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by

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On the Depth of the Fiber Cone of Filtrations

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1 Introduction

Let $(A, \mathfrak{m})$ be a noetherian local ring with maximal ideal $\mathfrak{m}$ and I an ideal of A. The fiber cone of I (with respect to $\mathfrak{m}$) is defined as $F_{\mathfrak{m}}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$, which is a noetherian graded algebra over the residue field $A/\mathfrak{m}$. This graded object encodes several informations on I. For instance, when the residue field is infinite its dimension coincides with the minimal number of generators of any minimal reduction of I, that is the analytic spread of I. Also the Hilbert function of $F_{\mathfrak{m}}(I)$ provides the minimal number of generators of the powers of I. See [26] for basic information concerning these facts.

The arithmetical properties of fiber cones have been scantily studied. If I is generated by a regular sequence, or more in general by a family of analytically independent elements, $F_{\mathfrak{m}}(I)$ is trivially Cohen-Macaulay. On the other hand, C. Huneke and J. Sally proved that if A is Cohen-Macaulay and I is $\mathfrak{m}$-primary with reduction number one then $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay, see [11]. Also, M. Morales and A. Simis have shown in [19] that the

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fiber cone of the defining ideal of a monomial curve in $\mathbb{P}^3$ lying on a quadric is always Cohen-Macaulay. That result was extended by P. Gimenez [2] to monomial varieties of codimension two whose Rees algebra is presented by an ideal generated by elements of degree at most two.

A more general approach to the Cohen-Macaulayness of the fiber cone was made by K. Shah in [27], where the case of equimultiple ideals with reduction number at most two was considered. In particular, it was proven there that if A is Cohen-Macaulay and I is an equimultiple ideal with reduction number one the fiber cone of I is always Cohen-Macaulay, see also [28] for another related result. The method followed by Shah to study the Cohen-Macaulayness of fiber cones is the following (we always assume that $A/\mathfrak{m}$ is infinite): Let I be an ideal of A and J a minimal reduction of I. Then $F_{\mathfrak{m}}(I)$ is a finite extension of $F_{\mathfrak{m}}(J)$ (which is, in fact, a Noether normalization of $F_{\mathfrak{m}}(I)$). Since J is generated by a family of analytically independent elements, $F_{\mathfrak{m}}(J)$ is a polynomial ring over the residue field $A/\mathfrak{m}$, hence $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay if and only if it is free as $F_{\mathfrak{m}}(J)$-module, see [20]. Now, if J is generated by a regular sequence and the reduction number of I is at most two, Shah is able to find conditions which guaranty the existence of a basis of $F_{\mathfrak{m}}(I)$ as a $F_{\mathfrak{m}}(J)$-module.

The first goal of the present paper was to extend the above results to ideals of any reduction number. We achieve it, since as a consequence of the basic criterion we prove in Theorem 2.8 we may formulate the following:

Theorem 1.1 *Let $(A, \mathfrak{m})$ be a noetherian local ring and $I \subsetneq A$ an ideal. Let J be a minimal reduction of I and denote by $r_J(I)$ the reduction number of I with respect to J. Assume that J is generated by a regular sequence and that $J \cap I^n = JI^{n-1}$ for all $1 \leq n \leq r_J(I)$. Then, $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay if and only if $J \cap \mathfrak{m}I^n = J\mathfrak{m}I^{n-1}$ for all $1 \leq n \leq r_J(I)$.*

This result recovers what Shah proved for reduction number at most two, but our approach is completely different. Namely, a minimal system of generators of a minimal reduction of I provides a system of parameters of $F_{\mathfrak{m}}(I)$ composed by homogeneous elements of degree one. Our conditions may then be viewed as a "mixed-Valabrega-Valla" like criterion to this family of homogeneous elements of degree one be a regular sequence in $F_{\mathfrak{m}}(I)$. In order to prove it, we introduce a filtration of submodules of A whose associated graded module can be thought as an intermediate of the associated graded ring of I and the fiber cone of I, while to controle depths we use a slight generalization of the modified Koszul complexes studied by S. Huckaba and T. Marley in [9]. Furthermore, this point of view allows us to extend Shah's results not only to ideals of any reduction number but also to noetherian filtrations which include the filtration of the integral closures of the powers an ideal.

All the above technical results, including our basic criterion Theorem 2.8, are developed in Section 2, while Section 3 is devoted to study fiber cones of equimultiple good filtrations. In particular, we give a criterion for an integrally closed $\mathfrak{m}$-primary ideal whose second Hilbert coefficient (resp. normalized second Hilbert coefficient) is equal to one having Cohen-Macaulay fiber cone (resp. normalized fiber cone), Theorem 3.10 (resp. Theorem 3.12). This allows us to characterize for which 2-dimensional rational or elliptic sigularities the normalized fiber cone is Cohen-Macaulay, see Theorem 3.13. In Section 4 we prove that if A is Cohen-Macaulay and I is an analytic deviation one ideal with reduction number one the fiber cone of I is always Cohen-Macaulay. Finally, in Section 5 we apply all the above results to determine in each case the minimal number of generators of the powers or the integral closures of the powers of the ideal.

2 Modified Koszul complexes and the depth of the fiber cone

Let A be a commutative noetherian ring, I an ideal of A, and E an A-module. Let $\mathbf{E} = (E_n)_{n\geq 0}$ be a filtration of A-submodules of E : $E = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_n \supseteq \cdots$. Recall that $\mathbf{E}$ is said to be an I-filtration if $IE_n \subseteq E_{n+1}$ for all $n \geq 0$. Then, for a given family $a_1, \ldots, a_k$ of elements in I the A-module $\overline{\mathbf{E}} = \bigoplus_{n\geq 0} E_n$ can be viewed as a graded module over the polynomial ring $A[T_1, \ldots, T_k]$ by means of the multiplication

$$f(T_1, \ldots, T_k)\, m_i = f(a_1, \ldots, a_k)\, m_i, \quad m_i \in E_i\,,$$

see [15]. Thus we can consider $K(T_1,\ldots,T_k;\overline{\mathbf{E}})$ the graded Koszul complex of $T_1,\ldots,T_k$ with coefficients in $\overline{\mathbf{E}}$. It's component of degree n

$$K_n(T_1,\ldots,T_k;\overline{\mathbf{E}}): \quad 0 \to E_{n-k} \to E_{n-k-1}^{\binom{k}{k-1}} \to \cdots \to E_{n-2}^{\binom{k}{2}} \to E_{n-1}^{\binom{k}{1}} \to E_n \to 0$$

is a subcomplex of the Koszul complex $K(a_1,\ldots,a_k;E)$ of $a_1,\ldots,a_k$ with coefficients in E. Hence there exists an exact sequence of complexes

$$0 \to K_n(T_1,\ldots,T_k;\overline{\mathbf{E}}) \to K(a_1,\ldots,a_k;E) \to C(a_1,\ldots,a_k;\mathbf{E};n) \to 0$$

where $C(a_1,\ldots,a_k;\mathbf{E};n) = K(a_1,\ldots,a_k;E)/K_n(T_1,\ldots,T_k;\overline{\mathbf{E}})$.

Definition 2.1 *The complex $C(a_1,\ldots,a_k;\mathbf{E};\mathbf{n})$ is the n-th modified Koszul complex of $a_1,\ldots,a_k$ with coefficients in $\mathbf{E}$.*

Note that from the natural exact sequence

$$0 \to K(T_1,\ldots,T_{k-1};\overline{\mathbf{E}}) \to K(T_1,\ldots,T_k;\overline{\mathbf{E}}) \to K(T_1,\ldots,T_{k-1};\overline{\mathbf{E}})(-1) \to 0$$

and taking components of degree n we get the exact sequence of modified Koszul complexes

$$0 \to C(a_1,\ldots,a_{k-1};\mathbf{E};n) \to C(a_1,\ldots,a_k;\mathbf{E};n) \to C(a_1,\ldots,a_{k-1};\mathbf{E};n-1)(-1) \to 0,$$

as well as the long exact sequence of homologies

$$\begin{gathered}\cdots \longrightarrow H_i(C(a_1,\ldots,a_{k-1};\mathbf{E};n)) \longrightarrow H_i(C(a_1,\ldots,a_k;\mathbf{E};n)) \longrightarrow \\ H_{i-1}(C(a_1,\ldots,a_{k-1};\mathbf{E};n-1)) \xrightarrow{\cdot\pm a_k} H_{i-1}(C(a_1,\ldots,a_{k-1};\mathbf{E};n)) \longrightarrow \cdots .\end{gathered}$$

Similarly to [9, Lemma 3.2], where the case of Hilbert filtrations is considered, one can explicitely compute some homologies of modified Koszul complexes. We omit the proof.

Lemma 2.2 *(i)* $H_0(C(a_1,\ldots,a_k;\mathbf{E};n)) = E/(E_n + (a_1,\ldots,a_k)E)$.

(ii) $H_k(C(a_1,\ldots,a_k;\mathbf{E};n)) = (E_{n-k+1} :_E (a_1,\ldots,a_k))/E_{n-k}$.

(iii) If $a_1,\ldots,a_k$ is a regular sequence in E then

$$H_1(C(a_1,\ldots,a_k;\mathbf{E};n)) = ((a_1,\ldots,a_k)E \cap E_n)/(a_1,\ldots,a_k)E_{n-1}.$$

Now, let us consider a filtration $\mathcal{I} = (I_n)_{n\geq 0}$ of ideals of A, i.e. a sequence of ideals $A = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$ such that $I_n I_m \subseteq I_{n+m}$ for all $n, m \geq 0$. (For an ideal I we shall also denote by I its adic filtration.) A filtration $\mathbf{E}$ of an A-module E is said to be $\mathcal{I}$-compatible if $I_n E_m \subseteq E_{n+m}$ for all $n, m \geq 0$. (Observe that if I is an ideal, $\mathbf{E}$ is an I-filtration if and only if $\mathbf{E}$ is I-compatible.) Then, $G(\mathbf{E}) = \bigoplus_{n\geq 0} E_n/E_{n+1}$ is in a natural way a graded module over $G(\mathcal{I}) = \bigoplus_{n\geq 0} I_n/I_{n+1}$, the associated graded ring of $\mathcal{I}$.

Let $a \in I_1$ and denote by a^* its image in $I_1/I_2 \hookrightarrow G(\mathcal{I})$. Recall that a filtration $\mathcal{I}$ of ideals A is said to be noetherian if $R(\mathcal{I}) = \bigoplus_{n\geq 0} I_n t^n \subseteq A[t]$, the Rees ring of $\mathcal{I}$, is noetherian. Then, the associated graded ring of $\mathcal{I}$ is also noetherian. By a straightforward adaptation of [9, Proposition 3.3, Lemma 3.4, and Proposition 3.5] we obtain the following:

Proposition 2.3 *Assume that $\mathcal{I}$ is a noetherian filtration and that $G(\mathbf{E})$ is finitely generated as $G(\mathcal{I})$-module. Let $a_1, \ldots, a_k$ be a family of elements in I_1. Then:*

(i) $\operatorname{depth}_{(a_1^*,\ldots,a_k^*)} G(\mathbf{E}) = \min\{j \mid H_{k-j}(C(a_1,\ldots,a_k;\mathbf{E};n)) \neq 0$ *for some* $n\}$.

(ii) If $H_j(C(a_1,\ldots,a_k;\mathbf{E};n)) = 0$ for all n, then $H_i(C(a_1,\ldots,a_k;\mathbf{E};n)) = 0$ for all $i \geq j$ and n.

(iii) (Generalized Valabrega-Valla) $a_1^*, \ldots, a_k^*$ is a regular sequence in $G(\mathbf{E})$ if and only if $a_1, \ldots, a_k$ is a regular sequence in E and $(a_1,\ldots,a_k)E \cap E_n = (a_1,\ldots,a_k)E_{n-1}$ for all $n \geq 1$.

Now we turn to the main object of this paper. Let $\mathcal{I}$ be a filtration of ideals of A and H an ideal of A which contains I_1.

Definition 2.4 *The graded ring $F_H(\mathcal{I}) = \bigoplus_{n\geq 0} I_n/HI_n$ is the fiber cone of $\mathcal{I}$ with respect to H.*

Assume that $\mathbf{E}$ is an $\mathcal{I}$-compatible filtration. Then we can consider the graded $F_H(\mathcal{I})$-module defined by $F_H(\mathbf{E}) = \bigoplus_{n\geq 0} E_n/HE_n$. On the other hand the following filtration of E:

$$\mathbf{E}^H = (E_n^H)_{n\geq 0}, \text{ where } E_n^H = HE_{n-1} \text{ if } n \geq 1,\ E_0^H = E$$

is an I_1-filtration, hence $G(\mathbf{E}^H)$ is a graded $G(I_1)$-module. Finally, for any given element $a \in I_1$ denote by a^*, a^0, and a' its image in $I_1/I_2 \hookrightarrow G(\mathcal{I})$, $I_1/HI_1 \hookrightarrow F_H(\mathcal{I})$, and $I_1/I_1^2 \hookrightarrow G(I_1)$, respectively.

Our next proposition relates by means of an exact sequence, the Koszul complex with coefficients in the fiber cone with the modified Koszul complexes with coefficients in $\mathbf{E}$ and $\mathbf{E}^H$.

Proposition 2.5 *Let $a_1, \ldots, a_k$ be elements in I_1. There exists an exact sequence of complexes*

$$0 \to K_n(a_1^0, \ldots, a_k^0; F_H(\mathbf{E})) \to C(a_1, \ldots, a_k; \mathbf{E}^H; n+1) \to C(a_1, \ldots, a_k; \mathbf{E}; n) \to 0$$

where $K_n(a_1^0, \ldots, a_k^0; F_H(\mathbf{E}))$ is the n-th component of the graded Koszul complex of $a_1^0, \ldots, a_k^0$ with coefficients in $F_H(\mathbf{E})$.

Proof. Straightforward after explicitely writing each complex.

Proposition 2.6 *Assume that $\mathcal{I}$ is noetherian and $G(\mathbf{E})$ finitely generated as a $G(\mathcal{I})$-module. Let $a_1, \ldots, a_k$ be elements in I_1. If* $\operatorname{depth}_{(a_1^*, \ldots, a_k^*)} G(\mathbf{E}) \geq r$ *then*

$$H_j(K(a_1^0, \ldots, a_k^0; F_H(\mathbf{E})))_n \simeq H_j(C(a_1, \ldots, a_k; \mathbf{E}; n+1)) \ \textit{for all } j > k - r, \ \textit{for all } n.$$

Proof. From the exact sequence in Proposition 2.5 we get a long exact sequence of homology

$$\begin{gathered} \cdots \longrightarrow H_{i+1}(C(a_1, \ldots, a_k; \mathbf{E}; n)) \longrightarrow H_i(C(a_1^0, \ldots, a_k^0; F_H(\mathbf{E})))_n \longrightarrow \\ H_i(C(a_1, \ldots, a_k; \mathbf{E}^H; n+1)) \longrightarrow H_i(C(a_1, \ldots, a_k; \mathbf{E}; n)) \longrightarrow \cdots . \end{gathered}$$

If $\operatorname{depth}_{(a_1^*, \ldots, a_k^*)} G(\mathbf{E}) \geq r$ then by Proposition 2.3(i) $H_j(C(a_1, \ldots, a_k; \mathbf{E}; n)) = 0$ for all $j > k - r$ and n, hence $H_j(K(a_1^0, \ldots, a_k^0; F_H(\mathbf{E})))_n \simeq H_j(C(a_1, \ldots, a_k; \mathbf{E}^H; n+1))$ for all $j > k - r$ and n too.

If we assume that $\operatorname{depth}_{(a_1^*, \ldots, a_k^*)} G(\mathbf{E})$ is the biggest possible we obtain:

Proposition 2.7 *Under the hypothesis of Proposition 2.6, if* $\operatorname{depth}_{(a_1^*, \ldots, a_k^*)} G(\mathbf{E}) = k$ *then* $\operatorname{depth}_{(a_1^0, \ldots, a_k^0)} F_H(\mathbf{E}) = \operatorname{depth}_{(a_1', \ldots, a_k')} G(\mathbf{E}^H)$.

Proof. Apply Proposition 2.3(i) and Proposition 2.6.

Finally, writing the above proposition in terms of the generalized Valabrega-Valla we get:

Theorem 2.8 *Let A be a noetherian ring, $\mathcal{I}$ a noetherian filtration of ideals of A, and* $\mathbf{E}$ *a filtration of submodules of an A-module E. Suppose that* $\mathbf{E}$ *is $\mathcal{I}$-compatible and $G(\mathbf{E})$ finitely generated as a $G(\mathcal{I})$-module, and let $a_1, \ldots, a_k$ be a family of elements in I_1. Assume that*

(i) $a_1, \ldots, a_k$ is a regular sequence in A.

(ii) $(a_1, \ldots, a_k)E \cap E_n = (a_1, \ldots, a_k)E_{n-1}$ for all $n \geq 1$.

Then, $\mathrm{depth}_{(a_1^0, \ldots, a_k^0)} F_H(\mathbf{E}) = k$ *if and only if $(a_1, \ldots, a_k)E \cap HE_n = (a_1, \ldots, a_k)HE_{n-1}$ for all $n \geq 1$.*

Proof. Apply Proposition 2.7 and Proposition 2.3(iii).

3 The case of equimultiple good filtrations

From now on $(A, \mathfrak{m})$ will be a d-dimensional local ring with an infinite residue field. Let I be an ideal of A and $\mathcal{I} = (I_n)_{n \geq 0}$ a filtration of ideals of A. Recall that $\mathcal{I}$ is said to be I-good if $\mathcal{I}$ is an I-filtration and there exists n_0 such that $II_n = I_{n+1}$ for all $n \geq n_0$. More generally, $\mathcal{I}$ is said to be good if it is I-good for some ideal $I \subseteq A$, equivalently if $\mathcal{I}$ is I_1-good. Note that every good filtration is noetherian.

Adic filtrations are good, and it is well known that if A is analytically unramified the filtration given by the integral closures of the powers of an ideal I is also good. On the other hand, if A is Cohen-Macaulay and $\widetilde{I}$ denotes the Ratliff-Rush closure of I, the filtration $(\widetilde{I^n})_{n \geq 0}$ is good when $\mathrm{ht}(I) > 0$.

In [7] reductions of good filtrations were studied. Let $\mathcal{J} = (J_n)_{n \geq 0} \leq \mathcal{I} = (I_n)_{n \geq 0}$ be two filtrations of A. Following [21], it is said that $\mathcal{J}$ is a reduction of $\mathcal{I}$ if there exists a positive integer m such that $I_n = I_{n-1}J_1 + \cdots + I_{n-m}J_m$ for all $n \gg 0$. It was observed in

[21] that minimal reductions of filtrations never exist. On the contrary, in [7] it is proved that minimal good reductions of good filtrations always exist. Let $\mathcal{J}$ be a good filtration which is a reduction of a good filtration $\mathcal{I}$. We then say that $\mathcal{J}$ is a minimal (good) reduction of $\mathcal{I}$ if does not properly contain any good filtration which is a reduction of $\mathcal{I}$.

Proposition 3.1 *([7, Proposition 2.6]) Let $\mathcal{I}$ be a good filtration. Then $\mathcal{J}$ is a minimal (good) reduction of $\mathcal{I}$ if and only if $\mathcal{J} = (J^n)_{n\geq 0}$, where J is a minimal reduction of I_1. In particular, minimal (good) reductions of $\mathcal{I}$ always exist.*

By abuse of language we then say that the ideal J is a minimal reduction of $\mathcal{I}$. Most of the properties of minimal reductions of ideals can be in this way extended to good filtrations. Define $s(\mathcal{I}) = \dim F_{\mathfrak{m}}(\mathcal{I})$ to be the analytical spread of I. As a natural generalization of the classical result of Northcott and Rees we have the following:

Proposition 3.2 *([7, Lemma 2.8]) Let $\mathcal{I}$ be a good filtration, and for a given element $a \in I_1$ denote by a^0 its image in $I_1/\mathfrak{m}I_1 \hookrightarrow F_{\mathfrak{m}}(\mathcal{I})$. The following are then equivalent:*

(i) $a_1, \ldots, a_s$ is a minimal system of generators of a minimal reduction of $\mathcal{I}$.

(ii) $a_1^0, \ldots, a_s^0$ is a system of parameters of $F_{\mathfrak{m}}(\mathcal{I})$.

It is easy to see [7, Lemma 2.7] that for a good filtration $\mathcal{I}$, $s(\mathcal{I}) = \dim G(\mathcal{I})/\mathfrak{m}G(\mathcal{I})$. Next lemma shows that if $\operatorname{depth} F_{\mathfrak{m}}(\mathcal{I}) > 0$ then we also have the equality $F_{\mathfrak{m}}(\mathcal{I}) = G(\mathcal{I})/\mathfrak{m}G(\mathcal{I})$.

Lemma 3.3 *Let $\mathcal{I}$ be a good filtration. If* $\operatorname{depth} F_{\mathfrak{m}}(\mathcal{I}) > 0$ *then $I_{n+1} \subseteq \mathfrak{m}I_n$ for all $n \geq 0$ and $F_{\mathfrak{m}}(\mathcal{I}) = G(\mathcal{I})/\mathfrak{m}G(\mathcal{I})$. In particular, if $I_1 = \mathfrak{m}$ then $\mathcal{I} = (\mathfrak{m}^n)_{n\geq 0}$.*

Proof. Consider the exact sequence

$$0 \to \bigoplus_{n\geq 0}(\mathfrak{m}I_n + I_{n+1})/\mathfrak{m}I_n \to F_{\mathfrak{m}}(\mathcal{I}) \to G(\mathcal{I})/\mathfrak{m}G(\mathcal{I}) \to 0\,.$$

Since $\mathcal{I}$ is good $I_{n+1} \subseteq \mathfrak{m}I_n$ for $n \gg 0$, hence $\dim(\bigoplus_{n\geq 0}(\mathfrak{m}I_n + I_{n+1})/\mathfrak{m}I_n) = 0$. If $\operatorname{depth} F_{\mathfrak{m}}(\mathcal{I}) > 0$ this implies that $\bigoplus_{n\geq 0}(\mathfrak{m}I_n + I_{n+1})/\mathfrak{m}I_n = 0$, thus $I_{n+1} \subseteq \mathfrak{m}I_n$ for all

$n \geq 0$. Now, if $I_1 = \mathfrak{m}$ we have $I_1 I_n \subseteq I_{n+1} \subseteq \mathfrak{m} I_n = I_1 I_n$ for all $n \geq 0$, that is, $I_{n+1} = I_1 I_n$ for all $n \geq 0$. This means that $\mathcal{I} = (\mathfrak{m}^n)_{n \geq 0}$.

Let $\mathcal{I}$ be a good filtration and J a reduction of $\mathcal{I}$. Similarly to the adic case we define the reduction number of $\mathcal{I}$ with respect to J as the number $r_J(\mathcal{I}) = \min\{n \mid I_{m+1} = JI_m$ for all $m \geq n\}$, and the reduction number of $\mathcal{I}$ as $r(\mathcal{I}) = \min\{r_J(\mathcal{I}) \mid J$ a minimal reduction of $\mathcal{I}\}$. If A is analytically unramified, $\mathcal{I} = (\overline{I^n})_{n \geq 0}$ and J is a minimal reduction of I we set $\overline{r}_J(I) = r_J(\mathcal{I})$, and similarly for $\overline{r}(I)$, the normalized reduction number of I. We also say that $\mathcal{I}$ is equimultiple if $\mathrm{ht}(I) = s(I)$. Note that if A is Cohen-Macaulay this implies that J is generated by a regular sequence.

The following result extends [27, Theorem 1 and Theorem 2], where only adic filtration with reduction number one and two were considered. Our result is valid for good filtrations of any reduction number, as well as it shows that the conditions there are also necessary.

Theorem 3.4 *Let $\mathcal{I}$ be a good filtration and J a minimal reduction of $\mathcal{I}$. Assume that*

(i) J is generated by a regular sequence, and

(ii) $J \cap I_n = JI_{n-1}$ for all $1 \leq n \leq r_J(\mathcal{I})$.

Then $F_{\mathfrak{m}}(\mathcal{I})$ is Cohen-Macaulay if and only if $J \cap \mathfrak{m} I_n = J\mathfrak{m} I_{n-1}$ for all $1 \leq n \leq r_J(\mathcal{I})$.

Proof. Let $J = (a_1, \ldots, a_s)$, $s = s(\mathcal{I})$. Then, $a_1^0, \ldots, a_s^0$ is a system of parameters of $F_{\mathfrak{m}}(\mathcal{I})$. On the other hand, the equalities $J \cap I_n = JI_{n-1}$ are satisfied for all $1 \leq n \leq r_J(\mathcal{I})$ by hypothesis, and also for $n > r_J(\mathcal{I})$ since then $I_n = JI_{n-1}$. Thus, by Theorem 2.8 $F_{\mathfrak{m}}(\mathcal{I})$ is Cohen-Macaulay if and only if $J \cap \mathfrak{m} I_n = J\mathfrak{m} I_{n-1}$ for all $n \geq 1$. The statement is then clear taking into account that $J \cap \mathfrak{m} I_n = J\mathfrak{m} I_{n-1}$ for $n > r_J(\mathcal{I})$ trivially.

Remark 3.5 Note that in [27, Theorem 2] (adic case with reduction number two) it was used the stronger condition $I^2\mathfrak{m} = J\mathfrak{m} I$ instead $J \cap \mathfrak{m} I^2 = J\mathfrak{m} I$.

Remark 3.6 Let $\mathcal{I} = (I_n)_{n \geq 0}$ be a good filtration with $r(\mathcal{I}) \leq 1$ and let J be a minimal reduction of $\mathcal{I}$ such that $r_J(\mathcal{I}) = r(\mathcal{I})$. Then $I_n = J^{n-1} I_1$ for all $n \geq 1$ and $I_n = J^{n-1} I_1 \subseteq I_1^{n-1} I_1 \subseteq I_1^n \subseteq I_n$. Therefore $I_n = I_1^n$ for all $n \geq 0$ and $\mathcal{I}$ is the I_1-adic filtration.

Corollary 3.7 *[27, Corollary 1(a)] Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and I an equimultiple ideal with $r(I) \leq 1$. Then $F_{\mathfrak{m}}(I)$ is Cohen Macaulay.*

Proof. Since A is Cohen-Macaulay and I is equimultiple any minimal reduction of I is generated by a regular sequence. Let J be a such one with $r_J(I) = r(I)$. Then, condition (ii) in Theorem 3.4 holds trivially. On the other hand, since any minimal system of generators of J is part of a minimal system of generators of I we have that $J \cap \mathfrak{m}I = J\mathfrak{m}$. Consequently, $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay by Theorem 3.4.

Example 3.8 Let $A = K[[X, Y, Z, T]]/(T^2, ZT, XZ - YT) = K[[x, y, z, t]]$ where K is a field with $|K| = \infty$. A is a 2-dimensional Cohen-Macaulay local ring. Let $I = (\,(x + z)^2, yt(x + z)\,)$ and $J = (\,(x + z)^2\,)$. Then $\operatorname{ht}(I) = 1$ and $I^2 = JI$, thus since I is not a complete intersection I is equimultiple with $r(I) = 1$. By Corollary 3.7 $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay. Note that A/I is not Cohen-Macaulay, see [6, Remark (22.21)], and neither $G(I)$ is by [6, Proposition (25.1)].

Proposition 3.9 *Let $\mathcal{I} = (I_n)_{n \geq 0}$ be a good filtration with I_1 $\mathfrak{m}$-primary. Assume that $G(\mathcal{I})$ is Cohen-Macaulay and let J be a minimal reduction of $\mathcal{I}$. The following are then equivalent:*

(i) $F_{\mathfrak{m}}(\mathcal{I})$ is Cohen-Macaulay.

(ii) $J \cap \mathfrak{m}I_n = J\mathfrak{m}I_{n-1}$ for all $1 \leq n \leq r_J(\mathcal{I})$.

Proof. Let $J = (a_1, \ldots, a_d)$. Then, $a_1^*, \ldots, a_d^*$ is a system of parameters of $G(\mathcal{I})$, and so it is a regular sequence since $G(\mathcal{I})$ is Cohen-Macaulay. Now apply Proposition 2.3(iii) and Theorem 3.4.

Assume now that I is $\mathfrak{m}$-primary. Let $H_I(n) = \operatorname{length}(A/I^n)$ and $P_I(n)$ its Hilbert function and its Hilbert polynomial respectively. Then, $H_I(n) = P_I(n)$ for $n \gg 0$ and $P_I(n)$ may be written as

$$P_I(n) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n + d - i - 1}{d - i}.$$

Suppose furthermore that A is Cohen-Macaulay. It's a well known result of Northcott that $e_1(I) \geq e_0(I) - \text{length}(A/I) \geq 0$ and that $e_1(I) = 0$ if and only if I is generated by a system of parameters of A. It has also been proved by Huneke [10, Theorem 2.1] and Ooishi [24, Theorem 3.3], independently, that $e_1(I) = e_0(I) - \text{length}(A/I)$ if and only if $r_J(I) \leq 1$ for any minimal reduction J of I. More recently, Sally [25] proved that if $d \geq 2$ and $e_2(I) \neq 0$, then $e_1(I) = e_0(I) - \text{length}(A/I) + 1$ if and only if for some (every) minimal reduction J of I, $r_J(I) = 2$ and $\text{length}(I^2/JI) = 1$. And that if such conditions hold then $e_2(I) = 1$, see also [9]. Moreover, Sally also noted that the equality $e_0(I) - \text{length}(A/I) + 1 = e_1(I)$ does not imply $e_2(I) \neq 0$. Finally, Itoh [13] has proven that if $d \geq 2$ and I is integrally closed, $e_2(I) = 1$ if and only if $e_1(I) = e_0(I) - \text{length}(A/I) + 1$.

For such kind of equimultiple ideals which reduction number two we may then formulate the following:

Theorem 3.10 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and I a $\mathfrak{m}$-primary ideal. Suppose that I is integrally closed and $e_2(I) = 1$. The following are then equivalent:*

(i) $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay.

(ii) $\mu(JI) < \mu(I^2)$ for any minimal reduction J of I.

(iii) $\mu(JI) = \mu(I^2) - 1$ for any minimal reduction J of I.

(Here, $\mu(\cdot)$ denotes the minimal number of generators.)

Proof. Let J be any minimal reduction of I. By [13, Corollary 14], $r_J(I) = 2$, $\text{length}(I^2/JI) = 1$, and $G(I)$ is Cohen-Macaulay. Hence by Proposition 3.9, $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay if and only if $J \cap \mathfrak{m}I^n = J\mathfrak{m}I^{n-1}$ for $n = 1, 2$. On the other hand, $\mathfrak{m}I^2 \subseteq JI$ because $\text{length}(I^2/JI) = 1$, hence the conditions we have to check are equivalent to $J \cap \mathfrak{m}I = J\mathfrak{m}$ and $\mathfrak{m}I^2 = J\mathfrak{m}I$, where the first equality holds because J is a minimal reduction of I.

Consider now the exact sequence

$$0 \longrightarrow JI \longrightarrow I^2 \longrightarrow I^2/JI \longrightarrow 0.$$

Tensorizing by $A/\mathfrak{m}$ we then get the exact sequence

$$JI/\mathfrak{m}JI \xrightarrow{\phi} I^2/\mathfrak{m}I^2 \longrightarrow I^2/JI + \mathfrak{m}I^2 \longrightarrow 0\,.$$

Since $I^2/JI + \mathfrak{m}I^2 = I^2/JI \simeq A/\mathfrak{m}$ we have that $\mathfrak{m}JI = \mathfrak{m}I^2$ if and only if ϕ is injective if and only if $\mathrm{rk}_{A/\mathfrak{m}}(JI/\mathfrak{m}JI) = \mathrm{rk}_{A/\mathfrak{m}}(I^2/\mathfrak{m}I^2) - 1$ if and only if $\mathrm{rk}_{A/\mathfrak{m}}(JI/\mathfrak{m}JI) < \mathrm{rk}_{A/\mathfrak{m}}(I^2/\mathfrak{m}I^2)$.

Let now $(A, \mathfrak{m})$ be an analytically unramified local ring, and I an ideal of A. As we have already mentioned the filtration $(\overline{I^n})_{n\geq 0}$ defined by the integral closures of the powers of I is good. Suppose furthermore that I is $\mathfrak{m}$-primary and let $\overline{H}_I(n) = \mathrm{length}_A(A/\overline{I^n})$ the normalized Hilbert function of I, and $\overline{P}_I(n)$ its normalized Hilbert polynomial. Then $\overline{H}_I(n) = \overline{P}_I(n)$ for large n and $\overline{P}_I(n)$ may be written as

$$\overline{P}_I(n) = \sum_{i=0}^{d}(-1)^i\,\overline{e}_i(I)\binom{n+d-i-1}{d-i}.$$

Suppose moreover that A is Cohen-Macaulay of dimension $d \geq 2$ and I is a parameter ideal. In [14, Proposition 10 and Proposition 12], Itoh has shown that $\overline{e}_1(I) - \mathrm{length}(\overline{I}/I) \geq \mathrm{length}(\overline{I^2}/I\overline{I})$, and that equality holds if and only if $\overline{r}_I(I) \leq 2$. Furthermore, $\overline{e}_2(I) \geq \overline{e}_1(I) - \mathrm{length}(\overline{I}/I)$, and equality also holds if and only if $\overline{r}_I(I) \leq 2$.

The following lemma may be obtained from the above by means of easy arguments, and so we omit the proof. Note that if $d = 2$ it can also be obtained from [10].

Lemma 3.11 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ which is analytically unramified. I a $\mathfrak{m}$-primary ideal, and J a minimal reduction of I. Then:*

(i) If $\overline{e}_2(I) = 0$. $\overline{r}_J(I) \leq 1$ (and $\overline{I}$ is normal).

(ii) If $\overline{e}_2(I) = 1$. $\overline{r}_J(I) = 2$ and $\mathrm{length}(\overline{I^2}/J\overline{I}) = 1$.

Moreover, if I is integrally closed and $\overline{e}_2(I) \leq 1$ then $r_J(I) \leq \overline{r}_J(I)$, $e_2(I) \leq \overline{e}_2(I)$, and both equalities hold if and only if I is normal.

Let $\mathcal{I} = (\overline{I^n})_{n\geq 0}$ and denote by $\overline{F}_{\mathfrak{m}}(I) = \bigoplus_{n\geq 0} \overline{I^n}/\mathfrak{m}\overline{I^n}$ the normalized fiber cone of I. ($\overline{F}(A)$ if $I = \mathfrak{m}$). Similarly to Theorem 3.10 we may obtain:

Theorem 3.12 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with dimension $d \geq 2$ which is analytically unramified and I a $\mathfrak{m}$-primary ideal. If $\overline{e}_2(I) = 1$ the following are then equivalent:*

(i) $\overline{F}_{\mathfrak{m}}(I)$ is Cohen-Macaulay.

(ii) $\mu(J\overline{I}) < \mu(\overline{I^2})$ for any minimal reduction J of I.

(iii) $\mu(J\overline{I}) = \mu(\overline{I^2}) - 1$ for any minimal reduction J of I.

Moreover, if $I = \mathfrak{m}$, $\overline{F}(A)$ is Cohen-Macaulay if and only if $\mathfrak{m}$ is normal.

Proof. First of all note that $G(\mathcal{I})$ is Cohen-Macaulay by [12, Proposition 3]. We may then proceed as in the proof of Theorem 3.10. Finally assume $I = \mathfrak{m}$. If $\overline{F}(A)$ is Cohen-Macaulay then $\mathfrak{m}$ is normal by Lemma 3.3. Conversely, if $\mathfrak{m}$ is normal then $\overline{F}(A) = G(\mathcal{I}) = G(\mathfrak{m})$ which is Cohen-Macaulay as we have already noted.

Recall now that if $(A, \mathfrak{m})$ is an analytically unramified local ring of dimension d and I is $\mathfrak{m}$-primary, the normal genus of I is defined as $\overline{g}(I) = \overline{e}_d(I)$, and the arithmetical genus of A as $p_g(A)$ =sup $\{\overline{g}(I) | I$ is $\mathfrak{m}-$primary$\}$, see [22]. Assume in addition that A is 2-dimensional and Cohen-Macaulay. It can be then shown that for any $\mathfrak{m}$-primary ideal $I \subsetneq A$, the normal genus $\overline{g}(I)$ equals to length$(H^1(X, \mathcal{O}_X))$ where $X = Proj(R(I))$, [22, Theorem 3.1]. If A is also analytically normal then by Lipman [17] there exists a desingularization $Y \rightarrow Spec(A)$, and the integer length$(H^1(Y, \mathcal{O}_Y))$ does not depend on Y and equals to $H(A) =$ sup $\{$length$(H^1(Z, \mathcal{O}_Z)) \mid Z \rightarrow Spec(A)$ a proper birational map with Z normal$\}$. Furthermore, if A is normal such a desingularization can be obtained by blowing up an $\mathfrak{m}$-primary ideal. Therefore, if A is analytically normal we have that $p_g(A) = H(A)$. The ring A is said to be a rational singularity if $H(A) = 0$, and elliptic if $H(A) = 1$.

As a consequence of Theorem 3.12 we may characterize for which rational or elliptic singularities the normalized fiber cone is Cohen-Macaulay. Namely:

Theorem 3.13 *Let $(A, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring which is analytically normal. If A is a rational singularity, then $\overline{F}(A)$ is Cohen-Macaulay. If A is an elliptic singularity the following are equivalent:*

(i) $\overline{F}(A)$ *is Cohen-Macaulay.*

(ii) $\mathfrak{m}$ *is normal.*

(iii) $\mathfrak{m}$ *is normal and* $r(\mathfrak{m}) = 1$, *or* $r(\mathfrak{m}) = 2$.

Proof. If $\overline{F}(A)$ is Cohen-Macaulay then $\mathfrak{m}$ is normal because of Lemma 3.3. Assume now that $\mathfrak{m}$ is normal. Then $r(\mathfrak{m}) = \overline{r}(\mathfrak{m})$ by Lemma 3.11 and so $r(\mathfrak{m}) \leq 2$ since $\overline{g}(\mathfrak{m}) \leq 1$.

For $(iii) \Rightarrow (i)$ we only need to show that if $r(\mathfrak{m}) = 2$ then $\overline{F}(A)$ is Cohen-Macaulay. But, by Lemma 3.11 we have that $\mathfrak{m}$ is normal with $\overline{e}_2(\mathfrak{m}) = 1$, hence by Theorem 3.12 $\overline{F}(A)$ is Cohen-Macaulay.

Example 3.14 (i) Let $A = \mathbb{C}[[x, y, z]]/(x^4 + y^4 + z^2)$ (the local ring of an elliptic complex surface singularity of type $\widetilde{E}_7$). It's easy to see that in this case $r(\mathfrak{m}) = 1$ and $\mathfrak{m}$ is not normal, hence $\overline{F}(A)$ is not Cohen-Macaulay. (Note that $r(\mathfrak{m}) \neq \overline{r}(\mathfrak{m})$.)

(ii) Let $A = \mathbb{C}[[x, y, z]]/(x^3 + y^3 + z^3)$ (type $\widetilde{E}_6$). Then $r(\mathfrak{m}) = 2$, thus $\overline{F}(A)$ is Cohen-Macaulay by Theorem 3.13.

(iii) Let $A = \mathbb{C}[[x, y, z]]/(x^6 + y^3 + z^2)$ (type $\widetilde{E}_8$). Now we have $r(\mathfrak{m}) = 1$, and it can be shown that $\mathfrak{m}$ is normal. Therefore, $\overline{F}(A)$ is Cohen-Macaulay.

4 The case of analytic deviation one ideals

Let I be an ideal of A. Recall that the analytic deviation of I is defined as the difference $\mathrm{ad}(I) = s(I) - \mathrm{ht}\,(I)$. The arithmetical properties of blow up rings of ideals with small deviation have been deeply investigated in recent years, see e. g. [8,3,4,30,5], and the results are particularly good when the reduction number of I is also small. Our main goal in this section is to show that the fiber cone of any analytic deviation one ideal with reduction number less or equal to one is Cohen-Macaulay.

Next lemma summarizes most of the properties concerning minimal reductions of analytic deviation one ideals we shall use. We say that I is generically a complete intersection if $\mu(I_\mathfrak{p}) = \mathrm{ht}\,(I)$ for all $\mathfrak{p} \in \mathrm{Min}(A/I)$.

Lemma 4.1 *Let $(A, \mathfrak{m})$ be a local Cohen-Macaulay ring, and I a generically complete intersection ideal of A with $\mathrm{ht}(I) = h$. Assume $\mathrm{ad}(I) = 1$ and let J be a minimal reduction of I such that $I^2 = JI$. Then, there are elements $a_1, \dots, a_{h+1}$ in J such that:*

(i) $J = (a_1, \dots, a_{h+1})$ *and* $a_1, \dots, a_h$ *is a regular sequence.*

(ii) $I_{\mathfrak{p}} = (a_1, \dots, a_h)_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \mathrm{Min}(A/I)$.

(iii) $((a_1, \dots, a_h)^m : a_{h+1}^n) \cap I^m = (a_1, \dots, a_h)^m$ *for all* n, m.

(iv) If $h \geq 1$, $(a_1, \dots, a_h)^i \cap I^n = (a_1, \dots, a_h)^i I^{n-i}$ *for all* $n \geq 1$, $i = 1, \dots, n-1$.

Proof. For (i) and (ii) see [30, Lemma 2.2]. By [8, Remark 2.1 (iii) and Lemma 2.5 (ii)] we get (iii) and (iv).

Now we state the main result of this section.

Theorem 4.2 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and I an ideal of A. Assume that I is generically a complete intersection, $\mathrm{ad}(I) = 1$, and $r(I) \leq 1$. Then, $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay.*

Proof. By induction on $h = \mathrm{ht}(I)$. Assume $h = 0$ and let $J = (a_1)$ a minimal reduction of I as in Lemma 4.1. Since $\dim F_{\mathfrak{m}}(I) = 1$ it suffices to see that $a_1^0 \in I/\mathfrak{m}I \hookrightarrow F_{\mathfrak{m}}(I)$ is not a zero divisor. This is equivalent to see that $(\mathfrak{m}I^{n+1} : a_1) \cap I^n = \mathfrak{m}I^n$ for all $n \geq 0$, which is clear if $n = 0$ since a_1 is part of a minimal system of generators of I. Assume $n > 0$ and let $x \in (\mathfrak{m}I^{n+1} : a_1) \cap I^n$. Since $r(I) \leq 1$ we have $I^{n+1} = a_1 I^n$, so $xa_1 \in \mathfrak{m}a_1 I^n$ and there exists $y \in \mathfrak{m}I^n$ such that $a_1(x - y) = 0$. By Lemma 4.1(iii) $x - y = 0$ and so $(\mathfrak{m}I^{n+1} : a_1) \cap I^n = \mathfrak{m}I^n$.

Assume now that $h \geq 1$ and let $J = (a_1, \dots, a_{h+1})$ be a minimal reduction of I as in Lemma 4.1. First we are going to see that $a_1^0, \dots, a_h^0$ is a regular sequence in $F_{\mathfrak{m}}(I)$. By Theorem 2.8 it suffices to show that (i): $(a_1, \dots, a_h) \cap I^{n+1} = (a_1, \dots, a_h)I^n$, and (ii): $(a_1, \dots, a_h) \cap \mathfrak{m}I^{n+1} = (a_1, \dots, a_h)\mathfrak{m}I^n$, for all $n \geq 0$. (i) being a direct consequence of Lemma 4.1(iv), let us prove (ii). If $n = 0$ then $(a_1, \dots, a_h) \cap \mathfrak{m}I = (a_1, \dots, a_h)\mathfrak{m}$ because $a_1, \dots, a_h$ is part of a minimal system of parameters of I. Assume $n > 0$. Since

$r(I) \leq 1$ then $(a_1, \ldots, a_h) \cap \mathfrak{m} I^{n+1} = (a_1, \ldots, a_h) \cap \mathfrak{m} J I^n = (a_1, \ldots, a_h) \cap (\mathfrak{m}(a_1, \ldots, a_h) I^n + \mathfrak{m} a_{h+1} I^n) = \mathfrak{m}(a_1, \ldots, a_h) I^n + ((a_1, \ldots, a_h) \cap \mathfrak{m} a_{h+1} I^n)$. Thus it suffices to see that $(a_1, \ldots, a_h) \cap \mathfrak{m} a_{h+1} I^n \subseteq \mathfrak{m}(a_1, \ldots, a_h) I^n$. Let $x a_{h+1} \in (a_1, \ldots, a_h)$ with $x \in \mathfrak{m} I^n$. Then $x \in I \cap (a_1, \ldots, a_h) : a_{h+1}$ and by Lemma 4.1(iii) $x \in (a_1, \ldots, a_h)$. Hence $x \in \mathfrak{m} I^n \cap (a_1, \ldots, a_h) = (a_1, \ldots, a_h) \mathfrak{m} I^{n-1}$ by induction. Consequently, $x a_{h+1} \in \mathfrak{m}(a_1, \ldots, a_h) I^n$ as we wanted to see.

Now set $B = A/(a_1, \ldots, a_h)$, $\overline{\mathfrak{m}} = \mathfrak{m}/(a_1, \ldots, a_h)$, and $\overline{I} = I/(a_1, \ldots, a_h)$. Then B is Cohen-Macaulay, $\mathrm{ad}(I) = 1$, and $r(\overline{I}) \leq 1$. Furthermore, $F_{\mathfrak{m}}(I)/(a_1^0, \ldots, a_h^0) \simeq F_{\overline{\mathfrak{m}}}(\overline{I})$, hence by induction $F_{\overline{\mathfrak{m}}}(\overline{I})$ is Cohen-Macaulay and so $F_{\mathfrak{m}}(I)$ is.

Remark 4.3 Note that one can find analytic deviation one ideals with $r(I) = 1$ such that the associated graded ring $G(I)$ is not Cohen-Macaulay, see [30, Example].

Remark 4.4 By means of Theorem 4.2 we may also recover the case of monomials curves in $\mathbb{P}^3$ lying on a quadric. Indeed, by [19, Proposition (3.1.2)] the defining ideal of such a curve has analytic deviation and reduction number one.

5 The Hilbert function of Cohen-Macaulay fiber cones

In this section we want to describe the behaviour of the Hilbert function of Cohen-Macaulay fiber cones. So assume $(A, \mathfrak{m})$ is a local ring, $\mathcal{I} = (I_n)_{n \geq 0}$ a good filtration of ideals of A, and let J be a minimal reduction of $\mathcal{I}$. There exists a finite morphism of graded rings defined by

$$\begin{array}{ccc} F_{\mathfrak{m}}(J) & \xrightarrow{\Phi} & F_{\mathfrak{m}}(\mathcal{I}) \\ \overline{a} \in J^n/\mathfrak{m} J^n & \mapsto & \varphi(\overline{a}) = a^0 \in I_n/\mathfrak{m} I_n \end{array}$$

(note that Φ is injective in degree one since $J \cap \mathfrak{m} I_1 = \mathfrak{m} J$). Furthermore, J is generated by a family of analytically independent elements and so $F_{\mathfrak{m}}(J)$ is a polynomial ring in $\mu(J) = s(\mathcal{I})$ variables. As a consequence we have that $F_{\mathfrak{m}}(\mathcal{I})$ is a Cohen-Macaulay ring if and only if $F_{\mathfrak{m}}(I)$ is free as a $F_{\mathfrak{m}}(J)$-module. In particular, the multiplicity $e(F_{\mathfrak{m}}(I))$ equals to its rank, and since $J F_{\mathfrak{m}}(\mathcal{I}) = \bigoplus_{n \geq 1} (J + \mathfrak{m} I_n)/\mathfrak{m} I_n$ is generated by a system of parameters of $F_{\mathfrak{m}}(\mathcal{I})$ with $\dim\, (F_{\mathfrak{m}}(\mathcal{I})/J F_{\mathfrak{m}}(\mathcal{I})) = 0$ we obtain $e(F_{\mathfrak{m}}(I)) = \mathrm{length}(F_{\mathfrak{m}}(\mathcal{I})/J F_{\mathfrak{m}}(\mathcal{I}))$.

We summarize the above considerations in the following lemma.

Lemma 5.1 *Let $(A, \mathfrak{m})$ be a local ring and $\mathcal{I}$ a good filtration of ideals of A. Let J be a minimal reduction of $\mathcal{I}$. Then:*

(i) $F_{\mathfrak{m}}(\mathcal{I})$ is Cohen-Macaulay if and only if $F_{\mathfrak{m}}(\mathcal{I})$ is free as a graded $F_{\mathfrak{m}}(J)$-module.

(ii) If $F_{\mathfrak{m}}(\mathcal{I})$ is Cohen-Macaulay then

$$\mathrm{rk}_{F_{\mathfrak{m}}(J)}(F_{\mathfrak{m}}(\mathcal{I})) = e(F_{\mathfrak{m}}(\mathcal{I})) = \mathrm{length}(F_{\mathfrak{m}}(\mathcal{I})/JF_{\mathfrak{m}}(\mathcal{I})).$$

Remark 5.2 Observe that if $F_{\mathfrak{m}}(\mathcal{I})$ is Cohen-Macaulay then the morphism Φ must be injective, hence $J^n \cap \mathfrak{m}I_n = \mathfrak{m}J^n$ for all $n \geq 0$.

Now we may give the following behaviour of the Hilbert function of a fiber cone which is Cohen-Macaulay. It generalizes [27, Theorem 6] to good filtrations.

Theorem 5.3 *Let $(A, \mathfrak{m})$ be a local ring and $\mathcal{I} = (I_n)_{n\geq 0}$ a good filtration of ideals of A. Let J be a minimal reduction of $\mathcal{I}$, $r = r_J(\mathcal{I})$, and $s = s(\mathcal{I})$. If $F_{\mathfrak{m}}(\mathcal{I})$ is Cohen-Macaulay then*

$$\begin{aligned}\mu(I_n) &= \sum_{i=0}^{r}(\mu(I_i) - \mathrm{length}(JI_{i-1}/JI_{i-1} \cap \mathfrak{m}I_i))\binom{n+s-i-1}{s-1} \\ &= \sum_{i=0}^{\infty}(\mu(I_i) - \mathrm{length}(JI_{i-1}/JI_{i-1} \cap \mathfrak{m}I_i))\binom{n+s-i-1}{s-1}.\end{aligned}$$

Proof. Consider the family of elements in $F_{\mathfrak{m}}(\mathcal{I})$ given by $\left\{1, \{a^0_{i_1}, \dots, a^0_{i_{r_i}}\}\right\}_{i=1,\dots,r}$, where $1 \in A/\mathfrak{m}$ and $\{\overline{a^0_{i_1}}, \dots, \overline{a^0_{i_{r_i}}}\}$ is a basis of the $A/\mathfrak{m}$-vector space $I_i/\mathfrak{m}I_i + JI_{i-1}$, for all $i = 1, \dots, r$. It's then clear that $\left\{1, \{a^0_{i_1}, \dots, a^0_{i_{r_i}}\}\right\}_{i=1,\dots,r}$ is a system of generators of $F_{\mathfrak{m}}(\mathcal{I})$ as a $F_{\mathfrak{m}}(J)$-module which is a basis because its cardinal equals to $\mathrm{length}(F_{\mathfrak{m}}(\mathcal{I})/JF_{\mathfrak{m}}(\mathcal{I})) = \mathrm{rk}_{F_{\mathfrak{m}}(J)}(F_{\mathfrak{m}}(\mathcal{I}))$. Let us denote by $b^i_j = a^0_{i_j}$, for all $i = 1, \dots, r$ and $j = 1, \dots, r_i$. Then, $F_{\mathfrak{m}}(\mathcal{I}) = \bigoplus_{i=1}^{r}\left(\bigoplus_{j=1}^{r_i} b^i_j F_{\mathfrak{m}}(J)\right)$ and taking the piece of degree n we get $I_n/\mathfrak{m}I_n = \bigoplus_{i=0}^{r}(b^i_1, \dots, b^i_{r_i})F_{\mathfrak{m}}(J)_{n-i}$ for all $n \geq 0$. Since $\mathrm{length}(F_{\mathfrak{m}}(J)_{n-i}) = \binom{n+s-i-1}{s-1}$ we have that $\mu(I_n) = \mathrm{length}(I_n/\mathfrak{m}I_n) = \sum_{i=0}^{r} r_i\binom{n+s-i-1}{s-1}$. Furthermore, $r_i = \mathrm{length}(I_i/\mathfrak{m}I_i + JI_{i-1}) = \mathrm{length}((I_i/\mathfrak{m}I_i)/(\mathfrak{m}I_i + JI_{i-1}/\mathfrak{m}I_i)) = \mathrm{length}(I_i/\mathfrak{m}I_i) - \mathrm{length}(\mathfrak{m}I_i + JI_{i-1}/\mathfrak{m}I_i) = \mu(I_i) - \mathrm{length}(JI_{i-1}/JI_{i-1} \cap \mathfrak{m}I_i)$. And the sum can be extended to ∞ since $I_i = JI_{i-1}$ for $i > r$.

Remark 5.4 As a by-product of the above result we also get that if $F_{\mathfrak{m}}(\mathcal{I})$ is Cohen-Macaulay then $r_J(\mathcal{I})$ and length$(JI_{i-1}/JI_{i-1} \cap \mathfrak{m}I_i)$ are independent of J for all $i \geq 0$. The part concerning the independence of the reduction number can also be deduced for adic filtrations from [16, Proposition 4.25]. See [7,18,29] for other related results.

Remark 5.5 It may be proven in the adic case that the converse of Theorem 5.3 also holds, see [1]. The same idea can be applied for the case of good filtrations: Assuming that particular form for the Hilbert function of the fiber cone and after a straightforward computation one gets that the multiplicity of $F_{\mathfrak{m}}(\mathcal{I})$ coincides with the length of $F_{\mathfrak{m}}(\mathcal{I})/JF_{\mathfrak{m}}(\mathcal{I})$, hence $F_{\mathfrak{m}}(\mathcal{I})$ being Cohen-Macaulay. We thank J. Verma for explaining us that.

For the particular case of $\mathfrak{m}$-primary good filtrations whose associated graded ring is Cohen-Macaulay we may express the Hilbert function of the fiber cone in a better way. As a consequence we also give and affirmative answer to what Shah conjectured in [27, Question 3(d)].

Corollary 5.6 *Let $(A, \mathfrak{m})$ be a local ring and $\mathcal{I} = (I_n)_{n\geq 0}$ a good filtration such that I_1 is $\mathfrak{m}$-primary. Let J be a minimal reduction of $\mathcal{I}$ and $r = r_J(\mathcal{I})$. If $G(\mathcal{I})$ and $F_{\mathfrak{m}}(\mathcal{I})$ are Cohen-Macaulay then*

$$\mu(I_n) = \sum_{i=0}^{r}(\mu(I_i) - \mu(JI_{i-1}))\binom{n+d-i-1}{d-1} \text{ for all } n \geq 0.$$

Furthermore, $\mu(I_i) - \mu(JI_{i-1}) \geq 0$ for all $i = 0, \ldots, r$.

Proof. By Proposition 3.9 $JI_{i-1} \cap \mathfrak{m}I_i \subseteq J \cap \mathfrak{m}I_i = J\mathfrak{m}I_{i-1} \subseteq JI_{i-1} \cap \mathfrak{m}I_i$ for all $0 \leq i \leq r$, thus $JI_{i-1} \cap \mathfrak{m}I_i = J\mathfrak{m}I_{i-1}$ for all $0 \leq i \leq r$. Now, apply Theorem 5.3 and note that $\mu(I_i) - \mu(JI_{i-1}) = \text{length}(I_i/\mathfrak{m}I_i + JI_{i-1}) \geq 0$ for all $i = 0, \ldots, r$.

Corollary 5.7 *Let $(A, \mathfrak{m})$ be a local ring and I an ideal of A with $r(I) = 1$. Let $s = s(I)$ and assume $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay (for instance, if A is Cohen-Macaulay and I is equimultiple). Then*

$$\mu(I^n) = \binom{n+s-1}{s-1} + (\mu(I) - s)\binom{n+s-2}{s-1} \textit{ for all } n \geq 0.$$

Proof. Apply Theorem 5.3 taking into account that if J is a minimal reduction of I then $J \cap \mathfrak{m}I = \mathfrak{m}J$.

If I is an analytic deviation one ideal with small reduction number we then obtain:

Corollary 5.8 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and I an ideal of A with* $\mathrm{ht}(I) \geq 1$. *Assume I is generically a complete intersection, $ad(I) = 1$, and $r(I) = 1$. Set $h = \mathrm{ht}(I)$. Then*

$$\mu(I^n) = \binom{n+h}{h} + (\mu(I) - h - 1)\binom{n+h-1}{h} \text{ for all } n \geq 0.$$

Proof. $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay by Theorem 4.2.

As for $\mathfrak{m}$-primary ideals whose second Hilbert coefficient is equal to one we record the following nice formula if $d = 2$.

Corollary 5.9 *Let $(A, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring and I a $\mathfrak{m}$-primary ideal which is integrally closed. Assume $e_2(I) = 1$ and $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay. Then*

$$\mu(I^n) = n\mu(I) \text{ for all } n \geq 0.$$

Proof. Let J be a minimal reduction of I. By Lemma 3.11 $r_J(I) = 2$ and $G(I)$ is Cohen-Macaulay. Furthermore, by Theorem 3.10 $\mu(JI) = \mu(I^2) - 1$. Hence by Corollary 5.6 we get $\mu(I^n) = \binom{n+1}{1} + (\mu(I) - 2)\binom{n}{1} + \binom{n-1}{1} = n\mu(I)$ for all $n \geq 0$.

Similarly, if the second normalized Hilbert coefficient is one we obtain:

Corollary 5.10 *Let $(A, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring which is analytically unramified and I a $\mathfrak{m}$-primary ideal. Assume $\overline{e}_2(I) = 1$ and $\overline{F}_{\mathfrak{m}}(I$ is Cohen-Macaulay. Then*

$$\mu(\overline{I^n}) = n\mu(\overline{I}) \text{ for all } n \geq 0.$$

Finally, we may also recover the following result concerning the minimal number of generators of the powers of the maximal ideal of an elliptic surface singularity.

Corollary 5.11 *[23, Corollary 3.6] Let $(A, \mathfrak{m})$ be a local ring and assume A is an elliptic surface singularity. Then, either $\mu(\mathfrak{m}^n) = e(A)n + 1 = (\mathrm{emb}(A) - 1)n$ for all $n \geq 0$, or $\mu(\mathfrak{m}^n) = e(A)n = \mathrm{emb}(A)n$ for all $n \geq 0$.*

Proof. Since A is elliptic $\overline{g}(\mathfrak{m}) \leq 1$ and by Lemma 3.11, $r(\mathfrak{m}) \leq 2$. If $r(\mathfrak{m}) = 1$ then $G(\mathfrak{m})$ is Cohen-Macaulay by Corollary 3.7, hence by Corollary 5.7 $\mu(\mathfrak{m}^n) = (\mathrm{emb}(A) - 1)n + 1 = e(A)n + 1$ for all $n \geq 0$. If $r(\mathfrak{m}) = 2$ then $\mathfrak{m}$ is normal by Lemma 3.11, and $\overline{F}(A) = G(\mathfrak{m})$ is Cohen-Macaulay by Theorem 3.12. Now apply Corollary 5.10 to get $\mu(\mathfrak{m}^n) = e(A)n = \mathrm{emb}(A)n$ for all $n \geq 0$.

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