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LOCALLY FINITE QUASIVARIETIES OF MV-ALGEBRAS

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Locally finite quasivarieties of MV-algebras.*

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Abstract

In this paper we show that every locally finite quasivariety of MV-algebras is finitely generated and finitely based. To see this result we study critical MV-algebras. We also give axiomatizations of some of these quasivarieties.

Introduction

In [4, 5] C.C.Chang introduced MV-algebras in order to give an algebraic counterpart of the Łukasiewicz's many valued propositional calculus. In fact, the class of all MV-algebras, in a termwise equivalent presentation named Wajsberg algebras, is the equivalent variety semantics, in the sense of [1], of this calculus (see [17]).

From the equivalence between MV-algebras and many valued Łukasiewicz's logic, it is easy to see that finitary extensions of Łukasiewicz's propositional calculus correspond to subquasivarieties of MV-algebras, and axioms and rules of the calculus correspond with equations and quasiequations, respectively. Hence, finite axiomatizable finitary extensions of Łukasiewicz's propositional calculus correspond with finite axiomatizable quasivarieties of MV-algebras.

In this paper, we study finite axiomatizability of locally finite quasivarieties of MV-algebras. Concretely, we show, in Section 2, that locally finite quasivarieties and finitely generated quasivarieties of MV-algebras coincide, Theorem 2.13, and that they are finitely axiomatizable, Theorem 2.15. To prove these results, we give a characterization of critical MV-algebras, Theorem 2.11, and we see that any locally finite quasivariety of MV-algebras is generated by critical MV-algebras, Theorem 2.9. By using a result of [11], we deduce the finite axiomatizability of these quasivarieties.

In section 3, we give some examples of locally finite quasivarieties of MV-algebras, and we give an effective axiomatization for each considered quasivariety .

We include a preliminary section, Section 1, containing basic definitions, results and notation used in the paper.

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1 Definitions and first properties.

An **MV-algebra** is an algebra $\langle A, \oplus, \neg, 0\rangle$ of type $(2,1,0)$ satisfying the following equations:

MV1. $(x \oplus y) \oplus z \approx x \oplus (y \oplus z)$

MV2. $x \oplus y \approx y \oplus x$

MV3. $x \oplus 0 \approx x$

MV4. $\neg(\neg x) \approx x$

MV5. $x \oplus \neg 0 \approx \neg 0$

MV6. $\neg(\neg x \oplus y) \oplus y \approx \neg(x \oplus \neg y) \oplus x$

By taking $y = \neg 0$ in MV6, we deduce:

MV7. $x \oplus \neg x \approx \neg 0$.

Therefore, if we set $1 = \neg 0$ and $x \odot y = \neg(\neg x \oplus \neg y)$, then $\langle A, \oplus, \odot, \neg, 0, 1\rangle$ satisfies all axioms given in [14, Lemma 2.6], and hence the above definition of MV-algebras is equivalent to Chang's definition [4]. We denote by $\mathbb{W}$, the class of all MV-algebras. $\mathbb{W}$ is a variety since it is an equational class.

Given $\mathbf{G} = \langle G, \wedge, \vee, +, -, 0\rangle$ a *lattice ordered abelian group* and $u \in G \quad u > 0$, we define the algebra $\Gamma(\mathbf{G}, u) = \langle [0,u], \oplus, \neg, 0\rangle$ where

- $[0,u] = \{a \in G : 0 \leq a \leq u\}$,
- $a \oplus b = u \wedge (a + b)$, $\neg a = u - a$ and $0 = 0^{\mathbf{G}}$.

Then $\Gamma(\mathbf{G}, u)$ is an MV-algebra. In fact any MV-algebra is isomorphic to the unit segment of some lattice ordered abelian group. Concretely, the category of MV-algebras is equivalent to the category of lattice ordered abelian groups with strong unit (See [6],[14]).

The following MV-algebras play an important role in the paper.

- $\mathbf{[0,1]} = \Gamma(\mathbf{R}, 1)$, where $\mathbf{R}$ is the totally ordered group of the reals.
- $\mathbf{[0,1]} \cap \mathbf{Q} = \Gamma(\mathbf{Q}, 1) = \langle \{\frac{k}{m} : k \leq m < \omega\}, \oplus, \neg, 0\rangle$, where $\mathbf{Q}$ is the totally ordered abelian group of the rationals and ω represents the set of all natural numbers.

For every $0 < n < \omega$

- $\mathbf{Ł}_n = \Gamma(\mathbf{Q}_n, 1) = \langle \{\frac{k}{n} : 0 \leq k \leq n\}, \oplus, \neg, 0\rangle$, where $\mathbf{Q}_n = \{\frac{k}{n} : k \in \mathbb{Z}\}$ is a subgroup of $\mathbf{Q}$ and $\mathbb{Z}$ is the set of all integers.

- $\mathbf{L}_n^\omega = \Gamma(\mathbf{Q}_n \otimes \mathbf{Z}, (1,0)) = \langle\{(\frac{k}{n}, i) : (0,0) \leq (\frac{k}{n}, i) \leq (1,0)\}, \oplus, \neg, 0\rangle$, where $\mathbf{Z}$ is the totally ordered group of the integers and $\mathbf{Q}_n \otimes \mathbf{Z}$ is the lexicographic product of $\mathbf{Q}_n$ and $\mathbf{Z}$.

The following theorem states some well-known results on MV-algebras.(See for instance [6]).

Theorem 1.1

1. *Every simple MV-algebra is isomorphic to a subalgebra of* $[\mathbf{0,1}]$.
2. *Every finite simple MV-algebra is isomorphic to* $\mathbf{L}_n$ *for some* $n \in \omega$.
3. *Every finite MV-algebra is isomorphic to a direct product of finite simple MV-algebras.*
4. $\mathbf{L}_n \subseteq \mathbf{L}_m$ *if and only if* $n|m$. □

If we define $0x = 0$ and for each $n \in \omega$ $(n+1)x = x \oplus nx$, then

Lemma 1.2 [18, Lemma 2.2.] $\mathbf{L}_n$ *is embeddable into an MV-algebra* $\mathbf{A}$ *if, and only if, there is an element* $a \in \mathbf{A}$ *such that* $(n-1)(\neg a) = a$. *Moreover* $a \neq 1^{\mathbf{A}}$. □

We denote by $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_R$ and $\mathbb{P}_U$ the operators *isomorphic image, homomorphic image, substructure, direct product, reduced product and ultraproduct* respectively. We recall that a class $\mathbb{K}$ of algebras is a **variety** if and only if it is closed by $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. And a class $\mathbb{K}$ of algebras is a **quasivariety** if and only if it is closed by $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}_R$, or equivalently, by and $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_U$. Given a class $\mathbb{K}$ of algebras, the variety generated by $\mathbb{K}$, denoted by $\mathbb{V}(\mathbb{K})$, is the least variety containing $\mathbb{K}$. Similarly, the quasivariety generated by a class $\mathbb{K}$, which we denote by $\mathbb{Q}(\mathbb{K})$, is the least quasivariety containing $\mathbb{K}$. We also recall that a class $\mathbb{K}$ of algebras is a variety if and only if it is an equational class, and $\mathbb{K}$ is a quasivariety if and only if it is a quasi-equational class.

2 Locally finite quasivarieties and critical algebras.

An algebra $\mathbf{A}$ is **locally finite** if and only if every finitely generated subalgebra is finite. A class $\mathbb{K}$ is **locally finite** if and only if every member of $\mathbb{K}$ is locally finite. A variety, or a quasivariety, is **finitely generated** if it is generated by a finite set of finite algebras.
We recall some basic properties of locally finite varieties and quasivarieties.

Theorem 2.1 [3, page 70] *Let* $\mathbb{K}$ *be a finite set of finite algebras. Then* $\mathbb{V}(\mathbb{K})$ *is a locally finite variety.* □

Theorem 2.2 [3, page 69] *A variety* $\mathbb{K}$ *is locally finite if and only if*

$$|X| < \omega \text{ implies } |\mathrm{F}_{\mathbb{K}}(\bar{X})| < \omega$$

where $\mathrm{F}_{\mathbb{K}}(\bar{X})$ *is the free algebra with respect* $\mathbb{K}$. $\square$

From the above theorem we can deduce:

Theorem 2.3 *Let* $\mathbb{K}$ *be a quasivariety. The following conditions are equivalent:*

1. $\mathbb{K}$ *is a locally finite quasivariety*
2. $\mathbb{V}(\mathbb{K})$ *is a locally finite variety.*
3. $\mathbb{K}$ *is contained in a locally finite variety.*

Proof : $1 \Rightarrow 2$: Assume that $\mathbb{K}$ is locally finite. Since $\mathbb{K}$ is a quasivariety, we have that $\mathrm{F}_{\mathbf{V}(\mathbb{K})}(\bar{X}) \in \mathbb{K}$. Therefore if $|X| < \omega$ then $|\mathrm{F}_{\mathbf{V}(\mathbb{K})}(\bar{X})| < \omega$. And by Theorem 2.2 we obtain that $\mathbb{V}(\mathbb{K})$ is locally finite.
$2 \Rightarrow 3$: It is trivial, since $\mathbb{K} \subseteq \mathbb{V}(\mathbb{K})$.
$3 \Rightarrow 1$: Since any subclass of a locally finite class of algebras is also locally finite, from 3 we trivially obtain 1 $\square$

Corollary 2.4 *Every finitely generated quasivariety is locally finite.* $\square$

We want to obtain all locally finite varieties and quasivarieties of MV-algebras. First we recall which are the varieties of MV-algebras.

Theorem 2.5 [13, Theorem 4.11] $\mathbb{K}$ *is a proper subvariety of* $\mathbb{W}$ *if and only if there exist two disjoint finite subsets* I, J *of natural numbers such that*

$$\mathbb{K} = \mathbb{V}(\mathbf{L}_i \ i \in I, \mathbf{L}_j^{\omega} \ j \in J).$$

From the above we have:

Lemma 2.6 *Let* $\mathbb{K}$ *be a variety of MV-algebras.* $\mathbb{K}$ *is a locally finite variety if and only if* $\mathbb{K} = \mathbb{V}(\mathbf{L}_{n_1}, \ldots \mathbf{L}_{n_r})$ *for some* $n_1, \ldots, n_r \in \omega$

Proof : By Theorem 2.1, for every $r < \omega$ and any $n_1, \ldots, n_r \in \omega$, $\mathbb{V}(\mathbf{L}_{n_1}, \ldots \mathbf{L}_{n_r})$ is a locally finite variety.
If $\mathbb{K} = \mathbb{W}$, then, since $[\mathbf{0}, \mathbf{1}]$ is not locally finite, $\mathbb{K}$ is not locally finite. If a proper variety $\mathbb{K}$ is not of the form $\mathbb{V}(\mathbf{L}_{n_1}, \ldots \mathbf{L}_{n_r})$ for some $n_1, \ldots, n_r \in \omega$, then by Theorem 2.5 we have that $\mathbb{K} = \mathbb{V}(\mathbf{L}_i \ i \in I, \mathbf{L}_j^{\omega} \ j \in J)$ with $J \neq \emptyset$. Hence there is $j \in J$ such that $\mathbf{L}_j^{\omega} \in \mathbb{K}$ and, since $\mathbf{L}_j^{\omega}$ is not locally finite, $\mathbb{K}$ is not a locally finite variety. $\square$

From Lemma 2.6 and Theorem 2.3 we can deduce:

Corollary 2.7 *A quasivariety of MV-algebras is locally finite if and only if it is a subquasivariety of a variety of the form* $\mathbb{V}(\mathbf{L}_{n_1}, \ldots \mathbf{L}_{n_r})$ *for some* $n_1, \ldots, n_r \in \omega$. $\square$

A **critical** algebra is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras. In order to study critical algebras we need to recall a general result of Model Theory.

Theorem 2.8 [3, page 213] *Every algebra is embeddable into an ultraproduct of its finitely generated subalgebras.* □

The interest of critical algebras is given by the following result, which is mentioned in [10, page 128], but no proof is given.

Theorem 2.9 *Every locally finite quasivariety is generated by its critical algebras.*

Proof : Let $\mathbb{K}$ be a locally finite quasivariety and let $\mathbf{A} \in \mathbb{K}$. By Theorem 2.8, if $\mathcal{F} = \{\mathbf{B} \subseteq \mathbf{A} : \mathbf{B} \text{ is finitely generated}\}$, then $\mathbf{A} \in \mathbb{ISP}_U(\mathcal{F})$. Since $\mathbb{K}$ is locally finite we have that $\mathbf{A} \in \mathbb{ISP}_U(\{\mathbf{B} \subseteq \mathbf{A} : \mathbf{B} \text{ is finite}\}) \subseteq \mathbb{Q}(\mathbb{K}_{fin})$, where $\mathbb{K}_{fin}$ is the class of all finite algebras in $\mathbb{K}$. Hence $\mathbb{K} = \mathbb{Q}(\mathbb{K}_{fin})$.
Let $\mathbf{A} \in \mathbb{K}_{fin}$ then we claim that $\mathbb{Q}(\mathbf{A}) = \mathbb{Q}(\{\mathbf{B} \subseteq \mathbf{A} : \mathbf{B} \text{ is critical }\})$. To prove the claim we proceed by induction over the cardinal of A. If $|A| = 1$, then $\mathbf{A}$ is already critical and we trivially have that $\mathbb{Q}(\mathbf{A}) = \mathbb{Q}(\{\mathbf{B} \subseteq \mathbf{A} : \mathbf{B} \text{ is critical }\})$.
Assume $|A| = n$. If $\mathbf{A}$ is critical, then $\mathbb{Q}(\mathbf{A}) = \mathbb{Q}(\{\mathbf{B} \subseteq \mathbf{A} : \mathbf{B} \text{ is critical }\})$.
In other case, we have that $\mathbb{Q}(\mathbf{A}) = \mathbb{Q}(\{\mathbf{B} \subseteq \mathbf{A} : \mathbf{B} \neq \mathbf{A}\})$. Since for $\mathbf{B} \subsetneq \mathbf{A}$, $|B| < n$, by inductive hypothesis we have that $\mathbb{Q}(\mathbf{B}) = \mathbb{Q}(\{\mathbf{C} \subseteq \mathbf{B} : \mathbf{C} \text{ is critical }\})$.
Hence,

$$\begin{aligned} \mathbb{Q}(\mathbf{A}) &= \mathbb{Q}(\{\mathbf{B} : \mathbf{B} \subsetneq \mathbf{A}\}) \\ &= \mathbb{Q}(\{\mathbf{C} \subseteq \mathbf{B} : \mathbf{C} \text{ is critical and } \mathbf{B} \subsetneq \mathbf{A}\}) \\ &= \mathbb{Q}(\{\mathbf{B} \subseteq \mathbf{A} : \mathbf{B} \text{ is critical }\}). \end{aligned}$$

and the claim is proved. Thus we have that

$$\mathbb{K} = \mathbb{Q}(\mathbb{K}_{fin}) \subseteq \mathbb{Q}(\{\mathbf{B} \subseteq \mathbf{A} : \mathbf{B} \text{ is critical and } \mathbf{A} \in \mathbb{K}_{fin}\}) = \mathbb{Q}(\mathbb{K}_{Crit}) \subseteq \mathbb{K}$$

where $\mathbb{K}_{Crit}$ is the class of all critical algebras in $\mathbb{K}$. □
Our next purpose is to characterize critical MV-algebras. We need a previous result.

Lemma 2.10 *If $\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}$ is embeddable into $\prod_{j \in J} \text{Ł}_{m_j}$ where the set $\{m_j : j \in J\}$ is finite then*

1. *For every $i < l$ there exists $j \in J$ such that $n_i | m_j$.*

2. *For every $j \in J$ there exists $i < l$ such that $n_i | m_j$.*

Proof : 1) If $\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}$ is embeddable into $\prod_{j \in J} \text{Ł}_{m_j}$, then

$$\text{Ł}_{n_1} \times \cdots \times \text{Ł}_{n_l} \in \mathbb{V}(\prod_{j \in J} \text{Ł}_{m_j}) = \mathbb{V}(\{\text{Ł}_{m_j} ; j \in J\}).$$

Hence, for every $i < l$, $\text{Ł}_{n_i} \in \mathbb{V}(\{\text{Ł}_{m_j}; j \in J\})$. Since $\{m_j : j \in J\}$ is finite, from a result due to Jónsson [3, page 149], we deduce that the class subdirectly irreducible members of $\mathbb{V}(\{\text{Ł}_{m_j}; j \in J\})$ is $\mathbb{I}(\{\text{Ł}_n : \exists j \in J\ \text{Ł}_n \subseteq \text{Ł}_{m_j}\})$. Since Ł_{n_i} is simple, therefore subdirectly irreducible, for every $i < l$ there exists $j \in J$ such that $\text{Ł}_{n_i} \subseteq \text{Ł}_{m_j}$, and by 4 of Theorem 1.1 $n_i | m_j$.
2) For each $j \in J$ consider the natural projection: $\pi_j : \prod_{j\in J} \text{Ł}_{m_j} \longrightarrow \text{Ł}_{m_j}$. Let $\gamma: \text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}} \to \prod_{j\in J} \text{Ł}_{m_j}$ be an embedding, then for every $j \in J$ $\gamma_j = \pi_j \circ \gamma$ is an homomorphism from $\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}$ to Ł_{m_j}. Hence

$$\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}/Ker(\gamma_j) \cong \gamma_j(\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}) \subseteq \text{Ł}_{m_j}$$

So, $\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}/Ker(\gamma_j)$ is simple, and by [6, Theorem 4.1.19] we have that $Ker(\gamma_j)$ is a maximal congruence relation of $\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}$. From [7, Lemma 2.3] (see also [16]) and the fact that all Ł_n's are simple, it can be deduced that there is $k < l$ such that

$$Ker(\gamma_j) = \text{Ł}^2_{n_0} \times \cdots \times \text{Ł}^2_{n_{k-1}} \times \Delta_{\text{Ł}_{n_k}} \times \text{Ł}^2_{n_{k+1}} \times \cdots \times \text{Ł}^2_{n_{l-1}}.$$

Hence, for every $j \in J$ there exists $k < l$ such that

$$\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}/Ker(\gamma_j) \cong \text{Ł}_{n_k} \subseteq \text{Ł}_{m_j}.$$

Thus $n_k | m_j$. □
Finally we give a characterization of all critical MV-algebras.

Theorem 2.11 *An MV-algebra* $\mathbf{A}$ *is critical if and only if* $\mathbf{A}$ *is isomorphic to a finite MV-algebra* $\text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}$ *satisfying the following conditions:*

1. *For every* $i, j < l$, $i \neq j$ *implies* $n_i \neq n_j$.
2. *Consider the correspondence* $D : \omega \to \mathcal{P}(\omega): n \mapsto D(n) = \{d < \omega : d | n\}$. *Then there is at most one* n_i, $i < l$, *such that* $|D(n_i) \cap \{n_j : j < l\}| > 1$.

Proof : Assume that $\mathbf{A} = \text{Ł}_{n_0} \times \cdots \times \text{Ł}_{n_{l-1}}$ satisfies conditions 1) and 2). First, we will show the following:
Claim: *Every proper subalgebra of* $\mathbf{A}$ *is embeddable into a subalgebra of* $\mathbf{A}$ *of the form* $\text{Ł}_{d_0} \times \cdots \times \text{Ł}_{d_{l-1}}$, *where* $d_i | n_i$ *for each* $i < l$ *and there exists* $j < l$ *such that* $d_j \neq n_j$.
Proof of the claim: Let $\mathbf{B}$ be a proper subalgebra of $\mathbf{A}$. Since $\mathbf{A}$ is finite $\mathbf{B}$ is also finite and by Theorem 1.1, $\mathbf{B}$ is isomorphic to $\text{Ł}_{p_0} \times \cdots \times \text{Ł}_{p_{r-1}}$. For each $i < l$ consider the natural projection: $\pi_i : \mathbf{A} \to \text{Ł}_{n_i}$, if for all $i < l$, we write $\gamma_i = \pi_i \restriction_{\mathbf{B}}$, then we can assume that $\mathbf{B}$ is embeddable into $\gamma_0(\mathbf{B}) \times \cdots \times \gamma_{l-1}(\mathbf{B})$. Moreover, since $\gamma_i(\mathbf{B}) \subseteq \text{Ł}_{n_i}$, we have $\gamma_i(\mathbf{B}) = \text{Ł}_{d_i}$ for some $d_i | n_i$.

Assume that $\gamma_i(\mathbf{B}) = \mathbf{Ł}_{n_i}$ for each $i < l$. Then, for every $i < l$, $\mathbf{B}/Ker(\gamma_i) \cong \mathbf{Ł}_{n_i}$ and since $\mathbf{Ł}_{n_i}$ is simple, $Ker(\gamma_i)$ is a maximal congruence relation of $\mathbf{B}$. From [7, Lemma 2.3] (see also [16]) and the fact that all $\mathbf{Ł}_n$'s are simple, it can be deduced that there is $k < r$ such that

$$Ker(\gamma_i) = \text{Ł}^2_{p_0} \times \cdots \times \text{Ł}^2_{p_{k-1}} \times \Delta_{\text{Ł}_{p_k}} \times \text{Ł}^2_{p_{k+1}} \times \cdots \times \text{Ł}^2_{p_{r-1}}.$$

Hence, for every $i < l$ there exists $k < r$ such that $\mathbf{Ł}_{p_k} = \mathbf{Ł}_{n_i}$. By condition (1), $i \neq j$ implies $n_i \neq n_j$, so $l \leq r$ and

$$\mathbf{B} \cong \mathbf{Ł}_{n_0} \times \cdots \times \mathbf{Ł}_{n_{l-1}} \times \mathbf{Ł}_{m_l} \times \cdots \times \mathbf{Ł}_{m_{r-1}} = \mathbf{A} \times \mathbf{Ł}_{m_l} \times \cdots \times \mathbf{Ł}_{m_{r-1}}$$

that implies $|\mathbf{A}| \leq |\mathbf{B}|$, which contradicts that $\mathbf{B}$ is a proper subalgebra of $\mathbf{A}$. And the claim is proved.
Suppose that $\mathbf{A} \in \mathbb{Q}(\{\mathbf{B} \subsetneq \mathbf{A}\})$, then

$$\mathbf{A} \in \mathbb{ISPP}_U(\{\mathbf{Ł}_{d_0} \times \cdots \times \mathbf{Ł}_{d_{l-1}} : \ \forall i \ d_i | n_i; \ \exists k \ d_k \neq n_k\}).$$

Since $\{\mathbf{Ł}_{d_0} \times \cdots \times \mathbf{Ł}_{d_{l-1}} : \ \forall i \ d_i | n_i; \ \exists k \ d_k \neq n_k\}$ is a finite set of finite MV-algebras, we have that $\mathbf{A} \in \mathbb{ISP}(\{\mathbf{Ł}_{d_0} \times \cdots \times \mathbf{Ł}_{d_{l-1}} : \ \forall i \ d_i | n_i; \ \exists k \ d_k \neq n_k\})$. Thus $\mathbf{Ł}_{n_0} \times \cdots \times \mathbf{Ł}_{n_{l-1}}$ is embeddable into $\prod\limits_{k<n} (\mathbf{Ł}_{d_{0,k}} \times \cdots \times \mathbf{Ł}_{d_{l-1,k}})^{\alpha_k}$ where

$$\{\mathbf{Ł}_{d_{0,k}} \times \cdots \times \mathbf{Ł}_{d_{l-1,k}} : k < n\} \subseteq \{\mathbf{Ł}_{d_0} \times \cdots \times \mathbf{Ł}_{d_{l-1}} : \ \forall i \ d_i | n_i; \ \exists k \ d_k \neq n_k\}.$$

Since the set $\{d_{t,k} : t < l; k \leq n\}$ is finite, we can apply Lemma 2.10.
If there exists $i, j < l$ such that $i \neq j$ and $n_i | n_j$, then by conditions 1) and 2), n_j is unique. By Lemma 2.10, there exists $\mathbf{Ł}_{d_{t,m}}$ such that $n_j | d_{t,m}$. That is, there exists

$$\mathbf{Ł}_{d_{0,m}} \times \cdots \times \mathbf{Ł}_{d_{l-1,m}} \in \{\mathbf{Ł}_{d_0} \times \cdots \times \mathbf{Ł}_{d_{l-1}} : \ \forall i \ d_i | n_i; \ \exists k \ d_k \neq n_k\}$$

such that $n_j | d_{t,m}$ for some $t < l$. By condition 2), n_j does not divide any n_i other than itself. Therefore, since $d_{t,m}$ is a divisor of n_t, we have that $n_t = d_{t,m} = n_j$ and by 1), $t = j$. Since

$$\mathbf{Ł}_{d_{0,m}} \times \cdots \times \mathbf{Ł}_{d_{l-1,m}} \in \{\mathbf{Ł}_{d_0} \times \cdots \times \mathbf{Ł}_{d_{l-1}} : \ \forall i \ d_i | n_i; \ \exists k \ d_k \neq n_k\},$$

there exists $r \neq j < l$ such that $d_{r,m} | n_r$ and $d_{r,m} \neq n_r$. By 2) of Lemma 2.10, there exists $s, r < l$ such that $s \neq r < l$ and $n_s | d_{r,m}$. Thus $n_s | n_r$, $n_i | n_j$, $r \neq s$, $i \neq j$ and $r \neq j$, which contradicts condition 2).
If for all $1 \leq i, j \leq l$ such that $i \neq j$, $n_i \nmid n_j$, then the same argument follows by taking any n_j, $j < l$.
Since $\mathbf{A}$ is finite and $\mathbf{A} \notin \mathbb{Q}(\{\mathbf{B} \subsetneq \mathbf{A}\})$, $\mathbf{A}$ is critical.
Conversely, if $\mathbf{A}$ is a critical MV-algebra, then $\mathbf{A}$ is finite and by Theorem 1.1, we can suppose, without loss of generality, that

$$\mathbf{A} = {\mathbf{Ł}_{n_0}}^{m_0} \times \cdots \times {\mathbf{Ł}_{n_{k-1}}}^{m_{k-1}}$$

for some $n_0, \dots, n_{k-1}, m_0, \dots, m_{k-1} \in \omega$ and $n_i \neq n_j$ when $i \neq j$. If not all m_i's are equal to 1, then the correspondence

$$\alpha\colon (a(0), \dots, a(k-1)) \mapsto \alpha(a) = \Big((\overbrace{a(0), \dots, a(0)}^{m_0}), \dots, (\overbrace{a(k-1), \dots, a(k-1)}^{m_{k-1}})\Big)$$

defines an isomorphism from $\mathbf{L}_{n_0} \times \dots \times \mathbf{L}_{n_{k-1}}$ onto a proper subalgebra of $\mathbf{A}$. Let $m = max\{m_0, \dots, m_{k-1}\}$, then the correspondence

$$\beta\colon \mathbf{L}_{n_0}{}^{m_0} \times \dots \times \mathbf{L}_{n_{k-1}}{}^{m_{k-1}} \to \mathbf{L}_{n_0}^m \times \dots \times \mathbf{L}_{n_{k-1}}^m$$

such that for every $r < k$,

$$\beta((b(0), \dots, b(k-1)))(r) = (b(r)(1), \dots, b(r)(m_r), \overbrace{b(r)(1), \dots, b(r)(1)}^{m-m_r}),$$

gives an embedding from $\mathbf{A}$ into $\mathbf{L}_{n_0}{}^m \times \dots \times \mathbf{L}_{n_{k-1}}{}^m \cong (\mathbf{L}_{n_0} \times \dots \times \mathbf{L}_{n_{k-1}})^m$. Thus $\mathbf{A} \in \mathbb{Q}(\mathbf{L}_{n_0} \times \dots \times \mathbf{L}_{n_{k-1}})$. Since $\mathbf{A}$ is critical, we have $m_0, \dots, m_{k-1} = 1$. Hence it satisfies condition 1). Suppose condition 2) fails, then there exist $j < r < k$ such that

$$|D(n_j) \cap \{n_m : m < k\}| > 1 \text{ and } |D(n_r) \cap \{n_m : m < k\}| > 1.$$

Thus there exist $i \neq j$ and $s \neq r$ such that $n_i | n_j$, $n_s | n_r$ and $j \neq r$. Since $n_i | n_j$, we have that the correspondence that maps

$$(a(0), \dots, a(j-1), a(j+1), \dots, a(k-1))$$

to

$$(a(0), \dots, a(j-1), a(i), a(j+1), \dots, a(k-1)).$$

defines an isomorphism from $\mathbf{L}_{n_0} \times \dots \times \mathbf{L}_{n_{j-1}} \times \mathbf{L}_{n_{j+1}} \times \dots \times \mathbf{L}_{n_{k-1}}$ onto a proper subalgebra of $\mathbf{A}$.
Similarly the algebra $\mathbf{L}_{n_0} \times \dots \times \mathbf{L}_{n_{r-1}} \times \mathbf{L}_{n_{r+1}} \times \dots \times \mathbf{L}_{n_{k-1}}$ is isomorphic to a proper subalgebra of $\mathbf{A}$. Finally, observe that $\mathbf{L}_{n_0} \times \dots \times \mathbf{L}_{n_{k-1}}$ is embeddable into

$$\mathbf{L}_{n_1}{}^2 \times \dots \times \mathbf{L}_{n_{j-1}}{}^2 \times \mathbf{L}_{n_j} \times \mathbf{L}_{n_{j+1}}{}^2 \times \dots \times \mathbf{L}_{n_{r-1}}{}^2 \times \mathbf{L}_{n_r} \times \mathbf{L}_{n_{r+1}}{}^2 \times \dots \times \mathbf{L}_{n_k}{}^2,$$

by means of the correspondence δ defined as:

$$\delta(a(0), \dots, a(k-1))(i) = \begin{cases} (a(i), a(i)) & \text{if } i \neq j, r \\ a(i) & \text{if } i = j, r \end{cases}.$$

Therefore
$\mathbf{A} \in \mathbb{Q}(\mathbf{L}_{n_0} \times \dots \times \mathbf{L}_{n_{j-1}} \times \mathbf{L}_{n_{j+1}} \times \dots \times \mathbf{L}_{n_{k-1}}, \mathbf{L}_{n_0} \times \dots \times \mathbf{L}_{n_{r-1}} \times \mathbf{L}_{n_{r+1}} \times \dots \times \mathbf{L}_{n_{k-1}})$
in contradiction with the fact that $\mathbf{A}$ is critical. $\square$

Corollary 2.12 *The number of non isomorphic critical MV-algebras in a proper variety of MV-algebras is finite.* □

Proof : If $\mathbb{K}$ is a proper variety of MV-algebras, then it is shown in [13] and [6] that $\mathbb{K}$ contains a finite number of $\mathbf{L}_n's$. Let $M = \{n \in \omega : \mathbf{L}_n \in \mathbb{K}\}$, clearly $|M| < \omega$. By Theorem 2.11, all critical algebras in $\mathbb{K}$ are:

$$\mathbb{I}(\{\mathbf{L}_{n_1} \times \cdots \times \mathbf{L}_{n_l} : \text{ satisfying (1) and (2) of Theorem 2.11 and } n_i \in M\}).$$

Since $|M|$ is finite we have that the number of non isomorphic critical MV-algebras in $\mathbb{K}$ is finite. □

From the above result we deduce :

Theorem 2.13 *A quasivariety of MV-algebras is locally finite if and only if it is finitely generated.*

Proof : Let $\mathbb{K}$ be a locally finite quasivariety of MV-algebras, by Corollary 2.7, $\mathbb{V}(\mathbb{K})$ is a proper subvariety of $\mathbb{W}$, thus, applying Corollary 2.12 the number of non isomorphic critical MV-algebras in $\mathbb{K}$ is finite. By Theorem 2.9, $\mathbb{K}$ is generated by its critical algebras, therefore, since a quasivariety is closed under the operation of isomorphic images, $\mathbb{K}$ is finitely generated.
The converse is given by Corollary 2.4 □
In general locally finite quasivarieties are not finitely axiomatizable, not even finitely generated quasivarieties are finitely axiomatizable. For instance: Let $\mathbb{K} = \mathbb{Q}(\mathbf{A})$ where $\mathbf{A} = \langle\{0,1,2\}, f, g\rangle$ is of type (1,1) with f and g defined by $f(0) = 1, g(0) = 2$ and $f(x) = g(x) = x$ for $x \neq 0$. Due to Gorbunov [12], $\mathbb{K}$ is not finitely axiomatizable while it is finitely generated.(see also [8, page 149]).
We will show that locally finite quasivarieties of MV-algebras are finitely axiomatizable. For this we need the following result.

Lemma 2.14 [11, Lemma 4.2] *Let $\mathbb{M}$ be a locally finite quasivariety of finite type, then for a quasivariety $\mathbb{K}$ contained in $\mathbb{M}$ the following conditions are equivalent:*

1. *$\mathbb{K}$ is not finitely axiomatizable relative to $\mathbb{M}$.*
2. *There exists an infinite sequence $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots$ of finite algebras of $\mathbb{M}$ satisfying:*
   - *(a) $|A_i| < |A_{i+1}|$ for all i;*
   - *(b) $\mathbf{A}_i \notin \mathbb{K}$ for all i;*
   - *(c) Every proper subalgebra of every $\mathbf{A}_i$ belongs to $\mathbb{K}$.* □

From the above result and Corollary 2.12 we deduce:

Theorem 2.15 *Every locally finite quasivariety of MV-algebras is finitely axiomatizable.*

Proof : Let $\mathbb{K}$ be a locally finite quasivariety of MV-algebras, by Corollary 2.7, $\mathbb{V}(\mathbb{K})$ is a proper locally finite subvariety of $\mathbb{W}$. Observe that any finite MV-algebra $\mathbf{A} \in \mathbb{V}(\mathbb{K})$ satisfying conditions (b) and (c) is critical. It follows from Corollary 2.12 that the number of non isomorphic critical MV-algebras in $\mathbb{V}(\mathbb{K})$ is finite. Therefore it is impossible to obtain an infinite sequence of finite MV-algebras of $\mathbb{V}(\mathbb{K})$ satisfying conditions $(a), (b)$ and (c). Thus, by Lemma 2.14, $\mathbb{K}$ is finitely axiomatizable relative to $\mathbb{V}(\mathbb{K})$. And since any variety of MV-algebras is finitely axiomatizable (see for instance [13], [6] and [9]), we have that $\mathbb{K}$ is finitely axiomatizable. □

3 Applications to axiomatization of concrete samples.

It is well known that every subvariety of MV-algebras is finitely axiomatizable. In fact, some effective axiomatizations are given in [9] and in [15]. To our concern, we only need to axiomatize locally finite varieties of MV-algebras. In [18] it is proved that the variety generated by $\mathbf{Ł}_n$ is finitely axiomatizable and it is axiomatized by MV1,...,MV6 plus a single axiom of the form $\varphi(x) \approx 1$, denoted by $v_n(x) \approx 1$. Moreover, *for every* $n_1, \ldots, n_r < \omega, \mathbb{V}(\mathbf{Ł}_{n_1}, \ldots, \mathbf{Ł}_{n_r})$ *is the subvariety of* $\mathbb{W}$ *defined by the equation* $v_{n_1}(x) \vee \cdots \vee v_{n_r}(x) \approx 1$ [18, Theorem 1.8]. Where $\vee$ is defined by $x \vee y = \neg(\neg x \oplus y) \oplus y$.
Given a quasivariety $\mathbb{K}$ of MV-algebras, we define the following class

$$\mathbb{K} : \mathbf{Ł}_n = \{\mathbf{A} \in \mathbb{K} : \mathbf{Ł}_n \notin \mathbb{IS}(\mathbf{A})\}.$$

From Lemma 1.2, the quasiequation $(n-1)(\neg x) \approx x \Rightarrow x \approx 1$ holds in an MV-algebra $\mathbf{A}$ if and only if $\mathbf{A}$ does not contain $\mathbf{Ł}_n$. Therefore $\mathbb{K} : \mathbf{Ł}_n$ is axiomatized by:

$$\{\text{axioms of } \mathbb{K}\} \cup \{(n-1)(\neg x) \approx x \Rightarrow x \approx 1\}.$$

Therefore, since it is a quasiequational class it is a quasivariety. It is easy to see that the following properties hold

3.1 $(\mathbb{K} : \mathbf{Ł}_n) : \mathbf{Ł}_m = (\mathbb{K} : \mathbf{Ł}_m) : \mathbf{Ł}_n = \mathbb{K} : \mathbf{Ł}_n \cap \mathbb{K} : \mathbf{Ł}_m = \mathbb{K} : \mathbf{Ł}_n, \mathbf{Ł}_m$.

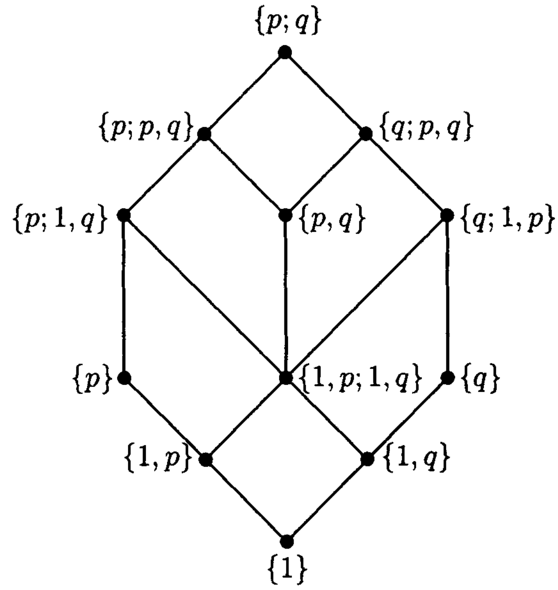
3.2 $\mathbb{K} : \mathbf{Ł}_n = \mathbb{K}$ *if and only if* $\mathbf{Ł}_n \notin \mathbb{K}$.

Our first example is to identify all quasivarieties contained in $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q)$, where p and q are two distinct prime natural numbers. Since the only divisors of p and q are $1, p, q$, by the characterization given in Theorem 2.11, we have that all critical MV-algebras contained in $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q)$ are

$$\mathbb{I}(\{\mathbf{Ł}_1, \mathbf{Ł}_p, \mathbf{Ł}_q, \mathbf{Ł}_1 \times \mathbf{Ł}_p, \mathbf{Ł}_1 \times \mathbf{Ł}_q, \mathbf{Ł}_p \times \mathbf{Ł}_q\}).$$

All subquasivarieties of $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q)$ are sketched in figure 1.

Figure 1: Lattice of all quasivarieties contained in $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q)$.



$\{n_1, \ldots, n_l; m_1, \ldots, m_k\}$ stands for $\mathbb{Q}(\mathbf{Ł}_{n_1} \times \cdots \times \mathbf{Ł}_{n_l}, \mathbf{Ł}_{m_1} \times \cdots \times \mathbf{Ł}_{m_k})$.

We want to give an effective axiomatization of all subquasivarieties of $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q)$. We know that the subvarieties are $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q)$, $\mathbb{V}(\mathbf{Ł}_p)$, $\mathbb{V}(\mathbf{Ł}_q)$ and $\mathbb{V}(\mathbf{Ł}_1)$, which can be axiomatized by MV1,...,MV6 plus $v_p(x) \vee v_q(x) \approx 1$, $v_p(x) \approx 1$, $v_q(x) \approx 1$ and $v_1(x) \approx 1$ respectively. In order to give axiomatizations for the other quasivarieties we state:

Theorem 3.3 *The following equalities hold:*

1. $\mathbb{Q}(\{\mathbf{Ł}_p, \mathbf{Ł}_p \times \mathbf{Ł}_q\}) = \mathbb{V}(\{\mathbf{Ł}_p, \mathbf{Ł}_q\}) : \mathbf{Ł}_q$.
2. $\mathbb{Q}(\{\mathbf{Ł}_q, \mathbf{Ł}_p \times \mathbf{Ł}_q\}) = \mathbb{V}(\{\mathbf{Ł}_p, \mathbf{Ł}_q\}) : \mathbf{Ł}_p$.
3. $\mathbb{Q}(\mathbf{Ł}_p \times \mathbf{Ł}_q) = \mathbb{V}(\{\mathbf{Ł}_p, \mathbf{Ł}_q\}) : \mathbf{Ł}_p, \mathbf{Ł}_q$.
4. $\mathbb{Q}(\mathbf{Ł}_1 \times \mathbf{Ł}_p) = \mathbb{V}(\mathbf{Ł}_p) : \mathbf{Ł}_p$.
5. $\mathbb{Q}(\mathbf{Ł}_1 \times \mathbf{Ł}_q) = \mathbb{V}(\mathbf{Ł}_q) : \mathbf{Ł}_q$.

Proof : 1) Since $\mathbf{Ł}_q$ is not contained in $\mathbf{Ł}_p$ and $\mathbf{Ł}_p \times \mathbf{Ł}_q$, then $\mathbb{Q}(\{\mathbf{Ł}_p, \mathbf{Ł}_p \times \mathbf{Ł}_q\}) \subseteq \mathbb{V}(\{\mathbf{Ł}_p, \mathbf{Ł}_q\}) : \mathbf{Ł}_q$. By 3.2, $\mathbb{V}(\{\mathbf{Ł}_p, \mathbf{Ł}_q\}) : \mathbf{Ł}_q$ is a proper subquasivariety of $\mathbb{V}(\{\mathbf{Ł}_p, \mathbf{Ł}_q\})$ and it is a locally finite subquasivariety. Hence, it suffices to show that all critical

MV-algebras of $\mathbb{V}(\{\mathbf{Ł}_p, \mathbf{Ł}_q\})$ not containing $\mathbf{Ł}_q$ belong to $\mathbb{Q}(\{\mathbf{Ł}_p, \mathbf{Ł}_p \times \mathbf{Ł}_q\})$. The only critical MV-algebras of $\mathbb{V}(\{\mathbf{Ł}_p, \mathbf{Ł}_q\})$ not containing $\mathbf{Ł}_q$, up to isomorphism, are

$$\{\mathbf{Ł}_1, \mathbf{Ł}_p, \mathbf{Ł}_1 \times \mathbf{Ł}_p, \mathbf{Ł}_1 \times \mathbf{Ł}_q, \mathbf{Ł}_p \times \mathbf{Ł}_q\} \subseteq \mathbb{Q}(\{\mathbf{Ł}_p, \mathbf{Ł}_p \times \mathbf{Ł}_q\})$$

2) can be obtained as 1).
3) follows from 1), 2) and 3.1.
4) Since $\mathbf{Ł}_1 \times \mathbf{Ł}_p \in \mathbb{V}(\mathbf{Ł}_p)$ and $\{\mathbf{Ł}_1, \mathbf{Ł}_1 \times \mathbf{Ł}_p\}$ are, up to isomorphism, the only critical MV-algebras in $\mathbb{V}(\mathbf{Ł}_p)$ which do not contain $\mathbf{Ł}_p$, we have $\mathbb{Q}(\mathbf{Ł}_1 \times \mathbf{Ł}_p) = \mathbb{V}(\mathbf{Ł}_p) : \mathbf{Ł}_p$.
5) is obtained as 4). □

Using the above theorem and Lemma 1.2, we can obtain an effective axiomatization of all quasivarieties listed above. We still have three subquasivarieties of $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q)$ without an effective axiomatization. Next result takes care of that.

Theorem 3.4 *If $r = \max\{p, q\}$, then*

1. *$\mathbb{Q}(\mathbf{Ł}_p, \mathbf{Ł}_1 \times \mathbf{Ł}_q)$ is axiomatized by the axioms of $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q) : \mathbf{Ł}_q$ and*

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow v_p(y) \approx 1.$$

2. *$\mathbb{Q}(\mathbf{Ł}_q, \mathbf{Ł}_1 \times \mathbf{Ł}_p)$ is axiomatized by the axioms of $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q) : \mathbf{Ł}_p$ and*

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow v_q(y) \approx 1.$$

3. *$\mathbb{Q}(\mathbf{Ł}_1 \times \mathbf{Ł}_p, \mathbf{Ł}_1 \times \mathbf{Ł}_q)$ is axiomatized by the axioms of $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q) : \mathbf{Ł}_q, \mathbf{Ł}_p$ and*

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow x \approx 1.$$

Proof : 1) Since $\mathbf{Ł}_p, \mathbf{Ł}_1 \times \mathbf{Ł}_q$ do not contain $\mathbf{Ł}_q$, then they satisfy all axioms of $\mathbb{V}(\mathbf{Ł}_p, \mathbf{Ł}_q) : \mathbf{Ł}_q$. Moreover, since for any $a \in \mathrm{Ł}_1 \times \mathrm{Ł}_q$,

$$ra = (1,1) \ \text{ implies } r(\neg a) \neq (1,1),$$

the quasiequation

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow v_p(y) \approx 1$$

is valid in $\mathbf{Ł}_1 \times \mathbf{Ł}_q$. Since $v_p(y) \approx 1$ is valid in $\mathbf{Ł}_p$, then $\mathbf{Ł}_p$ is also model of

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow v_p(y) \approx 1.$$

Therefore, all members of $\mathbb{Q}(\mathbf{Ł}_p, \mathbf{Ł}_1 \times \mathbf{Ł}_q)$ are models of the above quasiequation. Since the class of all models of the quasiequation

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow v_p(y) \approx 1$$

in $\mathbb{V}(\text{Ł}_p, \text{Ł}_q) : \text{Ł}_q$ is a locally finite quasivariety of MV-algebras, it suffices to show that all critical MV-algebras of $\mathbb{V}(\text{Ł}_p, \text{Ł}_q) : \text{Ł}_q$ satisfying the quasiequation are included in $\mathbb{Q}(\text{Ł}_p, \text{Ł}_1 \times \text{Ł}_q)$. The critical MV-algebras of $\mathbb{V}(\text{Ł}_p, \text{Ł}_q) : \text{Ł}_q$ are isomorphic to $\{\text{Ł}_1, \text{Ł}_p, \text{Ł}_1 \times \text{Ł}_p, \text{Ł}_1 \times \text{Ł}_q, \text{Ł}_p \times \text{Ł}_q\}$. Observe that they all satisfy the quasiequation, but $\text{Ł}_p \times \text{Ł}_q$. Take for instance $x = y = (\frac{p-1}{p}, \frac{q-1}{q})$. And since $\{\text{Ł}_1, \text{Ł}_p, \text{Ł}_1 \times \text{Ł}_p, \text{Ł}_1 \times \text{Ł}_q\} \subseteq \mathbb{Q}(\text{Ł}_p, \text{Ł}_1 \times \text{Ł}_q)$, 1) is proved.
2) is proved as 1).
3) Since $\text{Ł}_1 \times \text{Ł}_p, \text{Ł}_1 \times \text{Ł}_q \in \mathbb{V}(\text{Ł}_p, \text{Ł}_q) : \text{Ł}_q, \text{Ł}_p$ and there is no element a in $\text{Ł}_1 \times \text{Ł}_p$ or in $\text{Ł}_1 \times \text{Ł}_q$ which satisfies $ra = (1,1)$ and $r(\neg a) = (1,1)$, then $\text{Ł}_1 \times \text{Ł}_p, \text{Ł}_1 \times \text{Ł}_q$ satisfy the axioms.
Using a similar argument as in 1), we can show that $\{\text{Ł}_1, \text{Ł}_1 \times \text{Ł}_p, \text{Ł}_1 \times \text{Ł}_q\}$ are, up to isomorphism, the only critical MV-algebras in $\mathbb{V}(\text{Ł}_p, \text{Ł}_q) : \text{Ł}_q, \text{Ł}_p$ satisfying the quasiequation

$$rx \approx 1 \;\&\; r(\neg x) \approx 1 \Rightarrow x \approx 1,$$

since they belong to $\mathbb{Q}(\{\text{Ł}_1 \times \text{Ł}_p, \text{Ł}_1 \times \text{Ł}_q\})$, 3) is proved. □

Our second example are all quasivarieties contained in $\mathbb{V}(\text{Ł}_{p^r})$, where p is a prime natural number and r is a natural number.

Theorem 3.5 *The class of all critical algebras in $\mathbb{V}(\text{Ł}_{p^r})$ is*

$$\mathbb{I}\Big(\{\text{Ł}_{p^s} : s \leq r\} \cup \{\text{Ł}_{p^n} \times \text{Ł}_{p^m} : n < m \leq r\}\Big).$$

Proof : Since Ł_{p^s} with $s \leq r$ and $\text{Ł}_{p^n} \times \text{Ł}_{p^m}$ with $n < m \leq r$ belong to $\mathbb{V}(\text{Ł}_{p^r})$ and they satisfy conditions given in Theorem 2.11, they are critical.
To prove the other inclusion, consider $\mathbf{A} \in \mathbb{V}(\text{Ł}_{p^r})$ a critical MV-algebra. All divisors of p^r are p^s with $s \leq r$, hence, by condition 1) of Theorem 2.11, $\mathbf{A}$ is isomorphic to

$$\text{Ł}_{p^{n_1}} \times \cdots \times \text{Ł}_{p^{n_k}}$$

such that $n_i < n_j$ if $i < j$ and $n_i \leq r$ for all $1 \leq i \leq k$.
If $k > 2$, then $\mathbf{A}$ does not satisfy condition 2) of Theorem 2.11, therefore it would not be critical. Hence $k = 1$ or $k = 2$. □

From 2.10 we deduce the following properties:

Lemma 3.6 *Let $n_1, \ldots, n_k < \omega$ and $m_1, , \ldots, n_k < \omega$ such that $n_i < m_i$ for every $1 \leq i \leq k$.*

1. *$\text{Ł}_{p^{n_i}} \times \text{Ł}_{p^{m_i}} \subseteq \text{Ł}_{p^{n_j}} \times \text{Ł}_{p^{m_j}}$ if and only if $n_i \leq n_j$ and $m_i \leq m_j$.*
2. *$\text{Ł}_{p^{n_i}} \times \text{Ł}_{p^{m_i}} \subseteq \text{Ł}_{p^s} \times \text{Ł}_{p^s}$ if and only if $m_i \leq s$.*
3. *$\text{Ł}_{p^s} \subseteq \text{Ł}_{p^{n_j}} \times \text{Ł}_{p^{m_j}}$ if and only if $s \leq n_j$.* □

Theorem 3.7 *Every subquasivariety of $\mathbb{V}(\text{Ł}_{p^r})$ is of type:*

1. $\mathbb{Q}(\mathbf{Ł}_{p^s}) = \mathbb{V}(\mathbf{Ł}_{p^s})$ *where* $s \leq r$.

2. $\mathbb{Q}(\mathbf{Ł}_{p^{n_1}} \times \mathbf{Ł}_{p^{m_1}}, \ldots, \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}})$ *such that* $n_i < m_i \leq r$, *for every* $1 \leq i \leq k$ *and* $n_i < n_j$ *and* $m_i > m_j$ *if* $i < j$.

3. $\mathbb{Q}(\mathbf{Ł}_{p^{n_1}} \times \mathbf{Ł}_{p^{m_1}}, \ldots, \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}}, \mathbf{Ł}_{p^s})$ *such that* $n_i < s < m_i \leq n$, *for every* $1 \leq i \leq k$ *and* $n_i < n_j$ *and* $m_i > m_j$ *if* $i < j$.

Proof : Let $\mathbb{K}$ be a subquasivariety of $\mathbb{V}(\mathbf{Ł}_{p^r})$, by Theorem 2.13, $\mathbb{K}$ is finitely generated by critical MV-algebras, say $\mathbb{K} = \mathbb{Q}(\mathbf{A}_1, \ldots, \mathbf{A}_k)$. By Theorem 3.5, for every $1 \leq i \leq k$

$$\mathbf{A}_i \cong \mathbf{Ł}_{p^s} \text{ with } s \leq r \quad \text{or} \quad \mathbf{A}_i \cong \mathbf{Ł}_{p^n} \times \mathbf{Ł}_{p^m} \text{ with } n < m \leq r.$$

Therefore $\mathbb{K} = \mathbb{Q}(\{\mathbf{Ł}_{p^{s_1}}, \ldots, \mathbf{Ł}_{p^{s_j}}, \mathbf{Ł}_{p^{n_{j+1}}} \times \mathbf{Ł}_{p^{m_{j+1}}}, \ldots, \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}}\})$. Let

$$\mathfrak{G} = \{\mathbf{Ł}_{p^{s_1}}, \ldots, \mathbf{Ł}_{p^{s_j}}, \mathbf{Ł}_{p^{n_{j+1}}} \times \mathbf{Ł}_{p^{m_{j+1}}}, \ldots, \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}}\}.$$

We give a recursive procedure to find a subfamily of generators satisfying our conditions and generating $\mathbb{K}$:
First step: Let $m'_1 = \max\{s_1, \ldots, s_j, m_{j+1}, \ldots, m_k\}$. If $m'_1 = s_l$ for some $1 \leq l \leq j$ then, by Lemma 3.6 and Theorem 1.1,

$$\{\mathbf{Ł}_{p^{s_1}}, \ldots, \mathbf{Ł}_{p^{s_j}}, \mathbf{Ł}_{p^{n_{j+1}}} \times \mathbf{Ł}_{p^{m_{j+1}}}, \ldots, \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}}\} \subseteq \mathbb{Q}(\mathbf{Ł}_{p^{s_l}}).$$

Therefore $\mathbb{K} = \mathbb{Q}(\mathbf{Ł}_{p^{s_l}}) = \mathbb{V}(\mathbf{Ł}_{p^{s_l}})$.
Second step: If $m'_1 \neq s_l$ for every $1 \leq l \leq j$, let $n'_1 = \max\{n_i : m_i = m'_1\}$. Consider $\mathfrak{G}_1 = \{\mathbf{Ł}_{p^s} \in \mathfrak{G} : s > n'_1\} \cup \{\mathbf{Ł}_{p^n} \times \mathbf{Ł}_{p^m} \in \mathfrak{G} : n > n'_1\}$. Observe that if $\mathbf{Ł}_{p^s} \in \mathfrak{G}_1$, then $n'_1 < s < m'_1$ and that if $\mathbf{Ł}_{p^n} \times \mathbf{Ł}_{p^m} \in \mathfrak{G}_1$, then $n'_1 < n$ and $m < m'_1$. Thus, by Lemma 3.6, $\mathfrak{G} \setminus \mathfrak{G}_1 \subseteq \mathbb{Q}(\mathbf{Ł}_{p^{n'_1}} \times \mathbf{Ł}_{p^{m'_1}})$. Therefore

$$\mathbb{K} = \mathbb{Q}(\{\mathbf{Ł}_{p^{n'_1}} \times \mathbf{Ł}_{p^{m'_1}}\} \cup \mathfrak{G}_1) = \mathbb{Q}(\mathbf{Ł}_{p^{n'_1}} \times \mathbf{Ł}_{p^{m'_1}}) \vee \mathbb{Q}(\mathfrak{G}_1).$$

Since $\mathbf{Ł}_{p^{n'_1}} \times \mathbf{Ł}_{p^{m'_1}} \in \mathfrak{G} \setminus \mathfrak{G}_1$, then $|\mathfrak{G}_1| < |\mathfrak{G}|$.
At this point, we come back to first step, but using $\mathfrak{G}_1$ instead of $\mathfrak{G}$. And we proceed recursively.
This algorithm ends because $\mathfrak{G}$ is finite. At the end, we will have a finite sequence $\mathbf{Ł}_{p^{n'_1}} \times \mathbf{Ł}_{p^{m'_1}}, \ldots, \mathbf{Ł}_{p^{n'_l}} \times \mathbf{Ł}_{p^{m'_l}}$ and probably $\mathbf{Ł}_{p^{m'_{l+1}}}$ such that they satisfy conditions given in the statement of this theorem. □

Finally, we give an axiomatization for each subquasivariety of $\mathbb{V}(\mathbf{Ł}_{p^r})$.

Theorem 3.8

1. $\mathbb{V}(\mathbf{Ł}_{p^s})$ *with* $s \leq r$, *is axiomatized by*

(a) MV1,...,MV6 *and*

(b) $v_{p^s}(x) \approx 1$.

2. $\mathbb{Q}(\mathbf{Ł}_{p^{n_1}} \times \mathbf{Ł}_{p^{m_1}}, \ldots, \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}})$ *such that* $n_i < m_i \leq n$, *for every* $1 \leq i \leq k$ *and* $n_i < n_j$ *and* $m_i > m_j$ *if* $i < j$, *is axiomatized by:*

   (a) MV1,...,MV6,

   (b) $v_{p^{m_1}}(x) \approx 1$,

   (c) $(p^{n_k+1} - 1)(\neg x) \approx x \Rightarrow x \approx 1$ *and*

   (d) $(p^{n_{j-1}+1} - 1)(\neg x) \approx x \Rightarrow v_{p^{m_j}}(y) \approx 1$ *for every* $2 \leq j \leq k$.

3. $\mathbb{Q}(\mathbf{Ł}_{p^{n_1}} \times \mathbf{Ł}_{p^{m_1}}, \ldots, \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}}, \mathbf{Ł}_{p^s})$ *such that* $n_i < s < m_i \leq n$, *for every* $1 \leq i \leq k$ *and* $n_i < n_j$ *and* $m_i > m_j$ *if* $i < j$, *is axiomatized by:*

   (a) MV1,...,MV6,

   (b) $v_{p^{m_1}}(x) \approx 1$,

   (c) $(p^{s+1} - 1)(\neg x) \approx x \Rightarrow x \approx 1$,

   (d) $(p^{n_{j-1}+1} - 1)(\neg x) \approx x \Rightarrow v_{p^{m_j}}(y) \approx 1$ *for every* $2 \leq j \leq k$ *and*

   (e) $(p^{n_k+1} - 1)(\neg x) \approx x \Rightarrow v_{p^s}(y) \approx 1$.

Proof : 1) It is obvious by the remark given at the beginning of this section.
2) Since $\mathfrak{G} = \{\mathbf{Ł}_{p^{n_1}} \times \mathbf{Ł}_{p^{m_1}}, \ldots, \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}}\} \subseteq \mathbb{V}(\mathbf{Ł}_{p^{m_1}})$, the members of $\mathbb{Q}(\mathfrak{G})$ are models of (a) and (b).
Moreover, $\mathbf{Ł}_{p^{n_k+1}} \not\subseteq \mathbf{A}$ for each $\mathbf{A} \in \mathfrak{G}$, hence $\mathfrak{G} \subseteq \mathbb{V}(\mathbf{Ł}_{p^r}) : \mathbf{Ł}_{p^{n_k+1}}$ and every algebra in $\mathfrak{G}$ is a model of (c).
By Lemma 1.2, if $\mathbf{A} \in \mathfrak{G}$ and it does not contain $\mathbf{Ł}_{p^{n_{j-1}+1}}$, then it is a model of (d). On the other hand, if $\mathbf{A} \in \mathfrak{G}$ is such that $\mathbf{Ł}_{p^{n_{j-1}+1}} \subseteq \mathbf{A}$, then it belongs to the variety generated by $\mathbf{Ł}_{p^{m_j}}$, therefore it also satisfies (d).
Since every member in $\mathfrak{G}$ is a model of the equations and quasiequations $(a), (b), (c)$ and (d), each member of $\mathbb{Q}(\mathfrak{G})$ is also a model of $(a), (b), (c)$ and (d).
To see the other inclusion, observe that (a) and (b) axiomatize $\mathbb{V}(\mathbf{Ł}_{p^{m_1}})$, hence the class of all models of $(a), (b), (c)$ and (d) is a subquasivariety of $\mathbb{V}(\mathbf{Ł}_{p^{m_1}})$. Thus, it is a locally finite quasivariety of MV-algebras. By Theorem 2.9, it suffices to show that every critical MV-algebra in $\mathbb{V}(\mathbf{Ł}_{p^{m_1}})$ satisfying (c) and (d) belongs to $\mathbb{Q}(\mathfrak{G})$.
Let $\mathbf{A} \in \mathbb{V}(\mathbf{Ł}_{p^{m_1}})$ be a critical MV-algebra, by Theorem 3.5,

$$\mathbf{A} \cong \mathbf{Ł}_{p^s} \text{ with } s \leq m_1 \quad \text{or} \quad \mathbf{A} \cong \mathbf{Ł}_{p^n} \times \mathbf{Ł}_{p^m} \text{ with } n < m \leq m_1.$$

If $\mathbf{A}$ satisfies (c), then $\mathbf{Ł}_{p^{n_k+1}} \not\subseteq \mathbf{A}$. Hence

$$\mathbf{A} \cong \mathbf{Ł}_{p^s} \text{ with } s \leq n_k \quad \text{or} \quad \mathbf{A} \cong \mathbf{Ł}_{p^n} \times \mathbf{Ł}_{p^m} \text{ with } n < m \leq m_1 \text{ and } n \leq n_k.$$

$\mathbf{A} \cong \mathbf{Ł}_{p^s}$, with $s \leq n_k$, implies $\mathbf{Ł}_{p^s} \subseteq \mathbf{Ł}_{p^{n_k}} \times \mathbf{Ł}_{p^{m_k}}$. Thus $\mathbf{A} \in \mathbb{Q}(\mathfrak{G})$.
If $\mathbf{A} \cong \mathbf{Ł}_{p^n} \times \mathbf{Ł}_{p^m}$, with $n < m \leq m_1$ and $n \leq n_k$, we have several possibilities:
- $n \leq n_1$: since $m \leq m_1$, by Lemma 3.6, we have $\mathbf{Ł}_{p^n} \times \mathbf{Ł}_{p^m} \subseteq \mathbf{Ł}_{p^{n_1}} \times \mathbf{Ł}_{p^{m_1}}$. Hence, $\mathbf{A} \in \mathbb{Q}(\mathfrak{G})$.
- $n_1 < n \leq n_k$: there is a unique $2 \leq j \leq k$ such that $n_{j-1} < n \leq n_j$. Hence, $\mathbf{Ł}_{p^{n_{j-1}+1}} \subseteq \mathbf{A}$. Since $\mathbf{A}$ is a model of (d) and $\mathbf{Ł}_{p^{n_{j-1}+1}} \subseteq \mathbf{A}$, we have that $\mathbf{A}$ is a model of the equation $v_{p^{m_j}}(y) \approx 1$. Thus $\mathbf{A} \in \mathbb{V}(\mathbf{Ł}_{p^{m_j}})$ and $m \leq m_j$. Moreover, $n \leq n_j$ so, by Lemma 3.6, $\mathbf{Ł}_{p^n} \times \mathbf{Ł}_{p^m} \subseteq \mathbf{Ł}_{p^{n_j}} \times \mathbf{Ł}_{p^{m_j}}$ and $\mathbf{A} \in \mathbb{Q}(\mathfrak{G})$.
3) can be proved similarly to 2). □

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