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ON THE COHEN-MACAULAYNESS OF DIAGONAL SUBALGEBRAS OF THE REES ALGEBRA

by

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On the Cohen-Macaulayness of diagonal subalgebras of the Rees algebra

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1 Introduction

Let Y be a smooth closed subscheme of $\mathbb{P}_k^{n-1}$ defined by a homogeneous ideal $I \subset A = k[X_1, \ldots, X_n]$. Let X be obtained by blowing up $\mathbb{P}_k^{n-1}$ along Y. Denote by I_c the degree c part of I. For c large enough, there is an embedding of X in $\mathbb{P}_k^{N-1}$, where $N = \dim_k I_c$. The homogeneous coordinate ring of this embedding is the subalgebra $k[I_c]$ of A. On the other hand, it's well-known that $Proj(R_A(I^e)) = Proj(R_A(I))$, that is, powers of I blow-up to the same scheme X. So the rings $k[(I^e)_c]$ are coordinate rings of projective embeddings of X, for $c >> e > 0$.

We want to study the Cohen-Macaulay property of these rings. Considering the Rees algebra $R_A(I)$ endowed with a natural bigrading, one can obtain the above rings $k[(I^e)_c]$ as diagonals of $R_A(I)$ (see Section 2). A natural question is which properties of $R_A(I)$ are

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inherited by $k[(I^e)_c]$. Assume that $R_A(I)$ is Cohen-Macaulay. Do exist positive integers c, e such that $k[(I^e)_c]$ is Cohen-Macaulay?.

Our study is based on the paper by A. Conca, J. Herzog, N.V. Trung and G. Valla [2]. When I is a complete intersection ideal, they prove that there exist positive integers c, e such that $k[(I^e)_c]$ is Cohen-Macaulay and describe exactly for which c, e the rings $k[(I^e)_c]$ are Cohen-Macaulay (see [2], Theorem 4.6). When I is an equigenerated ideal, they also give a positive answer if $k[I_d]$ is Cohen-Macaulay and $a(k[I_d]) < 0$ (e.g., this holds when I is of linear type), and if I is a perfect ideal of codimension 2 such that I has a linear presentation matrix, $\mu(I) > n$ and I satisfies G_n (see [2], Corollaries 3.12, 3.13 and 3.14). Their results are obtained by studying a $\mathbb{Z}^2$-graded minimal free resolution of $R_A(I)$ over S, where S is a suitable polynomial ring endowed with a certain bigrading. If this resolution is "good" (see Definition 2.6 for the precise meaning) it is possible to give a positive answer to the question. They also conjecture that if $R_A(I)$ is Cohen-Macaulay then always exist positive integers c, e such that $k[(I^e)_c]$ is Cohen-Macaulay.

In Section 4, we will give a positive answer to the above conjecture. Our approach is based on a detailed study of the bigraded minimal free resolution of $R_A(I)$. We will see that if $R_A(I)$ is Cohen-Macaulay this resolution is always "good" and so the conjecture is true. Namely, let $A = k[X_1, ..., X_n]$ be a polynomial ring and I a homogeneous ideal of A. Suppose that I is minimally generated by forms $f_1, ..., f_r$ of degree $d_1, ..., d_r$ respectively and put $d = d_r \geq ... \geq d_1$, $u = \sum_{j=1}^r d_j$. Then:

Theorem (Theorem 4.4) *Assume that $R_A(I)$ is Cohen-Macaulay. Then $k[(I^e)_c]$ is Cohen-Macaulay for $c >> 0$ relatively to $e >> 0$. More explicitely, given $e > 0$ let*

$$\alpha = min\{(e-1)d + u - n, e(u-n)\}$$
$$\beta = min\{(e-1)d + u - d_1(r-1), e(u-d_1)\}$$

Then $k[(I^e)_c]$ is Cohen-Macaulay if $c > max\{\alpha, \beta, de\}$.

Moreover, if I is equigenerated then $k[(I^e)_c]$ is Cohen-Macaulay for all $c > max\{\alpha, de\}$. In particular, if I is equigenerated and $n \geq u$ then $k[(I^e)_c]$ is Cohen-Macaulay for all $c > de$.

Our main tool is the bigraded a-invariant of the Rees algebra. By computing this a-invariant (see Proposition 3.7), and then comparing with the results obtained in the key Proposition 3.6, we are able to bound the shifts appearing in the bigraded minimal free resolution of $R_A(I)$ when $R_A(I)$ is Cohen-Macaulay, and as a consequence one can see that the resolution is good.

Recently, S.D. Cutkosky and J. Herzog have proved a very general theorem about the Cohen-Macaulay property of the rings $k[(I^e)_c]$ when the scheme $Proj(R_A(I))$ is Cohen-Macaulay (a weaker assumption). In this case, other conditions are needed to ensure that there exists a constant f such that $k[(I^e)_c]$ is Cohen-Macaulay for $c \geq ef$ (see [3], Theorem 4.1).

We may also apply the relationship between the bigraded a-invariant and the shifts in a graded minimal free resolution proved in Proposition 3.6 to obtain an analogous result for the diagonal of a standard bigraded k-algebra. We get the following:

Proposition (Proposition 4.5) *Let R be a standard bigraded Cohen-Macaulay k-algebra. Assume that $\mathbf{a}(R) < 0$. Then R_Δ is a Cohen-Macaulay ring, for all the diagonals Δ.*

This result has been proved by E. Hyry for the $(1,1)$-diagonal of any standard bigraded algebra defined over a local ring in [9], Theorem 2.5.

2 Preliminaries

In this section, we fix some notation and recall several results of [2] which will play a fundamental role in the sequel . Throughout this paper k will be a field, $A = k[X_1, ..., X_n]$ the polynomial ring in n variables with the usual grading and I a homogeneous ideal of A.

Let $R = \bigoplus_{(u,v)\in\mathbb{N}^2} R_{(u,v)}$ be a $\mathbb{N}^2$-graded k-algebra and denote by $M^2(R)$ the category of $\mathbb{Z}^2$-graded R-modules. Given c, e positive integers, the set $\Delta := \{ (cl, el) \mid l \in \mathbb{Z}\} \subset \mathbb{Z}^2$ is called the (c, e)-diagonal of $\mathbb{Z}^2$. We may then define the diagonal subalgebra of R along Δ as $R_\Delta := \bigoplus_{l\in\mathbb{Z}} R_{(cl,el)} \subset R$. Similary, given a $\mathbb{Z}^2$-graded R-module L we define the diagonal submodule of L along Δ as $L_\Delta := \bigoplus_{l\in\mathbb{Z}} L_{(cl,el)}$. It can be easily seen that we have an exact functor $(\ )_\Delta : M^2(R) \to M^1(R_\Delta)$, where $M^1(R_\Delta)$ denotes the category of $\mathbb{Z}$-graded R_Δ-modules.

Assume that I is minimally generated by homogeneous polynomials $f_1, ..., f_r$ of degrees $d_1, ..., d_r$ respectively, and put $d = d_r \geq ... \geq d_1$, $u = \sum_{j=1}^r d_j$. In this situation, we can consider the Rees algebra of I: $R_A(I) = \bigoplus_{n \geq 0} I^n t^n \subset A[t]$ with the $\mathbb{N}^2$-grading given by $R_A(I)_{(i,j)} = (I^j)_i$. On the other hand, let $S = k[X_1, ..., X_n, Y_1, ..., Y_r]$ be the polynomial ring in $n+r$ variables with the $\mathbb{N}^2$-grading obtained by giving deg $X_i = (1,0)$ for $i = 1, ..., n$, deg $Y_j = (d_j, 1)$ for $j = 1, ..., r$. Then we get an epimorphism of $\mathbb{N}^2$-graded algebras defined by :

$$\begin{array}{ccc} S & \longrightarrow & R_A(I) \\ X_i & \mapsto & X_i \\ Y_j & \mapsto & f_j t \end{array}$$

The next result gives a relationship between $k[(I^e)_c]$ and the Rees algebra $R_A(I)$. If $c \geq de$ we obtain that $k[(I^e)_c]$ is isomorphic to the (c,e)- diagonal of $R_A(I)$. By this reason, it will be useful to study the functor $(\ )_\Delta$.

Proposition 2.1 *([2], Section 1)*

(i) S_Δ *is a Cohen-Macaulay ring and* $\dim S_\Delta = n + r - 1$.

(ii) If $c \geq de$ *then* $R_A(I)_\Delta \cong k[(I^e)_c]$.

(iii) If $c \geq de + 1$ *then* $\dim k[(I^e)_c] = n$.

From now we will always assume that $c \geq de + 1$.

Let $0 \to D_t \to \ldots \to D_1 \to D_0 = S \to R_A(I) \to 0$ be a $\mathbb{Z}^2$ -graded minimal free resolution of $R_A(I)$ over S. For every p, D_p is a direct sum of S-modules of the type $S(a,b)$, where $S(a,b)$ is the twisted module S with shifting degree (a,b). The central idea in [2] is to obtain information about $R_A(I)_\Delta$ from this resolution and the modules $S(a,b)_\Delta$. We will denote by m_S and m_{S_Δ} the homogeneous maximal ideals of S and S_Δ respectively.

From the computation of the local cohomology of the modules $S(a,b)_\Delta$ it is easy to see that dim $S(a,b)_\Delta = n+r-1$ ([2], Corollary 3.2). As a consequence, and applying [2], Proposition 2.8 we obtain:

Proposition 2.2 *Let L be a finitely generated $\mathbb{Z}^2$-graded S-module and let*

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to L \to 0$$

be a $\mathbb{Z}^2$-graded minimal free resolution of L over S. Assume that $(D_p)_\Delta$ is a Cohen-Macaulay module, for $p = 0, ..., t$. Then

$$H^q_{m_{S_\Delta}}(L_\Delta) \cong (H^{q+1}_{m_S}(L))_\Delta, \ \forall q \geq 0.$$

In order to apply the above proposition we need study the Cohen-Macaulayness of the diagonals of the twisted modules $S(a,b)$.

Definition 2.3 *For any $\mathbb{Z}$-graded module E, we define:* $\operatorname{supp} E = \{\, l \in \mathbb{Z} \mid E_l \neq 0\}$

Proposition 2.4 *([2], Lemmas 3.1 and 3.3) Let $a, b \in \mathbb{Z}$. Then :*

(i) $H^q_{m_{S_\Delta}}(S(a,b)_\Delta) = 0, \ \forall q \neq n, r, n+r-1.$

(ii) $\operatorname{supp} H^r_{m_{S_\Delta}}(S(a,b)_\Delta) = \{\, s \in \mathbb{Z} \mid \frac{(b+r)d-u-a}{c-ed} \leq s \leq -\frac{b+r}{e}\}.$

(iii) $\operatorname{supp} H^n_{m_{S_\Delta}}(S(a,b)_\Delta) = \{\, s \in \mathbb{Z} \mid \frac{-b}{e} \leq s \leq \frac{bd-a-n}{c-ed}\}.$

Definition 2.5 *We say that a property holds for $c >> 0$ relatively to $e >> 0$ if there exists e_0 such that for all $e \geq e_0$ there exists a positive integer $c(e)$ depending on e such that this property holds for all (c,e) with $c \geq c(e)$.*

Definition 2.6 *Let L be a finitely generated $\mathbb{Z}^2$-graded S-module and*

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to L \to 0$$

a $\mathbb{Z}^2$-graded minimal free resolution of L over S. We say that the resolution is good if every module D_p is a direct sum of modules $S(a,b)$ such that $S(a,b)_\Delta$ are Cohen-Macaulay for $c >> 0$ relatively to $e >> 0$.

In this paper we are interested in studying the relationship between the Cohen-Macaulay property of a finitely generated $\mathbb{Z}^2$- graded S-module L and that of its diagonal L_Δ. Note that if L has a good resolution we can apply Proposition 2.2 for large Δ. In particular, if L is Cohen-Macaulay this property will be inherited by L_Δ.

The last result of this section gives necessary and sufficient numerical conditions for the Cohen-Macaulayness of the modules $S(a,b)_\Delta$ for $c >> 0$ relatively to $e >> 0$. Applying Proposition 2.4, we obtain:

Proposition 2.7 *([2], Corollary 3.5) Let $a, b \in \mathbb{Z}$. Then $S(a,b)_\Delta$ is a Cohen-Macaulay module for $c >> 0$ relatively to $e >> 0$ if and only if a, b satisfy one of the following conditions:*

(i) $b \leq -r$ *and* $(b+r)d - u - a > 0$,

(ii) $-r < b < 0$,

(iii) $b \geq 0$ *and* $bd - a - n < 0$.

3 On multigraded a-invariants

A. BASIC DEFINITIONS

In this section we recall some definitions of the theory of multigraded rings. The basic source is [7]. Let $S = \bigoplus_{\mathbf{n} \in \mathbb{N}^r} S_{\mathbf{n}}$ be a $\mathbb{N}^r$-graded noetherian ring. Let us denote by $S^+ = \bigoplus_{\mathbf{n} \neq \mathbf{0}} S_{\mathbf{n}}$, $d = \dim S$ and $M^r(S)$ the category of r-graded S-modules.

Given a group morphism $\varphi : \mathbb{Z}^r \to \mathbb{Z}^q$ such that $\varphi(\mathbb{N}^r) \subset \mathbb{N}^q$, we can define $S^\varphi := \bigoplus_{\mathbf{m} \in \mathbb{N}^q} (\bigoplus_{\varphi(\mathbf{n}) = \mathbf{m}} S_{\mathbf{n}})$. We may think S^φ as the ring S endowed with a different grading. Similary, given an r-graded S-module L, we define $L^\varphi := \bigoplus_{\mathbf{m} \in \mathbb{Z}^q} (\bigoplus_{\varphi(\mathbf{n}) = \mathbf{m}} L_{\mathbf{n}})$. Then $(\quad)^\varphi : M^r(S) \to M^q(S^\varphi)$ is an exact functor. Considering $\varphi_i : \mathbb{Z}^r \to \mathbb{Z}$ defined by $\varphi_i(\mathbf{n}) = n_i$, with $\mathbf{n} = (n_1, ..., n_r)$, we will denote by $S_i = S^{\varphi_i}$, $L_i = L^{\varphi_i}$.

We will say that S is defined over a local ring if $S_0 = A$ is a local ring. Then S has a unique homogeneous maximal ideal $\mathcal{M} = \mathfrak{m} \oplus S^+$, where $\mathfrak{m}$ is the maximal ideal of A.

The next lemma shows that the local cohomology modules behave well under a change of grading.

Lemma 3.1 *([7].Lemma 1.1) Let S be a $\mathbb{N}^r$-graded noetherian ring defined over a local ring. Let $\mathcal{M}$ be the homogeneous maximal ideal of S. Let $\varphi : \mathbb{Z}^r \to \mathbb{Z}^q$ be a morphism such that $\varphi(\mathbb{N}^r) \subset \mathbb{N}^q$. For every r-graded S-module L, we have*

$$(H^i_{\mathcal{M}}(L))^{\varphi} = H^i_{\mathcal{M}^{\varphi}}(L^{\varphi}),\ \forall i.$$

Definition 3.2 *Let S be a $\mathbb{N}^r$-graded noetherian ring defined over a local ring and M the homogeneous maximal ideal of S. Let $\mathcal{M}_j = \mathcal{M}^{\varphi_j}$, for $j = 1, ..., r$. Let*

$$a_j = a(S_j) = \max\{\ m \in Z \mid [H^d_{\mathcal{M}_j}(S_j)]_m \neq 0\}.$$

We define the multigraded a-invariant of S as $\mathbf{a}(S) = (a_1, ..., a_r)$.

Using Lemma 3.1 we obtain that

$$a_j = \max\{\ m \in Z \mid \exists \mathbf{n} \in \mathbb{Z}^r : \varphi_j(\mathbf{n}) = m, [H^d_{\mathcal{M}}(S)]_{\mathbf{n}} \neq 0\}.$$

In analogous way, given a finitely generated r-graded S-module L, we can define the multigraded a-invariant of L. If dim $L = l$, let

$$b_j = \max\{\ m \in Z \mid \exists \mathbf{n} \in \mathbb{Z}^r : \varphi_j(\mathbf{n}) = m, [H^l_{\mathcal{M}}(L)]_{\mathbf{n}} \neq 0\}.$$

Then $\mathbf{a}(L) = (b_1, ..., b_r)$ is called the multigraded a-invariant of L.

B. ON THE COMPUTATION OF MULTIGRADED a-INVARIANTS

Let S be a $\mathbb{N}^r$-graded noetherian k-algebra and $\mathcal{M}$ the homogeneous maximal ideal of S. Assume that we have an exact sequence of finitely generated graded S-modules

$$\ldots \to D_t \to \ldots \to D_1 \to D_0 \to 0$$

such that Im $(D_{p+1}) \subset \mathcal{M}D_p$, for all $p \geq 0$. Let us denote by $\{\ v_{p,i}\}$ the set of degree vectors of a minimal homogeneous system of generators of D_p. Note that this set is uniquely determined because it can be obtained as the homogeneous components of the vector space $D_p \otimes_S k$ which are not zero. We set $\mathbf{m}_p = \min_{\leq_{lex}}\{ v_{p,i}\}$ and $\mathbf{M}_p = \max_{\leq_{lex}}\{ v_{p,i}\}$, where $\leq_{lex}$ is the lexicographic order. Let us denote by $n_p^j = \min_i\{ v_{p,i}^j\}$, $N_p^j = \max_i\{ v_{p,i}^j\}$, where $v_{p,i}^j$is the j-th component of the vector $v_{p,i}$, $\mathbf{n}_p = (n_p^1, ..., n_p^r)$, $\mathbf{N}_p = (N_p^1, ..., N_p^r)$ and let us consider also $\leq$ the partial order in $\mathbb{Z}^r$ defined coefficientwise. Then we have:

Lemma 3.3 *(i)* $\mathbf{n}_p \leq \mathbf{n}_{p+1}$.

(ii) $\mathbf{m}_p <_{lex} \mathbf{m}_{p+1}$

Proof. Let $C_p = Coker(D_{p+1} \to D_p)$, $\forall p \geq 1$. Then there are short exact sequences:

$$0 \to C_{p+2} \to D_{p+1} \to C_{p+1} \to 0, \ \ \forall p \geq 0.$$

Applying the functor $- \bigotimes_S k$, we get exact sequences:

$$C_{p+2}/\mathcal{M}C_{p+2} \to D_{p+1}/\mathcal{M}D_{p+1} \to C_{p+1}/\mathcal{M}C_{p+1} \to 0, \ \ \forall p \geq 0.$$

It holds that $C_{p+2} \subset \mathcal{M}D_{p+1}$ and so the first arrow is the zero morphism. Therefore we get isomorphisms

$$D_{p+1}/\mathcal{M}D_{p+1} \to C_{p+1}/\mathcal{M}C_{p+1}.$$

Let us denote by $\{ e_{p,i}\}$ a minimal homogeneous system of generators of D_p with $\deg(e_{p,i}) = v_{p,i}$ and by $f : D_{p+1} \to D_p$. From the last isomorphism it follows that $f(e_{p+1,i}) \neq 0$, for all i. Let us fix $i = l$. We can write $f(e_{p+1,l}) = \sum_i \lambda_i e_{p,i}$, where λ_i are homogeneous elements of $\mathcal{M}$. Set $deg(\lambda_i) = (\lambda_i^1, ..., \lambda_i^r) \in \mathbb{N}^r$ and note that $deg(\lambda_i) \neq \mathbf{0}$ if $\lambda_i \neq 0$. Looking the j-th component of the degree, we get $v_{p+1,l}^j \geq \min_l\{ v_{p,i}^j\} = n_p^j$, and so $n_{p+1}^j \geq n_p^j$, for all j. To show ii) we will prove that $\mathbf{v}_{p+1,l} >_{lex} \mathbf{m}_p$. It is clear that $v_{p+1,l}^1 \geq \min_l\{ v_{p,i}^1\} = m_p^1$. If $v_{p+1,l}^1 > m_p^1$, then we get $\mathbf{v}_{p+1,l} >_{lex} \mathbf{m}_p$. Otherwise, $v_{p,i}^1 = m_p^1$ and $\lambda_i^1 = 0$ for all i such that $\lambda_i \neq 0$. So we get $v_{p+1,l}^2 \geq \min_i\{ v_{p,i}^2 \mid v_{p,i}^1 = m_p^1\} = m_p^2$. By repeating this argument, we obtain the result since there exist i, j such that $\lambda_i^j > 0$.

Let L be a finitely generated r-graded S-module and assume that there exists a finite r-graded minimal free resolution of L over S:

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to L \to 0$$

For every $p = 0, \dots, t$, let $D_p = \bigoplus_i S(a^1_{p,i}, \dots, a^r_{p,i})$. In the next lemma we study the shifts appearing in this resolution. For the $\mathbb{Z}$-graded case, see for instance [4], Exercise 20.19. Note that, with the notation introduced before, $n^j_p = \min_i\{-a^j_{p,i}\}$, $N^j_p = \max_i\{-a^j_{p,i}\}$, $\mathbf{m}_p = \min_{\leq_{lex}}\{(-a^1_{p,i}, \dots, -a^r_{p,i})\}$ and $\mathbf{M}_p = \max_{\leq_{lex}}\{(-a^1_{p,i}, \dots, -a^r_{p,i})\}$.

Lemma 3.4 *(i)* $\mathbf{n}_p \leq \mathbf{n}_{p+1}$.

(ii) $\mathbf{m}_p <_{lex} \mathbf{m}_{p+1}$.

(iii) If S is a Cohen-Macaulay ring and L is a Cohen-Macaulay S-module, then $\mathrm{N}_p \leq \mathrm{N}_{p+1}$ and $\mathbf{M}_p <_{lex} \mathbf{M}_{p+1}$.

Proof. We get (i) and (ii) from the Lemma 3.3. Let us prove (iii). Denote by K_S the canonical module of S. Note that it exists because S is a finitely generated k-algebra. If L is a Cohen-Macaulay module, dualizing by K_S we get an r-graded exact sequence:

$$0 \to Hom_S(D_0, K_S) \to \dots \to Hom_S(D_t, K_S) \to Ext^t_S(L, K_S) \to 0.$$

Let $D^*_p = Hom_S(D_p, K_S) = \bigoplus_i K_S(-a^1_{p,i}, \dots, -a^r_{p,i})$, $\forall p$. One can easily see that $Im(D^*_p) \subset \mathcal{M} D^*_{p+1}$.

Let $\{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ the set of degree vectors of a minimal homogeneous system of generators of K_S. Denote by $\mathbf{a}_{p,i} = (a^1_{p,i}, \dots, a^r_{p,i})$. Then the vectors $\mathbf{a}_{p,i} + \mathbf{b}_l$ are the degrees of a minimal homogeneous system of generators of D^*_p. Let us consider $\widetilde{\mathbf{m}}_p = \min_{\leq_{lex}}\{\mathbf{a}_{p,i} + \mathbf{b}_l\} = -\mathbf{M}_p + \min_{\leq_{lex}}\{\mathbf{b}_l\}$ and $\tilde{n}^j_p = \min_{i,l}\{a^j_{p,i} + b^j_l\} = -N^j_p + \min_l\{b^j_l\}$. Applying Lemma 3.3, we obtain that $\tilde{n}^j_{p+1} \leq \tilde{n}^j_p$ and $\widetilde{\mathbf{m}}_{p+1} <_{lex} \widetilde{\mathbf{m}}_p$ and so $\mathrm{N}_{p+1} \geq \mathrm{N}_p$ and $\mathbf{M}_{p+1} >_{lex} \mathbf{M}_p$.

Remark 3.5 For $j = 1, \dots, r$, let us consider the following order:

$$(s_1, \dots, s_r) \leq_j (t_1, \dots, t_r) \iff (s_j, \dots, s_r, s_1, \dots, s_{j-1}) \leq_{lex} (t_j, \dots, t_r, t_1, \dots, t_{j-1}).$$

Note that $\leq_1$ is the lexicographic order. Then Lemma 3.4 is also true if we define $\mathbf{m}_p = min_{\leq_j}\{(-a^1_{p,i}, \dots, -a^r_{p,i})\}$, $\mathbf{M}_p = max_{\leq_j}\{(-a^1_{p,i}, \dots, -a^r_{p,i})\}$.

The next result gives a formula for the multigraded a-invariant of L by means of $\mathbf{a}(S)$ and the shifts appearing in the resolution (see [1], Example 3.6.15 for the $\mathbb{Z}$-graded case).

Proposition 3.6 *Assume that S is a Cohen-Macaulay k-algebra and L a Cohen-Macaulay S-module. Then $a(L_j) = a(S_j) + \max_i\{-a^j_{t,i}\}$, for all $j = 1, ..., r$, i.e.* $\mathbf{a(L)} = \mathbf{a(S)} + \mathbf{N_t}$.

Proof. Let $d = \dim S$, $l = \dim L$ and $\mathcal{M}$ the homogeneous maximal ideal of S. From the graded minimal free resolution of L

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to L \to 0$$

and using that $t = d - l$ and S is a Cohen-Macaulay ring, we obtain the graded exact sequence:

$$0 \to H^l_{\mathcal{M}}(L) \to H^d_{\mathcal{M}}(D_t) \to H^d_{\mathcal{M}}(D_{t-1}).$$

Note that $H^d_{\mathcal{M}}(D_p)= \bigoplus_i[H^d_{\mathcal{M}}(S)]_{(a^1_{p,i},...,a^r_{p,i})}$. In degree $(\alpha_1, ..., \alpha_r)$, we get the exact sequence:

$$0 \to [H^l_{\mathcal{M}}(L)]_{(\alpha_1,...,\alpha_r)} \to \bigoplus_i [H^d_{\mathcal{M}}(S)]_{(\alpha_1+a^1_{t,i},...,\alpha_r+a^r_{t,i})} \to \bigoplus_i [H^d_{\mathcal{M}}(S)]_{(\alpha_1+a^1_{t-1,i},...,\alpha_r+a^r_{t-1,i})}$$

Let $\gamma_1 = a(S_1) + \max_i\{-a^1_{t,i}\}$. We will prove that $a(L_1) = \gamma_1$.

Given $\alpha_1 > \gamma_1$, then $\alpha_1 + a^1_{t,i} > a(S_1) + \max_i\{-a^1_{t,i}\} + a^1_{t,i} \geq a(S_1)$. So $H^d_{\mathcal{M}}(S)_{(\alpha_1+a^1_{t,i},...,\alpha_r+a^r_{t,i})} = 0$, for all i. Therefore $H^l_{\mathcal{M}}(L)_{(\alpha_1,...,\alpha_r)} = 0$ and so $a(L_1) \leq \gamma_1$.

Let $\beta_1 := a(S_1)$ and $(\beta_2, ..., \beta_r) := \max_{\leq_{lex}}\{(\alpha_2, ..., \alpha_r) \in \mathbb{Z}^{r-1} \mid H^d_{\mathcal{M}}(S)_{(\beta_1,\alpha_2,...,\alpha_r)} \neq 0\}$. Then we consider $\gamma_j := \beta_j + M^j_t$. We will see that $H^l_{\mathcal{M}}(L)_{(\gamma_1,...,\gamma_r)} \neq 0$. First note that for i such that $(-a^1_{t,i}, ..., -a^r_{t,i}) = (M^1_t, ..., M^r_t)$ then $H^d_{\mathcal{M}}(S)_{(\gamma_1+a^1_{t,i},...,\gamma_r+a^r_{t,i})} = H^d_{\mathcal{M}}(S)_{(\beta_1,\beta_2,...,\beta_r)} \neq 0$. On the other hand, by the Lemma 3.4 we have $(-a^1_{t-1,i}, ..., -a^r_{t-1,i}) \leq_{lex} M_{t-1} <_{lex} M_t$, for all i. Let l such that $-a^j_{t-1,i} = M^j_t, \forall j < l$ and $-a^l_{t-1,i} < M^l_t$. Then $\gamma_j + a^j_{t-1,i} = \beta_j, \forall j < l$ and $\gamma_l + a^l_{t-1,i} > \beta_l$, thus $(\gamma_1 + a^1_{t-1,i}, ..., \gamma_r + a^r_{t-1,i}) >_{lex} \beta$. From this it follows that $H^d_{\mathcal{M}}(S)_{(\gamma_1+a^1_{t-1,i},...,\gamma_r+a^r_{t-1,i})} = 0$, for all i. Therefore $H^l_{\mathcal{M}}(L)(\gamma_1, ..., \gamma_r) \neq 0$ and $a(L_1) = a(S_1) + \max_i\{-a^1_{t,i}\}$. The proof is similar for $j = 2, ..., r$ (use Remark 3.5).

C. THE BIGRADED a-INVARIANT OF THE REES ALGEBRA

Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring and $I \subset A$ a homogeneous ideal. Denote by R the Rees algebra of I and by G the associated graded ring of I. We can consider R and G as bigraded rings in the following way:

$$R_{(i,j)} = (I^j)_i$$
$$G_{(i,j)} = (I^j)_i/(I^{j+1})_i$$

Let denote by $\mathbf{a}(R) = (a(R_1), a(R_2))$ the bigraded a-invariant of R and by $a(G)$ the usual a-invariant of G. We set M the homogeneous maximal ideal of R. Then:

Proposition 3.7 *(i)* $a(R_2) = -1$, $a(R_1) \leq -n$.

(ii) Assume that R is Cohen-Macaulay and $a(G) < -1$. Then $H_M^{n+1}(R)_{(-n,-1)} \neq 0$ and so $a(R_1) = -n$.

Proof. We set $R^+ = \bigoplus_{j>0}(I^j)_i$. We have two bigraded exact sequences:

$$0 \to R^+ \to R \to A \to 0$$

$$0 \to R^+(0,1) \to R \to G \to 0.$$

For every (i,j), we get exact sequences:

$$\ldots \to H_M^n(A)_{(i,j)} \to H_M^{n+1}(R^+)_{(i,j)} \to H_M^{n+1}(R)_{(i,j)} \to 0$$

$$\ldots \to H_M^n(G)_{(i,j)} \to H_M^{n+1}(R^+)_{(i,j+1)} \to H_M^{n+1}(R)_{(i,j)} \to 0.$$

Note that $A_{(i,j)} = 0$ if $j \neq 0$ and so $H_M^n(A)_{(i,j)} = 0$ if $j \neq 0$.

As $R_2 = \bigoplus_j(\bigoplus_i(I^j)_i) = \bigoplus_j I^j$ is the Rees algebra with the usual $\mathbb{Z}$- grading, $a(R_2) = -1$. We want to determine $a(R_1) = max\{\ i \mid \exists j : H_M^{n+1}(R)_{(i,j)} \neq 0\}$. Suppose that $H_M^{n+1}(R)_{(i,j)} \neq 0$. As $a(R_2) = -1$, we have $j \leq -1$. From the second exact sequence, $H_M^{n+1}(R^+)_{(i,j+1)} \neq 0$. If $j+1 < 0$, from the first exact sequence, we have $H_M^{n+1}(R)_{(i,j+1)} \cong H_M^{n+1}(R^+)_{(i,j+1)} \neq 0$. Repeating this argument, we obtain $H_M^{n+1}(R^+)_{(i,0)} \neq 0$. As $H_M^{n+1}(R)_{(i,0)} = 0$, $H_M^n(A)_{(i,0)} \neq 0$ and so $i \leq a(A) = -n$. From this it follows that $a(R_1) \leq -n$.

Assume now that R is Cohen-Macaulay and $a(G) < -1$. If $H_M^{n+1}(R)_{(-n,-1)} = 0$, from the second exact sequence we get $H_M^{n+1}(R^+)_{(-n,0)} = 0$. As R is Cohen-Macaulay, we get from the first one that $H_M^n(A)_{(-n,0)} = 0$, which is a contradiction.

Remark 3.8 Note that in the proof of the Proposition 3.7 (ii) it is enough to assume that $H_M^n(G)_{(-n,-1)} = 0$ and $H_M^n(R)_{(-n,0)} = 0$.

Remark 3.9 Let us consider the group morphism $\psi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\psi(i,j) = i+j$. By Lemma 3.1, $H_M^{n+1}(R^\psi)_l = \bigoplus_{i+j=l} H_M^{n+1}(R)_{(i,j)}$. Therefore, applying Proposition 3.7 we get $a(R^\psi) \leq -n-1$. If R is Cohen-Macaulay and $a(G) < -1$ we know that $H_M^{n+1}(R)_{(-n,-1)} \neq 0$ and so $a(R^\psi) = -n-1$.

4 The main result

Let S the polynomial ring in n+r variables introduced in section 2 and let

$$0 \to D_t \to \ldots \to D_1 \to D_0 = S \to R_A(I) \to 0$$

be a $\mathbb{Z}^2$ -graded minimal free resolution of $R_A(I)$ over S. For every p, D_p is a direct sum of S-modules of the type $S(a,b)$.

Assuming that $R_A(I)$ is Cohen-Macaulay, we will give bounds of the shifts (a,b) in this resolution. This will be done by using the values of the a-invariant of the $R_A(I)$ computed in section 3. Recall that I is minimally generated by homogeneous polynomials $f_1, ..., f_r$ of degrees $d_1, ..., d_r$ respectively , and put $d = d_r \geq ... \geq d_1$ and $u = \sum_{j=1}^r d_j$.

Proposition 4.1 *Assume that the Rees algebra $R_A(I)$ is Cohen-Macaulay. Let*

$$0 \to D_{r-1} \to \ldots \to D_1 \to D_0 = S \to R_A(I) \to 0$$

be a $\mathbb{Z}^2$ -graded minimal free resolution of $R_A(I)$ over S. Given $p \geq 1$, for every shift (a,b) of a free sumand of D_p we have

(i) $a \leq 0$, $b \leq 0$, $a \leq d_1 b$.

(ii) $-a-b \leq u+p$.

(iii) $-a \leq u+p-(r-1)$.

(iv) $-r < b < 0$.

Proof. It is clear that $a \leq 0$, $b \leq 0$, $a \leq d_1 b$. Let us prove the other ones. For (ii) consider the morphism $\psi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\psi(i,j) = i+j$. Applying the functor $(\;)^{\psi}$, we get a $\mathbb{Z}$-graded minimal free resolution of $R_A(I)^{\psi}$ over S^{ψ}. Note that for every shift (a,b), $S(a,b)^{\psi} = S^{\psi}(a+b)$. Moreover $a(S^{\psi}) = -n-u-r$ and $a(R^{\psi}) \leq -n-1$ (see Remark 3.9). Given (a,b) in D_p, we obtain by Lemma 3.4 and Proposition 3.6:

$$-a-b \leq \max\{\,-\alpha-\beta \mid (\alpha,\beta) \text{ in } D_p\} \leq \max\{\,-\alpha-\beta \mid (\alpha,\beta) \text{ in } D_{r-1}\} + p-(r-1) =$$
$$= a(R^{\psi}) - a(S^{\psi}) + p-(r-1) \leq u+p.$$

The proof of (iii) is analogous by considering $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\varphi((i,j)) = i$.

Let (a,b) be a shift in D_p, with $p \geq 1$. By Lemma 3.4 and Proposition 3.6 we have: $-b \leq \max\{\,-\beta \mid (\alpha,\beta) \text{ in } D_p\} \leq \max\{\,-\beta \mid (\alpha,\beta) \text{ in } D_{r-1}\} = a(R_2)-a(S_2) = -1+r < r$. Therefore we get $b > -r$. Moreover we know the first homomorphism in the resolution:

$$\begin{array}{ccc} S & \longrightarrow & R_A(I) \\ X_i & \mapsto & X_i \\ Y_j & \mapsto & f_j t \end{array}$$

So, if (a,b) appears in D_1, b must to be less than 0 and we obtain (iv).

Remark 4.2 When I is a complete intersection ideal, all the shifts in the resolution may be explicitely computed. In fact, by the Eagon-Northcott complex if (a,b) appears in D_p, (a,b) is of the type:

$$a = -d_{j_1} - \dots - d_{j_{p+1}}, b = -m$$

where $1 \leq j_1 \leq \dots \leq j_{p+1} \leq r, 1 \leq m \leq p$ (see [2], Lemma 4.1). Note that b takes all the values between $-r$ and 0 and the bounds of Proposition 4.1 (ii), (iii) are sharp if $p = r-1$.

Corollary 4.3 *Assume that the Rees algebra $R_A(I)$ is Cohen-Macaulay. Then $R_A(I)$ has a good $\mathbb{Z}^2$-graded minimal free resolution.*

Proof. As a consequence of Proposition 4.1 (iv) and using Proposition 2.7, we have that $R_A(I)$ has a good $\mathbb{Z}^2$-graded minimal free resolution.

Now we state our main result:

Theorem 4.4 *Assume that the Rees algebra $R_A(I)$ is Cohen-Macaulay. Then $k[(I^e)_c]$ is Cohen-Macaulay for $c >> 0$ relatively to $e >> 0$. More explicitely, given $e > 0$ let :*

$$\alpha = \min\{(e-1)d + u - n, e(u-n)\}$$
$$\beta = \min\{(e-1)d + u - d_1(r-1), e(u-d_1)\}$$

Then $k[(I^e)_c]$ is Cohen-Macaulay if $c > \max\{\alpha, \beta, de\}$.

Moreover, if I is an equigenerated ideal, then $k[(I^e)_c]$ is Cohen-Macaulay for all $c >$ $\max\{\alpha, de\}$. *In particular, if I is an equigenerated ideal and $n \geq u$ then $k[(I^e)_c]$ is Cohen-Macaulay for all $c > de$.*

Proof. By Corollary 4.3 we have that $R_A(I)$ has a good $\mathbb{Z}^2$-graded minimal free resolution and so, by Proposition 2.2, $k[(I^e)_c]$ is Cohen-Macaulay for $c >> 0$ relatively to $e >> 0$.

By virtue of Proposition 2.1, $(D_0)_\Delta = S_\Delta$ is Cohen-Macaulay for all Δ. We want to determine positive integers c, e such that $(D_p)_\Delta$ is Cohen-Macaulay for all $p = 1, ..., r$. Using Proposition 2.4, we are looking for c, e such that $H^r_{m_{S_\Delta}}(S(a,b)_\Delta) = H^n_{m_{S_\Delta}}(S(a,b)_\Delta) = 0$ for every shift (a, b) that appears in the resolution, in other words such that the sets

$$X = \{ s \in \mathbb{Z} \mid \frac{(b+r)d - u - a}{c - ed} \leq s \leq -\frac{b+r}{e}\} = \emptyset,$$

$$Y = \{ s \in \mathbb{Z} \mid \frac{-b}{e} \leq s \leq \frac{bd - a - n}{c - ed}\} = \emptyset,$$

for all (a, b) in the resolution. We know that $b > -r$ by Proposition 4.1 . So, if $s \in X$ then $s \leq -1$. One can see that if $c > e(u - d_1)$ then by Proposition 4.1 (i) it holds $\frac{(b+r)d-u-a}{c-ed} > -\frac{b+r}{e}$ and so $X = \emptyset$. Also if $c > (e-1)d + u - d_1(r-1)$ one gets $\frac{(b+r)d-u-a}{c-ed} > -1$ and so $X = \emptyset$. Therefore, if $c > \beta$ then $X = \emptyset$. Note that if I is an equigenerated ideal (i.e.

$d = d_r = \ldots = d_1$) then $(b+r)d - u - a = bd - a \geq 0$ and X is always empty. In analogous way, if $c > \alpha$ then $Y = \emptyset$.

The same methods may be used to study the diagonals of standard bigraded algebras.

Proposition 4.5 *Let R be a standard bigraded Cohen-Macaulay k-algebra. Assume that* $\mathbf{a}(R) < 0$. *Then R_Δ is a Cohen-Macaulay ring, for all the diagonals Δ.*
(Here $\mathbf{a}(R) < 0$ means $a(R_1) < 0$ and $a(R_2) < 0$.)

Proof. Let $R = k[s_1, \ldots, s_n, t_1, \ldots, t_r]$ be a standard bigraded k-algebra, where s_i, t_j are homogeneous elements with $\deg s_i = (1,0)$, $\deg t_j = (0,1)$. Let us consider the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ with the bigraded structure given by $\deg X_i = (1,0)$, $\deg Y_j = (0,1)$. Let

$$0 \to D_t \to \ldots \to D_1 \to D_0 = S \to R \to 0$$

be a $\mathbb{Z}^2$-graded minimal free resolution of R over S. Note that $t = n + r - \dim R$. For all the shifts (a,b) in D_p, $p \geq 1$, we have (Lemma 3.4 and Proposition 3.6):

$$-a \leq \max\{ -\alpha \mid (\alpha,\beta) \text{ in } D_t \} = a(R_1) - a(S_1) < n$$

$$-b \leq \max\{ -\beta \mid (\alpha,\beta) \text{ in } D_t \} = a(R_2) - a(S_2) < r.$$

Therefore, $a > -n$ and $b > -r$. Let us see that the resolution is good. We note that for all $p \geq 1$ and (a,b) in D_p we have $a \leq 0$, $b \leq 0$ and $(a,b) \neq (0,0)$. So, if $b = 0$ then $a < 0$. Thus, every shift (a,b) in D_p holds $-n < a < 0$ or $-r < b < 0$. By Proposition 2.7, the resolution is good. Moreover, we will see that for all c, e $H^n_{m_S}((D_p)_\Delta) = H^r_{m_S}((D_p)_\Delta) = 0$. In our case $d = d_r = \ldots = d_1 = 0, u = 0$ and we have

$$\operatorname{supp} H^r_{m_{S_\Delta}}(S(a,b)_\Delta) = \{ s \in \mathbb{Z} \mid \frac{-a}{c} \leq s \leq -\frac{b+r}{e} \}$$

$$\operatorname{supp} H^n_{m_{S_\Delta}}(S(a,b)_\Delta) = \{ s \in \mathbb{Z} \mid \frac{-b}{e} \leq s \leq \frac{-a-n}{c} \}$$

Using $-r < b \leq 0$ and $-n < a \leq 0$ it is easy to check that these supports are empty.

Remark 4.6 Let L be a finitely generated $\mathbb{Z}^2$-graded S-module. For $i = 1, 2$, let us denote by $L_i = L^{\varphi_i}$, where $\varphi_i : \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $\varphi_i(\mathbf{n}) = n_i$. By considering the initial degree of L_i, $indeg(L_i) = min\{ j \mid [L_i]_j \neq 0\}$, then one can define the initial degree of L $\mathbf{indeg}(L) := (indeg(L_1), indeg(L_2))$.

Assume that L is a Cohen-Macaulay S-module. A natural question is when there exists a diagonal Δ such that L_Δ is Cohen-Macaulay. This problem has already been studied when L is the Rees algebra or a standard bigraded k-algebra. By using the same methods, if $\mathbf{a}(L)$ is good enough one can ensure that L has a good resolution, and so L_Δ is Cohen-Macaulay for $c >> 0$ relatively to $e >> 0$. Namely, given $e >> 0$ (depending on $indeg(L_2)$) one can find α depending on e, $\mathbf{a}(L)$ and $\mathbf{indeg}(L)$ such that L_Δ is Cohen-Macaulay if $c > \alpha$.

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References

[1] W. Bruns, J. Herzog, *Cohen Macaulay rings*, Cambridge studies in advanced mathematics, 39, 1993.

[2] A. Conca, J. Herzog, N.V. Trung, G. Valla, *Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces*, preprint (1995).

[3] S.D. Cutkosky, J. Herzog, *Cohen-Macaulay coordinate rings of blowup schemes*, preprint (1996).

[4] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, GTM 150, 1994.

[5] S. Goto, K. Watanabe, *On graded rings I*, J.Math.Soc.Japan 30 (1978), 179-213.

[6] S. Goto, K. Watanabe, *On graded rings II*, Tokyo J.Math.1 (1978), 237-261.

[7] M. Herrmann, E. Hyry, J. Ribbe, *On the Cohen-Macaulay and Gorenstein properties of multigraded Rees algebras*, Manuscripta Math. 79 (1993), 343-377.

[8] M. Herrmann, S. Ikeda, U. Orbanz, *Equimultiplicity and blowing up*, Berlin-Heidelberg, Springer, 1988.

[9] E. Hyry, *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*, preprint(1996).

[10] A. Simis, N.V. Trung, G. Valla, *The diagonal subalgebra of a blow-up algebra*, preprint(1995).

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