g''(z) &= \frac{w^2(f''(z)(w - f(z)) + 2f'(z)^2)}{(w - f(z))^3} && \to g''(0) = f''(0) + \frac{2}{w}
\end{aligned}$$

Then, we obtain:

$$g(z) = z + \left(a_2 + \frac{1}{w}\right) z^2 + \cdots$$

Observe that $g(0) = 0$ and $g'(0) = 1$, so $g \in S$. By the Bierberbach theorem,

$$\left| a_2 + \frac{1}{w} \right| \leq 2.$$

Finally, using the triangle inequality:

$$\left| \frac{1}{w} \right| \leq \left| \frac{1}{w} + a_2 \right| + |a_2| \leq 2 + 2 = 4, \quad \text{and therefore} \quad |w| \geq \frac{1}{4}.$$

The equality $|w| = \frac{1}{4}$ holds if and only if equality holds when applying Bieberbach's theorem, and it happens if and only if g is a rotation of the Koebe function. Hence, by definition of g, if and only if f is a rotation of the Koebe function. $\square$

3.2.2 Theorems of distortion and growth

Now, we shall study the most relevant consequences that result from Bieberbach's inequality.

This theorem has further implications in the geometric theory of conformal mapping. One important consequence is the Koebe distortion theorem, which provides sharp upper and lower bounds for $|f'(z)|$ as f ranges over the class S. But, before we enunciate it, let's state a theorem which gives us a estimate that will be used in the distortion theorem and other related results.

Theorem 3.7. *For each $f \in S$,*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}, \quad |z| = r < 1. \tag{3.2}$$

Proof. Let $z_0 \in \mathbb{D}$, and consider the authomorphism of $\mathbb{D}$:

$$\varphi_{z_0}(w) = \frac{w + z_0}{1 + w\bar{z_0}}, \quad \varphi_{z_0}(0) = z_0.$$

Since $\varphi_{z_0}(w)$ is univalent, the composition $(f \circ \varphi_{z_0})(w)$, which first transfers the origin to $z_0 \in \mathbb{D}$, and then to $f(z_0)$, is also univalent. Let's normalize it so that it belongs to S. First, we compute the first two coefficients of the Taylor development of $(f \circ \varphi_{z_0})$:
First, $(f \circ \varphi_{z_0})(0) = f(z_0)$.
Second, since $(f \circ \varphi_{z_0})'(w) = f'(\varphi_{z_0}(w))\varphi'_{z_0}(w) = f'(\varphi_{z_0}(w))\dfrac{1-|z_0|^2}{(1+w\bar{z_0})^2}$, we get $(f \circ \varphi_{z_0})'(0) = f'(z_0)(1-|z_0|^2)$.
Therefore,

$$(f \circ \varphi_{z_0})(w) = f(z_0) + f'(z_0)(1-|z_0|^2)w + \cdots .$$

From this, we can see that the function h:

$$h(w) = \frac{f(\varphi_{z_0}(w)) - f(z_0)}{f'(z_0)(1-|z_0|^2)} \tag{3.3}$$

fulfills the normalization, so it belongs to S (proof of Property 3, page 13). Now, we compute the next term of Taylor's development of h at the origin. We see that:

$$h''(w) = \frac{f''(\varphi_{z_0}(w))\varphi'_{z_0}(w)^2 + f'(\varphi_{z_0}(w))\varphi''_{z_0}(w)}{f'(z_0)(1-|z_0|^2)}, \quad \text{and so}$$

$$h''(0) = \frac{f''(\varphi_{z_0}(0)\varphi'_{z_0}(0)^2 + f'(\varphi_{z_0}(0))\varphi''_{z_0}(0)}{f'(z_0)(1-|z_0|^2)} = (1-|z_0|^2)\frac{f''(z_0)}{f'(z_0)} - 2\bar{z_0}.$$

Therefore,

$$h(w) = z + \frac{1}{2}\left[(1-|z_0|^2)\frac{f''(z_0)}{f'(z_0)} - 2\bar{z_0}\right]z^2 + \cdots .$$

Since $h \in S$, by Bieberbach's theorem,

$$\left|\frac{1}{2}\left[(1-|z_0|^2)\frac{f''(z_0)}{f'(z_0)} - 2\bar{z}_0\right]\right| \leq 2.$$

Replacing z_0 by z, multiplying by 2 and dividing by $(1-|z|^2)$, we get:

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\bar{z}}{(1-|z|^2)}\right| \leq \frac{4}{(1-|z|^2)}.$$

Finally, multiplying by $|z|$, we obtain the inequality we are looking for:

$$\left|z\frac{f''(z)}{f'(z)} - \frac{2|z|^2}{(1-|z|^2)}\right| \leq \frac{4|z|}{(1-|z|^2)}.$$

□

Once we've introduced this inequality, let's state the distortion theorem:

Theorem 3.8. (Distortion Theorem). *For each $f \in S$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1.$$

For any $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Proof. With the change of variable $z = re^{\imath\theta}$ in the inequality of Theorem 3.4, we have:

$$\left|e^{\imath\theta}\frac{f''(re^{\imath\theta})}{f'(re^{\imath\theta})} - \frac{2r}{(1-r^2)}\right| \leq \frac{4}{(1-r^2)}.$$

On the other hand, observe that:

$$\left|e^{\imath\theta}\frac{f''(re^{\imath\theta})}{f'(re^{\imath\theta})} - \frac{2r}{(1-r^2)}\right| = \left|\frac{\partial}{\partial r}\log[(1-r^2)f'(re^{\imath\theta})]\right|.$$

We prove it:

$$\frac{\partial}{\partial r}\log[(1-r^2)f'(re^{\imath\theta})] = \frac{-2rf'(re^{\imath\theta}) + (1-r^2)f''(re^{\imath\theta})e^{\imath\theta}}{(1-r^2)f'(re^{\imath\theta})} = \frac{-2r}{1-r^2} + \frac{f''(re^{\imath\theta})e^{\imath\theta}}{f'(re^{\imath\theta})}.$$

Since $f'(0) = 1$, we can integrate along the ray from 0 to z, and get:

$$\begin{aligned}\left|\log[(1-r^2)f'(re^{\imath\theta})]\right| &= \left|\int_0^r \frac{\partial}{\partial\rho}\log[(1-\rho^2)f'(\rho e^{\imath\theta})]d\rho\right| \\ &\leq \int_0^r \left|\frac{\partial}{\partial\rho}\log[(1-\rho^2)f'(\rho e^{\imath\theta})]\right| d\rho \\ &= \int_0^r \left|e^{\imath\theta}\frac{f''(\rho e^{\imath\theta})}{f'(\rho e^{\imath\theta})} - \frac{2\rho}{(1-\rho^2)}\right| \\ &\leq \int_0^r \frac{4}{1-\rho^2}d\rho = 2\log\left(\frac{1+r}{1-r}\right).\end{aligned}$$

using the inequality (2) in the last step.

From this, it follows that:

$$-2\log\left(\frac{1+|z|}{1-|z|}\right) \le \log[(1-|z|^2)|f'(z)|] \le 2\log\left(\frac{1+|z|}{1-|z|}\right).$$

Exponentiating:

$$\left(\frac{1-|z|}{1+|z|}\right)^2 \le (1-|z|^2)|f'(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^2.$$

Finally, diving by $(1-|z|^2)$, we obtain:

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}.$$

□

Using the distortion theorem, it is possible to find, as well, sharp lower and upper bounds for $|f(z)|$. This result is called the Growth theorem, and it is necessary, among other things, to prove the compactness of the class S, as mentioned before.

Theorem 3.9. (Growth Theorem). *For each $f \in S$,*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Proof. Let's start with the upper bound. Let $f \in S$, and consider $z = re^{\imath\theta}$, with $0 < r < 1$. By the fundamental theorem of calculus:

$$f(z) = \int_0^z f'(z)dz = \int_0^r f'(\rho e^{\imath\theta})e^{\imath\theta}d\rho.$$

Then, applying the distortion theorem, we obtain:

$$|f(z)| = \left|\int_0^r f'(\rho e^{\imath\theta})e^{\imath\theta}d\rho\right| \le \int_0^r |f'(\rho e^{\imath\theta})||e^{\imath\theta}|d\rho \le \int_0^r \frac{1+\rho}{(1-\rho)^3}d\rho = \frac{r}{(1-r)^2}.$$

Now, we need to prove the lower estimate.

Observe that, if $|f(z)| \ge \frac{1}{4}$, the bound holds trivially, since $\frac{r}{(1+r)^2} < \frac{1}{4}$ for $0 < r < 1$.

Now, if $|f(z)| \leq \dfrac{1}{4}$, the Koebe One-quarter theorem implies that the radial segment from 0 to $f(z)$ lies entirely in the range of f.
We parametrize this segment as:

$$\gamma : [0,1] \to \mathbb{C}, \quad \gamma(t) = tf(z).$$

Let C be the preimage of γ:

$$C : [0,1] \to \mathbb{D}, \quad C(t) = f^{-1}(\gamma(t)).$$

Since C is a simple arc from 0 to z, we can write:

$$f(z) = \int_C f'(\zeta)d\zeta$$

If we do the change of variable $\zeta = C(t)$, we see that:

$$f'(\zeta)d\zeta = f'(C(t))C'(t)dt.$$

Then,

$$C'(t) = (f^{-1})'(\gamma(t))\gamma'(t) = \frac{1}{f'(C(t))}f(z).$$

Therefore, we obtain:

$$\int_C f'(\zeta)d\zeta = \int_0^1 f'(C(t))\frac{1}{f'(C(t))}f(z)dt = \int_0^1 f(z)dt = f(z).$$

This means that $f'(\zeta)d\zeta$ has constant sign along C. Therefore, the distortion theorem gives:

$$|f(z)| = \int_C |f'(\zeta)||d\zeta| \geq \int_0^r \frac{1-\rho}{(1+\rho)^3}d\rho = \frac{r}{(1+r)^2}.$$

Finally, let's discuss the equality. Notice that equality in either of the bounds implies equality when applying the distortion theorem in both cases. And as we've proved before, equality holds in the distortion theorem if and only if f is a suitable rotation of the Koebe function. □

To end this section, let's show a theorem that mixes up both distortion and growth theorems. It can be seen as well as a Growth theorem for the logarithm's derivative.

Theorem 3.10. *For each $f \in S$,*

$$\frac{1-r}{1+r} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+r}{1-r}, \quad |z| = r < 1.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Proof. Although the inequality mixes both theorems, it cannot be established by a direct combination of them. So, consider the function $h \in S$, the disk automorphism of f given in (3.3). By the growth theorem:

$$\frac{|w|}{(1+|w|)^2} \leq |h(w)| \leq \frac{|w|}{(1-|w|)^2}, \quad w \in \mathbb{D}.$$

In partircular, taking $w = -z_0$,

$$\frac{|z_0|}{(1+|z_0|)^2} \leq |h(-z_0)| \leq \frac{|z_0|}{(1-|z_0|)^2}.$$

As

$$h(-z_0) = \frac{f(\varphi_{z_0}(-z_0)) - f(z_0)}{(1-|z_0|^2)f'(z_0)} = \frac{-f(z_0)}{(1-|z_0|^2)f'(z_0)},$$

we have

$$|h(-z_0)| = \frac{|f(z_0)|}{(1-|z_0|^2)|f'(z_0)|}.$$

Hence:

$$\frac{(1-|z_0|^2)|z_0|}{(1+|z_0|)^2} \leq \left|\frac{f(z_0)}{f'(z_0)}\right| \leq \frac{(1-|z_0|^2)|z_0|}{(1-|z_0|)^2}.$$

Finally, simplifying and dividing by $|z_0|$, we obtain:

$$\frac{1-|z_0|}{1+|z_0|} \leq \left|\frac{f(z_0)}{z_0 f'(z_0)}\right| \leq \frac{1+|z_0|}{1-|z_0|}.$$

Since z_0 is arbitrary, the inequality is proved.
Now, let's see the case of equality:

Observe that if $f(z) = k(z) = \dfrac{z}{(1-z)^2}$ the Koebe function, then:

$$\frac{zk'(z)}{k(z)} = \frac{z\dfrac{1+z}{(1-z)^3}}{\dfrac{z}{(1-z)^2}} = \frac{1+z}{1-z}.$$

Thus, $k(z)$ offers cases of equality. Now, it remains to be seen that rotations of the Koebe function provide the only cases of equality.

Suppose first that the lower bound is attained for some $f \in S$ at some point $\zeta \in \mathbb{D}$. Then, the function $h = f \circ \varphi_\zeta$ satisfies:

$$|h(-\zeta)| = \frac{|\zeta|}{(1-|\zeta|)^2},$$

and, by the growth theorem, we infer that

$$h(z) = \frac{z}{\left(1 + \frac{\bar{\zeta}}{|\zeta|}z\right)^2}.$$

Now, consider:

$$z = \varphi_\zeta(w) = \frac{w - \zeta}{1 - \bar{\zeta}w}, \quad \text{or} \quad w = \varphi_\zeta^{-1}(z) = \frac{z + \zeta}{1 + \bar{\zeta}z},$$

and let's define:

$$G(w) = h\left(\frac{w - \zeta}{1 - \bar{\zeta}w}\right).$$

In order to see the relation between G and f, we compute:

$$\begin{aligned} G(w) - G(0) = h\left(\frac{w-\zeta}{1-\bar{\zeta}w}\right) - h(-\zeta) &= \frac{f\left(\varphi_\zeta\left(\frac{w-\zeta}{1-\bar{\zeta}w}\right)\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} - \frac{-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} \\ &= \frac{f\left(\varphi_\zeta\left(\frac{w-\zeta}{1-\bar{\zeta}w}\right)\right)}{(1-|\zeta|^2)f'(\zeta)} = \frac{f(w)}{(1-|\zeta|^2)f'(\zeta)}. \end{aligned}$$

We notice that $f(w)$ is a constant multiple of $G(w) - G(0)$. Moreover, we have that:

$$\begin{aligned} G(w) &= \frac{\frac{w-\zeta}{1-\bar{\zeta}w}}{\left(1 + \frac{\bar{\zeta}}{|\zeta|}\frac{w-\zeta}{1-\bar{\zeta}w}\right)^2} = \frac{(w-\zeta)(1-\zeta w)}{\left(1 - \bar{\zeta}w + \frac{\bar{\zeta}w}{|\zeta|} - |\zeta|\right)^2} \\ &= \frac{w - \zeta - \bar{\zeta}w^2 + |\zeta|^2 w}{(1-|\zeta|)^2\left(1 + \frac{\bar{\zeta}}{|\zeta|}w\right)^2}. \\ G(0) &= \frac{-\zeta}{(1-|\zeta|)^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} G(w) - G(0) &= \frac{1}{(1-|\zeta|)^2}\left(\frac{w - \zeta - \bar{\zeta}w^2 + |\zeta|^2 w}{(1 + \frac{\bar{\zeta}}{|\zeta|}w)^2} + \zeta\right) \\ &= \frac{w(1 + |\zeta|^2 - 2|\zeta|)}{(1-|\zeta|)^2\left(1 + \frac{\bar{\zeta}}{|\zeta|}w\right)^2} = \frac{(1+|\zeta|)^2}{(1-|\zeta|)^2}\frac{w}{\left(1 + \frac{\bar{\zeta}}{|\zeta|}w\right)^2}. \end{aligned}$$

As the lower bound of the inequality was attained for f, then:

$$\frac{(1-|\zeta|)}{(1+|\zeta|)} = \frac{\zeta f'(\zeta)}{f(\zeta)} e^{\imath\theta}, \quad \theta \in [0, 2\pi).$$

Suppose, without loss of generality, that $e^{\imath\theta} = 1$. Then,

$$G(w) - G(0) = \frac{f(\zeta)(1+|\zeta|)}{\zeta f'(\zeta)(1-|\zeta|)} \frac{w}{\left(1+\frac{\bar{\zeta}}{|\zeta|}w\right)^2}.$$

Equating both expressions, we have:

$$\frac{f(w)}{(1-|\zeta|^2)f'(\zeta)} = \frac{f(\zeta)(1+|\zeta|)}{\zeta f'(\zeta)(1-|\zeta|)} \frac{w}{\left(1+\frac{\bar{\zeta}}{|\zeta|}w\right)^2}.$$

Hence,

$$f(w) = \frac{f(\zeta)(1+|\zeta|)^2}{\zeta} \frac{w}{\left(1+\frac{\bar{\zeta}}{|\zeta|}w\right)^2} = C_\zeta \frac{w}{\left(1+\frac{\bar{\zeta}}{|\zeta|}w\right)^2}.$$

Thus, f is a rotation of the Koebe function. With the same argument for the upper bound, we conclude that equality in both bounds occurs if and only if f is a rotation of the Koebe function. $\square$

Chapter 4

The Bieberbach conjecture

In 1916, after proving the inequality $|a_2| \leq 2$ for $f \in S$, and observing that the Koebe function played a special role in multiple results related to the class S as being the extremal function, Bieberbach though that $k(z)$ could maximize as well $|a_n|$, for every n. Thus, he came up with the well-known Bieberbach conjecture:

Theorem 4.1. (Bieberbach's Conjecture). *The coefficients of each function $f \in S$ satisfy:*

$$|a_n| \leq n, \quad \textit{for} \quad n = 2, 3...$$

Strict inequality holds for every n unless f is a rotation of the Koebe function.

This problem, however, is more difficult to solve than it may seem. To give a first hint that this is so, let's try to find a bound for the coefficients using basic knowledge of complex analysis.

Assume the $f \in S$. By Cauchy's inequalities,

$$|a_n| = \left| \frac{f^n(0)}{n!} \right| \leq \frac{\max\limits_{|z|=r} |f(z)|}{r^n}.$$

Now, using the Growth theorem, we have:

$$\max_{|z|=r} |f(z)| = \frac{r}{(1-r)^2}.$$

Therefore,

$$|a_n| \leq \frac{\frac{r}{(1-r)^2}}{r^n} = \frac{1}{r^{n-1}(1-r)^2}.$$

In order to optimize this bound for $r < 1$, let's maximize its denominator.
Let:

$$g(r) = r^{n-1}(1-r)^2.$$

Then,

$$g'(r) = (n-1)r^{n-2}(1-r)^2 - r^{n-1}2(1-r) = (1-r)r^{n-2}((n-1)(1-r)-2r).$$

So,

$$g'(r) = 0 \quad \text{if and only if} \quad r = 0, \quad r = 1, \quad \text{or} \quad r = \frac{n-1}{n+1}.$$

Observe that $g(0) = g(1) = 0$, so the maximum is at $r = \dfrac{n-1}{n+1}$. Therefore;

$$\begin{aligned} |a_n| \le \frac{1}{r^{n-1}(1-r)^2} &= \frac{1}{\left(\dfrac{n-1}{n+1}\right)^{n-1}\left(1-\dfrac{n-1}{n+1}\right)^2} = \frac{1}{4}\frac{(n+1)^{n+1}}{(n-1)^{n-1}} \\ &= \frac{1}{4}(n+1)^2\left(\frac{n+1}{n-1}\right)^{n-1} = \frac{1}{4}(n+1)^2\left(1+\frac{2}{n-1}\right)^{n-1}. \end{aligned}$$

Let us see now that:

$$(n+1)^2\left(1+\frac{2}{n-1}\right)^{n-1} < n^2e^2,$$

or equivalently,

$$\left(\frac{n+1}{n}\right)^2\left(1+\frac{2}{n-1}\right)^{n-1} < e^2.$$

Consider $h(t) = \left(1+\dfrac{1}{t}\right)^2\left(1+\dfrac{2}{t-1}\right)^{n-1}$. Since $\lim_{t\to\infty} h(t) = e^2$, to prove the inequality, it is enough to see that $h(t)$ is an increasing function.

If we apply the logarithm to h, and make the change of variable $x = t-1$, we obtain the function:

$$f(x) = 2\log\left(1+\frac{1}{x+1}\right) + x\log\left(1+\frac{2}{x}\right).$$

Its derivative is:

$$\begin{aligned} f'(x) &= \frac{2}{\left(1+\dfrac{1}{x+1}\right)}\left(-\frac{1}{(x+1)^2}\right) + \log\left(1+\frac{2}{x}\right) + x\frac{-\dfrac{2}{x^2}}{\left(1+\dfrac{2}{x}\right)} \\ &= \frac{-2}{(x+2)(x+1)} + \log\left(1+\frac{2}{x}\right) - \frac{2}{x+2}. \end{aligned}$$

As $\log(1+t) \ge t$ for $t \in (0, \frac{1}{2})$,

$$f'(x) \ge \frac{-2}{(x+2)(x+1)} + \frac{2}{x} - \frac{2}{x+2} = \frac{2}{x} - \frac{2}{x+1} > 0.$$

Therefore, we deduce that the function f is a increasing function, which implies that $h(n)$ is, too.

With all this, we come to the conclusion that:

$$|a_n| \leq \frac{n^2 e^2}{4}.$$

This is quite far from the estimate we are looking for.

4.1 Partial results

Many attempts have been made to prove this conjecture, in different directions.

It has been proved for some specific n: In 1923, the mattematician Karl Loewner proved that $|a_3| \leq 3$. It was considered one of the most important steps towards the proof of the BC, as the way to prove the general case, through differential equations, came from this demonstration. More than 30 years passed without progress, until in 1955, P.Garabedian and M.Schiffer came up with the proof that $|a_4| \leq 4$. The sixth and fifth coefficient theorems followed in 1968 by Pederson and Ozawa, and in 1972 by Pederson and Schiffer, respectively.

4.1.1 The Littlewood theorem

On the other hand, in 1925 the mathematician John E. Littlewood proved the first result for all $n \geq 2$ that showed that the order of magnitude of the coefficients was the one stipulated by the conjecture.

Theorem 4.2. (Littlewood's Theorem). *The coefficients of a function $f \in S$ satisfy*

$$|a_n| \leq en, \quad \textit{for} \quad n \geq 2.$$

Before we prove it, we shall introduce a lemma, stated by Littlewood, which we will use in the proof:

Lemma 4.3. (Littlewood's integral inequality) *For each function $f \in S$,*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{\imath\theta})| d\theta \leq \frac{r}{1-r}, \quad 0 \leq r < 1.$$

Proof. Given $f \in S$, we consider its square root transformation:

$$h(z) = \sqrt{f(z^2)} = \sum_{n=1}^{\infty} c_n z^n.$$

Applying the growth theorem to f, we have:

$$|f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

Since $h(z)^2 = f(z^2)$, for $|z| = r < 1$:

$$|h(z)|^2 = |f(z^2)| \le \frac{r^2}{(1-r^2)^2}, \quad \text{hence} \quad |h(z)| \le \frac{r}{(1-r^2)}.$$

This means that h maps the disk $|z| < r$ conformally onto a domain D_r, which lies in the disk $|w| < \dfrac{r}{(1-r^2)}$. Let A_r be the area of D_r. Then, we see that A_r is no greater than the area of the disk:

$$A_r \le \pi \frac{r^2}{(1-r^2)^2}.$$

Let's calculate A_r more precisely. Let $w = h(z)$. Then,

$$A_r = \iint_{D_r} dm(w) = \iint_{D_r} |h'(z)|^2 dm(z),$$

where

$$dm(w) = \frac{\imath}{2}(dw \wedge d\bar{w}) = \frac{\imath}{2}(h'(z)dz \wedge \overline{h'(z)}d\bar{z}) = |h'(z)|^2 dm(z).$$

Now, letting $z = re^{\imath\theta}$, we have:

$$\begin{aligned} A_r &= \int_0^{2\pi}\int_0^r \left| \sum_{n=1}^{\infty} nc_n \rho^{n-1} e^{(n-1)\imath\theta} \right|^2 \rho d\rho d\theta = \int_0^{2\pi}\int_0^r \sum_{n=1}^{\infty} n^2 |c_n|^2 \rho^{2n-1} d\rho d\theta \\ &= \frac{1}{2}\sum_{n=1}^{\infty} n|c_n|^2 r^{2n-1} \int_0^{2\pi} d\theta = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n-1}. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} n|c_n|^2 r^{2n} \le \frac{r^2}{(1-r^2)^2}, \quad 0 \le r < 1.$$

If we integrate both parts of the inequality from 0 to r, we have:

$$\int_0^r \sum_{n=1}^{\infty} n|c_n|^2 \rho^{2n-1} d\rho = \sum_{n=1}^{\infty} n|c_n|^2 \int_0^r \rho^{2n-1} d\rho = \sum_{n=1}^{\infty} |c_n|^2 \frac{r^{2n}}{2}$$

$$\int_0^r \frac{\rho^2}{(1-\rho^2)^2} d\rho = \int_1^{1-r^2} -\frac{1}{2u^2} du = \frac{1}{2}\frac{r^2}{1-r^2}, \quad \text{where} \quad u = 1-\rho^2,$$

from which we infer:

$$\sum_{n=1}^{\infty} n|c_n|^2 \frac{r^{2n}}{2n} \le \frac{1}{2}\frac{r^2}{1-r^2}, \quad \text{hence} \quad \sum_{n=1}^{\infty} |c_n|^2 r^{2n} \le \frac{r^2}{1-r^2}.$$

Finally, using the definition of $h(z)$:

$$\sum_{n=1}^{\infty} |c_n|^2 r^{2n} = |h(re^{\imath\theta})|^2 = |f(r^2 e^{2\imath\theta})| = \frac{1}{2\pi}\int_0^{2\pi} |f(r^2 e^{2\imath\theta})| d\theta \le \frac{r^2}{1-r^2},$$

where the last inequality is equivalent to:

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{\imath\theta})| d\theta \le \frac{r}{1-r}.$$

$\square$

After this, we prove Littlewood's theorem:

Proof. We know, by the Cauchy theorem, that:

$$a_n = \frac{1}{2\pi\imath}\int_{|z|=r} \frac{f(z)}{z^{n+1}} dz, \quad r < 1.$$

In absolute value, and changing $z = re^{\imath\theta}$, we have:

$$|a_n| = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(re^{\imath\theta})}{r^n e^{n\imath\theta}} d\theta\right| \le \frac{1}{2\pi r^n}\int_0^{2\pi} |f(re^{\imath\theta})| d\theta.$$

Now, using the previous estimation, we obtain:

$$|a_n| \le \frac{1}{2\pi r^n}\int_0^{2\pi} |f(re^{\imath\theta})| d\theta = \frac{1}{r^n} M_1(r,f) \le \frac{1}{r^n}\frac{r}{1-r} = \frac{1}{(1-r)r^{n-1}}.$$

To find the minimum value of the bound, we maximize the denominator.
Let:

$$g(r) = r^{n-1}(1-r), \quad r \in [0,1].$$

Then,

$$g'(r) = (n-1)r^{n-2}(1-r) - r^{n-1} = r^{n-2}((n-1)(1-r) - r)$$

So, the maximum is attained at $r = 1 - \dfrac{1}{n}$. Thus, if we substitute in g, we obtain:

$$|a_n| \le \frac{1}{(1-r)r^{n-1}} = \frac{1}{\left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n}} = n\left(\frac{n}{n-1}\right)^{n-1} = n\left(1+\frac{1}{n-1}\right)^{n-1} < en.$$

$\square$

This bound was improved over the years. In 1974, A. Baernstein improved Littlewood's integral inequality by showing that k had the largest possible integral mean of every order, i.e, for each $f \in S$ and each real number p:

$$\int_0^{2\pi} |f(re^{\imath\theta})|^p d\theta \leq \int_0^{2\pi} |k(re^{\imath\theta})|^p d\theta.$$

Therefore, if we study the case $p = 1$, we have:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |f(re^{\imath\theta})| d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{re^{\imath\theta}}{(1-re^{\imath\theta})^2}\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{r}{|1-re^{\imath\theta}|^2} d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi} \frac{1}{(1-re^{\imath\theta})(1-re^{-\imath\theta})} d\theta.\end{aligned}$$

Changing $\zeta = e^{\imath\theta}$, and applying the Residue Theorem, we obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{\imath\theta})| d\theta \leq \frac{r}{2\pi i}\int_{|\zeta|=1} \frac{1}{(1-r\zeta)(\zeta-r)} d\zeta = \frac{r}{1-r^2}.$$

4.1.2 The BC for some subclasses of S

The conjecture has been proved, as well, for certain subclasses of S. We will focus on these two: the subclass of starlike functions, and the subclass of functions with real coefficients.

Definition 4.4. *A set $E \subset \mathbb{C}$ is said to be* ***starlike*** *with respect to a point $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E.*

A starlike function is a conformal mapping of the unit disk onto a domain starlike with respect to the origin.

Let S^* denote the subclass of starlike functions. Before we prove the conjecture for $f \in S^*$, we shall introduce two useful results:

Lemma 4.5. (Carathéodory's Lemma). *Suppose that φ is a holomorphic function in $\mathbb{D}$, $\varphi(0) = 1$, $\Re\{\varphi(z)\} > 0$ in $\mathbb{D}$, and*

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

Then,

$$|c_n| \leq 2, \quad \textit{for} \quad n \geq 1, \quad \textit{and} \quad |\varphi(z)| \leq \frac{1+r}{1-r}, \quad \textit{for} \quad |z| = r < 1.$$

Proof. By Cauchy's Integral formula, we have:

$$c_n = \frac{1}{2\pi\imath}\int_{|z|=r}\frac{\varphi(z)}{z^{n+1}}dz = \frac{1}{2\pi}\int_0^{2\pi}\frac{\varphi(re^{\imath\theta})}{r^n e^{n\imath\theta}}d\theta.$$

Hence,

$$c_n r^n = \frac{1}{2\pi}\int_0^{2\pi}\frac{\varphi(re^{\imath\theta})}{e^{n\imath\theta}}d\theta, \quad n \geq 0. \tag{4.1}$$

On the other hand, as $\varphi(z)$ is holomorphic in $\mathbb{D}$, $\varphi(z)z^{n-1}$ is too for all $n \geq 1$, and by the Cauchy theorem:

$$\int_{|z|=r}\varphi(z)z^{n-1}dz = 0, \quad n \geq 1.$$

Writing $z = re^{\imath\theta}$, we have:

$$\int_0^{2\pi}\varphi(re^{\imath\theta})r^{n-1}e^{(n-1)\imath\theta}\imath re^{\imath\theta}d\theta = \imath r^n\int_0^{2\pi}\varphi(re^{\imath\theta})e^{n\imath\theta}\imath d\theta = 0.$$

In particular:

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(re^{\imath\theta})e^{n\imath\theta}d\theta = 0, \quad n \geq 1. \tag{4.2}$$

Now, in (4.1), we replace the integrand by its complex conjugate and add the result to (4.2). Since $\varphi + \overline{\varphi} = 2U$, $U = \Re\{\varphi\}$, we get:

$$c_n r^n = \frac{1}{\pi}\int_0^{2\pi}U(re^{\imath\theta})e^{-n\imath\theta}d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi}U(re^{\imath\theta})e^{-n\imath\theta}d\theta,$$

whence

$$\begin{aligned}
|c_n|r^n &\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U(re^{\imath\theta})e^{-n\imath\theta}|d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi}U(re^{\imath\theta})d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\varphi(re^{\imath\theta}) + \overline{\varphi(re^{\imath\theta})}}{2}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(re^{\imath\theta})d\theta + \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(re^{\imath\theta})d\theta} \\
&= \varphi(0) + \overline{\varphi(0)} = c_0 + \bar{c_0} = 2.
\end{aligned}$$

using the Cauchy inequalities.

Finally, letting $r \to 1$, we have:

$$|c_n| \leq 2, \quad \text{for every} \quad n \geq 1.$$

The other equality follows:

$$\begin{aligned}
|\varphi(z)| &= \left|\sum_{n=0}^{\infty}c_n z^n\right| = \left|1 + \sum_{n=1}^{\infty}c_n z^n\right| \leq 1 + \sum_{n=1}^{\infty}|c_n||z^n| \leq 1 + 2\sum_{n=1}^{\infty}r^n \\
&= 1 + 2\frac{r}{1-r} = \frac{1+r}{1-r}, \quad |z| = r < 1.
\end{aligned}$$

Observe that both inequalities are sharp, since:

$$\text{If} \quad \varphi(z) = \frac{1+r}{1-r} = 1 + 2\sum_{n=1}^{\infty} r^n, \quad \text{then} \quad c_n = 2, \quad n \geq 1.$$

$\square$

Lemma 4.6. *Let $f \in S$. Then, $f(\mathbb{D})$ is starlike with respect to the origin if and only if*

$$\Re\left[z\frac{f'(z)}{f(z)}\right] > 0.$$

Proof. Suppose first that f is starlike with respect to the origin. Then, let's see that f maps each subdisk $|z| < \rho < 1$ onto a starlike domain, i.e, $g(z) = f(\rho z)$ is starlike in $\mathbb{D}$. To prove it, we must show that for each fixed $t \in (0,1)$ and for each $z \in \mathbb{D}$, the point $tg(z)$ is in the range of g.

Let $w_t : \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function in $\mathbb{D}$, such that:

$$tf(z) = f(w_t(z)), \quad \text{for all} \quad z \in \mathbb{D}.$$

Observe that $|w_t(z)| \leq 1$ and for $z = 0$, $0 = tf(0) = f(w_t(0))$, so $w_t(0) = 0$. Therefore, by the Schwarz lemma,

$$|w_t(z)| \leq |z|, \quad z \in \mathbb{D}.$$

Now, we have:

$$tg(z) = tf(\rho z) = f(w_t(\rho z)) = f(\rho\frac{1}{\rho}w_t(\rho z)) = g(w^*_{t,\rho}(z)),$$

where $w^*_{t,\rho}(z) = \frac{w_t(\rho z)}{\rho}$, with $|w^*_{t,\rho}(z)| = \left|\frac{w_t(\rho z)}{\rho}\right| \leq \left|\frac{\rho z}{\rho}\right| = |z|$. Thus, f maps each circle $|z| = \rho < 1$ onto a curve C_ρ that bounds a starlike domain.

Knowing this, we can affirm that $\arg(f(z))$ increases as z moves around the circle $|z| = \rho$. In other words,

$$\frac{\partial}{\partial\theta}\{arg(f(\rho e^{\imath\theta})\} \geq 0. \tag{4.3}$$

Recall that:

$$\frac{\partial}{\partial z}(\Im[g(z)]) = \frac{\partial}{\partial z}\left(\frac{g(z) - \overline{g(z)}}{2\imath}\right) = \frac{g'(z) - \overline{g'(z)}}{2\imath} = \Im\left[\frac{\partial}{\partial z}g(z)\right].$$

Using this, and that $\log(f(\rho e^{\imath\theta})) = |f(\rho e^{\imath\theta})| + \imath \arg(f(\rho e^{\imath\theta}))$, we obtain:

$$\frac{\partial}{\partial\theta}\{\arg(f(\rho e^{\imath\theta})\} = \frac{\partial}{\partial\theta}\{\Im[\log(f(\rho e^{\imath\theta}))]\} = \Im\left[\frac{\partial}{\partial z}\log(f(\rho e^{\imath\theta}))\right].$$

Computing the derivative, we see:

$$\frac{\partial}{\partial z}\log(f(\rho e^{\imath\theta})) = \frac{f'(\rho e^{\imath\theta})\imath\rho e^{\imath\theta}}{f(\rho e^{\imath\theta})} = \frac{f'(z)\imath z}{f(z)}, \quad z = \rho e^{\imath\theta}.$$

Hence,

$$\Im\left[\frac{\partial}{\partial z}\log(f(\rho e^{\imath\theta}))\right] = \Im\left[\frac{f'(z)\imath z}{f(z)}\right] = \Re\left[\frac{f'(z)z}{f(z)}\right].$$

Notice that if we see that (4.3) is strict, we will have shown this implication.

Let's define the function:

$$u(z) = -\Re\left[\frac{f'(z)z}{f(z)}\right].$$

Observe that $u(z)$ is an harmonic function, and $u(z) \leq 0$. Then, by the maximum principle for harmonic functions,

$$\max_{z\in\overline{\mathbb{D}}} u(z) = \max_{z\in\partial\mathbb{D}} u(z) \leq 0.$$

Since u is not constant, this implies that, for any point z with $|z| < 1$, $u(z) < 0$. Therefore, by definition of u,

$$\Re\left[\frac{f'(z)z}{f(z)}\right] > 0, \quad \text{for} \quad |z| < 1.$$

Now, let's prove the converse.

Suppose that $\Re\left[\frac{f'(z)z}{f(z)}\right] > 0$. Since $f \in S$, we know that $f(\mathbb{D})$ is simply connected, so we can define the function:

$$\arg(f(z)) = \phi(r,\theta), \quad \text{where} \quad z = re^{\imath\theta},$$

uniquely by analytic continuation, with the initial condition $\phi(0,\theta) = \theta$. As $\phi(r,\theta)$ is differentiable and $f(z) \neq 0$ for $z \neq 0$, we can define a single branch of the logarithm so that:

$$\begin{aligned}\frac{\partial}{\partial\theta}\phi(r,\theta) &= \frac{\partial}{\partial\theta}\Im[\log(f(z))] = \Im\left[\frac{\partial}{\partial\theta}\log(f(z))\right] = \Im\left[\frac{f'(z)}{f(z)}\frac{\partial z}{\partial\theta}\right] \\ &= \Im\left[\frac{f'(z)}{f(z)}\imath z\right] = \Re\left[z\frac{f'(z)}{f(z)}\right] > 0, \quad \text{by hypothesis.}\end{aligned}$$

We obtain that $\phi(r,\theta)$ is an increasing function of θ for each fixed value of r. Hence, when z describes the circle $|z| = r$ once in a positive sense, $f(z)$ traces a curve C_r once in a positive sense. We can parametrize C_r by the angle:

$$R = F(\phi)$$

where $F(\phi)$ is a real analytic function of period 2π. In conclusion, we clearly see that every point of this curve is visible from the origin, which implies that the set of all points that lie on the curves C_s, $0 \le s \le r < 1$ is starlike with respect to the origin, so it is $f(\mathbb{D})$. $\square$

After these two results, let's prove the Bieberbach conjecture for starlike domains. This was proved by Rolf Nevanlinna in 1920.

Theorem 4.7. *The coefficients of a function $f \in S^*$ satisfy*

$$|a_n| \le n, \quad \text{for} \quad n = 2, 3, ...$$

Strict inequality holds for all n unless f is a rotation of the Koebe function.

Proof. Given $f \in S^*$, define the function:

$$\varphi(z) = \frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

By Lemma 4.6, we have that $\Re\left[z\frac{f'(z)}{f(z)}\right] > 0$, so $\Re[\varphi(z)] > 0$. Thus, using Carathéodory's lemma (Lemma 4.5), we obtain that $|c_n| \le 2$ for all $n \ge 2$.

Now, if we multiply both sides of the equality by $f(z)$, we get:

$$zf'(z) = \varphi(z)f(z)$$

Expressing $\varphi(z)$ and $f(z)$ by their power series, and equating the coefficients according to the degree of z, we have:

$$z\left(1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right) = \left(1 + \sum_{n=1}^{\infty} c_n z^n\right)\left(z + \sum_{n=2}^{\infty} a_n z^n\right).$$

$$\begin{aligned} n = 2: &\quad 2a_2 = a_2 + c_1 \\ n = 3: &\quad 3a_3 = a_3 + c_2 + c_1 a_2 \\ n = 4: &\quad 4a_4 = a_4 + c_3 + c_1 a_3 + c_2 a_2 \\ \dots & \end{aligned}$$

So, in general,

$$n a_n = a_n + \sum_{k=1}^{n-1} c_{n-k} a_k. \tag{4.4}$$

Using this, let's show by induction the inequality $|a_n| \le n$, for $n \ge 2$.

- Initial case: For $n = 2$, we have: $2a_2 = a_2 + c_1$, so $a_2 = c_1$. Then by the Carathéodory lemma, $|a_2| = |c_1| \leq 2$.

- Inductive case: Suppose that we've proved $|a_k| \leq k$, for $k = 2, ..., n-1$. We want to see that $|a_n| \leq n$. Then, from (4.4) follows,

$$(n-1)|a_n| = \left| \sum_{k=1}^{n-1} c_{n-k} a_k \right| \leq \sum_{k=1}^{n-1} |c_{n-k}||a_k|.$$

  As we know that $|c_n| \leq 2$ for $n \geq 2$, and $|a_k| \leq k$, for $k = 2, ..., n-1$:

$$(n-1)|a_n| \leq \sum_{k=1}^{n-1} |c_{n-k}||a_k| \leq \sum_{k=1}^{n-1} 2k = 2\sum_{k=1}^{n-1} k = 2\frac{n(n-1)}{2} = n(n-1),$$

  from which we obtain $|a_n| \leq n$.

We still need to prove that strict inequality holds for all n unless f is a rotation of the Koebe function.

The Bieberbach theorem claims that if f is not a rotation of the Koebe function, then $|a_2| < 2$. Then, we deduce, by looking at the previous induction, that if f is not a rotation of the Koebe function, $|a_n| < n$ for all $n \geq 2$. □

Finally, let's show the conjecture for the case of functions with real coefficients. This result was proved by Jean Dieudonné and Werner W. Rogosinski in 1931, though using quite different methods.

Denote by S_r the subclass of functions f in S with real Taylor coefficients. Then,

Theorem 4.8. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + ...$ be in S_r. Then,*

$$|a_n| \leq n, \quad \textit{for} \quad n \geq 2.$$

Proof. A first observation is that, since the coefficients a_n are real, $a_n = \bar{a_n}$, and:

$$f(\bar{z}) = \bar{z} + \sum_{n\geq 2} a_n \bar{z}^n = \bar{z} + \sum_{n\geq 2} \overline{a_n z^n} = \overline{f(z)}, \quad \text{for all} \quad z \in \mathbb{D}.$$

Therefore, the image domain $f(\mathbb{D})$ is symmetric with respect to the real axis. In particular, we see that, since f is univalent in $\mathbb{D}$, and $f(z) = f(\bar{z})$, it implies that that $f(z)$ is real if and only if z is real.

Moreover, as $f'(0) = 1$, f maps the upper half of $\mathbb{D}$ to the upper half of $f(\mathbb{D})$. The same happens with the lower part of $\mathbb{D}$.

Now, if we write f as a power series, using polar coordinates and with $a_1 = 1$, we have:

$$\begin{aligned} f(z) = f(re^{\imath\theta}) &= \sum_{n=1}^{\infty} a_n r^n e^{n\imath\theta} = \sum_{n=1}^{\infty} a_n r^n (\cos n\theta + \imath \sin n\theta) \\ &= \sum_{n=1}^{\infty} a_n r^n \cos n\theta + \imath \sum_{n=1}^{\infty} a_n r^n \sin n\theta = u(z) + \imath v(z), \end{aligned}$$

u and v real-valued functions of z.

Let's compute the integral:

$$\int_{-\pi}^{\pi} \sin n\theta \, v(re^{\imath\theta}) d\theta = \int_{-\pi}^{\pi} \sin n\theta \left(\sum_{k\geq 1} a_k r^k \sin k\theta \right) d\theta = \sum_{k\geq 1} a_k r^k \int_{-\pi}^{\pi} \sin k\theta \sin n\theta d\theta.$$

Writing the sinus in its exponential form,

$$\int_{-\pi}^{\pi} \sin k\theta \sin n\theta d\theta = \frac{1}{4} \int_{-\pi}^{\pi} (-e^{(n+k)\imath\theta} + e^{(k-n)\imath\theta} + e^{-(k-n)\imath\theta} - e^{-(k+n)\imath\theta}) d\theta.$$

Then, using (3.1) (see Section 3.2) we have:

$$\int_{-\pi}^{\pi} \sin n\theta \, v(re^{\imath\theta}) d\theta = \begin{cases} \pi a_n r^n & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

Thus, for each $r < 1$, we obtain the equality:

$$|a_n r^n| = \frac{1}{\pi} \left| \int_{-\pi}^{\pi} \sin n\theta \, v(re^{\imath\theta}) d\theta \right|.$$

Since the integrand is an even function of θ, for being product of two odd functions, and f maps the upper half of $\mathbb{D}$ to the upper half of $f(\mathbb{D})$, so $v(re^{\imath\theta}) \geq 0$ when $\theta \in [0, \pi]$, we can write:

$$\begin{aligned} |a_n r^n| &= \frac{2}{\pi} \left| \int_0^{\pi} \sin n\theta \, v(re^{\imath\theta}) d\theta \right| \leq \frac{2}{\pi} \int_0^{\pi} |\sin n\theta| |v(re^{\imath\theta})| d\theta \\ &= \frac{2}{\pi} \int_0^{\pi} |\sin n\theta| v(re^{\imath\theta}) d\theta. \end{aligned}$$

To finish the proof, we need a result from real analysis, that is:

$$|\sin n\theta| \leq n \sin\theta, \quad 0 \leq \theta \leq \pi, \quad n = 1, 2, \cdots.$$

We prove it by induction:

1. Initial case: For $n = 1$, it is clear that $|\sin\theta| \leq \sin\theta$.

2. Inductive case: Suppose that $|\sin n\theta| \leq n \sin\theta$. Let's show it for $n+1$. By trigonometric inequalities, we have:

$$\begin{aligned}|\sin{(n+1)\theta}| &= |\sin n\theta \cos\theta + \cos n\theta \sin\theta| \\ &\leq |\sin n\theta||\cos\theta| + |\cos n\theta||\sin\theta| \leq |\sin n\theta||\sin\theta|,\end{aligned}$$

as $|\cos n\theta| \leq 1$ for all n and θ. Now, by inductive hypotesis,

$$|\sin{(n+1)\theta}| \leq |\sin n\theta||\sin\theta| \leq n\sin\theta + \sin\theta = (n+1)\sin\theta.$$

Given this, we obtain:

$$|a_n r^n| \leq \frac{2}{\pi} n \int_0^{\pi} \sin\theta v(re^{\imath\theta}) d\theta = \frac{1}{\pi} n \int_0^{2\pi} \sin\theta v(re^{\imath\theta}) d\theta = n a_1 = n.$$

As $r < 1$ is arbitrary, letting r tend to 1, we prove the theorem. $\square$

Chapter 5

The general proof

The general proof of Bieberbach's conjecture was given by Louis De Branges in 1984. Following Loewner's idea, he considered the functions $f \in S$ as the initial condition of a parametric family of analytic and univalent functions that satisfied the Loewner differential equation. Let's explain this in more detail.

Consider a function $f \in S$, and let ϕ be a conformal map from $\mathbb{C}^\infty \setminus \overline{\mathbb{D}}$ to $\mathbb{C}^\infty \setminus \overline{f(\mathbb{D})}$, such that $\phi(\infty) = \infty$.

Consider the curve $\partial D(0, e^t)$, $t \geq 0$, and let U_t be the domain enveloped by $\phi(\partial D(0, e^t))$ and containing 0. Then, we see that $(U_t)_{t \geq 0}$ is a family of increasing simply connected domains, where $U_s \subset U_t$, for $s < t$.

By the Riemman Mapping Theorem, for each $t \geq 0$, there exists a unique conformal mapping $f_t : \mathbb{D} \to U_t$ such that $f_t(0) = 0$, and $f_t'(0) > 0$. By reparametrization of t, we can assume that $f_t'(0) = e^t$. Therefore, they have the form:

$$f_t(z) = f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad a_n(t) \in \mathbb{C}, \quad 0 \leq t < +\infty. \tag{5.1}$$

These families $\{f_t\}_{t \geq 0}$ of univalent maps $f_t : \mathbb{D} \to \mathbb{C}$ satisfying $f_t(0) = 0$, and $f_t'(0) = e^t$, and such that $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$, for $0 \leq s < t \leq \infty$ are called **Loewner chains**. A key example of a Loewner chain, is the family:

$$f_t(z) = e^t \frac{z}{(1-z)^2}, \tag{5.2}$$

of dilated Koebe functions. Notice that the image $f_t(\mathbb{D})$ of each f_t is the domain $\mathbb{C} \setminus (-\infty, -\frac{e^t}{4}]$, which is clearly monotone increasing in t.

Now, let's verify that the family of function (5.1) satisfies the **Loewner differential equation**:

$$\partial_t f_t(z) = z f_t'(z) p_t(z) \tag{5.3}$$

with initial condition $f_t(z)|_{t=0} = f_0(z) = f$, and where $p_t(z)$ is a suitable family of holomorphic functions in $\mathbb{D}$, with $\Re[p_t(z)] > 0$, and $p_t(0) = 1$, $0 \leq t < +\infty$.

Form the relation $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$, we can define the univalent function:

$$\varphi_{t,s} = f_t^{-1} \circ f_s : \mathbb{D} \to \mathbb{D}, \tag{5.4}$$

called transition function. We observe that $|\varphi_{s,t}(z)| \le 1$, and $\varphi_{s,t}(0) = 0$, so by the Scharwz's lemma,

$$|\varphi_{s,t}(z)| \le |z|, \quad z \in \mathbb{D}.$$

With this, we know that the transformation $\dfrac{z - \varphi_{s,t}(z)}{z + \varphi_{s,t}(z)}$ has positive real part. Normalizing the function so that its value in $z = 0$ is 1, we obtain the functions:

$$p_{s,t}(z) = \frac{1 + e^{s-t}}{1 - e^{s-t}} \frac{z - \varphi_{s,t}(z)}{z + \varphi_{s,t}(z)}, \quad p_{s,t}(z) : \mathbb{D} \to \mathbb{H} = \{\Re[z] > 0\}, \quad p_{s,t}(0) = 1.$$

These holomorphic functions $p : \mathbb{D} \to \mathbb{H}$ are called **Herglotz functions** (see Carathéodory's lemma (Lemma 4.5) for some properties of these functions).

On the other hand, it can be verified that f_t is Lipschitz with respect to t, i.e, for each z, there exists a constant $C(z) > 0$ such that:

$$|f_t(z) - f_s(z)| \le C(z)|t - s|.$$

This constant $C(z)$ can be taken uniformly for each compact $K \subset \mathbb{D}$. Hence, by this and the Growth theorem, we can deduce that f_t is differentiable a.e $t \ge 0$ for all $z \in \mathbb{D}$. So, as s approaches t we have:

$$f_s(z) = f_t(z) + \partial_t f_t(z)(s - t) + o(|s - t|).$$

Now, from the equality (5.4) and using Taylor again, we get:

$$\begin{aligned}
\varphi_{s,t}(z) &= f_t^{-1}[f_t(z) + \partial_t f_t(z)(s - t) + o(|s - t|)] \\
&= z + (f_t^{-1})'(f_t(z))(s - t)\partial_t f_t(z) + o(|s - t|) \\
&= z + \frac{1}{f_t'(z)}(s - t)\partial_t f_t(z) + o(|s - t|).
\end{aligned}$$

Thus, for $s \sim t$:

$$\begin{aligned}
p_{s,t}(z) &= \frac{1 + e^{s-t}}{1 - e^{s-t}} \frac{z - \varphi_{s,t}(z)}{z + \varphi_{s,t}(z)} = \frac{2}{t - s} \frac{\frac{1}{f_t'(z)}(t - s)\partial_t f_t(z)}{2z}(1 + o(1)) \\
&= \frac{1}{z} \frac{\partial_t f_t(z)}{f_t'(z)}(1 + o(1)).
\end{aligned}$$

Finally, computing the limit, we obtain the equation (5.3):

$$p_t(z) = \lim_{s \to t^-} p_{s,t}(z) = \frac{1}{z} \frac{\partial_t f_t(z)}{f_t'(z)},$$

where p_t is a Herglotz function with $p_t(0) = 1$.

Observe, for example, that the Loewner chain (5.2) solves the Loewner equation with the Herglotz function: $p_t(z) = \frac{1-z}{1+z}$. We see it:

$$\begin{aligned} \partial_t k_t(z) &= e^t \frac{z}{(1-z)^2} \\ k_t'(z) &= e^t \left(\frac{(1-z)^2 + 2z(1-z)}{(1-z)^4} \right) = e^t \left(\frac{(1-z)+2z}{(1-z)^3} \right) \end{aligned}$$

Hence,

$$z k_t'(z) P_t(z) = z\, e^t \left(\frac{(1-z)+2z}{(1-z)^3} \right) \frac{1-z}{1+z} = e^t \frac{z}{(1-z)^2} = \partial_t k_t(z).$$

From the differential equation (5.3), we can extract some information about the coefficients. This is the approach started by Loewner and explotated by de Branges.

Let $f_t(z) = e^t z + a_2(t) z^2 + a_3(t) z^3 + \cdots$. As f_t is differenciable in almost every t for each z, and is locally uniformly continuous in z, we see from Cauchy's formula

$$a_n(t) = \frac{1}{2\pi\imath} \int_{|\zeta|=r} \frac{f_t(\zeta)}{\zeta^{n+1}} d\zeta,$$

that the $a_n(t)$ are differenciable a.e t. So, denoting

$$p_t(z) = c_0(t) + c_1(t) z + c_2(t) z^2 + \cdots, \quad c_0(t) = 1.$$

we can rewrite Loewner's differential equation (5.3) as:

$$\begin{aligned} \partial_t f_t(z) &= \partial_t a_1(t) z + \partial_t a_2(t) z^2 + \partial_t a_3(t) z^3 + \cdots \\ z f_t'(t) p_t'(z) &= z(a_1(t) + 2a_2(t) z + 3a_3(t) z^2 + \cdots)(c_0(t) + c_1(t) z + c_2(t) z^2 + \cdots) \\ &= (a_1(t) z + 2a_2(t) z^2 + 3a_3(t) z^3 + \cdots)(c_0(t) + c_1(t) z + c_2(t) z^2 + \cdots) \end{aligned}$$

Equating the coefficients with respect to the degree of z, we obtain:

$$\begin{aligned} n = 2: &\quad \partial_t a_2(t) = a_1(t) c_1(t) + 2a_2(t) = 2a_2(t) + c_1(t) e^t \\ n = 3: &\quad \partial_t a_3(t) = e^t c_2(t) + 2a_2(t) c_1(t) + 3a_3(t) \\ \cdots & \end{aligned}$$

In general,

$$\partial_t a_n(t) = \sum_{j=1}^{n} j a_j(t) c_{n-j}(t), \quad n \geq 2.$$

For instance, for the Loewner chain (5.2) it is immediate that $a_n(t) = ne^t$, and it can be verified that $c_n(t) = 2(-1)^n$ for $n \geq 1$, solve these equations. Writing

$$\frac{1-z}{1+z} = \sum_{n=0}^{\infty}(-z)^n + \sum_{n=1}^{\infty}(-z)^n = 1 + \sum_{n=1}^{\infty} 2(-1)^n z^n,$$

we see that $c_n = 2(-1)^n$, and by Carathéodory's lemma (Lemma 4.5), the function is extremal.

Moreover, these equalities allow us to deduce, with not at all immediate methods, some cases of the Bieberbach conjecture.

Let's see it for $n = 2$: The equality above for $n = 2$ is equivalent to:

$$\frac{\partial}{\partial t}[e^{-2t} a_2(t)] = e^{-t} c_1(t).$$

If we apply the distortion theorem to $e^t f_t \in S$, we obtain:

$$|a_2(t)| = \left| \frac{f_t''(0)}{2} \right| = O(e^t).$$

Using this,

$$\int_0^{\infty} e^{-t} c_1(t) dt = [e^{-2t} a_2(t)]_{t=0}^{t=\infty} = 0 - a_2(0) = -a_2(0).$$

The Carathéodory lemma (Lemma 4.5) implies that $|c_1(t)| \leq 2$. Therefore,

$$|a_2| = |a_2(0)| \leq \int_0^{\infty} e^{-t} |c_1(t)| dt = 2 \int_0^{\infty} e^{-t} dt = 2.$$

Once we've exposed the starting point of de Branges' proof, let's illustrate the line of argument he followed to get to the BC. For that, we must introduce two conjectures: the Robertson conjecture (1936) and the Milin conjecture (1971).

Let S^2 denote the subclass of square root transformations (property 4, Chapter 3):

$$f_2(z) = \sqrt{f(z^2)} = \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1}, \quad f(z) \in S, \quad b_1 = 1 \tag{5.5}$$

of the class S.

Theorem 5.1. (Robertson's Conjecture). *If a function $f_2 \in S^2$, then:*

$$\sum_{k=1}^{n} |b_{2k-1}|^2 \leq n, \quad \text{for} \quad n = 1, 2, \cdots.$$

Equality holds if and only if f_2 is a rotation of the Koebe function.

It is not difficult to see that the Robertson conjecture implies the Bieberbach conjecture. If we write the coefficients of a function $f \in S$ in terms of the coefficients of a function $f_2 \in S^2$, using that $f(z^2) = f_2(z)^2$, we get:

$$a_n = \sum_{k=1}^{n} b_{2k-1} b_{2(n+1-k)}, \quad n = 1, 2, \cdots, \quad b_1 = 1.$$

Now, using the Cauchy inequality, we see:

$$|a_n| \leq \sum_{k=1}^{n} |b_{2k-1}|^2, \quad n = 2, 3, \cdots, \quad b_1 = 1,$$

which implies the BC.

Now, let's denote the logarithmic coefficients of a function $f \in S$, generated by the Taylor expansion as:

$$\log\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty} d_n z^n, \quad \log(1) = 0. \tag{5.6}$$

Theorem 5.2. (Milin's Conjecture) *If $f \in S$,*

$$\sum_{k=1}^{n} (k|d_n|^2 - \frac{4}{k})(n - k + 1) \leq 0, \quad n = 1, 2, \cdots.$$

Equality holds if and only if f is a rotation of the Koebe function.

It has been shown, as well, that the Milin conjecture implies the Robertson conjecture. We see it:

The mathematicians Lebedev and Milin found the inequality:

$$\sum_{k=1}^{n+1} |b_{2k-1}|^2 \leq (n+1) \exp\left(\frac{1}{4(n+1)} \sum_{k=1}^{n} (k|d_k|^2 - \frac{4}{k})(n - k + 1)\right), \quad n = 1, 2, \cdots,$$

that relates the coefficients of functions (5.5) in S^2, and functions (5.6), with $f \in S$. From Milin's conjecture, we have:

$$\sum_{k=1}^{n} (k|d_k|^2 - \frac{4}{k})(n - k + 1) \leq 0, \quad n = 1, 2, \cdots.$$

Thus, for a big value of n, we observe that the exponent of e tends to 0, so the exponential tends to 1, and we obtain:

$$\sum_{k=1}^{n+1} |b_{2k-1}|^2 \leq (n+1), \quad n = 1, 2, \cdots.$$

which implies the Robertson conjecture.

De Branges realized that the coefficients of (5.6) obeyed more tractable equations. Hence, what de Branges did was prove the Milin conjecture for all $n \geq 1$, and from this, Robertson's and Bieberbach's conjectures were followed. He did the following:

He considered the Loewner chain (5.1) explained above, and denoted its logarithmic coefficients by the Taylor expansion as:

$$\log\left(\frac{f_t(z)}{e^t z}\right) = \sum_{n=1}^{\infty} d_n(t) z^n, \quad |z| < 1, \quad 0 \leq t < +\infty,$$

where $d_0(t) = \log(1) = 0$ and $d_1(t) = e^{-t} a_2(t)$, $a_2(0) = a_2$.

Then, with the help of the Loewner equation, de Branges proved the general inequality:

$$\sum_{k=1}^{n} \left(k|d_k(t)|^2 - \frac{4}{k}\right) \sigma_k(t) \leq 0, \quad 0 \leq t < +\infty, \tag{5.7}$$

for any integer $n \geq 1$, and where $\sigma_k(t)$ are the de Branges functions:

$$\sigma_k(t) = k \sum_{\nu=0}^{n-k} (-1)^{\nu} \frac{(2k+\nu+1)_{\nu}(2k+2\nu+2)_{n-k-\nu}}{(k+\nu)\nu!(n-k-\nu)!} e^{-\nu t - kt},$$

for $0 \leq t < +\infty$, $k = 1, \cdots, n$, and

$$(a)_{\nu} = a(a+1)\cdots(a+\nu-1), \quad (a)_0 = 1, \quad \nu = 1, 2, \cdots,$$

for any arbitrary number a.

These functions were the unique solution of the system of differential equations:

$$\sigma_k(t) - \sigma_k(t) = -\frac{\sigma_k'(t)}{k} - \frac{\sigma_{k+1}'(t)}{k+1}, \quad 0 \leq t < +\infty, \quad k = 1, \cdots, n, \quad \sigma_{n+1}(t) = 0,$$

with initial condition $\sigma_k(0) = n - k + 1$, for $k = 1, \cdots, n$.

Thus, for $t = 0$, he deduced from this initial condition and the inequality (5.7) that the Milin conjecture held, where $d_k = d_k(0)$ were the coefficients in (5.6) for the function $f(z, 0) = f(z)$.

Chapter 6

Conclusion

In this work, we have studied the holomorphic and univalent functions in proper simply connected domains of $\mathbb{C}$. More specifically, we have developed the theory of functions belonging to the class S, since due to Riemann's theorem, we have seen that the results related to this class can be extrapolated to arbitrary simply connected domains. Thus, we have explained in detail some of the most important results, such as the Area theorem, the Bieberbach theorem, the Koebe one-quarter theorem, or the distortion and growth theorems. Finally, we have faced one of the main problems of the theory of univalent functions, the Bieberbach conjecture, first showing some of its partial results, and ending with the explanation of the main idea of de Branges' general proof.

After carrying out this work, we have been able to notice how an apparently simple property, such as univalence, can give rise to a multitude of important results, for various reasons.

On the one hand, results such as Riemann's theorem, or the BC, have served to relate two areas of mathematics: geometry and complex analysis, in the so-called geometric theory of functions, and from which it has been possible to see how analytic and geometric properties of functions reflect each other.

On the other hand, others such as the Bieberbach theorem, or, again, the BC, have led to the development of resolution methods, like the Loewner method, used in the proof for the 3rd coefficient estimate, and later in de Branges' general proof, or others such as the variational methods.

However, this subject is not closed but is still, at present, source of study. At the same time that previous problems are being resolved, new ones, as new estimates of coefficients for specific functions, or possible applications of the methods commented above, are appearing. Challenges that, for sure, spur my interest in continuing to investigate this rich matter, and to which I hope to be able, someday, to do my bit.

Annex: Biography

Ludwig Bieberbach (4 December 1886 - 1 September 1982) was a German mathematician born in Goddelau, into a well-off family.

Figure 6.1: Ludwig Bieberbach

Ludwig was taught by private tutors up to the age of eleven. In 1905, he entered the Humanistic Gymnasium in Bensheim, where he became interested in mathematics, inspired by an excellent mathematics teacher.

After his military service, he decided to go to the University of Göttingen, where its enthusiastic atmosphere for research had a great influence on Bieberbach. He attended the algebra course by Minkowski which had brought him there, but he was influenced even more strongly by Felix Klein and his lectures on elliptic functions. Another strong influence on the direction of Bieberbach's mathematical interests came from Paul Koebe, who was only four years older than Bieberbach. Koebe, an expert on complex function theory, became a dozent at Göttingen in 1907 and also encouraged Bieberbach towards analysis. It was under Klein's direction that Bieberbach researched into automorphic functions for his doctorate, which was awarded in 1910 for his thesis "Zur Theorie der automorphen Funktionen".

He began working as a Privatdozent at Königsberg in 1910. There, he worked out the details of his solution to the first part of Hilbert's eighteenth problem, which gave the young Bieberbach an international reputation. A few years after, in 1913, Bieberbach was appointed professor of mathematics in Basel, Switzerland. In the following year, he married Johanna (Hannah) Friederike Stoermer (1882-1955); with whom he had four sons, and moved to Frankfurt. It was while Bieberbach was at Frankfurt that he produced the Bieberbach Conjecture for which he is best known today. In 1921, he accepted the Berlin professorship. He continued to increase his influence in German mathematics, becoming secretary of the

Deutschen Mathematiker-Vereinigung (German Mathematical Society).

On 30 January 1933, Hitler came to power, and soon after this Bieberbach was converted to the views of the Nazis and energetically persecuted his Jewish colleagues, including Edmund Landau and his former coauthor Schur, dismissing them from their posts. That year, he joined the Sturmabteilung, and in 1937, the NSDAP. Bieberbach was heavily influenced by Theodore Vahlen, another German mathematician and anti-Semite, who along with him founded the "Deutsche Mathematik" ("German mathematics") movement and journal of the same name. The purpose of the movement was to encourage and promote a "German" (in this case meaning intuitionistic) style in mathematics. Bieberbach's and Vahlen's idea of having German mathematics was only part of a wider trend in the scientific community in Nazi Germany towards giving the sciences racial character.

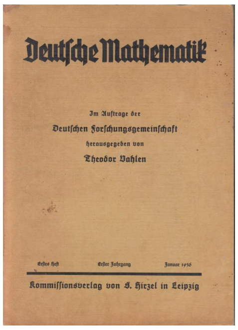

Figure 6.2: "Deutsche Mathematik" journal

After the end of World War II in 1945, Bieberbach lost all his positions because of his political involvement, being dismissed and arrested. Despite this, in 1949, Alexander Ostrowski invited him to lecture at Basel University, as he considered Bieberbach's political views irrelevant to his contributions to the field of mathematics.

In 1951, Bieberbach and Friedrich Wilhelm Levi were on a list to fill the second chair in Berlin. Both were in their 60s but had very different wartime experiences: Levi having been dismissed from the University of Leipzig in 1935 because he was Jewish, and Bieberbach having been the leading Nazi mathematician. Finally, Friedrich Levi was appointed.

After this, he continued to produce excellent books such as: "Theorie der geometrischen Konstruktionen" ("Theory of geometric constructions") in 1952, "Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage dargestellt" (Theory of ordinary differential equations presented on a function-theoretical basis) in 1953, "Einführung in die Theorie der Differentialgleichungen im reellen Gebiet" (Introduction to the theory of differential equations in the real domain) in 1956, or "Einführung in die analytische Geometrie" (Introduction to analytical geometry) in 1962.

Finally, Ludwig Bieberbach died in 1982.

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