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ANALYTICAL INVARIANTS OF CONFORMAL TRANSFORMATIONS. A DYNAMICAL SYSTEM APPROACH

by

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Analytical Invariants of Conformal Transformations. A dynamical system approach.

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Abstract

The paper is devoted to the problem of analytical classification of conformal maps of the form $f : z \mapsto z + z^2 + \dots$ in a neighborhood of the degenerate fixed point $z = 0$. It is shown that the analytical invariants, constructed in the works of Voronin and Ecalle, may be considered as a measure of the splitting for stable and unstable (semi-)invariant foliations associated with the fixed point. This splitting appears to be exponentially small with respect to the distance to the fixed point.

The problem of local classification of analytic morphisms on the complex plane $(\mathbf{C}, 0)$ seems to be solved completely at the present time. The classical result is that a morphism $f : z \mapsto \lambda z + \dots$ with $\lambda \neq 0$ and $|\lambda| \neq 1$ can be conjugated with its linear part $z \mapsto \lambda z$. The case of a resonant λ (there is $n \in \mathbf{N}$, such that $\lambda^n = 1$) was solved independently by J.Ecalle [1] and S.Voronin [2]. In this case there is a countable number of independent quantities which are invariant with respect to analytical changes of coordinates.

In the present paper we concentrate our attention on the case $\lambda = 1$ and repeat the results of [2], using a dynamical system approach to the problem. This gives us a geometric interpretation of the obstacles for the analytic conjugation, which are quite similar to the phenomenon of the splitting of separatrices in 2D Hamiltonian systems. In the latter class of systems the splitting leads to a chaotic behavior of trajectories. In the case of analytic maps on the complex plane it provides obstacles for analyticity only.

In the present paper we give a new method for construction of the basic solutions for the Abel equation (see Section 2). This method is based on the theory of finite-difference equations proposed in [3, 4]. In Section 3 the relation between analytic invariant and dynamics is discussed. And in Section 4 a numerical method for computing the analytic invariant for a given map is provided.

1 Analytical classification by Voronin

First we recall some results from the paper [2]. Let f denote a function analytic at zero,

$$f : z \mapsto z + z^2 + az^3 + O(z^4). \tag{1}$$

The problem is when for two given functions of this form, f and $\tilde{f}$, there is a diffeomorphism h, such that $\tilde{f} = h \circ f \circ h^{-1}$ in a neighborhood of $z = 0$.

Any map f with $f'(0) = 1$ and $f''(0) \neq 0$ can be transformed to the form (1) by a linear h. The coefficient a is invariant with respect to any h, which can be represented as a (formal) power series in z, $h(z) = z + \ldots$. There is not any other formal invariant since an application of the form (1) can be formally conjugated to the polynomial

$$z \mapsto z + z^2 + az^3, \tag{2}$$

but, of course, the corresponding series diverges in general.

In [2] a complete set of analytic invariants was constructed with the help of two basic solutions A_1 and A_2 of the Abel equation

$$A(f(z)) = A(z) + 1. \tag{3}$$

The functions A_1 and A_2 are analytic respectively in the domains

$$S_1 = \{\, |\arg z| < \pi - \delta \,\} \cap D_r, \qquad S_2 = \{\, |\arg(-z)| < \pi - \delta \,\} \cap D_r,$$

where $D_r = \{|z| < r\}$. It is assumed that r is sufficiently small. Moreover,

$$A_k(z) = -1/z + o(1/z) \tag{4}$$

on their domains. In particular, A_k transform a sectorial neighborhood of the origin into a sectorial neighborhood of the infinity. Consider the function $\Phi(t) = A_2 \circ A_1^{-1}(t)$. Due to (4) and (3) it is well defined for sufficiently large $|\mathrm{Im}\, t|$ and

$$\Phi(t+1) = \Phi(t) + 1. \tag{5}$$

Thus, it may be continued into two half-planes $\Pi_\pm = \{\zeta \in \mathbf{C} : \pm\mathrm{Im}\zeta > R\}$ provided the constant R is sufficiently large. Let

$$\Phi_\pm = A_2 \circ A_1^{-1}\Big|_{\Pi_\pm}. \tag{6}$$

The functions A_1 and A_2 are defined by (3) and (4) uniquely up to additive constants. Consequently, the functions $\Phi_\pm$ are defined up to a substitution $\tilde{\Phi}_\pm(\zeta) = \Phi_\pm(\zeta + a_1) + a_2$, where a_1 and a_2 are arbitrary complex numbers. This relation provides an equivalence in the set of pairs of analytical functions. The equivalence class of (Φ_+, Φ_-) is said to be an *analytical invariant* of the function f.

Indeed, let h be an analytical function at the origin, $h'(0) = 1$, and consider the function $\tilde{f} = h \circ f \circ h^{-1}$. The solutions of the Abel equation, which correspond to $\tilde{f}$, are given by $\tilde{A}_k = A_k \circ h$. The analytical invariant of the function $\tilde{f}$, given by

$$\tilde{\Phi} = \tilde{A}_2 \circ \tilde{A}_1^{-1} = (A_2 \circ h) \circ (A_1 \circ h)^{-1} = A_2 \circ A_1^{-1} = \Phi,$$

coincides with the analytical invariant of the function f.

Theorem 1 [2]. *Two functions of the form (1) are analytically conjugated in a neighborhood of zero if and only if their analytical invariants coincide.*

The equations (5) and (4) imply that the functions $\Phi_\pm$ may be represented in the form

$$\Phi_\pm(t) = t + b_0^\pm + \sum_{k=1}^{\infty} b_k^\pm e^{\pm 2i\pi kt}, \tag{7}$$

where the Fourier series converge and go to zero as $|\mathrm{Im}\, t| \to \infty$ in the half-planes $\Pi_\pm$, respectively. In [2] it was shown, that for any pair of analytical functions of the form (7) it is possible to construct a function (1), such that the pair is a representative of its analytical invariant.

Example. An important example is the function

$$f_0(z) = \frac{z}{1-z} = z + z^2 + z^3 + \dots . \tag{8}$$

In this case the Abel equation has an obvious globally defined solution

$$A(z) = -\frac{1}{z}.$$

We may choose $A_+(z) = A_-(z) = A(z)$. Consequently, $\Phi(t) = t$. That is the analytic invariant of this function is trivial, in the sense that $b_k^\pm = 0$ for all $k \geq 1$.

In the coordinate $t = -1/z$ the map f_0 is a translation $t \mapsto t + 1$. The trajectories belong to straight lines $\mathrm{Im}\,(t) = \mathrm{const}$. Coming back to the variable z we see that these lines are transformed into circles, which pass through the origin, with horizontal tangent on it. These circles are invariant with respect to the map f_0.

An arbitrary map of the form (1) has a similar (local topological) structure of trajectories since it may be topologically conjugated with f_0 [5].

2 Construction of inverse Abel functions

Instead of the Abel equation we will study the equation

$$z(t+1) = f(z(t)) \tag{9}$$

for the inverse function $z = A^{-1}$.

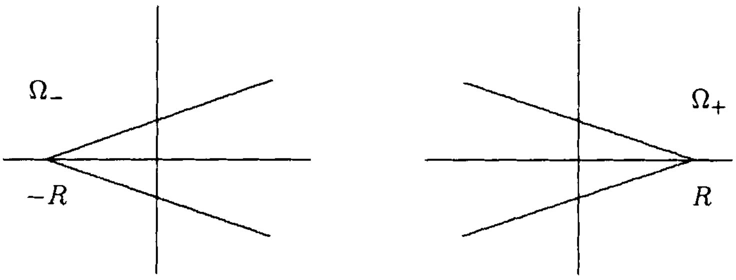

Figure 1: Domains of definition for the inverse Abel functions, Ω_- and Ω_+.

Lemma 2 *The equation (9) has a unique formal solution of the form*

$$z(t) = -\frac{1}{t} + \frac{(1-a)\log t}{t^2} + \sum_{k=3}^{\infty} \frac{p_k(\log t)}{t^k}, \tag{10}$$

where p_k stands for a polynomial of degree less than k. In the case of $a = 1$ all p_k are of degree zero, i.e., they are constant.

The proof of Lemma 2 is straightforward (see e.g. [1]).

Lemma 3 *For any $\delta_0 > 0$ there is a positive number R, such that the equation (9) has two solutions, $z^-(t)$ and $z^+(t)$, analytic in the sectors $\Omega_- = \{\,|\arg(z+R)| > \delta_0\,\}$ and $\Omega_+ = \{\,|\arg(z-R) - \pi| > \delta_0\,\}$, respectively, and which have the asymptotic expansions (10).*

The rest of the section contains the proof of Lemma 3. First, we need some elementary facts on the theory of finite-difference equations, which afford us to rewrite the finite-difference equation (9) as an "integral" one and use a contraction map principle in properly chosen functional space.

2.1 Linear finite-difference equations

In the present section we study the problem of inverting the linear finite-difference operator

$$L(w)(t) = w(t+1) - w(t) - \frac{2}{t}w(t).$$

This operator plays the role of an approximation for the variational equation associated with (9). First, we need a solution of the homogeneous equation $L(w_0) = 0$. Obviously,

$$w_0(t+1) = \left(1 + \frac{2}{t}\right) w_0(t).$$

We look for a solution of this equation in the form $w_0(t) = e^{u_0(t)}$. The substitution into the equation gives

$$u_0(t+1) - u_0(t) = \log\left(\frac{t+2}{t}\right).$$

The function $u_0(t) = \log((t+1)t)$ satisfies the last equation, and we immediately get the desired solution for the homogeneous equation:

$$w_0(t) = t(t+1). \tag{11}$$

Now we can solve the nonhomogeneous equation

$$Lw = f,$$

where f is a known function, which decreases fast enough at infinity. We use the method of variation of parameters looking for the solution in the form

$$w(t) = C(t)w_0(t).$$

Substitution into the equation gives

$$(C(t+1) - C(t))w_0(t+1) = f(t).$$

This equation has two solutions

$$C^-(t) = \sum_{k=1}^{\infty} \frac{f(t-k)}{w_0(t+1-k)}, \qquad C^+(t) = -\sum_{k=0}^{\infty} \frac{f(t+k)}{w_0(t+1+k)}.$$

Taking into account (11) we get the final expression for the solutions of the nonhomogeneous equation

$$w^-(t) = t(t+1)\sum_{k=1}^{\infty} \frac{f(t-k)}{(t+1-k)(t+2-k)}, \tag{12}$$

$$w^+(t) = -t(t+1)\sum_{k=0}^{\infty} \frac{f(t+k)}{(t+1+k)(t+2+k)}. \tag{13}$$

The series converge provided $f = O(t^d)$ with $d < -2$ for $|t| \to \infty$ and then $w = O(t^{d-1})$. Let $\mathcal{X}_d(\Omega^\pm)$ denote the Banach space of analytic functions in $\Omega^\pm$, which are continuous in the closure, with the norm

$$||f||_d = \sup_{\Omega^\pm} |t^d f(t)|.$$

It is easy to see that the equations (12) and (13) define two bounded linear operators $L^{-1}_{d,\pm} : \mathcal{X}_d(\Omega^\pm) \to \mathcal{X}_{d-1}(\Omega^\pm)$, which solve the equation $Lw = f$ in the corresponding functional spaces.

2.2 Proof of Lemma 3

We look for the solution of the equation (9) in the form

$$z_0^{\pm}(t) = -\frac{1}{t} + \frac{(1-a)\log t}{t^2} + w(t).$$

The first two terms in the right-hand side of the last equation satisfy the equation (9) with an error $O(t^{-4})$. Thus, substituting into the equation (9) we get

$$w(t+1) = \left(1 + \frac{2}{t}\right) w(t) + g(t) + f(w(t), t), \tag{14}$$

where

$$g(t) = O(t^{-4}), \qquad f(0,t) = 0, \qquad \frac{\partial f}{\partial w}(0,t) = O(t^{-2}\log t).$$

Inverting the linear operator we obtain from (14)

$$w(t) = \left(L_{d,\pm}^{-1} g\right)(t) + \left(L_{d,\pm}^{-1} f(w(\cdot), \cdot\right)(t).$$

Choosing $d = 3 - \delta$, $\delta \in (0,1)$, it is not difficult to prove the existence and uniqueness of a solution for the last equation by the contraction map principle. An analogous estimate may be found in [3, 4].

3 Invariant foliations

Considering the images of the lines $\mathrm{Im}\, t = \mathrm{const}$ with respect to $z^{\pm}$, we obtain (semi) invariant foliations. We say that z^- defines the unstable foliation, and z^+ defines the stable one. Originally, the unstable solution of the equation (9), z^-, is defined on the domain Ω_- only. But if the function f is entire, the function z^- may be analytically continued on the whole complex plane by iterating the equation (9). The restriction of the map z^- on Ω_- is one-to-one onto its image, but, in general, that is not true for the analytical continuation. The case of z^+ is different, since one has to continue the function z^+ to the left from its original domain of definition, Ω_+ (see Fig. 1). To do that one needs an inverse function f^{-1}. The map f is invertible in a neighborhood of $z = 0$, but it may have no global inverse.

For $f(z) = f_0(z) = z/(1-z)$ we have $z^+(t) = z^-(t) = -1/t$, and the invariant foliations consist of circles and cover the whole complex plane $\mathbf{C}$. Of course, the stable and unstable foliations coincide in this case.

In general, the stable and unstable foliations do not coincide (Fig. 2): the lines, leaving a neighborhood of the origin in a regular way, do not come back in the same way. The difference consists of the phase shift, described by the constants $b_0^{\pm}$ from (7), and oscillations, governed by the Fourier series. The

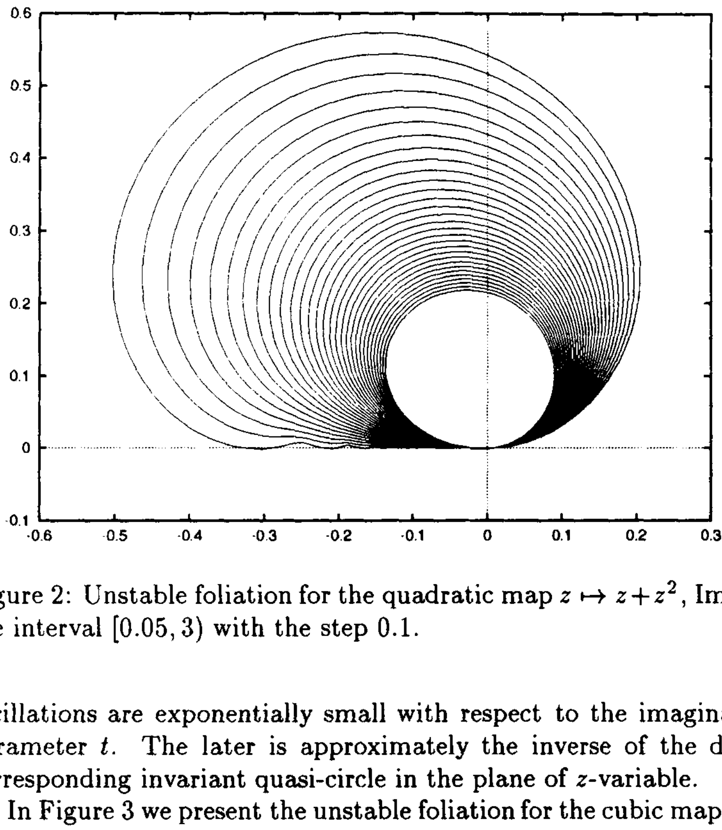

Figure 2: Unstable foliation for the quadratic map $z \mapsto z+z^2$, $\operatorname{Im} t$ goes through the interval $[0.05, 3)$ with the step 0.1.

oscillations are exponentially small with respect to the imaginary part of the parameter t. The later is approximately the inverse of the diameter of the corresponding invariant quasi-circle in the plane of z-variable.

In Figure 3 we present the unstable foliation for the cubic map $z \mapsto z+z^2+z^3$. It is seen that for small values of $\operatorname{Im} t$ the corresponding lines of unstable foliation have self intersections due to the passage near the critical point $z = (-1+\sqrt{2})/3$. The line, which passes through the critical point has cusps.

The lines, which start in Ω_- with small value of $|\operatorname{Im} t|$, may have a complicated continuation as, for example, in Figure 4.

Now we study the splitting of the stable and unstable foliations, i.e., we compare the lines, which coincide formally. Consider $z^+(t)$ and $z^-(t)$ for $\operatorname{Im} t = \sigma$, $\sigma \gg 1$. We assume that the branches of log are chosen in such a way, that the asymptotic expression (10) gives the same values for $z^+(t)$ and $z^-(t)$ in the upper half-plane. The last hypothesis makes no restriction, since the change of the branch is equivalent to a substitution $t \mapsto t + (1-a)2i\pi k$, $k \in \mathbf{Z}$. The difference $z^+(t) - z^-(t)$ decreases faster than any power of t^{-1} as $\operatorname{Im} t \to +\infty$. In particular, that implies that the corresponding representative of the analytic invariant has no constant term:

$$\Phi_+(t) = t + \sum_{k=1}^{\infty} b_k^+ e^{2i\pi k t}. \tag{15}$$

The abovementioned construction establishes a natural one-to-one correspon-

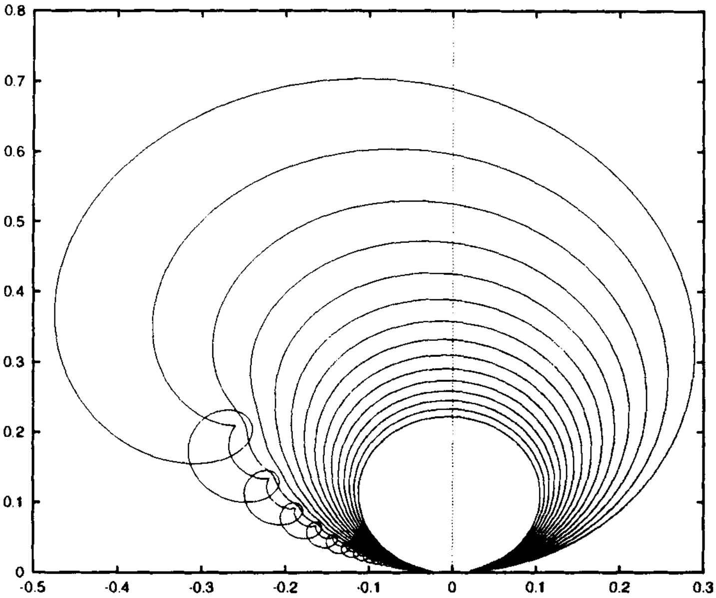

Figure 3: Unstable foliation for the cubic map $z \mapsto z + z^2 + z^3$: $\operatorname{Im} t \in (1.8, 4.8)$.

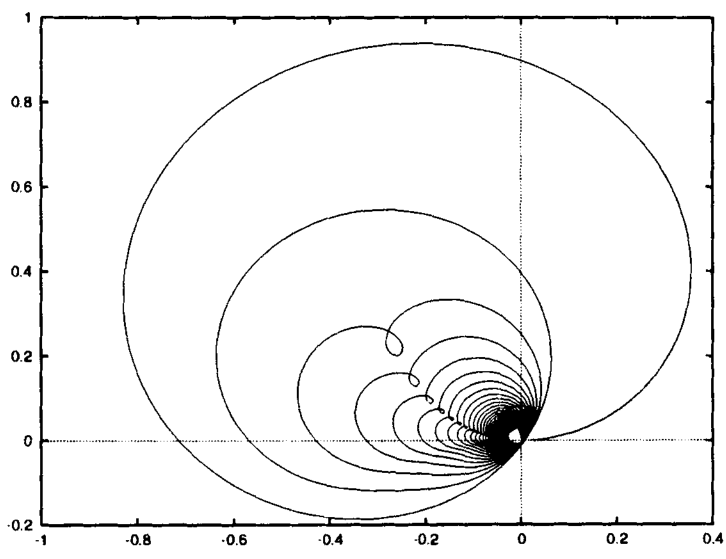

Figure 4: Continuation of the line of the unstable foliation for the cubic map $z \mapsto z + z^2 + z^3$, $\operatorname{Im} t = 1.6$.

dence between the lines of the stable and unstable foliations.

Proposition 4 *Let the map f have the following coefficients in the expansion (15)*

$$b_0^+ = b_1^+ = \ldots = b_{n-1}^+ = 0, \qquad b_n^+ \neq 0,$$

for some positive integer n, then for all sufficiently large $\sigma > 0$ the lines $z^+(t)$ and $z^-(t)$, $\mathrm{Im}\, t = \sigma$, intersect along $2n$ trajectories of the map f, and for any of these trajectories the intersection angle is given by

$$\alpha = 2\pi n |b_n^+| e^{-2\pi n\sigma} + O(e^{-2\pi(n+1)\sigma}).$$

If $b_n^+ = 0$ for all n, then the stable and unstable foliations coincide for $\mathrm{Im}\, t > 0$.

Note, that the angle has the same value at all points of a homoclinic trajectory, because the map f is conformal. The iterates of a homoclinic point accumulate to the origin, consequently, if $b_n^+ \neq 0$ for some n, then the line $z^-(t)$ (as well as $z^+(t)$), $\mathrm{Im}\, t = \sigma \gg 1$, is a closed real-analytic line (asymptotically a circle), but it is not differentiable at the origin.

It is remarkable, that for any map f of the form $f : z \mapsto z + z^2 + \ldots$ there are only two possibilities:

1. the angle is exponentially small with respect to $\sigma = \mathrm{Im}\, t$ and behaves asymptotically like $2\pi n |b_n^+| e^{-2\pi n\sigma}$,

2. it is identically zero and the stable and unstable foliations coincide.

The splitting of the stable and unstable foliations produces no chaotic behavior of trajectories: the map f is topologically conjugated to $z \mapsto z/(1-z)$.

An analogous study may be done in the lower half-plane $\mathrm{Im}\, t < 0$. The type of the splitting in the lower half-plane is independent of the behavior in the upper half-plane, and any combination may appears.

Proof of Proposition 4. We fix $\delta > 0$ and consider the sector $\delta < \arg z < \pi - \delta$. We take $t = A_2 = (z^+)^{-1}$ as a coordinate in this sector. Then the stable foliation is given by $\mathrm{Im}\, t = \sigma$ and the unstable foliation can be represented in the parametric form $t = \Phi_+(i\sigma + \tau)$, $\tau \in \mathbf{R}$. The line of the unstable foliation intersects $\mathrm{Im}\, t = \sigma$ provided

$$\mathrm{Im}\, (\Phi_+(i\sigma + \tau)) = \sigma.$$

Taking into account (15) we get

$$\mathrm{Im} \left(\sum_{k=n}^{\infty} b_k^+ e^{-2\pi k\sigma + 2i\pi k\tau} \right) = 0,$$

where we used that $b_1^+ = \ldots = b_{n-1}^+ = 0$. Since $b_n^+ \neq 0$ we get

$$\tau = -\frac{\arg b_1^+}{2\pi n} + \frac{k}{2n} + O(e^{-2\pi\sigma}). \qquad (16)$$

Since in the coordinate t the map f acts as $t \mapsto t+1$, we have $2n$ distinct homoclinic trajectories.

Since the line of the stable foliation is horizontal the intersection angle at a homoclinic point is given by

$$\tan\alpha = \frac{\operatorname{Im}\dot\Phi_+(i\sigma+\tau)}{\operatorname{Re}\dot\Phi_+(i\sigma+\tau)},$$

where τ is given by (16) and the point denotes the derivative with respect to τ. Using again (15) we get

$$\tan\alpha = \frac{\operatorname{Im}\sum_{k=n}^{\infty} b_k^+ 2i\pi k e^{-2\pi k\sigma + 2i\pi k\tau}}{1 + \operatorname{Re}\sum_{k=n}^{\infty} b_k^+ 2i\pi k e^{-2\pi k\sigma+2i\pi k\tau}} = \pm 2\pi n|b_n^+|e^{-2\pi n\sigma} + O(e^{-2\pi(n+1)\sigma}).$$

Since the map z^+ is conformal it preserves angles, and the angle of intersection of the lines of the stable and unstable foliations is described by the same formula in both z- and t- variables.

4 Computation of analytical invariants

We investigate numerically the analytical invariants for some maps.

The analytic invariant (6) is given by the function

$$\Phi(t) = A_2 \circ z^-(t).$$

The functions A_2 may be fixed by the following asymptotic condition

$$A_2 = -\frac{1}{z} + (1-a)\log z + \sum_{n\geq 1} a_n z^n, \qquad z \to 0, \quad z \in S_2. \qquad (17)$$

The coefficients a_n are defined uniquely from the Abel equation (3). If $f(z)$ is real for real z then these coefficients are real numbers. Comparing this asymptotic with (10) we obtain that $b_0^{\pm} = \pm(1-a)\pi$.

The other coefficients $b_k^{\pm}$ can not be obtained from these asymptotic expansions only because the Fourier series in (7) decrease exponentially fast with respect to $|\operatorname{Im} t|$.

We note that

$$\Phi(t) = A_2 \circ f^{2n} \circ z^-(t-n) - n \tag{18}$$

for all $n \in \mathbb{N}$. Here f^{2n} denotes the composition of $2n$ functions f.

The sequence $f^{2n} \circ z^-(t-n) = f^n \circ z^-(t)$ goes to zero in the sector S_2 as n goes to infinity provided $|\mathrm{Im}\, t|$ is not too small. Consequently, we can evaluate the function $\Phi(t)$ in the following way: choose n sufficiently large and calculate $z^-(t-n)$ using a finite part of the sum (10); apply $2n$ times the map f to the obtained point; use a finite part of the asymptotic expansion (17) to get the approximate value of $\Phi(t)$.

Fourier coefficients in (7) can be evaluated using the usual formula for Fourier coefficients

$$b_k^\pm = \exp(2\pi\sigma k) \int_0^1 (\Phi_\pm(\pm i\sigma + \tau) - (\pm i)\sigma - \tau) \exp(\mp 2\pi i k\tau)\, d\tau \tag{19}$$

where $\sigma > 0$ is a fixed number.

In the computations we used the asymptotic formula (17) with the error term being $O(z^4)$. The integral in (19) was computed by rectangular formula with 16 equidistant nodes. This method provides sufficiently high accuracy and is stable with respect to σ and n.

The results of the computation of the first Fourier coefficient for the polynomial map $z \mapsto z + z^2$ using different values of σ and n are given in the following table.

σ	n	$\lvert c_1^+ \rvert$	$\arg c_1^+$
3.5	100	22350579.12	2.9733955458
4.5	100	22350589.2	2.9733953515
6.0	100	22350580.22	2.9733953361
3.5	500	22350579.22	2.9733955202

Moreover, we find out numerically that $|b_1^+|$ of the polynomial (2) depends on a asymptotically as const $\cdot \exp(-2\pi^2 a)$ for $a \to -\infty$.

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