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HIGHER BOTT CHERN FORMS AND BEILINSON'S REGULATOR

by

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INTRODUCTION.

Let X be a smooth complex variety and let E be a vector bundle over X. Given a hermitian metric h on E we can define an explicit representative of the Chern character of E ([B-C]). Let E_X denote the differential graded commutative algebra of complex valued differential forms on X, and let $E_{X,\mathbb{R}}$ denote the subalgebra of real forms.

Let D be the unique connection of E satisfying

(1) D preserves h.
(2) If $U \subset X$ is an open subset and s is a holomorphic section of $E|_U$, then Ds is of pure type $(1,0)$.

Let $K = D^2$ be the curvature form. Let us write

$$\widetilde{\mathrm{ch}}_0(E,h) = \operatorname{tr}\exp(-K) \in \bigoplus_p (2\pi i)^p E^{p,p}_{X,\mathbb{R}}.$$

The form $\widetilde{\mathrm{ch}}_0(E,h)$ is closed. Its cohomology class, denoted $\mathrm{ch}_0(E)$, is the Chern character of E and is independent of the metric h.

Chern classes for higher algebraic K-theory were introduced by Gillet in [Gi]. These classes are defined on any cohomology theory satisfying certain properties. In the particular case where the class is the Chern character class and the cohomology theory is absolute Hodge cohomology, the map obtained is called Beilinson's regulator map. We will give a description of this map in terms of hermitian metrics.

The Chern character class on the K_0 is additive for exact sequences. Nevertheless one cannot make a consistent choice of representatives of the Chern character that behave additively for exact sequences. An analogous statement is true for higher K-theory. Following Schechtman ([Sch]) the lack of additivity of the representatives of the Chern character for K_i is responsible for the Chern character for K_{i+1}.

For instance, let

$$\overline{\xi} : 0 \to (E',h') \to (E,h) \to (E'',h'') \to 0,$$

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be an exact sequence of hermitian vector bundles. Then the Chern character classes satisfy

$$\mathrm{ch}_0(E) = \mathrm{ch}_0(E') + \mathrm{ch}_0(E'').$$

Nevertheless, in general

$$\widetilde{\mathrm{ch}}_0(E,h) \neq \widetilde{\mathrm{ch}}_0(E',h') + \widetilde{\mathrm{ch}}_0(E'',h'').$$

In the case when h' and h'' are the induced metrics, Bott and Chern ([B-C]) have defined a differential form $\widetilde{\mathrm{ch}}_1(\overline{\xi})$, that will be called the Bott Chern form of $\overline{\xi}$, such that

$$(1) \qquad -2\partial\overline{\partial}\,\widetilde{\mathrm{ch}}_1(\overline{\xi}) = \widetilde{\mathrm{ch}}_0(E',h') + \widetilde{\mathrm{ch}}_0(E'',h'') - \widetilde{\mathrm{ch}}_0(E,h).$$

Note that the normalization factor we use is different from the normalization factor used in the original paper. The forms $\widetilde{\mathrm{ch}}_1(\overline{\xi})$ are natural and well defined only up to $\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}$.

Bismut, Gillet and Soulé ([B-G-S], [G-S]) have given a different construction of Bott Chern forms that can be applied to the case when h' and h'' are not the induced metrics. These Bott Chern forms are also well defined only up to $\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}$.

Bott-Chern forms measure the lack of additivity of the Chern character forms. And, when X is proper, Gillet and Soulé ([G-S]) have given an explicit description of Beilinson's regulator for $K_1(X)$ in terms of Bott Chern forms. Moreover, in the same paper they have used Bott Chern forms to define arithmetic K_0 groups of arithmetic varieties. Deligne has suggested ([De]) the existence of higher arithmetic K-groups. In the definition of these groups, higher Bott Chern forms as presented in this paper might play a central role.

Following Schechtman's idea, the lack of additivity of Bott Chern forms should allow us to define second order Bott Chern forms that give a description of Beilinson's regulator map for the K_2. And we can repeat this process to obtain Beilinson's regulator map for all the K groups.

In this direction, for X is proper, the second author ([Wan]) has defined higher Bott Chern forms for exact n-cubes. The Bott Chern form of a n-cube measures the lack of additivity of the Bott Chern forms of the faces of this cube, generalizing equation (1). Higher Bott Chern forms provide characteristic classes from K-theory to real absolute Hodge cohomology. Moreover, in [Wan] it is proved that, if one can extend naturally higher Bott Chern forms to the non proper case, then these characteristic classes agree with Beilinson's regulator map.

In this paper we shall give a variant of Wang's original construction that can be easily extended to the non-proper case. Thus we obtain a description of Beilinson's regulator map in terms of differential forms. An interesting feature of the construction given here is that we obtain well defined Bott Chern forms and not only modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}$.

Paralel results in the framework of multiplicative K-theory have been obtained by Karoubi in [K1] and [K2].

Throughout the paper all vector bundles will be algebraic and we shall use the equivalent notion of locally free sheaf.

The plan of the paper is as follows. In §1 we recall the definition of real absolute Hodge cohomology. We shall also show that real absolute Hodge cohomology can be computed by means of a complex composed by forms defined on $X \times (\mathbb{P}^1)^n$, $n \geq 0$. Higher Bott Chern forms will live in this complex.

In §2 we introduce and study some properties of smooth at infinity hermitian metrics. Over a non proper smooth complex variety, to compute real absolute Hodge cohomology, one needs to impose logarithmic conditions at infinity to the differential forms. Thus we cannot use arbitrary hermitian metrics because they will produce differential forms with arbitrary singularities at infinity. The use of smooth at infinity hermitian metrics ensures that Bott Chern forms have the right behaviour at infinity.

In §3 we recall the notion of exact metrized n-cubes and define higher Bott Chern forms. These forms live in $X \times (\mathbb{P}^1)^*$.

In §4 we use higher Bott Chern forms to define Chern character classes from higher K-theory to real absolute Hodge cohomology.

In §5 we prove that the higher Chern character defined in §4 agrees with Beilinson's regulator.

In §6 we recall several complexes that compute real absolute Hodge cohomology and homology. Using them we give, for X proper, two different versions of higher Bott-Chern forms which are defined on X. The first one, obtained using the Thom-Whitney simple, is multiplicative. The second one agrees with classical Bott-Chern forms and with the original definition due to Wang.

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§1 Absolute Hodge cohomology I.

In this section we shall recall the definition of real absolute Hodge cohomology [Be] of a smooth complex algebraic variety X. By a smooth complex variety we shall mean a smooth separated scheme of finite type over $\mathbb{C}$. We shall also construct a complex, composed by forms on $X \times (\mathbb{P}^1)^n$, $n \geq 0$, whose cohomology is the real absolute Hodge cohomology of X.

(1.1) Let $\overline{X}$ be a smooth proper complex variety. Let $Y \subset \overline{X}$ be a normal crossing divisor and let us write $X = \overline{X} - Y$. Let $E^*_{\overline{X}}$ be the differential graded algebra of differential forms on $\overline{X}$, and let $E^*_{\overline{X}}(\log Y)$ be the differential graded algebra of C^∞ complex differential forms on $\overline{X}$ with logarithmic singularities along Y (see [Bu 1]). The algebra $E^*_{\overline{X}}(\log Y)$ has a real structure, $E^*_{\overline{X}}(\log Y)_{\mathbb{R}}$, a weigh filtration W defined over $\mathbb{R}$ and a Hodge filtration F. Moreover the cohomology of this algebra gives us the cohomology of X with its real mixed Hodge structure.

Let us denote by $\widehat{W}$ the décalée filtration of W. That is

$$\widehat{W}_r E^n_{\widetilde{X}}(\log Y) = \{x \in W_{r-n} E^n_{\widetilde{X}}(\log Y) \mid dx \in W_{r-n-1} E^{n+1}_{\widetilde{X}}(\log Y)\}.$$

We write

$$E^*_{\log}(X) = \varinjlim_{(\widetilde{X}_\alpha, Y_\alpha)} E^*_{\widetilde{X}_\alpha}(\log Y_\alpha),$$

where the limit is taken along all the smooth compactifications $\widetilde{X}_\alpha$ of X with $Y_\alpha = \widetilde{X}_\alpha - X$ a normal crossing divisor. Then $E^*_{\log}(X)$ is a differential graded algebra and it has an induced real structure, a weight filtration and a Hodge filtration. Moreover the map

$$(E^*_{\widetilde{X}}(\log Y)_{\mathbb{R}}, \widehat{W}) \longrightarrow (E^*_{\log}(X)_{\mathbb{R}}, \widehat{W})$$

is a filtered quasi-isomorphism and the map

$$(E^*_{\widetilde{X}}(\log Y), \widehat{W}, F) \longrightarrow (E^*_{\log}(X), \widehat{W}, F)$$

is a bifiltered quasi-isomorphism.

(1.2) Let us write

$$\mathfrak{H}^*(X,p) = s((2\pi i)^p \widehat{W}_{2p} E^*_{\log}(X)_{\mathbb{R}} \oplus \widehat{W}_{2p} \cap F^p E^*_{\log}(X) \xrightarrow{u} \widehat{W}_{2p} E^*_{\log}(X)),$$

where $u(r,f) = f - r$ and s denotes the simple of a morphism of complexes, i.e. the cône shifted by one. The differential of this complex will be denoted by $d_{\mathfrak{H}}$.

The real absolute Hodge cohomology of X ([Be]) is

$$H^n_{\mathcal{H}}(X, \mathbb{R}(p)) = H^n(\mathfrak{H}(X,p)).$$

(1.3) A cubical or cocubical object (see [G-N-P-P]) is an object modeled on the cube in the same way as a simplicial or cosimplicial object is modeled on the simplex. Let $(\mathbb{P}^1_{\mathbb{C}})^{\cdot}$ be the cocubical scheme which in degree n is $(\mathbb{P}^1_{\mathbb{C}})^n$, the n-fold product of the complex projective line. The faces and degeneracies

$$\begin{aligned} d^i_j &: (\mathbb{P}^1_{\mathbb{C}})^n \longrightarrow (\mathbb{P}^1_{\mathbb{C}})^{n+1},\ i = 1, \dots, n+1,\ j = 0,1 \\ s^i &: (\mathbb{P}^1_{\mathbb{C}})^n \longrightarrow (\mathbb{P}^1_{\mathbb{C}})^{n-1},\ i = 1, \dots, n, \end{aligned}$$

are given by

$$\begin{aligned} d^i_0(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, (0:1), x_i, \dots, x_n) \\ d^i_1(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, (1:0), x_i, \dots, x_n) \\ s^i(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n). \end{aligned}$$

(1.4) The complexes $\mathfrak{H}^*(X \times (\mathbb{P}^1)^\cdot, p)$ form a cubical complex. We shall write

$$\begin{aligned} s_i &= (\mathrm{Id} \times s^i)^*, \\ d_i^j &= (\mathrm{Id} \times d_j^i)^*. \end{aligned}$$

Let us denote by $\mathfrak{H}_{\mathbb{P}}^{*,*}(X,p)$ the associated double complex. That is

$$\mathfrak{H}_{\mathbb{P}}^{r,n}(X,p) = \mathfrak{H}^r(X \times (\mathbb{P}^1)^{-n}, p),$$

with differentials

$$\begin{aligned} d' &= d_{\mathfrak{H}}, \\ d'' &= \sum (-1)^{i+j} d_i^j. \end{aligned}$$

(1.5) We want to obtain from $\mathfrak{H}_{\mathbb{P}}^{*,*}(X,p)$, a complex which computes the absolute Hodge cohomology of X. On the one hand, since we are using a cubical theory we need to factor out by the degenerate elements (see [Mas]). On the other hand, we need to kill all cohomology classes coming from the projective spaces.

Let us denote by $p_0 : X \times (\mathbb{P}^1)^n \longrightarrow X$ the projection over the first factor and by $p_i : X \times (\mathbb{P}^1)^n \longrightarrow \mathbb{P}^1$, $i = 1, n$, the projection over the i-th projective line.

Let ω be the standard Kähler form over $\mathbb{P}^1$. Let $\omega_i = p_i^* \omega \in E_{\log}^*(X \times (\mathbb{P}^1)^n)$. For an element

$$x = (r, f, \eta) \in \mathfrak{H}^r(X \times (\mathbb{P}^1)^n, p),$$

we shall write

$$\begin{aligned} \omega_i \wedge x &= (\omega_i \wedge r, \omega_i \wedge f, \omega_i \wedge \eta) \\ &\in \mathfrak{H}^{r+2}(X \times (\mathbb{P}^1)^n, p+1). \end{aligned}$$

Definition 1.1. We shall denote by $\widetilde{\mathfrak{H}}^{*,*}(X,p)$ the double complex given by

$$\widetilde{\mathfrak{H}}^{r,n}(X,p) = \mathfrak{H}_{\mathbb{P}}^{r,n}(X,p) \Big/ \sum_{i=1}^{-n} s_i \left(\mathfrak{H}_{\mathbb{P}}^{r,n+1}(X,p)\right) \oplus \omega_i \wedge s_i \left(\mathfrak{H}_{\mathbb{P}}^{r-2,n+1}(X,p-1)\right).$$

We shall denote by $\widetilde{\mathfrak{H}}^*(X,p)$ the associated simple complex. The differential of this complex will be denoted by d.

In the definition of $\widetilde{\mathfrak{H}}^{r,n}(X,p)$, the first summand of the quotient is meant to kill the degenerate classes, whereas the second summand should kill the classes coming from the projective spaces. The next result shows that we have reached our objective.

Proposition 1.2. *The natural morphism of complexes*

$$\iota : \mathfrak{H}^*(X,p) = \widetilde{\mathfrak{H}}^{*,0}(X,p) \longrightarrow \widetilde{\mathfrak{H}}^*(X,p)$$

is a quasi-isomorphism.

Proof. Since $\widetilde{\mathfrak{H}}^*(X,p)$ is a simple complex associated to a double complex, there is a second quadrant spectral sequence with E_1 term

$$E_1^{r,n} = H^r(\widetilde{\mathfrak{H}}^{*,n}(X,p)).$$

When this spectral sequence converges, the limit is $H^*(\widetilde{\mathfrak{H}}^*(X,p))$. The following lemma shows that this spectral sequence converges and implies that ι is a quasi-isomorphism.

Lemma 1.3. *For $n < 0$ the cohomology of the complex $\widetilde{\mathfrak{H}}^{*,n}(X,p)$ is zero.*

Proof. For each j let us write

$$\widetilde{\mathfrak{H}}_j^{r,n}(X,p) = \mathfrak{H}_{\mathbb{P}}^{r,n}(X,p) \Big/ \sum_{i=1}^{j} s_i\left(\mathfrak{H}_{\mathbb{P}}^{r,n+1}(X,p)\right) \oplus \omega_i \wedge s_i\left(\mathfrak{H}_{\mathbb{P}}^{r-2,n+1}(X,p-1)\right).$$

Let us prove, by induction over j, that for $j \geq 1$

$$H^*(\widetilde{\mathfrak{H}}_j^{*,n}(X,p)) = 0.$$

For $j = 1$, $n \leq -1$, the complex $\widetilde{\mathfrak{H}}_1^{r,n}(X,p)$ is the cokernel of the monomorphism

$$\begin{array}{ccccc} \mathfrak{H}^*(X \times (\mathbb{P}^1)^{-n-1}, p) & \oplus & \mathfrak{H}^*(X \times (\mathbb{P}^1)^{-n-1}, p-1)[-2] & \longrightarrow & \mathfrak{H}^*(X \times (\mathbb{P}^1)^{-n}, p) \\ \alpha & \oplus & \beta & \longmapsto & s_1(\alpha) + \omega_1 \wedge s_1(\beta) \end{array}$$

But by the Dold-Thom isomorphism for absolute Hodge cohomology, the above morphism is a quasi-isomorphism. For $j > 1$, $n < -1$, $\widetilde{\mathfrak{H}}_j^{*,n}(X,p)$ is the cokernel of the monomorphism

$$\begin{array}{ccccc} \widetilde{\mathfrak{H}}_{j-1}^{*,n-1}(X,p) & \oplus & \widetilde{\mathfrak{H}}_{j-1}^{*,n-1}(X,p-1)[-2] & \longrightarrow & \widetilde{\mathfrak{H}}_{j-1}^{*,n}(X,p) \\ \alpha & \oplus & \beta & \longmapsto & s_j(\alpha)\omega_j \wedge s_j(\beta). \end{array}$$

By induction hypothesis, the source and the target of this morphism have zero cohomology. Therefore the cokernel also has zero cohomology.

§2 Smooth at infinity hermitian metrics.

In this section we introduce smooth at infinity hermitian metrics. For a smooth complex variety X and a locally free sheaf $\mathcal{F}$, a smooth at infinity hermitian metric is a metric that can be extended to a smooth metric over some compactification of $\mathcal{F}$. The interest of smooth at infinity hermitian metrics is that they provide representatives of Chern classes in absolute Hodge cohomology.

(2.1) Before defining smooth at infinity hermitian metrics, we shall study classes of compactifications of locally free sheaves.

Definition 2.1. Let X be a smooth complex variety and let $\mathcal{F}$ be a locally free sheaf over X. A compactification of $\mathcal{F}$ consists in a smooth compactification of X, $i : X \longrightarrow \widetilde{X}$, a locally free sheaf $\widetilde{\mathcal{F}}$ over $\widetilde{X}$ and an isomorphism $\varphi : \mathcal{F} \longrightarrow i^*\widetilde{\mathcal{F}}$.

A compactification of $\mathcal{F}$ will be denoted by $(i, \widetilde{X}, \widetilde{\mathcal{F}}, \varphi)$. Usually, we shall identify X with $i(X)$ and $\mathcal{F}$ with $\widetilde{\mathcal{F}}|_X$, and denote a compactification by $(\widetilde{\mathcal{F}}, \widetilde{X})$.

Proposition 2.2. *Let X be a smooth complex variety and let $\mathcal{F}$ be a locally free sheaf over X. Then there exists a compactification of $\mathcal{F}$.*

Proof. Let $X \longrightarrow \widetilde{X}_1$ be any compactification of X. Then there is a coherent sheaf $\widetilde{\mathcal{F}}_1$ on $\widetilde{X}_1$ such that $\widetilde{\mathcal{F}}_1|_X = \mathcal{F}$. By [Ro] (see also [Ri] and [N 1]) there is a proper modification $\psi : \widetilde{X} \longrightarrow \widetilde{X}_1$, which induces an isomorphism $\psi^{-1}(X) \longrightarrow X$, and such that $\widetilde{\mathcal{F}} = \psi^*(\widetilde{\mathcal{F}}_1)/\operatorname{Tor}(\psi^*(\widetilde{\mathcal{F}}_1))$ is a locally free sheaf. Moreover $\widetilde{\mathcal{F}}|_{\psi^{-1}(X)}$ is isomorphic to $\widetilde{\mathcal{F}}_1|_X$. Thus the induced map $i : X \longrightarrow \widetilde{X}$ is a compactification of X, and $\widetilde{\mathcal{F}}$ is a compactification of $\mathcal{F}$.

Definition 2.3. Let X be a smooth complex variety and let $\mathcal{F}$ be a locally free sheaf over X. Let $(i_1, \widetilde{X}_1, \widetilde{\mathcal{F}}_1, \varphi_1)$ and $(i_2, \widetilde{X}_2, \widetilde{\mathcal{F}}_2, \varphi_2)$ be two compactifications of $\mathcal{F}$. We say that $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ are equivalent if there exists a third compactification $(i_3, \widetilde{X}_3, \widetilde{\mathcal{F}}_3, \varphi_3)$ and morphisms $\psi_1 : \widetilde{X}_3 \longrightarrow \widetilde{X}_1$ and $\psi_2 : \widetilde{X}_3 \longrightarrow \widetilde{X}_2$ such that

1) $\psi_1 \circ i_3 = i_1$ and $\psi_2 \circ i_3 = i_2$.
2) There are isomorphisms $\alpha_1 : \widetilde{\mathcal{F}}_3 \longrightarrow \psi_1^*\widetilde{\mathcal{F}}_1$ and $\alpha_2 : \widetilde{\mathcal{F}}_3 \longrightarrow \psi_2^*\widetilde{\mathcal{F}}_2$ such that $i_3^*\alpha_1 \circ \varphi_3 = \varphi_1$ and $i_3^*\alpha_2 \circ \varphi_3 = \varphi_2$.

In order to simplify the notation, a class of equivalent compactifications of $\mathcal{F}$ will be denoted by a single symbol, for instance $\widetilde{\mathcal{F}}$. Moreover, if there is no danger of confusion, we shall denote by the same symbol the locally free sheaf which appears in any representative of this class.

(2.2) A compactification class induces uniquely determined compactification classes in quotients and subsheaves.

Theorem 2.4. *Let X be a smooth complex variety and let*

$$\xi : \ 0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \rightarrow 0$$

be an exact sequence of locally free sheaves over X. Then, for any compactification class $\widetilde{\mathcal{G}}$ of $\mathcal{G}$, there are uniquely determined compactification classes $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{H}}$ of $\mathcal{F}$ and $\mathcal{H}$ respectively, such that ξ extends to an exact sequence

$$\widetilde{\xi} : \ 0 \rightarrow \widetilde{\mathcal{F}} \xrightarrow{\widetilde{f}} \widetilde{\mathcal{G}} \xrightarrow{\widetilde{g}} \widetilde{\mathcal{H}} \rightarrow 0,$$

over a compactification $\widetilde{X}$ of X.

Proof. Let $\widetilde{X}_1$ be a compactification of X where $\widetilde{\mathcal{G}}$ is defined. Let $r = \operatorname{rk}\mathcal{H}$. Let $\mathbf{Grass}^r_{\widetilde{X}_1}(\widetilde{\mathcal{G}})$ be the Grassmanian of rank r quotients of $\widetilde{\mathcal{G}}$ ([G-D]). Let us denote by $\mathcal{U}$ the universal bundle on $\mathbf{Grass}^r_{\widetilde{X}_1}(\widetilde{\mathcal{G}})$. The exact sequence

$$\xi : \ 0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \rightarrow 0$$

induces a morphism

$$\varphi : X \longrightarrow \mathbf{Grass}^r_{\widetilde{X}_1}(\widetilde{\mathcal{G}}).$$

By resolution of singularities, there is a proper modification $\widetilde{X}$ of $\widetilde{X}_1$, which is a compactification of X and such that φ extends to a morphism

$$\widetilde{\varphi} : \widetilde{X} \longrightarrow \mathbf{Grass}^r_{\widetilde{X}_1}(\widetilde{\mathcal{G}}).$$

Then $\widetilde{\mathcal{H}} = \widetilde{\varphi}^*(\mathcal{U})$ is a compactification of $\mathcal{H}$, $\widetilde{\mathcal{F}} = \mathrm{Ker}(\widetilde{\mathcal{G}} \longrightarrow \widetilde{\mathcal{H}})$ is a compactification of $\mathcal{F}$ and ξ extends to an exact sequence

$$\widetilde{\xi} : \ 0 \rightarrow \widetilde{\mathcal{F}} \xrightarrow{\widetilde{f}} \widetilde{\mathcal{G}} \xrightarrow{\widetilde{g}} \widetilde{\mathcal{H}} \rightarrow 0.$$

The unicity follows from the fact that, since X is dense in $\widetilde{X}$, the morphism $\widetilde{\varphi}$ is unique.

Definition 2.5. Let X be a smooth complex variety and let

$$\xi : \ 0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \rightarrow 0$$

be an exact sequence of locally free sheaves over X. Let $\widetilde{\mathcal{G}}$ be a class of compactifications of $\mathcal{G}$. Then the classes of compactifications $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{H}}$, of $\mathcal{F}$ and $\mathcal{H}$ respectively, obtained in theorem 2.4 are called the induced compactifications.

(2.3) Let us introduce smooth at infinity hermitian metrics.

Definition 2.6. Let X be a smooth complex variety, let $\mathcal{F}$ be a locally free sheaf over X and let h be an hermitian metric on $\mathcal{F}$. We say that h is smooth at infinity if there exist a compactification $\widetilde{\mathcal{F}}$ of $\mathcal{F}$, and a smooth metric $\widetilde{h}$ on $\widetilde{\mathcal{F}}$ such that $\widetilde{h}|_X = h$.

A smooth at infinity hermitian metric determines univocally a compactification class.

Proposition 2.7. *Let X be a smooth complex variety and let $\mathcal{F}$ be a locally free sheaf on X. Let $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ be two compactifications of $\mathcal{F}$ and let $\widetilde{h}$ and $\widetilde{h}'$ be smooth metrics on $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$. If $\widetilde{h}|_X = \widetilde{h}'|_X$, then $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ are equivalent compactifications.*

Proof. We can assume that both compactifications are defined over the same variety $\widetilde{X}$. Let $\mathcal{K}_{\widetilde{X}}$ be the sheaf of rational functions over $\widetilde{X}$.

The identity on $\mathcal{F}$ induces morphisms

$$f : \widetilde{\mathcal{F}} \otimes \mathcal{K}_{\widetilde{X}} \longrightarrow \widetilde{\mathcal{F}}' \otimes \mathcal{K}_{\widetilde{X}},$$
$$f' : \widetilde{\mathcal{F}}' \otimes \mathcal{K}_{\widetilde{X}} \longrightarrow \widetilde{\mathcal{F}} \otimes \mathcal{K}_{\widetilde{X}},$$

which are inverses of each other. By symmetry it is enough to show that $f(\widetilde{\mathcal{F}}) \subset \widetilde{\mathcal{F}}'$.

Let U be a Zariski open subset of $\widetilde{X}$. A section $s \in \Gamma(U, \widetilde{\mathcal{F}}' \otimes \mathcal{K}_{\widetilde{X}})$ belongs to $\Gamma(U, \widetilde{\mathcal{F}}')$ if and only if $\widetilde{h}'(s(x)) < \infty$ for all $x \in U$. But if $s \in \Gamma(U, \widetilde{\mathcal{F}})$ then $\widetilde{h}'(f(s))|_{X \cap U} = \widetilde{h}(s)|_{X \cap U}$. Since $U \cap X$ is dense in U we have $\widetilde{h}'(f(s(x))) = \widetilde{h}(s(x)) < \infty$ for all $x \in U$.

Proposition 2.8. *Let*

$$\xi : 0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence of locally free sheaves on X and let h be a smooth at infinity metric on $\mathcal{F}$. Then the metrics h' and h'' induced by h in $\mathcal{F}'$ and $\mathcal{F}''$ are smooth at infinity.

Proof. Let $\widetilde{\mathcal{F}}$ be a compactification of $\mathcal{F}$ provided with a metric $\widetilde{h}$, such that $\widetilde{h}|_X = h$. By theorem 2.4. there are compactifications $\widetilde{\mathcal{F}}'$ and $\widetilde{\mathcal{F}}''$ such that ξ can be extended to an exact sequence

$$\widetilde{\xi} : 0 \longrightarrow \widetilde{\mathcal{F}}' \longrightarrow \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{F}}'' \longrightarrow 0.$$

Then the metric $\widetilde{h}$ induces smooth metrics $\widetilde{h}'$ and $\widetilde{h}''$ on $\widetilde{\mathcal{F}}'$ and $\widetilde{\mathcal{F}}''$. But the restrictions of $\widetilde{h}'$ and $\widetilde{h}''$ to X are h' and h''. Therefore these metrics are smooth at infinity.

Proposition 2.9. *Let $f : X \longrightarrow Y$ be a morphism between smooth complex varieties. Let $(\mathcal{F}, h)$ be a locally free sheaf over Y with h a smooth at infinity metric. Then (f^*h) is a smooth at infinity metric on the locally free sheaf $f^*\mathcal{F}$.*

Proof. Let $(\widetilde{Y}, \widetilde{\mathcal{F}})$ be a compactification of $(Y, \mathcal{F})$, such that there exists a hermitian metric $\widetilde{h}$ with $\widetilde{h}|_X = h$. Let $\widetilde{X}_1$ be any compactification of X. We shall denote by Γ the graph of f, and by $\overline{\Gamma}$ the adherence of Γ in $\widetilde{X}_1 \times \widetilde{Y}$. Let $\widetilde{X}$ be a resolution of singularities of $\overline{\Gamma}$ and let $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ be the induced morphism. Then $(\widetilde{X}, \widetilde{f}^*\widetilde{\mathcal{F}})$ is a compactification of $(X, f^*\mathcal{F})$ and $\widetilde{f}^*\widetilde{h}$ is a smooth metric such that $\widetilde{f}^*\widetilde{h}|_X = f^*h$. Therefore f^*h is smooth at infinity.

(2.4) Let us see that smooth at infinity hermitian metrics provide representatives of the Chern character classes in absolute Hodge cohomology. Let X be a smooth complex variety, $\mathcal{F}$ a locally free sheaf and h a smooth at infinity hermitian metric. Let $\widetilde{\mathcal{F}}$ be the compactification class of $\mathcal{F}$ determined by h, $\widetilde{X}$ a compactification of X where $\widetilde{\mathcal{F}}$ is defined, and $\widetilde{h}$ a smooth metric on $\widetilde{\mathcal{F}}$ extending h. Let K (resp. $\widetilde{K}$) be the curvature form of $(\mathcal{F}, h)$ (resp. $(\widetilde{\mathcal{F}}, \widetilde{h})$). Let us write

$$\widetilde{\mathrm{ch}}_0(\mathcal{F}, h) = \mathrm{Tr}\exp(-K),$$
$$\widetilde{\mathrm{ch}}_0(\widetilde{\mathcal{F}}, \widetilde{h}) = \mathrm{Tr}\exp(-\widetilde{K}).$$

These forms are closed. Moreover,

$$\widetilde{\mathrm{ch}}_0(\widetilde{\mathcal{F}}, \widetilde{h}) \in \bigoplus (2\pi i)^p E^{p,p}_{\widetilde{X},\mathbb{R}}.$$

Since $\widetilde{\mathrm{ch}}_0(\widetilde{\mathcal{F}}, \widetilde{h})|_X = \widetilde{\mathrm{ch}}_0(\mathcal{F}, h)$, then

$$\widetilde{\mathrm{ch}}_0(\mathcal{F}, h) \in \bigoplus_{p\geq 0} \left(W_0 E^{2p}_{\log}(X)_{\mathbb{R}} \cap W_0 \cap F^p E^{2p}_{\log}(X) \right).$$

Since this form is closed,

$$\widetilde{\mathrm{ch}}_0(\mathcal{F}, h) \in \bigoplus_{p\geq 0} \left(\widehat{W}_{2p} E^{2p}_{\log}(X)_{\mathbb{R}} \cap \widehat{W}_{2p} \cap F^p E^{2p}_{\log}(X) \right).$$

Thus the triple

$$\widetilde{\mathrm{ch}}_0(\mathcal{F}, h)_{\mathcal{H}} = (\widetilde{\mathrm{ch}}_0(\mathcal{F}, h), \widetilde{\mathrm{ch}}_0(\mathcal{F}, h), 0)$$

is a cycle of $\bigoplus_{p\geq 0} \mathfrak{H}^{2p}(X, p)$.

Proposition 2.10. *The cycle* $\widetilde{\mathrm{ch}}_0(\mathcal{F},h)_{\mathcal{H}}$ *represents the Chern character of* $\mathcal{F}$ *in absolute Hodge cohomology.*

Proof. If X is proper we have

$$H^{2p}_{\mathcal{H}}(X,\mathbb{R}(p)) = H^{p,p}(X,(2\pi i)^p\mathbb{R}).$$

Therefore the result follows from the classical description of the Chern character in terms of curvature forms. In the non proper case it follows from the functoriality of the Chern character.

§3 Exact n-cubes of locally free sheaves.

In this section we shall recall the notion of exact n-cube (see [Lo 2], [Wan]). To each metrized exact n-cube, $\overline{\mathcal{F}}$, which satisfies certain conditions, we shall associate a metrized locally free sheaf on $X\times(\mathbb{P}^1)^n$, called the n-th transgression of $\overline{\mathcal{F}}$. This transgression can be viewed as a homotopy between its vertexes. The Chern character form of the transgression will play the role of higher Bott Chern forms.

(3.1) First some notations. Let $\langle -1,0,1\rangle$ be the category associated to the ordered set $\{-1,0,1\}$. Let $\langle -1,0,1\rangle^n$ be its n-th cartesian power. By convention, the category $\langle -1,0,1\rangle^0$ has one element and one morphism.

Let $\mathfrak{E}$ be an exact category.

Definition 3.1. A n-cube of $\mathfrak{E}$, $\mathcal{F}$, is a functor from $\langle -1,0,1\rangle^n$ to $\mathfrak{E}$.

Definition 3.2. Given a n-cube $\mathcal{F}$, and numbers $i\in\{1,\dots,n\}$, $j\in\{-1,0,1\}$, then the $n-1$-cube, $\partial_i^j\mathcal{F}$ defined by

$$(\partial_i^j\mathcal{F})_{\alpha_1,\dots,\alpha_{n-1}} = \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},j,\alpha_i,\dots,\alpha_{n-1}}$$

is called a face of $\mathcal{F}$. Given a number $i\in\{1,\dots,n\}$ and a $n-1$-tuple $\alpha=(\alpha_1,\dots,\alpha_{n-1})\in\{-1,0,1\}^{n-1}$, the sequence

$$\partial_{i^c}^{\alpha}\mathcal{F} = \partial_n^{\alpha_{n-1}}\dots\partial_{i+1}^{\alpha_i}\partial_{i-1}^{\alpha_{i-1}}\dots\partial_1^{\alpha_1}\mathcal{F}$$

is called an edge of $\mathcal{F}$.

Explicitly, the edge $\partial_{i^c}^{\alpha}\mathcal{F}$ is

$$\mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},-1,\alpha_i,\dots,\alpha_{n-1}} \longrightarrow \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},0,\alpha_i,\dots,\alpha_{n-1}} \longrightarrow \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},1,\alpha_i,\dots,\alpha_{n-1}}.$$

Definition 3.3. A n-cube is called an exact n-cube if all its edges are short exact sequences.

We shall denote by $\underline{C}_n\mathfrak{E}$ the exact category of exact n-cubes. Observe that, for all non negative integers n, m, there is a natural isomorphism of categories $\underline{C}_n\underline{C}_m\mathfrak{E}\longrightarrow\underline{C}_{n+m}\mathfrak{E}$. In particular, an exact n-cube can be viewed as an exact sequence of exact $n-1$-cubes or as an exact $n-1$-cube of exact sequences.

The maps

$$\partial_i^j : \operatorname{Ob} \underline{C}_n \mathfrak{E} \longrightarrow \operatorname{Ob} \underline{C}_{n-1} \mathfrak{E},$$

are called face maps. The maps

$$s_i^j : \operatorname{Ob} \underline{C}_n \mathfrak{E} \longrightarrow \operatorname{Ob} \underline{C}_{n+1} \mathfrak{E}, \qquad \text{for } i = 1, \dots, n, \text{ and } j = -1, 1,$$

given by

$$s_i^j(\mathcal{F})_{\alpha_1,\dots,\alpha_{n+1}} = \begin{cases} 0, & \text{if } \alpha_i = j, \\ \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_{n+1}}, & \text{if } \alpha_i \neq j, \end{cases}$$

are called degeneracy maps. An exact n-cube $\mathcal{F} \in \operatorname{Im} s_i^j$ is called degenerate.

(3.2) We shall write $C_n \mathfrak{E} = \operatorname{Ob} \underline{C}_n \mathfrak{E}$ and $C\mathfrak{E} = \coprod C_n \mathfrak{E}$.

Assume that the category $\mathfrak{E}$ is small. To avoid set theoretical problems, in the sequel we shall always assume tacitly that we replace any large category by an equivalent small full subcategory. Observe that the diagram $C\mathfrak{E}$ behaves like a cubical diagram. We have replaced the category $\langle 0, 1 \rangle$ by the category $\langle -1, 0, 1 \rangle$. This motivates the following construction.

Let $\mathbb{Z}C_n\mathfrak{E}$ be the free abelian group generated by $C_n\mathfrak{E}$. And let the differential $d : \mathbb{Z}C_n\mathfrak{E} \longrightarrow \mathbb{Z}C_{n-1}\mathfrak{E}$ be given by

$$d = \sum_{i=1}^{n} \sum_{j=-1}^{1} (-1)^{i+j} \partial_i^j.$$

Let $D_n \subset \mathbb{Z}C_n\mathfrak{E}$ be the subgroup generated by the degenerate exact n-cubes. Then $dD_n \subset D_{n-1}$. Therefore the following definition makes sense.

Definition 3.4. The homology complex associated to $C\mathfrak{E}$ is

$$\widetilde{\mathbb{Z}}C\mathfrak{E} = \mathbb{Z}C\mathfrak{E}/D.$$

(3.3) For the remainder of the section, let us fix a smooth complex variety X. Let $\mathfrak{E}(X)$ be the exact category of locally free sheaves on X and let $\overline{\mathfrak{E}}(X)$ be the exact category of pairs $(\mathcal{F}, h)$, where $\mathcal{F} \in \operatorname{Ob} \mathfrak{E}(X)$ and h is a smooth at infinity hermitian metric on $\mathcal{F}$. The morphisms of this category are

$$\operatorname{Hom}_{\overline{\mathfrak{E}}(X)}((\mathcal{F}, h), (\mathcal{F}', h')) = \operatorname{Hom}_{\mathfrak{E}(X)}(\mathcal{F}, \mathcal{F}').$$

The forgetful functor $\overline{\mathfrak{E}}(X) \longrightarrow \mathfrak{E}(X)$ is an equivalence of categories. A quasi-inverse can be obtained by choosing a metric for each locally free sheaf.

For simplicity we shall write $C(X) = C\overline{\mathfrak{E}}(X)$. An element $\overline{\mathcal{F}} \in C_n(X)$ is called a metrized exact n-cube of locally free sheaves.

(3.4) For technical reasons we need to work with metrized exact n-cubes which have, in all the quotients, the induced metrics.

Definition 3.5. We shall say that a metrized exact n-cube, $\overline{\mathcal{F}} = \{(\mathcal{F}_\alpha, h_\alpha)\}$ has induced quotient metrics (an emi-n-cube for short) if, for each n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, and each i with $\alpha_i = 1$, the metric h_α is induced by the metric $h_{(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n)}$.

Let us see that there are enough emi-n-cubes. Let $\alpha \in \{-1, 0, 1\}^n$ be a n-tuple. We shall write $\alpha \leq 0$ if $\alpha_i \leq 0$ for all i.

Proposition 3.6. *Let $\mathcal{F}$ be an exact n-cube of locally free sheaves and, for all $\alpha \leq 0$, let h_α be a hermitian metric on $\mathcal{F}_\alpha$. Then there is a unique way to choose metrics h_α for all $\alpha \not\leq 0$, such that $\overline{\mathcal{F}} = \{(\mathcal{F}_\alpha, h_\alpha)\}$ is an emi-n-cube.*

Proof. The uniqueness is clear. For the existence, we have to see that, in each $\mathcal{F}_\alpha$, with $\alpha \not\leq 0$, all the possible induced metrics agree. This is guaranteed by the following result.

Lemma 3.7. *Let $\{E_{i,j}\}_{i,j=-1,0,1}$ be an exact 2-cube of complex vector spaces. Let h be a hermitian metric on $E_{0,0}$ and let $h_{1,0}$ and $h_{0,1}$ be the hermitian metrics in $E_{1,0}$ and $E_{0,1}$ induced by h. Then the metrics induced by $h_{1,0}$ and $h_{0,1}$ in $E_{1,1}$ agree.*

Proof. Let us identify $E_{-1,0}$ and $E_{0,-1}$ with their images in $E_{0,0}$. Then the metric $h_{1,0}$ in $E_{1,0}$ is induced by the isomorphism $E_{-1,0}^{\perp} \cong E_{1,0}$. Therefore we can identify $E_{1,0}$ with $E_{-1,0}^{\perp}$ and the morphism $E_{0,0} \longrightarrow E_{1,0}$ with the orthogonal projection. But the image of $E_{0,-1}$ by this orthogonal projection is $(E_{-1,0}+E_{0,-1}) \cap E_{-1,0}^{\perp}$. Therefore the metric in $E_{1,1}$ induced by $h_{1,0}$ is induced by the isomorphism $(E_{-1,0} + E_{0,-1})^{\perp} \cong E_{1,1}$. By symmetry, the same is true for the metric induced by $h_{0,1}$.

(3.5) Let $\mathbb{Z}C_{emi}(X)$ be the subcomplex of $\mathbb{Z}C(X)$ generated by the emi-n-cubes, and let D_{emi} be the subcomplex of $\mathbb{Z}C_{emi}(X)$ generated by the degenerate emi-n-cubes. We shall write

$$\widetilde{\mathbb{Z}}C_{emi}(X) = \mathbb{Z}C_{emi}(X)/\, D_{emi} \subset \widetilde{\mathbb{Z}}C(X).$$

To translate results about emi-n-cubes to all exact metrized n-cubes we need to construct a morphism of complexes

$$\widetilde{\mathbb{Z}}C(X) \longrightarrow \widetilde{\mathbb{Z}}C_{emi}(X).$$

If $\alpha \in \{-1,0,1\}^n$ with $\alpha_i > -1$, we shall write $\alpha - i = (\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_n)$. Let $\overline{\mathcal{F}} = \{(F_\alpha, h_\alpha)\} \in C_n(X)$. For $i = 1, \dots, n$ let $\lambda_i^1 \overline{\mathcal{F}}$ be defined by

$$\lambda_i^1 \overline{\mathcal{F}}_\alpha = \begin{cases} (\mathcal{F}_\alpha, h_\alpha), & \text{if } \alpha_i = -1, 0, \\ (\mathcal{F}_\alpha, h'_\alpha), & \text{if } \alpha_i = 1, \end{cases}$$

where h'_α is the metric induced by $h_{\alpha - i}$. Thus $\lambda_i^1 \overline{\mathcal{F}}$ has the same locally free sheaves as $\overline{\mathcal{F}}$, but we have replaced the metrics of the locally free sheaves of the face $\partial_i^1 \overline{\mathcal{F}}$, by the metrics induced by $\partial_i^0 \overline{\mathcal{F}}$.

Let $\lambda_i^2 \overline{\mathcal{F}}$ be the exact n-cube determined by

$$\begin{aligned} \partial_i^{-1} \lambda_i^2 \overline{\mathcal{F}} &= \partial_i^1 \overline{\mathcal{F}}, \\ \partial_i^0 \lambda_i^2 \overline{\mathcal{F}} &= \partial_i^1 \lambda_i^1 \overline{\mathcal{F}}, \\ \partial_i^1 \lambda_i^2 \overline{\mathcal{F}} &= 0. \end{aligned}$$

This n-cube measures in some sense the difference between $\overline{\mathcal{F}}$ and $\lambda_i^1 \overline{\mathcal{F}}$.

Let us write $\lambda_i \overline{\mathcal{F}} = \lambda_i^1 \overline{\mathcal{F}} + \lambda_i^2 \overline{\mathcal{F}}$, and let us denote by λ the map

$$\begin{array}{rccc} \lambda : & \mathbb{Z}C_n(X) & \longrightarrow & \mathbb{Z}C_n(X) \\ & \overline{\mathcal{F}} & \longmapsto & \begin{cases} \lambda_n \dots \lambda_1 \overline{\mathcal{F}}, & \text{if } n \geq 1, \\ \overline{\mathcal{F}}, & \text{if } n = 0. \end{cases} \end{array}$$

Then one can check the following properties:

(1) λ is a morphism of complexes.

(2) $\operatorname{Im}\lambda \subset \mathbb{Z}C_{emi}(X)$.

(3) $\lambda(D) \subset D_{emi}$.

Therefore this map induces a morphism of complexes

$$\lambda : \widetilde{\mathbb{Z}}C(X) \longrightarrow \widetilde{\mathbb{Z}}C_{emi}(X).$$

(3.6) Let $\overline{\mathcal{F}}$ be an emi-n-cube of locally free sheaves. We shall associate to it a locally free sheaf $\operatorname{tr}_n(\overline{\mathcal{F}})$ on $X \times (\mathbb{P}^1)^n$ which, roughly speaking, is a homotopy between the vertexes of $\overline{\mathcal{F}}$.

Let $((x_1 : y_1), \ldots, (x_n : y_n))$ be homogeneous coordinates of $(\mathbb{P}^1)^n$. Let $\mathcal{I}_{x_i}$ (resp. $\mathcal{I}_{y_i}$) be the sheaf of ideals in $X \times (\mathbb{P}^1)^n$ defined by the subvariety $x_i = 0$, (resp. $y_i = 0$). Let $p_0 : X \times (\mathbb{P}^1)^n \longrightarrow X$ and $p_i : X \times (\mathbb{P}^1)^n \longrightarrow \mathbb{P}^1$, $i = 1, \ldots, n$, be the projections. Then the maps

$$\mathcal{I}_{x_i} \xrightarrow{x_i^{-1}} p_i^* \mathcal{O}_{\mathbb{P}^1}(-1),$$
$$\mathcal{I}_{y_i} \xrightarrow{y_i^{-1}} p_i^* \mathcal{O}_{\mathbb{P}^1}(-1)$$

are isomorphisms. The sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$ has a metric induced by the standard metric on $\mathbb{C}^2$. We put in $\mathcal{I}_{x_i}$ and $\mathcal{I}_{y_i}$ the metrics induced by the above isomorphisms. By 2.9, these metrics are smooth at infinity.

For each pair of integers $i \in \{1, \ldots, n\}$ and $j \in \{-1, 0\}$, we write

$$\mathcal{I}_{i,j} = \begin{cases} \mathcal{I}_{y_i}, & \text{if } j = -1, \\ \mathcal{I}_{x_i}, & \text{if } j = 0. \end{cases}$$

For each $\alpha \in \langle -1, 0, 1\rangle^n$, with $\alpha \leq 0$, and for each $k \in \{1, \ldots, n\}$, with $\alpha_k = -1$, we write

$$\mathcal{J}_\alpha = \prod_{i=1}^{n} \mathcal{I}_{i,\alpha_i}^{-1} \subset \mathcal{K}_{X \times (\mathbb{P}^1)^n},$$
$$\mathcal{J}_{\alpha,k} = \prod_{i \neq k} \mathcal{I}_{i,\alpha_i}^{-1} \subset \mathcal{K}_{X \times (\mathbb{P}^1)^n},$$

where $\mathcal{K}_{X \times (\mathbb{P}^1)^n}$ is the sheaf of rational functions on $X \times (\mathbb{P}^1)^n$.

Given an n-tuple $\alpha \leq 0$ and an integer $k \in \{1, \ldots, n\}$, with $\alpha_k = -1$, we write $\alpha + k = (\alpha_1, \ldots, \alpha_k + 1, \ldots, \alpha_n)$. We have the inclusions

$$\mathcal{J}_{\alpha,k} \subset \mathcal{J}_\alpha,$$
$$\mathcal{J}_{\alpha,k} \subset \mathcal{J}_{\alpha+k}.$$

Let us denote by $\varphi_{\alpha,k} : \overline{\mathcal{F}}_\alpha \longrightarrow \overline{\mathcal{F}}_{\alpha+k}$ the morphism $\overline{\mathcal{F}}(\alpha \longrightarrow \alpha + k)$. Let ψ be the morphism

$$\psi : \bigoplus_{\alpha \leq 0} \bigoplus_{k | \alpha_k = -1} p_0^* \overline{\mathcal{F}}_\alpha \otimes \mathcal{J}_{\alpha,k} \longrightarrow \bigoplus_{\alpha \leq 0} p_0^* \overline{\mathcal{F}}_\alpha \otimes \mathcal{J}_\alpha,$$

which sends $s \otimes g \in p_0^* \overline{\mathcal{F}}_\alpha \otimes \mathcal{J}_{\alpha,k}$ to

$$\begin{aligned}\psi(s \otimes g) &= s \otimes g + \varphi_{\alpha,k}(s) \otimes g \\ &\in p_0^* \overline{\mathcal{F}}_\alpha \otimes \mathcal{J}_\alpha \oplus p_0^* \overline{\mathcal{F}}_{\alpha+k} \otimes \mathcal{J}_{\alpha+k}.\end{aligned}$$

The locally free sheaf $\bigoplus_{\alpha \leq 0} p_0^* \overline{\mathcal{F}}_\alpha \otimes \mathcal{J}_\alpha$ has a metric induced by the metrics of $\mathcal{I}_{x_i}$, $\mathcal{I}_{y_i}$ and $\overline{\mathcal{F}}_\alpha$. This metric is smooth at infinity.

Definition 3.8. The n-transgression of $\overline{\mathcal{F}}$ is the hermitian locally free sheaf

$$\operatorname{tr}_n(\overline{\mathcal{F}}) = \operatorname{Coker}(\psi),$$

with the metric induced by the metric of $\bigoplus_{\alpha \leq 0} p_0^* \mathcal{F}_\alpha \otimes \mathcal{J}_\alpha$. By proposition 2.8, this metric is smooth at infinity.

The following result follows directly from the definition.

Proposition 3.9. *Let $\overline{\mathcal{F}}$ be an emi-n-cube. Then there are isometries*

$$\begin{aligned}\operatorname{tr}_n(\overline{\mathcal{F}})|_{\{x_i=0\}} &\cong tr_{n-1}(\partial_i^0 \overline{\mathcal{F}}),\\ \operatorname{tr}_n(\overline{\mathcal{F}})|_{\{y_i=0\}} &\cong tr_{n-1}(\partial_i^{-1} \overline{\mathcal{F}}) \overset{\perp}{\oplus} tr_{n-1}(\partial_i^1 \overline{\mathcal{F}}).\end{aligned}$$

(3.7) Let us give an inductive construction of the transgressions. If $n = 1$, an emi-1-cube, $\overline{\mathcal{F}}$ is a short exact sequence

$$\overline{\mathcal{F}}_{-1} \xrightarrow{f} \overline{\mathcal{F}}_0 \longrightarrow \overline{\mathcal{F}}_1,$$

where the metric of $\overline{\mathcal{F}}_1$ is induced by the metric of $\overline{\mathcal{F}}_0$. Then $\operatorname{tr}_1(\overline{\mathcal{F}})$ is the cokernel of the map

$$\begin{array}{ccccc}\mathcal{F}_{-1} & \longrightarrow & \mathcal{F}_{-1} \otimes \mathcal{I}_{y_1}^{-1} & \oplus & \mathcal{F}_0 \otimes \mathcal{I}_{x_1}^{-1}\\ s & \longmapsto & s \otimes 1 & \oplus & f(s) \otimes 1.\end{array}$$

Observe that this is a minor modification of the locally free sheaf used by Bismut, Gillet and Soulé ([B-G-S], [G-S]) to construct Bott-Chern forms. In the definition given here, we avoid the use of partitions of unity, obtaining a natural construction. The price is to restrict ourselves to emi-n-cubes.

If $\overline{\mathcal{F}}$ is an emi-n-cube, let $\operatorname{tr}_1(\overline{\mathcal{F}})$ be the emi-$n-1$-cube over $X \times \mathbb{P}^1$ defined by:

$$\operatorname{tr}_1(\overline{\mathcal{F}})_\alpha = \operatorname{tr}_1(\partial_{n^c}^\alpha \overline{\mathcal{F}}).$$

Then we write

$$\operatorname{tr}_k(\overline{\mathcal{F}}) = \operatorname{tr}_1(\operatorname{tr}_{k-1}(\overline{\mathcal{F}})).$$

The hermitian locally free sheaf $\operatorname{tr}_n(\overline{\mathcal{F}})$ defined in this way coincides with the earlier definition. Thus the transgressions are simply an iteration of the construction of Bismut, Gillet and Soulé.

(3.8) For any homology complex A_*, we shall denote by A^* the cohomology complex defined by $A^k = A_{-k}$. Let us use the transgressions previously defined, to associate to every emi-n-cube a family of differential forms.

Definition 3.10. Let
$$\mathrm{ch}:\mathbb{Z}C^*_{emi}(X)\longrightarrow\bigoplus_p\widetilde{\mathfrak{H}}^*(X,p)[2p]$$
be the map given by
$$\mathrm{ch}(\overline{\mathcal{F}})=\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}}))_{\mathcal{H}},$$
where $\widetilde{\mathrm{ch}}_0(\cdot)_{\mathcal{H}}$ is as in (2.4).

Proposition 3.11. *The map* ch *is a morphism of complexes and factorizes through a unique morphism*
$$\mathrm{ch}:\widetilde{\mathbb{Z}}C^*_{emi}(X)\longrightarrow\bigoplus_p\widetilde{\mathfrak{H}}^*(X,p)[2p].$$

Proof. To see that it is a morphism of complexes, observe that, since the forms $\widetilde{\mathrm{ch}}_0(\cdot)_{\mathcal{H}}$ are closed,

$$\begin{aligned}d\,\mathrm{ch}(\mathrm{tr}_n(\overline{\mathcal{F}}))&=\sum_{i=1}^{n}\sum_{j=0}^{1}(-1)^{i+j}d_i^j\,\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}}))_{\mathcal{H}}\\&=\sum_{i=1}^{n}(-1)^i\,\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}}))_{\mathcal{H}}|_{\{x_i=0\}}+\sum_{i=1}^{n}(-1)^{i+1}\,\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}}))_{\mathcal{H}}|_{\{y_i=0\}}\\&=\sum_{i=1}^{n}(-1)^i\,\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}})|_{\{x_i=0\}})_{\mathcal{H}}+\sum_{i=1}^{n}(-1)^{i+1}\,\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}})|_{\{y_i=0\}})_{\mathcal{H}}.\end{aligned}$$

Therefore, by proposition 3.9,

$$\begin{aligned}d\,\mathrm{ch}(\mathrm{tr}_n(\overline{\mathcal{F}}))&=\sum_{i=1}^{n}\sum_{j=-1}^{1}(-1)^{i+j}\,\widetilde{\mathrm{ch}}_0(\mathrm{tr}_{n-1}(\partial_i^j\overline{\mathcal{F}}))_{\mathcal{H}}\\&=\mathrm{ch}(d\overline{\mathcal{F}}).\end{aligned}$$

To see the existence of the factorization, we have to show that, for a degenerate emi-n-cube $\overline{\mathcal{F}}$, we have $\mathrm{ch}(\overline{\mathcal{F}})=0$ in $\bigoplus\widetilde{\mathfrak{H}}^*(X,p)$. By symmetry we may assume that $\overline{\mathcal{F}}=s_n^j\overline{\mathcal{G}}$, with $j\in\{-1,1\}$ and $\overline{\mathcal{G}}$ an emi-$n-1$-cube.

If $j=1$, then $\mathrm{tr}_{n-1}(\overline{\mathcal{F}})$ is the exact sequence

$$0\rightarrow(\mathrm{Id}\times s^n)^*\,\mathrm{tr}_{n-1}(\overline{\mathcal{G}})\xrightarrow{\mathrm{Id}}(\mathrm{Id}\times s^n)^*\,\mathrm{tr}_{n-1}(\overline{\mathcal{G}})\rightarrow 0\rightarrow 0.$$

Therefore $\mathrm{tr}_n(\overline{\mathcal{F}})$ is the cokernel of the map

$$\begin{array}{ccccc}(\mathrm{Id}\times s^n)^*\,\mathrm{tr}_{n-1}(\overline{\mathcal{G}}) & \longrightarrow & (\mathrm{Id}\times s^n)^*\,\mathrm{tr}_{n-1}(\overline{\mathcal{G}})\otimes\mathcal{I}_{y_n}^{-1} & \oplus & (\mathrm{Id}\times s^n)^*\,\mathrm{tr}_{n-1}(\overline{\mathcal{G}})\otimes\mathcal{I}_{x_n}^{-1}\\ x & \longmapsto & x\otimes 1 & \oplus & x\otimes 1.\end{array}$$

But $\mathcal{I}_{y_n}^{-1}$ and $\mathcal{I}_{x_n}^{-1}$ are both isometric with $p_n^*\mathcal{O}(1)$. Hence this cokernel is isometric with $(\mathrm{Id}\times s^n)^*\,\mathrm{tr}_{n-1}(\overline{\mathcal{G}})\otimes p_n^*\mathcal{O}(2)$, where $\mathcal{O}(2)$ is provided with the standard metric. Thus

$$\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}}))_{\mathcal{H}}=(\mathrm{Id}\times s^n)^*\,\widetilde{\mathrm{ch}}_0(\mathrm{tr}_{n-1}(\overline{\mathcal{G}}))_{\mathcal{H}}+2\omega_n\wedge(\mathrm{Id}\times s^n)^*\,\widetilde{\mathrm{ch}}_0(\mathrm{tr}_{n-1}(\overline{\mathcal{G}}))_{\mathcal{H}}.$$

which is zero in $\bigoplus \widetilde{\mathfrak{H}}^*(X,p)$.

The case $j=-1$ is analogous.

Definition 3.12. We shall denote also by ch the composition

$$\widetilde{\mathbb{Z}}C^*(X) \xrightarrow{\lambda} \widetilde{\mathbb{Z}}C^*_{emi}(X) \xrightarrow{\mathrm{ch}} \bigoplus_p \widetilde{\mathfrak{H}}^*(X,p)[2p].$$

Definition 3.13. Let $\overline{\mathcal{F}}$ be a metrized exact n-cube. The form $\mathrm{ch}(\lambda(\overline{\mathcal{F}}))$ will be called the Bott Chern form of $\overline{\mathcal{F}}$ and will be denoted by $\widetilde{\mathrm{ch}}_{\mathrm{n}}(\overline{\mathcal{F}})_{\mathcal{H}}$.

§4 Higher characteristic classes.

The Chern character from K-theory to a suitable cohomology theory, such as absolute Hodge cohomology, is additive for exact sequences. Nevertheless, given a cochain complex which computes absolute Hodge cohomology, we cannot make a consistent choice of representatives of the Chern character that behaves additively. Following the ideas of Schechtman ([Sch]), the lack of additivity at the level of complexes, of the Chern character for K_n, gives us the Chern character for K_{n+1}.

In the previous section we have associated, to each metrized exact n-cube, a family of differential forms. The differential form associated to an n-cube measures the lack of additivity of the differential forms associated to its faces. In this section we shall see that this construction allows us to define higher Chern character classes from K-theory to absolute Hodge cohomology.

(4.1) Let us begin by reviewing the Waldhausen K-theory of a small exact category. We shall follow [Sch] (See also [Wal] or [Lo 1]).

For $n \in \mathbb{N}$, let $\mathrm{Cat}(n)$ denote the category associated with the ordered set $\{1,\dots,n\}$. Let M_n be the category of morphisms of $\mathrm{Cat}(n)$. That is

$$\mathrm{Ob}\, M_n = \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid 0 \le i \le j \le n\},$$

and $\mathrm{Hom}((i,j),(k,l))$ contains a unique element if $i \le k$ and $j \le l$ and is empty otherwise. The categories M_n form a cosimplicial category M.

For any category $\mathfrak{C}$, let us denote by $M_n\mathfrak{C}$ the category of functors from M_n to $\mathfrak{C}$.

Definition 4.1. Let $\mathfrak{E}$ be a small exact category and 0 a fixed zero object of $\mathfrak{E}$. Let $\underline{S}_n\mathfrak{E}$ be the full subcategory of $M_n\mathfrak{E}$, whose objects are the functors $M_n \longrightarrow \mathfrak{E}$, such that,

(1) for all i, $E_{i,i}=0$;
(2) for all $i \le j \le k$,

$$E_{i,j} \to E_{i,k} \to E_{j,k}$$

is a short exact sequence.

Let us write $S_n\mathfrak{E} = \mathrm{Ob}\,\underline{S}_n\mathfrak{E}$. We shall denote by $\underline{S}\mathfrak{E}$ or $\underline{S}^1\mathfrak{E}$ the simplicial exact category $\coprod \underline{S}_n\mathfrak{E}$, and by $S\mathfrak{E}$ or $S^1\mathfrak{E}$ the simplicial set $\mathrm{Ob}\,\underline{S}\mathfrak{E}$.

In other words we have:

$$\begin{aligned}
S_0\mathfrak{E} &= \{0\},\\
S_1\mathfrak{E} &= \operatorname{Ob}\mathfrak{E},\\
S_2\mathfrak{E} &= \{\text{exact sequences of } \mathfrak{E}\},\\
S_n\mathfrak{E} &= \left\{\begin{array}{c}\text{sequences of monomorphisms}\\ E_{0,1}\to E_{0,2}\to\cdots\to E_{0,n}\\ \text{with a choice of quotients}\\ E_{i,j}\cong E_{0,j}/E_{0,i}\end{array}\right\}.
\end{aligned}$$

In particular $S\mathfrak{E}$ is a pointed simplicial set. In the sequel we shall sometimes use the word space to denote simplicial sets.

For a space C, we shall denote by $|C|$ its geometric realization.

Proposition 4.2. (Cf. [Lo 2].) *There is a homotopy equivalence*

$$S\mathfrak{E} \cong BQ\mathfrak{E},$$

where Q denotes Quillen's Q-construction and B means classifying space. Therefore, for all $i \geq 0$,

$$K_i(\mathfrak{E}) = \pi_{i+1}\left(|S.\mathfrak{E}|, 0\right).$$

(4.2) Let us recall the notion of spectrum from [Th]. For any pointed space C, let us write ΣC for the suspension of C, and ΩC for the loop space of C. We shall use the same notation for topological spaces.

Definition 4.3. A prespectrum X is a sequence of pointed spaces X_n for all non-negative integers n, together with structure maps $\Sigma X_n \longrightarrow X_{n+1}$. These maps can be described by their adjoint $X_n \longrightarrow \Omega X_{n+1}$. A fibrant spectrum is a prespectrum such that all X_n are fibrant spaces and the structure maps $X_n \longrightarrow \Omega X_{n+1}$ are weak equivalences.

The space $S\mathfrak{E}$ is a piece of a prespectrum. To construct the other spaces that form the prespectrum, we write inductively

$$\begin{aligned}
\underline{S}^m\mathfrak{E} &= \underline{S}\underline{S}^{m-1}\mathfrak{E},\\
S^m\mathfrak{E} &= S\underline{S}^{m-1}\mathfrak{E}.
\end{aligned}$$

Then $\underline{S}^m$ is an exact m-simplicial category and S^m is a m-simplicial set. For a poly-simplicial set C let $\operatorname{diag}(C)$ denote its diagonal space. We shall denote by $|C| = |\operatorname{diag}(C)|$ its geometric realization.

Proposition 4.4. (Cf. [Sch].) *There are natural maps*

$$\varphi_m : \Sigma S^m\mathfrak{E} \longrightarrow S^{m+1}\mathfrak{E},$$

inducing homotopy equivalences

$$|S^m\mathfrak{E}| \cong \Omega\left|S^{m+1}\mathfrak{E}\right|.$$

As a consequence of this proposition, if we write $S^0\mathfrak{E} = \Omega S\mathfrak{E}$, then the sequence of spaces $\operatorname{diag}(S^m\mathfrak{E})$ is a prespectrum. Moreover, if we replace the above spaces by weakly equivalent fibrant spaces we shall obtain a fibrant spectrum. For instance, let us denote by Sing the singular functor (see [B-K]). Then, if we write

$$\mathbf{K}_m(\mathfrak{E}) = \operatorname{Sing}(|S^m\mathfrak{E}|),$$

the spaces $\mathbf{K}_m$ form a fibrant spectrum. By Proposition 4.2, the homotopy of this fibrant spectrum is the K-theory of $\mathfrak{E}$.

(4.3) For example, if X is a smooth complex variety, and we write $S^q(X) = S^q(\overline{\mathfrak{E}}(X))$, then he K-groups of X are

$$K_i(X) = \pi_{i+q}(|S^q(X)|, 0).$$

(4.4) Let us associate, to each element of $S_n\mathfrak{E}$, an exact metrized $n-1$-cube. We shall do so inductively. For $n = 1$, we write

$$\operatorname{Cub}(\{E_{i,j}\}_{0\leq i\leq j\leq 1}) = E_{0,1}.$$

Assume that we have defined $\operatorname{Cub} E$ for all $E \in S_m\mathfrak{E}$, with $m < n$. Let $E \in S_n\mathfrak{E}$. Then $\operatorname{Cub} E$ is the $n-1$-cube with

$$\begin{aligned}
\partial_1^{-1} \operatorname{Cub} E &= s_{n-2}^1 \dots s_1^1(E_{0,1}),\\
\partial_1^0 \operatorname{Cub} E &= \operatorname{Cub}(\partial_1 E),\\
\partial_1^{-1} \operatorname{Cub} E &= \operatorname{Cub}(\partial_0 E).
\end{aligned}$$

For instance, if $n = 2$, then $\operatorname{Cub}(\{E_{i,j}\}_{0\leq i\leq j\leq 2})$ is the short exact sequence

$$E_{0,1} \longrightarrow E_{0,2} \longrightarrow E_{1,2}.$$

On the other hand, if $n = 3$, then $\operatorname{Cub}(\{E_{i,j}\}_{0\leq i\leq j\leq 3})$ is the exact square

$$\begin{array}{ccccc}
E_{0,1} & \longrightarrow & E_{0,2} & \longrightarrow & E_{1,2}\\
\downarrow & & \downarrow & & \downarrow\\
E_{0,1} & \longrightarrow & E_{0,3} & \longrightarrow & E_{1,3}\\
\downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & E_{2,3} & \longrightarrow & E_{2,3}
\end{array}$$

All the faces of the $n-1$-cube $\operatorname{Cub} E$ can be computed explicitly.

Proposition 4.5. *Let $E \in S_n\mathfrak{E}$. Then, for $i = 1, \dots, n-1$, the faces of the $n-1$-cube $\operatorname{Cub} E$ are*

$$\begin{aligned}
\partial_i^{-1} \operatorname{Cub} E &= s_{n-2}^1 \dots s_i^1 \operatorname{Cub} \partial_{i+1} \dots \partial_n E,\\
\partial_i^0 \operatorname{Cub} E &= \operatorname{Cub} \partial_i E,\\
\partial_i^1 \operatorname{Cub} E &= s_{i-1}^{-1} \dots s_1^{-1} \operatorname{Cub} \partial_0 \dots \partial_{i-1} E.
\end{aligned}$$

By proposition 4.5 and using induction we have,

Corollary 4.6. *The $n-1$-cube* $\mathrm{Cub}\,E$ *is exact.*

Therefore we have a map $\mathrm{Cub} : S_n\mathfrak{C} \longrightarrow C_{n-1}\mathfrak{C}$.

(4.5) Let $\mathbb{Z}S\mathfrak{C}$ be the homological complex associated with the simplicial set $S\mathfrak{C}$. That is, $\mathbb{Z}S_n\mathfrak{C}$ is the free abelian group generated by $S_n\mathfrak{C}$, and the differential $d : \mathbb{Z}S_n\mathfrak{C} \longrightarrow \mathbb{Z}S_{n-1}\mathfrak{C}$ is given by

$$d = \sum_{i=0}^{n}(-1)^i\partial_i.$$

The map Cub can be extended by linearity to a map

$$\mathrm{Cub} : \mathbb{Z}S\mathfrak{C}[1] \longrightarrow \mathbb{Z}C\mathfrak{C}.$$

Note that this map is not a morphism of complexes. However, the map Cub induces a map also denoted by $\mathrm{Cub} : \mathbb{Z}S\mathfrak{C}[1] \longrightarrow \widetilde{\mathbb{Z}}C\mathfrak{C}$. And, since by proposition 4.5,

$$d\,\mathrm{Cub}\,E = \mathrm{Cub}\,dE + \text{degenerate elements},$$

we have:

Corollary 4.8. *The map* $\mathrm{Cub} : \mathbb{Z}S\mathfrak{C}[1] \longrightarrow \widetilde{\mathbb{Z}}C\mathfrak{C}$ *is a morphism of complexes.*

(4.6) We can obtain analogous maps for all the spaces $S^m\mathfrak{C}$. In particular, we have maps

$$\mathrm{Cub} : S_{n_1}\dots S_{n_m}\mathfrak{C} \longrightarrow C_{n_1-1}\dots C_{n_m-1}\mathfrak{C} \longrightarrow C_{n_1+\dots+n_m-m}\mathfrak{C}.$$

Let us denote by $\mathbb{Z}S^m\mathfrak{C}$ the chain complex that, in degree n, is the free abelian group generated by

$$\coprod_{n_1+\dots+n_m=n} S_{n_1}\dots S_{n_m}\mathfrak{C}.$$

The differential of this complex is the alternate sum of all the face maps. Note that this complex is homotopically equivalent to $\mathbb{Z}\,\mathrm{diag}(S^m\mathfrak{C})$. The induced map

$$\mathrm{Cub} : \mathbb{Z}S^m\mathfrak{C}[m] \longrightarrow \widetilde{\mathbb{Z}}C\mathfrak{C}$$

is also a morphism of complexes.

(4.7) We shall denote by $\mathbb{Z}S^*_m(X)$ the cohomological complex associated to the homological complex $\mathbb{Z}S^m_*(X)$.

Definition 4.9. The Chern character map is the composition

$$\mathbb{Z}S^*_m(X)[-m] \xrightarrow{\mathrm{Cub}} \widetilde{\mathbb{Z}}C^*(X) \xrightarrow{\lambda} \widetilde{\mathbb{Z}}C^*_{emi}(X) \xrightarrow{\mathrm{ch}} \bigoplus_p \widetilde{\mathfrak{H}}^*(X,p)[2p].$$

This map will also be denoted by ch. The Chern character classes are obtained by composing with the Hurewick map:

$$K_i(X) = \pi_{i+m}(S^m(X))) \rightarrow H_{i+m}(\mathbb{Z}S^m(X)) \rightarrow \bigoplus_p H^{2p-i}_{\mathcal{H}}(X,p).$$

§5 BEILINSON'S REGULATOR.

The aim of this section is to prove that the higher Chern character classes defined in §4 agree with Beilinson's regulator map.

(5.1) Let us begin by extending the definition of the map ch to the case of simplicial smooth complex varieties. To this end, we first recall the construction of absolute Hodge cohomology of $X = X.$ a smooth simplicial complex variety. For each p, the complexes $\widetilde{\mathfrak{H}}^*(X_n, p)$ form a cosimplicial complex as n varies. Let $\mathcal{N}\widetilde{\mathfrak{H}}^*(X., p)$ be the associated double complex and let us denote the simple complex by

$$\widetilde{\mathfrak{H}}^*(X,p) = s(\mathcal{N}\widetilde{\mathfrak{H}}^*(X.,p)).$$

Then

$$H^*_{\mathcal{H}}(X., \mathbb{R}(p)) = H^*(\widetilde{\mathfrak{H}}^*(X,p)).$$

For the definition of K-theory of simplicial schemes we shall follow [Sch]. We shall say that a smooth simplicial scheme $X = X.$ has finite dimension if there is an integer m such that

$$X = \mathrm{Sk}_m(X),$$

where $\mathrm{Sk}_m(X)$ is the m-th skeleton of X, that is, the simplicial scheme generated by $X_0, \dots, X_m$.

Let $X = X.$ be a simplicial scheme of finite dimension. The family of prespectrums $\{S(X_n)\}_n$ form a cosimplicial prespectrum $S(X.)$. Let $\mathbf{K}(X.)$ be a fibrant cosimplicial fibrant spectrum weakly equivalent to $S(X.)$. Then the K groups of X are defined as

$$K_i(X) = \pi_i(\mathrm{Tot}\,\mathbf{K}(X.)).$$

Since X has finite dimension, there is a convergent spectral sequence

$$E_1^{p,q} = K_{-q}(X_p) \Longrightarrow K_{-p-q}(X.).$$

Observe that for a given simplicial scheme $X.$, of finite dimension, it is not necessary to work with the whole spectrum. Let m be such that $X = \mathrm{Sk}_m X$. Let us choose an integer q, and let $\mathbf{K}_q(X.)$ be a fibrant cosimplicial fibrant space, weakly equivalent to $S^q(X.)$. If $q > m$ or $q > -i$, then

$$K_i(X.) = \pi_{i+q}(\mathrm{Tot}\,\mathbf{K}_q(X.)).$$

For an arbitrary simplicial scheme we write

$$\widehat{K}_*(X.) = \varprojlim_m K_*(\mathrm{Sk}_m(X.)).$$

Let X be a smooth simplicial complex variety of finite dimension. Since the map ch defined in section §4 gives us a morphism of complexes

$$\mathrm{ch} : s\mathcal{N}\mathbb{Z}S^*_q(X.)[q] \longrightarrow \bigoplus_p \widetilde{\mathfrak{H}}^*(X,p)[2p],$$

we can extend the definition of the Chern character to the simplicial case, obtaining maps:

$$\mathrm{ch} : K_i(X) \longrightarrow \bigoplus_p H_{\mathcal{H}}^{2p-i}(X, \mathbb{R}(p)).$$

If X does not have finite dimension, taking limits, we have also characteristic classes

$$\mathrm{ch} : \widehat{K}_i(X) \longrightarrow \bigoplus_p H_{\mathcal{H}}^{2p-i}(X, \mathbb{R}(p)).$$

Remark. 5.1. All the constructions needed to define the map ch can be extended to the case of a smooth simplicial scheme over $\mathbb{C}$ $X.$ such that each X_n is a (not necessarily finite) disjoint union of smooth complex varieties. For instance, by a compactification of X_n we shall mean a disjoint union of compactifications of each component of X_n.

(5.2) Beilinson ([Be]) has defined characteristic classes from K-theory to absolute Hodge cohomology. These classes are a particular case of the characteristic classes defined by Gillet ([Gi]) to any suitable cohomology theory. In particular, Beilinson's regulator is the Chern character in this theory. Let us denote by ρ Beilinson's regulator.

Then ch and ρ are natural transformations between contravariant functors. Both agree with the classical Chern character on the K_0 groups of smooth complex varieties. The aim of this section is to prove the following theorem.

Theorem 5.2. *Let X be a smooth complex variety. Let $\sigma \in K_i(X)$. Then* $\mathrm{ch}(\sigma) = \rho(\sigma)$.

Proof. Let $\mathfrak{U} = \{U_\alpha\}$ be an open covering of X. We shall denote by $\mathfrak{E}(X, \mathfrak{U})$ the full subcategory of $\mathfrak{E}(X)$ composed by the locally free sheaves on X whose restrictions to all U_α are free. We shall denote by $\overline{\mathfrak{E}}(X, \mathfrak{U})$ the category of hermitian vector bundles on X whose restrictions to all U_α are free. Let us write

$$K_i(X, \mathfrak{U}) = \pi_{i+1}(S\mathfrak{E}(X, \mathfrak{U})) = \pi_{i+1}(S\overline{\mathfrak{E}}(X, \mathfrak{U})).$$

Then

$$K_i(X) = \varinjlim_{\mathfrak{U}} K_i(X, \mathfrak{U}).$$

(5.2.1) Following Schechtman ([Sch]) we know that there is a simplicial scheme $\mathcal{B}P$, which is a classifying space for algebraic K-theory. More precisely, Schechtman proves the following result.

Theorem 5.3. (Schechtman) *There is a homotopy equivalence*

$$S\mathfrak{E}(X, \mathfrak{U}) \cong \underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P),$$

where $\underline{\mathrm{Hom}}$ *is the function space and* $N\mathfrak{U}$ *is the nerve of the covering.*

(5.2.2) Let $Y = Y.$ be a smooth simplicial scheme and let $f : Y \longrightarrow \mathcal{B}P$ be a map of simplicial schemes. Then f defines an element of $\pi_0\underline{\mathrm{Hom}}(Y, \mathcal{B}P)$. Let us denote by e_f the image of this element by the composition of the morphisms

$$\pi_0\underline{\mathrm{Hom}}(Y, \mathcal{B}P) \longrightarrow \pi_0 \operatorname{Tot}_n \underline{\mathrm{Hom}}(Y_n, \mathcal{B}P) \longrightarrow \pi_0 \operatorname{Tot}_n S\mathfrak{E}(Y_n, \{Y_n\}) \longrightarrow K_{-1}(Y).$$

Remark 5.4. The identity of $\mathcal{B}P$ defines an element, denoted by $e_{\mathcal{B}P} \in K_{-1}(\mathcal{B}P)$. Moreover, for f as above, $e_f = f^*(e_{\mathcal{B}P})$.

(5.2.3) The element $e_{\mathcal{B}P}$ is, in some sense, a universal element in K-theory. Since $e_{\mathcal{B}P} \in K_{-1}$, to exploit the universality of this element, we need to relate elements in K_n with elements in K_{-1}. This can be done using spheres.

Let $\sigma \in K_n(X)$. Then there is an open covering $\mathfrak{U}$ of X, such that $\sigma \in K_n(X, \mathfrak{U}) = \pi_{n+1}(\underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P))$. Therefore, since $\underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P)$ is fibrant, σ is represented by an element

$$\gamma_\sigma \in \underline{\mathrm{Hom}}(S^{n+1} \times N\mathfrak{U}, \mathcal{B}P) = \underline{\mathrm{Hom}}(S^{n+1}, \underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P)),$$

where S^{n+1} is the (pointed) simplicial $n+1$-dimensional sphere.

Lemma 5.5. *Let $Y = Y_.$ be a smooth simplicial complex variety. Then there are natural decompositions*

$$\widehat{K}_{-1}(S^{n+1} \times Y) = \widehat{K}_{-1}(Y) \oplus \widehat{K}_n(Y),$$
$$H^{2p+1}_{\mathcal{H}}(S^{n+1} \times Y, \mathbb{R}(p)) = H^{2p+1}_{\mathcal{H}}(Y, \mathbb{R}(p)) \oplus H^{2p-n}_{\mathcal{H}}(Y, \mathbb{R}(p)).$$

Moreover, the maps ch *and* ρ *are compatible with these decompositions.*

Proof. We may assume that Y has finite dimension because the general case is obtained taking the limit. Then

$$K_{-1}(S^{n+1} \times Y) = \pi_0(\mathrm{Tot}_\alpha(\mathrm{Tot}_\beta(\mathbf{K}(S^{n+1}_\alpha \times Y_\beta)))).$$

The spectral sequence associated with Tot_α has E_2-term:

$$E_2^{p,q} = \begin{cases} K_{-q}(Y), & \text{if } p = 0, n+1, \\ 0, & \text{if } p \neq 0, n+1. \end{cases}$$

Let us denote by $*$ the simplicial point. Since the spectral sequence of $* \times Y$ splits the spectral sequence of $S^{n+1} \times Y$, the above spectral sequence degenerates at the E_2-term, and the exact sequence obtained from this spectral sequence splits in a natural way.

The same argument works for cohomology. Moreover, since ch and ρ are natural transformations, they induce morphisms between the K-theoretical and the cohomological spectral sequences, proving the compatibility statement.

Let us denote by $pr : K_{-1}(S^{n+1} \times N\mathfrak{U}) \longrightarrow K_n(N\mathfrak{U})$ the projection. The precise meaning of the universality of $e_{\mathcal{B}P}$ is given by the following result.

Lemma 5.6. *In the group $K_n(N\mathfrak{U})$, the equality*

$$pr(\gamma_\sigma^*(e_{\mathcal{B}P})) = \sigma$$

holds.

Proof. By remark 5.4,

$$\mathrm{pr}(\gamma_\sigma^*(e_{\mathcal{B}P})) = \mathrm{pr}(e_{\gamma_\sigma}).$$

On the other hand, by the definition of γ_σ, the map

$$\pi_0(\underline{\mathrm{Hom}}(S^{n+1} \times N\mathfrak{U}, \mathcal{B}P)) \xrightarrow{\mathrm{pr}} \pi_{n+1}(\underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P))$$

sends the class of γ_σ to the class of σ. Therefore, since the diagram

$$\begin{array}{ccc} \pi_0(\underline{\mathrm{Hom}}(S^{n+1} \times N\mathfrak{U}, \mathcal{B}P)) & \longrightarrow & K_{-1}(S^{n+1} \times N\mathfrak{U}) \\ {\scriptstyle \mathrm{pr}}\downarrow & & {\scriptstyle \mathrm{pr}}\downarrow \\ \pi_{n+1}(\underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P)) & \longrightarrow & K_n(N\mathfrak{U}). \end{array}$$

is commutative we have that

$$\mathrm{pr}(\gamma_\sigma^*(e_{\mathcal{B}P})) = \sigma.$$

(5.2.4) By Remark 5.1, the map ch is defined for the simplicial scheme $\mathcal{B}P$. Moreover, by the naturality of ch and ρ and their compatibility with the map pr, we have

$$\begin{aligned} \mathrm{ch}(\sigma) &= \mathrm{pr}(\gamma_\sigma^*(\mathrm{ch}(e_{\mathcal{B}P}))), \\ \rho(\sigma) &= \mathrm{pr}(\gamma_\sigma^*(\rho(e_{\mathcal{B}P}))). \end{aligned}$$

Thus, to prove theorem 5.2, we are led to compare $\mathrm{ch}(e_{\mathcal{B}P})$ and $\rho(e_{\mathcal{B}P})$. For this comparison, we need to understand the cohomology of $\mathcal{B}P$. This cohomology has been computed by Schechtman ([Sch]). The simplicial scheme $\mathcal{B}P$ is the classifying space of a simplicial group $P_\cdot$, where $P_0 = *$ and $P_1 = \coprod_n GL(n)$. Thus it is a bisimplicial scheme $\mathcal{B}.P_\cdot$. The edge homomorphism of the spectral sequence associated to the second index gives us a morphism

$$d_H : H^{2p+1}_{\mathcal{H}}(\mathcal{B}P, \mathbb{R}(p)) \longrightarrow \prod_{n\geq 0} H^{2p}_{\mathcal{H}}(BGL(n), \mathbb{R}(p)).$$

Let us denote by $A = H^*_{\mathcal{H}}(\mathrm{Spec}\,\mathbb{C}, \mathbb{R}(*))$.

For each i, n let us denote by

$$c_{i,n} = c_i(E_n) \in H^{2i}_{\mathcal{H}}(BGL(n), \mathbb{R}(i)),$$

the i-th Chern class of the tautological vector bundle over $BGL(n)$. Then we have an isomorphism

$$H^*_{\mathcal{H}}(BGL(n), \mathbb{R}(*)) = A[c_{1,n}, \dots, c_{n,n}].$$

Let $s_{k,n} \in A[c_{1,n}, \dots, c_{n,n}]$ be the k-th Newton polynomial in the $c_{i,n}$. That is, $s_{k,n}/n!$ is the degree k term of the Chern character of the tautological vector bundle E_n. Let us write

$$s_k = (s_{k,0}, s_{k,1}, \dots) \in \prod_{n\geq 0} H^{2k}_{\mathcal{H}}(BGL(n), \mathbb{R}(k)).$$

Proposition 5.7. (Schechtman [Sch]) *There exist elements* $s_k^1 \in H_{\mathcal{H}}^{2k+1}(\mathcal{B}P, \mathbb{R}(k))$ *such that* $d_H(s_k^1) = s_k$ *and*

$$H_{\mathcal{H}}^*(\mathcal{B}P, \mathbb{R}(*)) = A[s_0^1, s_1^1, \dots].$$

(5.2.5) Since

$$H_{\mathcal{H}}^n(\operatorname{Spec}\mathbb{C}, \mathbb{R}(p)) \cong \begin{cases} \mathbb{R}, & \text{if } n = p = 0, \text{ or } n = 1,\ p > 0, \\ 0, & \text{otherwise,} \end{cases}$$

any element of $H_{\mathcal{H}}^{2k+1}(\mathcal{B}P, \mathbb{R}(k))$ can be written as

$$\alpha s_k^1 + \text{ decomposable elements,}$$

with $\alpha \in \mathbb{R}$. Moreover, since by the proof of 5.7 ([Sch]) the decomposable elements are mapped to 0 by d_H, we have

Corollary 5.8. *The group* $\operatorname{Ker} d_H \subset \bigoplus_k H_{\mathcal{H}}^{2k+1}(\mathcal{B}P, \mathbb{R}(k))$ *is generated by decomposable elements.*

(5.2.6) Schechtman computes the groups $\widehat{K}_*(\mathcal{B}P)$ in a similar way. In particular, there is also an edge homomorphism

$$d_K : \widehat{K}_{-1}(\mathcal{B}P) \longrightarrow \prod_n \widehat{K}_0(\mathcal{B}GL(n)).$$

Moreover, by the naturality of ch and ρ, they are compatible with the edge homomorphisms. In particular

$$d_H(\rho(e_{\mathcal{B}P})) = \rho(d_K(e_{\mathcal{B}P})), \text{ and } d_H(\mathrm{ch}(e_{\mathcal{B}P})) = \mathrm{ch}(d_K(e_{\mathcal{B}P})).$$

(5.2.7) Our next step will be to compare $d_H(\rho(e_{\mathcal{B}P}))$ with $d_H(\mathrm{ch}(e_{\mathcal{B}P}))$. To this end we shall see that, since the maps ch and ρ agree for the K_0 groups of smooth complex varieties then they also agree for the group $\widehat{K}_0(BGL(n))$.

Proposition 5.9. *Let* $\sigma \in \widehat{K}_0(BGL(n))$. *Then*

$$\mathrm{ch}(\sigma) = \rho(\sigma).$$

Proof. Let $Gr(n,k)$ be the Grassman manifold of dimension n linear subspaces of C^k and let $E(n,k)$ be the rank n tautological vector bundle. Let $\mathfrak{U}_k = \{U_\alpha\}$ be the standard trivialization of $E(n,k)$. Let us denote by $\psi : N\mathfrak{U}_k \longrightarrow Gr(n,k)$ the natural map and by $\varphi_k : N\mathfrak{U}_k \longrightarrow BGL(n)$ the classifying map. Since absolute Hodge cohomology can be computed as the cohomology of a Zariski sheaf, the map

$$\psi^* : H_{\mathcal{H}}^*(Gr(n,k), \mathbb{R}(*)) \longrightarrow H_{\mathcal{H}}^*(N\mathfrak{U}_k, \mathbb{R}(*))$$

is an isomorphism. Moreover, for each i_0 there is a number k_0, such that, for all $k \geq k_0$ and all $i \leq i_0$ the map

$$\varphi_k^* : H_{\mathcal{H}}^i(BGL(n), \mathbb{R}(*)) \longrightarrow H_{\mathcal{H}}^i(N\mathfrak{U}_k, \mathbb{R}(*))$$

is an isomorphism. But for $\sigma \in \widehat{K}_0(BGL(n))$ we have

$$\varphi_k^*(\mathrm{ch}(\sigma)) = \mathrm{ch}(\varphi_k^*(\sigma)) = \rho(\varphi_k^*(\sigma)) = \varphi_k^*\rho((\sigma)).$$

Since this is true for all k we have $\mathrm{ch}(\sigma) = \rho(\sigma)$.

Combining 5.8 and 5.9 we get:

Corollary 5.10. *The element* $\mathrm{ch}(e_{BP}) - \rho(e_{BP})$ *belongs to* $\mathrm{Ker}\, d_H$. *Therefore it is a sum of decomposable elements.*

(5.2.8) To exploit the fact that $\mathrm{ch}(e_{BP}) - \rho(e_{BP})$ is a sum of decomposable elements, we shall give a different description of how a class in $H^*_{\mathcal{H}}(\mathcal{B}P, \mathbb{R}(*))$ determines a map between K-theory and absolute Hodge cohomology.

For any smooth simplicial scheme over $\mathbb{C}$, X, and integers n, p, the complex

$$\mathcal{H}^*(X,n,p) = \tau_{\leq 0}\widetilde{\mathfrak{H}}^*(X,p)[n]$$

is a negatively graded cohomological complex. Let $\mathcal{H}_*(X,n,p)$ be the associated homological complex. Let us denote by $\mathcal{K}(X,n,p)$ the simplicial group obtained by Dold-Puppe from $\mathcal{H}_*(X,n,p)$. Then, for $i \geq 0$,

$$\pi_i\mathcal{K}(X,n,p) = H^{n-i}_{\mathcal{H}}(X, \mathbb{R}(p)).$$

Let us fix a smooth complex variety X, and $\mathfrak{U}$ an open covering of X. Let us denote by φ the tautological map

$$\varphi : N\mathfrak{U} \times \underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P) \longrightarrow \mathcal{B}P.$$

Given any class $x \in H^n_{\mathcal{H}}(\mathcal{B}P, \mathbb{R}(p))$, we have a class

$$\varphi^*(x) \in H^n_{\mathcal{H}}(N\mathfrak{U} \times \underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P), \mathbb{R}(p)) = \mathrm{Hom}_{Ho}(\underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P), \mathcal{K}(N\mathfrak{U}, n, p)).$$

For any $i > 0$, let us denote by $\pi_i(x)$ the induced map

$$\pi_i(x) : K_{i-1}(X, \mathfrak{U}) = \pi_i\underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P) \longrightarrow \pi_i\mathcal{K}(N\mathfrak{U}, n, p) = H^{n-i}_{\mathcal{H}}(X, \mathbb{R}(p)).$$

This construction can be extended to the case when X is a simplicial smooth complex manifold.

We have to show that this description agrees with the previous one. Such an agreement is guaranteed by the following result.

Lemma 5.11. *For* $x \in H^{2k+1}_{\mathcal{H}}(\mathcal{B}P, \mathbb{R}(p))$ *and* $\sigma \in K_{i-1}(X, \mathfrak{U})$ *we have*

$$\pi_i(x)(\sigma) = \mathrm{pr}(\gamma^*_\sigma(x)),$$

where γ_σ *is as in (5.2.3).*

Proof. Since the map $\pi_*(x)$ is natural, the same argument as for ch and ρ shows that

$$\pi_i(x)(\sigma) = \mathrm{pr}(\gamma^*_\sigma(\pi_0(x)(e_{BP}))).$$

But the map

$$\pi_0(x) : K_{-1}(\mathcal{B}P, \mathcal{B}P) = \pi_0\underline{\mathrm{Hom}}(\mathcal{B}P, \mathcal{B}P) \longrightarrow \pi_0\mathcal{K}(\mathcal{B}P, 2k+1, k) = H^{2k+1}(\mathcal{B}P, \mathbb{R}(k)),$$

sends the class of $f \in \pi_0 \underline{\mathrm{Hom}}(\mathcal{B}P, \mathcal{B}P)$ to $f^*(x)$. Since $e_{\mathcal{B}P}$ is represented by the identity map, we get

$$\pi_0(x)(e_{\mathcal{B}P}) = \mathrm{Id}^*(x) = x,$$

proving the lemma.

(5.2.9) The product structure in absolute Hodge cohomology is given by a morphism of complexes

$$\mathcal{H}^*(X, n, p) \otimes \mathcal{H}^*(X, m, q) \xrightarrow{\cup} \mathcal{H}^*(X, n+m, p+q),$$

which induces a map of spaces

$$\mathcal{K}^*(X, n, p) \times \mathcal{K}^*(X, m, q) \xrightarrow{\cup} \mathcal{K}^*(X, n+m, p+q).$$

The spaces $\mathcal{K}(X, n, p)$ are naturally pointed by the element 0. Moreover $0 \cup x = x \cup 0 = 0$. Therefore the above map of spaces factors through:

$$\mathcal{K}^*(X, n, p) \times \mathcal{K}^*(X, m, q) \longrightarrow \mathcal{K}^*(X, n, p) \wedge \mathcal{K}^*(X, m, q) \longrightarrow \mathcal{K}^*(X, n+m, p+q).$$

Lemma 5.12. *Let $x \in H^n_{\mathcal{H}}(\mathcal{B}P, \mathbb{R}(p))$ and $y \in H^m_{\mathcal{H}}(\mathcal{B}P, \mathbb{R}(q))$. Then for any $i > 0$ the map $\pi_i(x \cup y) = 0$.*

Proof. Let us write $E = \underline{\mathrm{Hom}}(N\mathfrak{U}, \mathcal{B}P)$. Then the map $\pi(x \cup y)$ can be factored as

$$\pi_i(E) \xrightarrow{\pi_i(\mathrm{diag})} \pi_i(E \wedge E) \rightarrow \pi_i(\mathcal{K}(N\mathfrak{U}, n, p) \wedge \mathcal{K}(N\mathfrak{U}, m, q)) \rightarrow \pi_i(\mathcal{K}(N\mathfrak{U}, u+m, p+q)).$$

But since $S^i \wedge S^i = S^{2i}$ and for $i > 0$, $\pi_i S^{2i} = 0$, the map $\pi_i(\mathrm{diag}) = 0$.

(5.2.9) We are ready to prove theorem 5.2. Let $i > 0$ and $\sigma \in K_{i-1}(X, \mathfrak{U})$. By lemma 5.11, we have that

$$\mathrm{ch}(\sigma) = \pi_i(\mathrm{ch}(e_{\mathcal{B}P}))(\sigma),$$
$$\rho(\sigma) = \pi_i(\rho(e_{\mathcal{B}P}))(\sigma).$$

Therefore

$$\mathrm{ch}(\sigma) - \rho(\sigma) = \pi_i(\mathrm{ch}(e_{\mathcal{B}P}) - \rho(e_{\mathcal{B}P}))(\sigma).$$

By corollary 5.10, $\mathrm{ch}(e_{\mathcal{B}P}) - \rho(e_{\mathcal{B}P})$ is a sum of decomposable elements. Therefore by lemma 5.12.

$$\mathrm{ch}(\sigma) = \rho(\sigma)$$

concluding the proof of the theorem.

(5.3) The same argument shows that, for a smooth simplicial complex variety X, an integer $i \geq 0$, and an element $\sigma \in \widehat{K}_i(X)$ then $\mathrm{ch}(\sigma) = \rho(\sigma)$. To prove the same result for $i < 0$ one can use an analogous argument using $\mathcal{B}^m P$ ([Sch]), with $m > -i$.

§6 Higher Bott-Chern forms.

The higher Bott-Chern forms introduced in §3 are differential forms defined on $X \times (\mathbb{P}^1)^*$. Nevertheless, the original Bott-Chern forms ([B-C]) and the higher Bott-Chern forms introduced by Wang in [Wa] are differential forms defined on X. The aim of this section is to relate both notions of higher Bott-Chern forms, in the case when X is a proper smooth complex variety. The main tool for this comparison will be an explicit quasi-isomorphism

$$\widetilde{\mathfrak{H}}^*(X,p) \longrightarrow \mathfrak{H}^*(X,p).$$

To this end we shall first introduce some complexes which compute absolute Hodge homology and cohomology.

(6.1) Let us begin by introducing the complex where the simplest Bott-Chern forms are defined. This complex is a minor modification of the complex used by Wang in [Wa] (see also [Bu 2]). The use of this complex has been suggested by Deligne in [De]. Let X be a proper smooth complex variety. We shall write

$$E^*_{\mathbb{R}}(X)(p) = (2\pi i)^p E^*_{\mathbb{R}}(X).$$

Definition 6.1. The complex $\mathfrak{W}^*(X,p)$ is defined by

$$\mathfrak{W}^n(X,p) = \begin{cases} E^{n-1}_{\mathbb{R}}(X)(p-1) \cap \bigoplus\limits_{\substack{p'+q'=n-1 \\ p'<p,\ q'<p}} E^{p',q'}(X), & \text{for } n \leq 2p-1, \\ E^{n}_{\mathbb{R}}(X)(p) \cap \bigoplus\limits_{\substack{p'+q'=n \\ p'\geq p,\ q'\geq p}} E^{p',q'}(X) \cap \operatorname{Ker} d, & \text{for } n = 2p, \\ 0, & \text{for } n > 2p. \end{cases}$$

If $x \in \mathfrak{W}^n(X,p)$ the differential $d_{\mathfrak{W}}$ is given by

$$d_{\mathfrak{W}} x = \begin{cases} -\pi(dx), & \text{for } n < 2p-1, \\ -2\partial\bar{\partial} x, & \text{for } n = 2p-1, \\ 0, & \text{for } n = 2p, \end{cases}$$

where

$$\pi : E^*(X) \longrightarrow E^*_{\mathbb{R}}(X)(p-1) \cap \bigoplus_{\substack{p'+q'=n-1 \\ p'<p,\ q'<p}} E^{p',q'}(X),$$

is the projection.

Proposition 6.2. *If X is a proper smooth complex variety, then*

$$H^*(\mathfrak{W}^*(X,p)) = H^*_{\mathcal{H}}(X, \mathbb{R}(p)).$$

Proof. Since X is proper,

$$H^n_{\mathcal{H}}(X,\mathbb{R}(p)) = \begin{cases} H^n_{\mathcal{D}}(X,\mathbb{R}(p)), & \text{for } n \leq 2p, \\ 0, & \text{for } n > 2p, \end{cases}$$

where $H^n_{\mathcal{D}}(X,\mathbb{R}(p))$, denotes real Deligne cohomology of X. Therefore the result follows from [Bu 2 §2].

As in [Bu 2], we have morphisms of complexes

$$\psi : \mathfrak{H}^*(X,p) \longrightarrow \mathfrak{W}^*(X,p)$$

and

$$\varphi : \mathfrak{W}^*(X,p) \longrightarrow \mathfrak{H}^*(X,p)$$

given by

$$\psi(a,f,\omega) = \begin{cases} \pi(\omega), & \text{for } n \leq 2p-1 \text{ and} \\ \sum_{i=p}^{n-p} a^{i,n-i} + \partial\omega^{p-1,n-p+1} + (-1)^p\overline{\partial\omega}^{p-1,n-p+1}, & \text{for } n \geq 2p, \end{cases}$$

and

$$\varphi(x) = \begin{cases} (\partial x^{p-1,n-p} - \bar{\partial} x^{n-p,p-1}, 2\partial x^{p-1,n-p}, x), & \text{for } n \leq 2p-1 \text{ and} \\ (x,x,0), & \text{for } n \geq 2p, \end{cases}$$

where, if $x \in E^*_X$, then $x = \sum x^{p,q}$ is the decomposition of x in terms of pure type. The morphisms φ and ψ are homotopy equivalences inverse to each other.

(6.2) In order to make the process of comparison clearer, we need an auxiliary complex to compute absolute Hodge cohomology, which is provided with a graded commutative and associative product. It can be obtained by means of the Thom-Whitney simple introduced by Navarro Aznar (see [N 2] for the general definition and properties of the Thom-Whitney simple).

Let L^*_1 be the differential graded commutative $\mathbb{R}$-algebra of algebraic forms over $\mathbf{A}^1_{\mathbb{R}}$. Explicitly $L^0_1 = \mathbb{R}[\epsilon]$ and $L^1_1 = \mathbb{R}[\epsilon]d\epsilon$. Let $\delta_0 : L^*_1 \longrightarrow \mathbb{R}$ (resp. δ_1) be the evaluation at 0 morphism (resp. evaluation at 1).

Definition 6.3. Let X be a smooth complex variety. The Thom-Whitney simple of the absolute Hodge complex, denoted by $\mathfrak{H}^*_{TW}(X,p)$, is the subcomplex of

$$\left((2\pi i)^p \widehat{W}_{2p} E^*_{\log}(X)_{\mathbb{R}} \oplus \widehat{W}_{2p} \cap F^p\, E^*_{\log}(X) \oplus (L^*_1 \underset{\mathbb{R}}{\otimes} \widehat{W}_{2p} E^*_{\log}(X))\right)$$

formed by the elements (r,f,ω) such that

$$\omega(0) = (\delta_0 \otimes \mathrm{Id})(\omega) = r,$$
$$\omega(1) = (\delta_1 \otimes \mathrm{Id})(\omega) = f.$$

Let E and I be the morphisms of complexes

$$\mathfrak{H}^*_{TW}(X,p) \underset{E}{\overset{I}{\rightleftarrows}} \mathfrak{H}^*(X,p)$$

given by

$$E(r,f,\omega) = (r,f,\epsilon \otimes f + (1-\epsilon) \otimes r + d\epsilon \otimes \omega),$$
$$I(r,f,\omega) = (r,f,\int_0^1 \omega),$$

where the integration symbol means formal integration with respect to the variable ϵ. These morphisms are homotopy equivalences (see [N 2]).

We shall denote by I' the composition

$$\mathfrak{H}^*_{TW}(X,*) \xrightarrow{I} \mathfrak{H}^*(X,*) \xrightarrow{\psi} \mathfrak{W}^*(X,*),$$

and by E' the composition

$$\mathfrak{W}^*(X,*) \xrightarrow{\varphi} \mathfrak{H}^*(X,*) \xrightarrow{E} \mathfrak{H}^*_{TW}(X,*).$$

The morphisms I' and E' are also homotopy equivalences inverse to each other.

We can define a product

$$\mathfrak{H}^n_{TW}(X,p) \otimes \mathfrak{H}^m_{TW}(X,q) \xrightarrow{\cup} \mathfrak{H}^{n+m}_{TW}(X,p+q),$$

by

$$(r,f,\omega) \cup (r',f',\omega') = (r \wedge r', f \wedge f', \omega \wedge \omega').$$

This product is associative, graded commutative and satisfies the Leibnitz rule. Therefore

$$\mathfrak{H}^*_{TW}(X,*) = \bigoplus_p \mathfrak{H}^*_{TW}(X,p)$$

is a differential associative graded commutative algebra. Moreover, the $\mathbb{R}$-algebra structure induced in $H^*_{\mathcal{H}}(X,\mathbb{R}(p))$ by this product coincides with the $\mathbb{R}$-algebra structure introduced by Beilinson ([Be]).

(6.3) Let us give the homology analogue of the last complex. This is done by means of currents. For a proper smooth complex variety X, let $D_{*,*}(X)$ be the double chain complex of complex currents over X, let $D_*(X)$ be the associated single complex, and let $D^{\mathbb{R}}_*(X)$ be the real subcomplex. We shall write

$$F_p D_*(X) = \bigoplus_{p' \leq p} D_{p',*}.$$

Let $\tau^{\geq 2p} D_*(X)$ be the subcomplex

$$\tau^{\geq 2p} D_n(X) = \begin{cases} D_n(X), & \text{if } n > 2p, \\ \mathrm{Ker}(d), & \text{if } n = 2p, \\ 0, & \text{if } n < 2p. \end{cases}$$

Since X is proper, the filtration τ plays the role of the weight filtration.

Let L_*^1 be the chain complex defined by $L_k^1 = L_1^{-k}$ (see 6.2). We shall denote by δ_0 and δ_1 the evaluation at 0 and 1 as in (6.2).

Definition 6.4. Let $\mathfrak{H}_*^{TW}(X,p)$ be the subcomplex of

$$\left((2\pi i)^{-p} \tau^{\geq 2p} D_*^{\mathbb{R}}(X) \oplus \tau^{\geq 2p} \cap F_p\, D_*(X) \oplus (L_*^1 \underset{\mathbb{R}}{\otimes} \tau^{\geq 2p} D_*(X))\right)$$

formed by the elements (r, f, ω) such that

$$\omega(0) = (\delta_0 \otimes \mathrm{Id})(\omega) = r,$$
$$\omega(1) = (\delta_1 \otimes \mathrm{Id})(\omega) = f.$$

The homology of the complex $\mathfrak{H}_*^{TW}(X,p)$ is the absolute Hodge homology of X.

(6.4) The last complex we introduce is an analogue of $\widetilde{\mathfrak{H}}^*(X,p)$, replacing $\mathfrak{H}^*(X,p)$ by $\mathfrak{H}_{TW}^*(X,p)$. We shall denote by $\mathfrak{H}_{\mathbb{P},TW}^{*,*}(X,p)$ the double complex given by

$$\mathfrak{H}_{\mathbb{P},TW}^{r,n}(X,p) = \mathfrak{H}_{TW}^{r}(X \times (\mathbb{P}^1)^{-n}, p),$$

with differentials

$$d' = d_{\mathfrak{H}},$$
$$d'' = \sum (-1)^{i+j} d_i^j.$$

Then the double complex $\widetilde{\mathfrak{H}}_{TW}^{*,*}(X,p)$ is given by

$$\widetilde{\mathfrak{H}}_{TW}^{r,n}(X,p) = \mathfrak{H}_{\mathbb{P},TW}^{r,n}(X,p) \Big/ \sum_{i=1}^{n} s_i \left(\mathfrak{H}_{\mathbb{P},TW}^{r,n+1}(X,p)\right) \oplus \omega_i \wedge s_i \left(\mathfrak{H}_{\mathbb{P},TW}^{r-2,n+1}(X,p-1)\right).$$

Finaly let $\widetilde{\mathfrak{H}}_{TW}^{*}(X,p)$ be the associated simple complex. The differential of this complex will be denoted by d.

Observe that the homotopy equivalences E and I induce homotopy equivalences

$$\widetilde{\mathfrak{H}}_{TW}^{*}(X,p) \underset{E}{\overset{I}{\rightleftarrows}} \widetilde{\mathfrak{H}}^{*}(X,p).$$

(6.5) In order to pull down forms in $X \times (\mathbb{P}^1)^n$ to X, we need some differential forms on $X \times (\mathbb{P}^1)^n$ which will play a role similar to the currents "integration along the standard simplex".

Let $(x : y)$ be homogeneous coordinates of $\mathbb{P}^1$, and let $t = x/y$ be the absolute coordinate of $\mathbb{P}^1$. Let us write $\mathbb{C}^* = \mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}$. Let

$$\begin{aligned}\lambda &= \frac{1}{2}E'(\log t\bar{t})\\ &= \frac{1}{2}\left(\frac{dt}{t} - \frac{d\bar{t}}{\bar{t}}, 2\frac{dt}{t}, (\varepsilon+1)\otimes\frac{dt}{t} + (\varepsilon-1)\otimes\frac{d\bar{t}}{\bar{t}} + d\varepsilon\otimes\log t\bar{t}\right)\\ &\in \mathfrak{H}^1_{TW}(\mathbb{C}^*, 1).\end{aligned}$$

Let us consider the open subset $(\mathbb{C}^*)^n \subset X \times (\mathbb{P}^1)^n$. Let us denote by by $p_i : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^*$, $i = 1, \dots, n$ the projections over the i-th factor. Let us write $\lambda_i = p_i^*\lambda$.

Definition 6.5. Let $W_n \in \mathfrak{H}^n_{TW}((\mathbb{C}^*)^n, n)$ be the form defined by

$$W_n = \lambda_1 \cup \cdots \cup \lambda_n.$$

(6.6) Since the forms W_n will play a central role, let us present a more explicit description. Let us write $W_n = (W_n^1, W_n^2, W_n^3)$. Then

$$\begin{aligned}W_n^1 &= \frac{1}{2^n}\bigwedge_{i=1}^n\left(\frac{dt_i}{t_i} - \frac{d\bar{t}_i}{\bar{t}_i}\right)\\ W_n^2 &= \bigwedge_{i=1}^n \frac{dt_i}{t_i}\\ W_n^3 &= \frac{1}{2^n}\bigwedge_{1=1}^n\left((\epsilon+1)\otimes\frac{dt_i}{t_i} + (\epsilon-1)\otimes\frac{d\bar{t}_i}{\bar{t}_i} + d\epsilon\otimes\log t_i\bar{t}_i\right).\end{aligned}$$

Let $\mathfrak{S}_n$ denote the symmetric group. Let us write, for $i = 0, \dots, n$,

$$P_n^i = \sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma\frac{dt_{\sigma(1)}}{t_{\sigma(1)}}\wedge\cdots\wedge\frac{dt_{\sigma(i)}}{t_{\sigma(i)}}\wedge\frac{d\bar{t}_{\sigma(i+1)}}{\bar{t}_{\sigma(i+1)}}\wedge\cdots\wedge\frac{d\bar{t}_{\sigma(n)}}{\bar{t}_{\sigma(n)}},$$

and, for $i = 1, \dots, n$,

$$S_n^i = \sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma\log(t_{\sigma(1)}\bar{t}_{\sigma(1)})\frac{dt_{\sigma(2)}}{t_{\sigma(2)}}\wedge\cdots\wedge\frac{dt_{\sigma(i)}}{t_{\sigma(i)}}\wedge\frac{d\bar{t}_{\sigma(i+1)}}{\bar{t}_{\sigma(i+1)}}\wedge\cdots\wedge\frac{d\bar{t}_{\sigma(n)}}{\bar{t}_{\sigma(n)}}.$$

Then we have

$$\begin{aligned}W_n^1 &= \frac{1}{2^n}\sum_{i=0}^n(-1)^{n-i}\frac{1}{i!(n-i)!}P_n^i,\\ W_n^3 &= \frac{1}{2^n}\sum_{i=0}^n\frac{(\epsilon+1)^i(\epsilon-1)^{n-i}}{i!(n-i)!}\otimes P_n^i + \frac{1}{2^n}\sum_{i=1}^n\frac{(\epsilon+1)^{i-1}(\epsilon-1)^{n-i}}{(i-1)!(n-i)!}d\epsilon\otimes S_n^i.\end{aligned}$$

(6.7) We are not as interested in the forms W_n, as in their associated currents. Let $\omega \in E^r_{(\mathbb{P}^1)^n}$. Let us denote by $[\omega] \in D_{2n-r}((\mathbb{P}^1)^n)$ the current defined by

$$[\omega](\varphi) = \frac{1}{(2\pi i)^n} \int_{(\mathbb{P}^1)^n} \varphi \wedge \omega.$$

If $a \otimes \omega \in L^*_1 \otimes E^r_{(\mathbb{P}^1)^n}$ we write

$$\begin{aligned}[a \otimes \omega] &= a \otimes [\omega] \\ &\in L^1_* \otimes D_{2n-r}((\mathbb{P}^1)^n).\end{aligned}$$

In this way we obtain a map

$$\begin{array}{ccc} \mathfrak{H}^r_{TW}((\mathbb{P}^1)^n, p) & \longrightarrow & \mathfrak{H}^{TW}_{2n-r}((\mathbb{P}^1)^n, n-p) \\ (r, f, \omega) & \longmapsto & ([r], [f], [\omega]) \end{array}$$

This definition can be extended to any locally integrable differential form.

Definition 6.6. We shall denote by $[W_n]$ the element of $\mathfrak{H}^{TW}_n((\mathbb{P}^1)^n, 0)$ given by

$$[W_n] = ([W^1_n], [W^2_n], [W^3_n]).$$

The following result exhibits the analogy between the currents "integration along the standard simplex" and the currents $[W_n]$.

Proposition 6.7. *The currents $[W_n]$ satisfy the relation*

$$d[W_n] = \sum_{i=1}^{n} \sum_{j=0,1} (-1)^{i+j} (d^i_j)_* [W_{n-1}].$$

Proof. Formally this proposition is the Leibnitz rule. To prove it we can work component by component. By a standard residue argument:

$$\begin{aligned} d[W^2_n] &= d\left[\bigwedge_{i=1}^{n} \frac{dt_i}{t_i}\right] \\ &= \sum_{i=1}^{n} \sum_{j=0} (-1)^{i+j} (d^i_j)_* [W^2_{n-1}]. \end{aligned}$$

By the same argument and taking some care with permutations one sees

$$d[P^i_n] = \sum_{k=1}^{n} \sum_{j=0}^{1} (-1)^{k+j} (d^k_j)_* (i[P^{i-1}_{n-1}] - (n-i)[P^i_{n-1}]),$$

$$d[S^i_n] = [P^i_n] + [P^{i-1}_n] + \sum_{k=1}^{n} \sum_{j=0}^{1} (-1)^{k+j} (d^k_j)_* ((i-1)[S^{i-1}_{n-1}] - (n-i)[S^i_{n-1}]).$$

The proposition follows from the above formulas and the explicit description of W^1_n and W^3_n given in (6.6).

(6.8) Acting component by component, the currents $[W_n]$ induce morphisms

$$[W_n] : \mathfrak{H}^{r,-n}_{\mathbb{P},TW}(X, p) = \mathfrak{H}^r_{TW}(X \times (\mathbb{P}^1)^n, p) \longrightarrow \mathfrak{H}^{r-n}_{TW}(X, p).$$

Lemma 6.8. *The morphisms $[W_n]$ factorize through morphisms*
$$[W_n] : \widetilde{\mathfrak{H}}^{r,n}_{TW}(X,p) \longrightarrow \mathfrak{H}^{r-n}_{TW}(X,p).$$

Proof. Let us denote by σ_i the automorphism of $(\mathbb{P}^1)^n$ given by
$$\sigma_i((x_1 : y_1), \dots, (x_i : y_i), \dots (x_n : y_n)) = ((x_1 : y_1), \dots, (y_i : x_i), \dots (x_n : y_n)).$$
Then $(\sigma_i)_*[W_n] = -[W_n]$, for $i = 1, \dots, n$. On the other hand, if
$$\eta \in s_i\left(\mathfrak{H}^{r,n+1}_{\mathbb{P},TW}(X,p)\right) \oplus \omega_i \wedge s_i\left(\mathfrak{H}^{r-2,n+1}_{\mathbb{P},TW}(X,p-1)\right)$$
then $(\sigma_i)^*\eta = \eta$. Therefore
$$[W_n]\eta = -(\sigma_i)_*[W_n]\eta = -[W_n](\sigma_i)^*\eta = -[W_n]\eta.$$
Hence $[W_n]\eta = 0$ proving the result.

Definition 6.9. Let W_{TW} be the morphism
$$W_{TW} : \widetilde{\mathfrak{H}}^*_{TW}(X,p) \longrightarrow \mathfrak{H}^*_{TW}(X,p)$$
given, for $\eta \in \widetilde{\mathfrak{H}}^{r,n}_{TW}(X,p)$, by
$$W_{TW}(\eta) = [W_n]\eta.$$

Proposition 6.10. *The morphism W_{TW} is a morphism of complexes. Moreover it is a quasi-isomorphism.*

Proof. The fact that is a morphism of complexes is a consequence of Proposition 6.7.

Let ι be the quasi-imorphism defined in Proposition 1.2. Let us write $\iota' = E \circ \iota \circ I$. Then ι' is also a quasi-isomorphism. Since $W_{TW} \circ \iota' = \mathrm{Id}$ we have that W_{TW} is also a quasi-isomorphism.

Definition. 6.11. Let us denote by W the morphism
$$W = I \circ W_{TW} \circ E : \widetilde{\mathfrak{H}}^*(X,p) \longrightarrow \mathfrak{H}^*(X,p).$$

Observe that W is also a quasi-isomorphism. Summarizing, we have the following diagram of complexes and quasi-isomorphisms.
$$\begin{tikzcd}
\widetilde{\mathfrak{H}}^*(X,p) \arrow[r, "W"] \arrow[d, "E"'] & \mathfrak{H}^*(X,p) \arrow[r, "\psi"] & \mathfrak{W}^*(X,p) \\
\widetilde{\mathfrak{H}}^*_{TW}(X,p) \arrow[r, "W_{TW}"] & \mathfrak{H}^*_{TW}(X,p) \arrow[u, "I"'] &
\end{tikzcd}$$

(6.9) The above diagram alow us to define different kinds of higher Bott Cher forms. For instance let us recover the original definition of higher Bott-Chern forms due to Wang ([Wan]) and the classical Bott-Chern forms.

Definition 6.12. Let $\overline{\mathcal{F}}$ be an exact metrized n-cube. We shall also call the Bott Chern form of $\overline{\mathcal{F}}$ the form
$$\widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})_W = \psi \circ W(\widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})_{\mathcal{H}}).$$

One may compute these forms directly using the following result.

Proposition 6.13. *Let $\overline{\mathcal{F}}$ be an emi-n-cube. Then*
$$\widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})_W = \frac{1}{(2\pi i)^n}\int_{(\mathbb{P}^1)^n} \widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}})) \wedge I'(W_n).$$

Proof. This result is consequence of the following facts

(1) The morphism I' is functorial.

(2) For any smooth complex variety Z, if $\omega \in \mathfrak{H}^{2p}_{TW}(Z,p)$ and $\eta \in \mathfrak{H}^{*}_{TW}(Z^*,*)$, then $I'(\omega \cup \eta) = I'(\omega) \wedge I'(\eta)$ (see [Bu 2]).

(3) $I' \circ E' = \mathrm{Id}$. Therefore
$$I'(E(\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}}))_{\mathcal{H}})) = I'(E'(\widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}})))) = \widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}})).$$

Up to a normalization factor, the formula given in Proposition 6.13 is the original definition due to Wang ([Wa]). To see this, let us compute explicitly $I'(W_n) \in \mathfrak{W}^n((\mathbb{C}^*)^n, n)$.

Proposition 6.14.
$$I'(W_n) = \frac{(-1)^n}{2n!}\sum_{i=1}^{n}(-1)^{i-1}S^i_n.$$

Proof. Since $W_n \in \mathfrak{H}^n_{TW}((\mathbb{C}^*)^n, n)$, by (6.1) and (6.2), we have
$$I'(W_n) = \pi\left(\int_0^1 W^3_n\right),$$
where the integral symbol means integration with respect to the variable ϵ, and π is the projection
$$\pi : E^{n-1}_{(\mathbb{C}^*)^n} \longrightarrow (2\pi i)^{n-1}E^{n-1}_{(\mathbb{C}^*)^n,\mathbb{R}}.$$
This projection is given by $\pi(z) = (z + (-1)^{n-1}\overline{z})/2$. Therefore
$$I'(W_n) = \frac{1}{2^{n+1}}\sum_{i=1}^{n}\int_0^1 \frac{(\epsilon+1)^{i-1}(\epsilon-1)^{n-i}}{(i-1)!(n-i)!}d\epsilon\left(S^i_n + (-1)^{n-1}\overline{S}^i_n\right).$$

But $\overline{S}^i_n = S^{n-i+1}_n$. Then, joining the terms with S^i_n, and taking into account that
$$(-1)^{n-1}\int_0^1 \frac{(\epsilon+1)^{n-i}(\epsilon-1)^{i-1}}{(i-1)!(n-i)!}d\epsilon = \int_{-1}^0 \frac{(\epsilon+1)^{i-1}(\epsilon-1)^{n-i}}{(i-1)!(n-i)!}d\epsilon,$$
we have that
$$I'(W_n) = \frac{1}{2^{n+1}}\sum_{i=1}^{n}\left(\int_{-1}^1 \frac{(\epsilon+1)^{i-1}(\epsilon-1)^{n-i}}{(i-1)!(n-i)!}d\epsilon\, S^i_n\right).$$
But
$$\int_{-1}^1 \frac{(\epsilon+1)^{i-1}(\epsilon-1)^{n-i}}{(i-1)!(n-i)!}d\epsilon = \frac{(-1)^{n+i-1}2^n}{n!},$$
proving the result.

The following result is a direct consequence of the definitions.

Proposition 6.15. *Let X be a proper smooth complex variety. Let*

$$\overline{\xi}: 0 \to \overline{\mathcal{F}} \to \overline{\mathcal{G}} \to \overline{\mathcal{H}} \to 0$$

be an exact sequence of locally free sheaves over X. Let us denote by $\widetilde{\mathrm{bc}}(\overline{\xi})$ the Bott Chern form of $\overline{\xi}$ as defined by Bismut, Gillet and Soulé ([B-G-S], [G-S]*). Then*

$$\widetilde{\mathrm{ch}}_1(\overline{\xi}) = -\frac{1}{2}\widetilde{\mathrm{bc}}\,\overline{\xi} \qquad \mathrm{mod}(\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}).$$

(6.10) The use of the Thom-Whitney simple for absolute Hodge cohomology, besides giving a way to construct the currents W_n, allows us to define a multiplicative theory of Bott-Chern forms.

Definition 6.16. Let $\overline{\mathcal{F}}$ be an exact metrized n-cube. We shall call the multiplicative Bott-Chern form of $\overline{\mathcal{F}}$ the form

$$\widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})_{TW} = W_{TW}(E(\widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})_{\mathcal{H}})).$$

In particular, if $\overline{\mathcal{F}}$ is a hermitian locally free sheaf, then

$$\widetilde{\mathrm{ch}}_0(\overline{\mathcal{F}})_{TW} = E(\widetilde{\mathrm{ch}}_0(\overline{\mathcal{F}})_{\mathcal{H}}).$$

On the other hand, if $\overline{\mathcal{F}}$ is an emi-n-cube, then

$$\widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})_{TW} = \frac{1}{(2\pi i)^n}\int_{(\mathbb{P}^1)^n} \widetilde{\mathrm{ch}}_0(\mathrm{tr}_n(\overline{\mathcal{F}}))_{TW} \cup W_n.$$

Definition 6.17. Let $\overline{\mathcal{F}}$ be a metrized exact n-cube and let $\overline{\mathcal{G}}$ be a metrized exact m-cube. Then $\overline{\mathcal{F}} \otimes \overline{\mathcal{G}}$ is the metrized exact $n+m$-cube given by

$$(\overline{\mathcal{F}} \otimes \overline{\mathcal{G}})_{i_1,\dots,i_{n+m}} = (\overline{\mathcal{F}})_{i_1,\dots,i_n} \otimes (\overline{\mathcal{G}})_{i_{n+1},\dots,i_{n+m}}$$

with the obvious morphisms and metrics.

Proposition 6.18. *Let $\overline{\mathcal{F}}$ (resp. $\overline{\mathcal{G}}$) be a metrized exact n-cube (resp. m-cube). Then*

$$\widetilde{\mathrm{ch}}_{n+m}(\overline{\mathcal{F}} \otimes \overline{\mathcal{G}})_{TW} = \widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})_{TW} \cup \widetilde{\mathrm{ch}}_m(\overline{\mathcal{G}})_{TW}.$$

Proof. We may assume that $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are emi-cubes.

Let $\pi_1 : X \times (\mathbb{P}^1)^{n+m} \longrightarrow X \times (\mathbb{P}^1)^n$ be the projection over the first n-projective lines and let $\pi_2 : X \times (\mathbb{P}^1)^{n+m} \longrightarrow X \times (\mathbb{P}^1)^m$ be the projection over the last m-projective lines.

Lemma 6.19. *Let $\overline{\mathcal{F}}$ (resp. $\overline{\mathcal{G}}$) be an emi-n-cube (resp. emi-m-cube). Then*

$$\operatorname{tr}_{n+m}(\overline{\mathcal{F}} \otimes \overline{\mathcal{G}}) = \pi_1^* \operatorname{tr}_n(\overline{\mathcal{F}}) \otimes \pi_2^* \operatorname{tr}_m(\overline{\mathcal{G}}).$$

Proof. By §3, (3.7), it is enough to show that, if $m \geq 1$, then

$$\operatorname{tr}_1(\overline{\mathcal{F}} \otimes \overline{\mathcal{G}}) = \overline{\mathcal{F}} \otimes \operatorname{tr}_1(\overline{\mathcal{G}}),$$

and if $m = 0$, then

$$\operatorname{tr}_1(\overline{\mathcal{F}} \otimes \overline{\mathcal{G}}) = \operatorname{tr}_1(\overline{\mathcal{F}}) \otimes \overline{\mathcal{G}}.$$

Since tr_1 is computed in each edge separately, it is enough to prove the case $n = 1$, $m = 0$, but this case follows directly from the definition.

Using lemma 6.19, the multiplicativity and functoriality of the Chern form and the definition of the forms W_n, we have:

$$\begin{aligned}
&\widetilde{\mathrm{ch}}_{\mathrm{n+m}}(\overline{\mathcal{F}} \otimes \overline{\mathcal{G}})_{TW} = \\
&\quad = \frac{1}{(2\pi i)^{n+m}} \int_{(\mathbb{P}^1)^{n+m}} \widetilde{\mathrm{ch}}_0(\pi_1^* \operatorname{tr}_n(\overline{\mathcal{F}}) \otimes \pi_2^* \operatorname{tr}_m(\overline{\mathcal{G}}))_{TW} \cup W_{n+m} \\
&\quad = \frac{1}{(2\pi i)^{n+m}} \int_{(\mathbb{P}^1)^{n+m}} \pi_1^* \widetilde{\mathrm{ch}}_0(\operatorname{tr}_n(\overline{\mathcal{F}}))_{TW} \cup \pi_2^* \widetilde{\mathrm{ch}}_0(\operatorname{tr}_m(\overline{\mathcal{G}}))_{TW} \cup \pi_1^* W_n \cup \pi_2^* W_m \\
&\quad = \widetilde{\mathrm{ch}}_{\mathrm{n}}(\overline{\mathcal{F}})_{TW} \cup \widetilde{\mathrm{ch}}_{\mathrm{m}}(\overline{\mathcal{G}})_{TW}.
\end{aligned}$$

References.

[Be] Beilinson, A. A., *Notes on absolute Hodge cohomology*, Contemporary Mathematics **55 I** (1986), 35–68.

[B-G-S] Bismut, J.M., Gillet, H. and Soulé, C., *Analytic torsion and holomorphic determinant bundles I: Bott Chern forms and analytic torsion,*, Comm. Math. Phis. **115** (1988), 49–78.

[B-K] Bousfield A.K. and Kan D.M., *Homotopy limits, completions and localizations. LNM 304*, Springer-Verlag, 1972.

[Bu 1] Burgos J. I., *A C^∞ logarithmic Dolbeault complex*, Compos. Math. **92** (1994), 61–86.

[Bu 2] Burgos J. I., *Arithmetic Chow rings and Deligne Beilinson cohomology*, J. of Algebraic Geometry (to appear).

[De] Deligne P., *Le déterminant de la cohomologie*, Contemporary Mathematics **67** (1987), 93–177.

[Gi] Gillet, H., *Riemann-Roch theorems for higher algebraic K-theory*, Advances in Math. **40** (1981), 203–289.

[G-S] Gillet, H. and Soulé, C., *Characteristic classes for algebraic vector bundles with hermitian metric I and II*, Annals of Math. **131** (1990), 163–238.

[G-D] Grothendieck, A. and Dieudonné, J.A., *Eléments de géométrie algébrique*, Springer-Verlag, 1971.

[G-N-P-P] Guillen, F., Navarro Aznar, V., Pascual-Gainza, P. and Puerta, F., *Hyperrésolutions cubiques et descente cohomologique. LNM 1335*, Springer-Verlag, 1988.

[Ka 1] Karoubi, M., *Homologie cyclique et K-théorie*, Astérisque **149** (1987).

[Ka 2] Karoubi, Théorie générale des classes caractéristiques secondaries., K-Theory **4**.

[Lo 1] Loday, J.L., *Homotopie des spaces de concordances (d'après F. Waldhausen)*, Sem. Bourbaki, 1977/78 N. 516, LNM 710, Springer-Verlag, 1979, pp. 187–205.

[Lo 2] Loday, J.L., *Spaces with finitely many non-trivial homotopy groups*, J. of Pure and Applied Algebra **24** (1982), 179–202.

[N 1] Navarro Aznar, V., *Sur les multiplicités de Schubert locales des faisceaux algébriques cohérents*, Comp. Math. **48** (1983), 311–326.

[N 2] Navarro Aznar, V., *Sur la théorie de Hodge-Deligne*, Invent. Math. **90** (1987), 11–76.

[Ri] Riemenschneider, O., *Characterizing Moishezon spaces by almost positive coherent analytic sheaves*, Math. Z. **123** (1971), 263–284.

[Ro] Rossi, H., *Picard variety of an isolated singular point*, Rice Univ. Studies **54** (1968), 63–73.

[Sch] Schechtman, *On the delooping of Chern character and Adams operations*, K-theory, arithmetic and geometry. LNM 1289, Springer-Verlag, 1987, pp. 265–319.

[Th] Thomason, R., *Algebraic K-theory and étalé cohomology*, Ann. Sci. Ec. Norm. Sup. **18** (1985), 437–552.

[Wal] Waldhausen, F., *Algebraic K-theory of generalized free products*, Ann. Math. **108** (1978), 135–256.

[Wan] Wang X., *Higher-order characteristic classes in arithmetic geometry*, Thesis Harvard, 1992.

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